RBF Meshfree Methods For The Black-Scholes PDE in Option Pricing

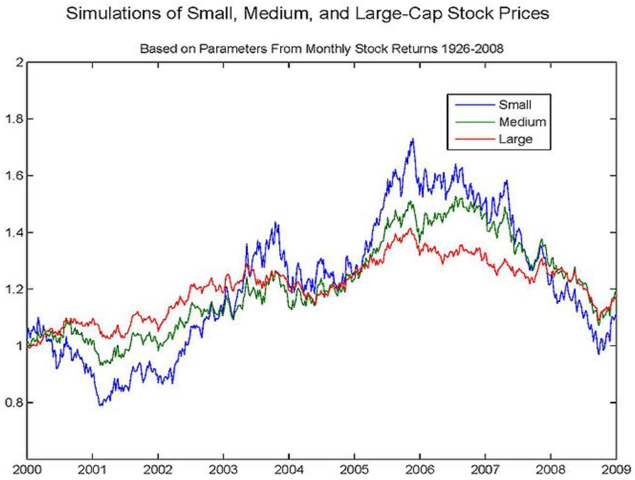
Nana Akwasi Abayie Boateng

Middle Tennessee State University

Thesis Advisor:
Abdul Q. M. Khaliq
Middle Tennessee State University

Outline

stock prices over time



$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + rs\frac{\partial v}{\partial s} - rv = 0$$

In 1997,the Nobel Price in Economics was awarded for the work that led to Black-Scholes option pricing theory.The goal of the theory of option pricing is to be able to compute a fair price for the option.

Black and Scholes,*The pricing of options and corporate liabilities*, J. pol. Econ.81,pp 637-659,1973

what are options

- An Option is a derivative product that represents a contract, which gives the holder or buyer the right to buy or sell at a prescribed price(the strike or exercise price) over a certain period of time or on a prescribed date(exercise date).
- Call Option
  Have option to buy asset S for price E at date T in the future.(Person who buys makes the choice)
- Put Option
  Have option to sell asset S for price E at date T in the future.(Person who sells makes the choice)

Standard Options

- European Option
  Option to buy or sell can be exercised only at date T

- American Option
  Option to buy or sell can be exercised once at anytime up to and including T

Brief History

- Black and Scholes in 1973 observed that the price of European call options by assuming a risk-neutrality of the underlying asset price,satisfies a lognormal partial differential equation of diffusiion type which is backward in time.
- The analytical solution of the European option exist.However,there is no analytical solution for the American option because of the free boundary condition(early exercise constraint).

Radial Basis Functions

- Let $x_1, x_2, ...x_N$ be a given set of distinct points in $\mathrm{R}^d, d \geq 1$.The basic idea behind the use of RBF's is that we interpolate the function by a linear combination of RBF's of the type as follows

$$F(x) = \sum_{j=1}^{N} c_j \phi(\|x - x_j\|)$$

where $\|.\|$ denotes the Euclidean norm,c_j are unknown scalars and ϕ denotes the radial basis function.

some well-known radial basis functions

- Multiquadratic $=\sqrt{r^2 + c^2}$
- Inverse Multiquadratic$=\frac{1}{\sqrt{r^2+c^2}}$
- Gaussian$=e^{-(cr)^2}$
- Cubic$=|r|^3$
- Thin plate spline$=r^2 \ln |r|$, where r denotes the distance between two nodes

Advantages

- Require neither domain nor surface discretization
- The formulation is similar for 2D and 3D problems
- Ease of learning
- Cost effective due to man-power reduction involved for the meshing

Disadvantages

- Ill-conditioning
- Dense matrices

Pricing the European Option

- $\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + rs\frac{\partial v}{\partial s} - rv = 0$
- r is the risk -free interest rate
- σ is the volatility of the stock
- $v(S, t)$ is the option value at time t and stock price S

Final conditions

$$v(s, T) = \begin{cases} max(E - S, 0), & \text{for a put} \\ max(S - E, 0), & \text{for a call} \end{cases}$$

E is the exercise price and T is the time to maturity

Boundary Condition

$$V(0,t) = E\exp(-r(T-t)) \quad V(S,t) \to 0 \quad \text{as} \quad S \to \infty \quad \text{for put}$$

$$V(0,t) = 0 \quad V(S,t) \to S \quad \text{as} \quad S \to \infty \quad \text{for call}$$

Analytical Solution(Black-Scholes Formula)

$$V(S,t) = E\exp(-r(T-t))N(-d_2) - SN(-d_1) \quad \text{for put}$$

$$V(S,t) = SN(d_1) - E\exp(-r(T-t))N(d_2) \quad \text{for call}$$

where N(.) is the cumulative distribution function of the standard normal distribution with

$$d_1 = \frac{\ln(\frac{S}{E})+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(\frac{S}{E})+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

American Option

In the region $0 \leq S \leq S_f(t)$

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP < 0$$

$$P = E - S$$

In the other region,$S_f(t) < S < \infty$

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0$$

$$P > E - S$$

Final and boundary conditions

$$P(S,T) = \max(E-S,0), \quad S \geq 0$$

$$\frac{\partial P}{\partial S}(S_f,t) = -1,$$

$$P(S_f(t),t) = max(E-S_f(t),0)$$

$$\lim_{S\to\infty} P(S,t) = 0$$

$$S_f(T) = E$$

$$P(S,t) = E-S, \quad 0 \leq S < S_f(t)$$

$$P(S,t) - max(E-S,0) \geq 0, \quad S \geq 0, \quad 0 \leq t \leq T$$

RBF Approximation

We consider a penalty method approach(Nielson *et al*(2002))

$$\frac{\partial P_\epsilon}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 P_\epsilon}{\partial S^2} + rs\frac{\partial P_\epsilon}{\partial S} - rP_\epsilon + \frac{\epsilon C}{P_\epsilon + \epsilon - q(s)} = 0$$

$$P(S,T) = max(E - S, O)$$

$$P(0,t) = E$$

$$P(S_\infty, t) = 0$$

RBF Approximation

$$\mathrm{P}_\epsilon(S,t) = \sum_{j=1}^{N} c_j(t)\phi(\|S - x_j\|)$$

This results in the system of(nonlinear)ODEs for the coefficients c_j (collected in the vector $\mathbf{c}$)

$$\Phi\dot{\mathbf{c}} + R\mathbf{c} + Q_c = 0$$

Here $R = \frac{1}{2}\sigma^2\Phi''_s + r\Phi'_s - r\Phi$

Entries of the matrices Φ,Φ'_s, and Φ''_s are given by

$$\Phi_{ij} = \phi(\|x_i - x_j\|), \Phi'_{s,ij} = x_i\phi'(\|x_i - x_j\|), \Phi''_{s,ij} = x_i\phi''(\|x_i - x_j\|)$$

The Theta Method Time Stepping Approach

We will use θ-method($\theta = 0$) for the time stepping which leads to the folowing equation

$$\Phi \frac{c^{n+1} - c^n}{k} + \theta R c^{n+1} + (1-\theta) R c^n + \theta Q(c^{n+1}) + (1-\theta) Q(c^n) = 0$$

After collocation points at the points $x_i, i = 1, ..., N$ coefficients

$c_j(T)$ are given as the solution of the linear system

$$\Phi c(T) = P$$

where Φ is as above and

$$P = [P_\epsilon(x_1, T), ... P_\epsilon(x_N, T)]^T$$

Single Asset American Option

Parameter [4]	Value
Minimum Asset Price	$S_0 = 0$
Maximum Asset Price	$S_\infty = 2$
Number of asset data points	$N = 101$
Number of time steps	$M = 100$
Time-step size	$k = 0.01$
Initial Time	$t = 0$
Expiration Time	$T = 1(year)$
Exercise price	$E = 1$
Risk-free interest rate	$r = 0.1$
Volatility	$\sigma = 0.2$
Shape parameter for Gaussian RBF	$c = 1.5$
Shape parameter Multiquadric RBF	$c = 1.0$
Shape parameter Inverse-Multiquadric RBF	$c = 1.5$
Regularization parameter	$\epsilon = 0.01$

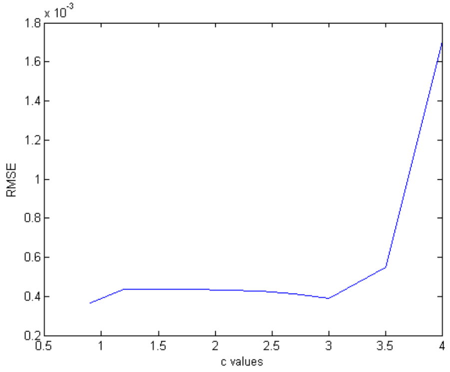


Figure: Comparison of RMSE for different c values using MQ-RBF at $N = 101$

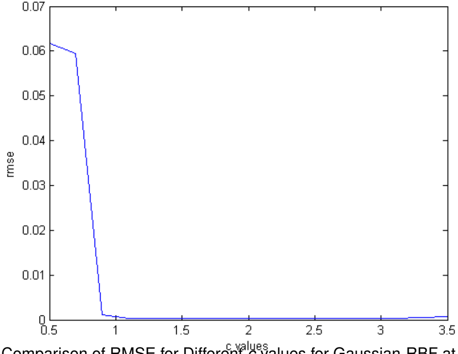


Figure: Comparison of RMSE for Different c values for Gaussian-RBF at $N = 101$ nodes

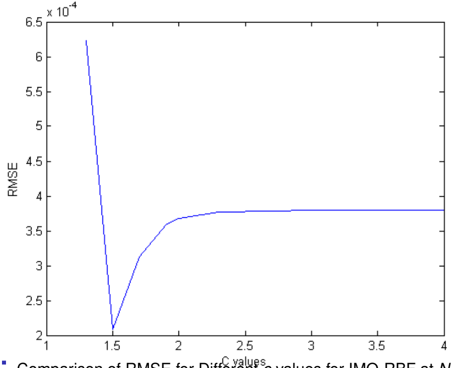


Figure: Comparison of RMSE for Different c values for IMQ-RBF at $N = 101$ nodes

S	FD 1001 [4]	RBF 41	RBF 81	RBF 101
0.6	0.4000037	4.000079284231482e-01	4.000137274388471e-01	4.000140155227794e-01
0.7	0.3001161	3.002190386780568e-01	3.002244584184045e-001	3.002248173328785e-01
0.8	0.2020397	2.026026872289588e-01	2.026217219929730e-01	2.026237471581654e-01
0.9	0.1169591	1.176250974697884e-01	1.176927181568679e-01	1.177005653395421e-01
1.0	0.0602833	5.952849811933896e-02	5.963331031716317e-02	5.964562793120096e-02
1.1	0.0293272	2.739145951270390e-02	2.748615753723662e-02	2.749734464846737e-02
1.2	0.0140864	1.210898358038832e-02	1.216998804034032e-02	1.217722986470464e-02
1.3	0.0070408	5.620025402083054e-03	5.650899377080217e-03	5.654603136082172e-03
1.4	0.0038609	3.034489511209617e-03	3.047863317966305e-03	3.049553028430248e-03
RMSE		3.800531000028380e-04	3.679472322155780e-04	3.665413983887557e-04
CPU time		7.800049999999814e-02	3.276021000000000e-01	9.516060999999993e-01
COND number		8.540711140948943e+03	9.371517158130049e+03	9.477187078272415e+03

Table: Values of American option at $t = 0$ using Gaussian-RBF with $c = 1.5$.

S	FD 1001 [4]	RBF 41	RBF 81	RBF 101
0.6	0.4000037	4.000136025947318e-01	4.000143544675322e-01	4.000144691188708e-01
0.7	0.3001161	3.002281861141670e-01	3.002328054598784e-01	3.002333928317463e-01
0.8	0.2020397	2.026990698419577e-01	2.027118053149325e-01	2.027132241451394e-01
0.9	0.1169591	1.180054049292558e-01	1.180378873827233e-01	1.180415024971369e-01
1.0	0.0602833	6.003360140092645e-02	6.011839322106172e-02	6.012808397308078e-02
1.1	0.0293272	2.777378229860611e-02	2.787819507876477e-02	2.789129963720338e-02
1.2	0.0140864	1.230516651466951e-02	1.240253754684245e-02	1.241824783091772e-02
1.3	0.0070408	5.682600710233188e-03	5.790463354382935e-03	5.817249701233540e-03
1.4	0.0038609	3.019767309641319e-03	3.204100755105077e-03	3.263693145839186e-03
RMSE		3.468133973660428e-04	3.255796482865938e-04	3.212911530746730e-04
CPU time		1.404009000000031e-01	2.652017000000058e-01	5.460034999999976e-01
COND number		1.350297755352479e+04	5.276831666999462e+04	8.204955512679130e+04

Table: Values of American option at $t = 0$ using Multiquadric-RBF with $c = 1.0$.

S	FD 1001 [4]	RBF 41	RBF 81	RBF 101
0.6	0.4000037	4.000081846731239e-01	3.999944677251256e-01	3.999848597023653e-01
0.7	0.3001161	3.002194386244130e-01	3.002110833603666e-01	3.002039340350107e-01
0.8	0.2020397	2.026110427716545e-01	2.026254392415477e-01	2.026245530783953e-01
0.9	0.1169591	1.176619409499281e-01	1.177464819756828e-01	1.177650983203513e-01
1.0	0.0602833	5.958535731817690e-02	5.973980242505235e-02	5.978150361976092e-02
1.1	0.0293272	2.745184583462177e-02	2.761876322846362e-02	2.767204819099156e-02
1.2	0.0140864	1.216693881001230e-02	1.231781400468144e-02	1.237667347724274e-02
1.3	0.0070408	5.680432372339327e-03	5.835817190838141e-03	5.909043172578131e-03
1.4	0.0038609	3.110299805121982e-03	3.332465656092971e-03	3.447089842679579e-03
RMSE		3.673986998156583e-04	3.341736920608062e-04	3.213851282534414e-04
CPU time		2.808017999999990e-01	1.060806800000002	1.388408899999998
COND number		1.511671497888172e+03	1.890239112015076e+03	2.011469801137463e+03

Table: Values of American option at $t = 0$ using Inverse-Multiquadric-RBF with $c = 1.5$.

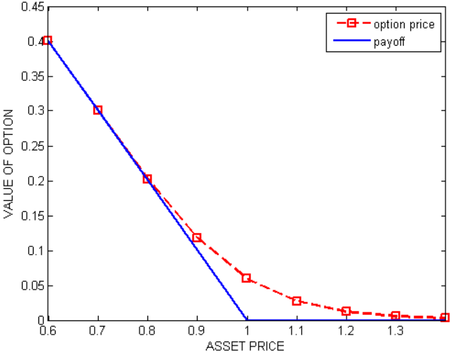


Figure: Payoff of the American put option using the Gaussian-RBF for $N = 101$ nodes

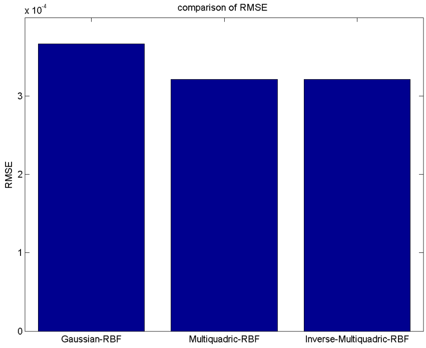


Figure: Comparison of RMSE between the 3 RBFs for $N = 101$

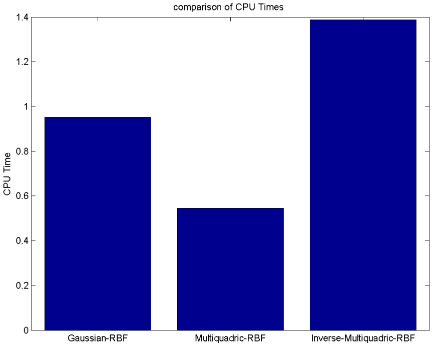


Figure: Comparison of CPU Times between the 3 RBFs for $N = 101$ nodes

Finite Difference Approximation

We replace the spatial derivatives in the Black-Scholes PDE by central difference approximation and use BE-ETD scheme for time stepping.

S	Option Value	FD1001 [4]
0.6	0.4001500	0.4000037
0.7	0.3002437	0.3001161
0.8	0.2049304	0.2020397
0.9	0.1288876	0.1169591
1.0	0.0746978	0.0602833
1.1	0.0399911	0.0293272
1.2	0.0199422	0.0140864
1.3	0.0093622	0.0070408
1.4	0.0041843	0.0038609
CPUTIME	7.2540	
RMSE	0.0075	

Table: RMSE for Finite Difference solution at $N = 2001$,$\epsilon = 10^{-4}$, $k = 0.001$.FD1001 is the reference solution obtained by Fausshauer and Khaliq(2004)[4] using a higher order difference approximation

Finite Difference Solutions

S	Option Value	FD1001 [4]
0.6	0.4015118	0.4000037
0.7	0.3028086	0.3001161
0.8	0.2100498	0.2020397
0.9	0.1334426	0.1169591
1.0	0.0778183	0.0602833
1.1	0.0419259	0.0293272
1.2	0.0211123	0.0140864
1.3	0.0100855	0.0070408
1.4	0.0046513	0.0038609
CPUTIME	1.0920	
RMSE	0.0098	

Table: Finite Difference solution at $N = 2001, \epsilon = 10^{-3}, k = 0.01$.

Finite Difference Solutions

S	Option Value	FD1001 [4]
0.6	0.4134769	0.4000037
0.7	0.3191524	0.3001161
0.8	0.2295821	0.2020397
0.9	0.1513232	0.1169591
1.0	0.0913208	0.0602833
1.1	0.0513920	0.0293272
1.2	0.0277877	0.0140864
1.3	0.0149302	0.0038609
1.4	0.0082378	0.0038609
CPUTIME	0.312002	
RMSE	0.0216	

Table: Finite Difference solution at $N = 2001$,$\epsilon = 10^{-2}$,$k = 0.1$.

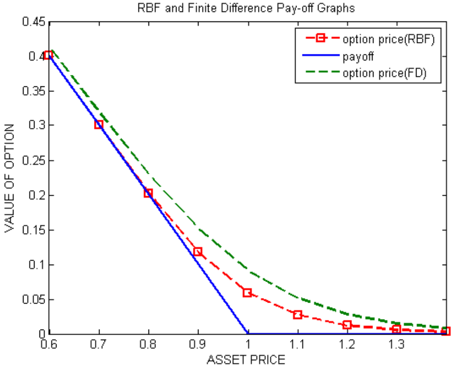


Figure: RBF $N = 101$, $k = 0.01$ and FD $N = 2001$,$k = 0.01$

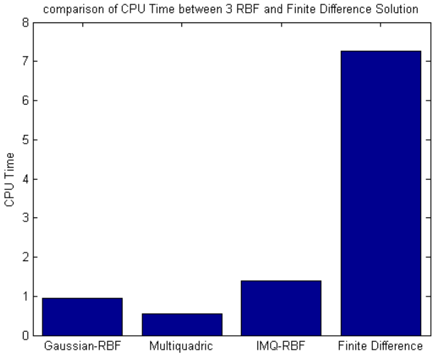


Figure: Comparison of CPU Times between the 3 RBFs for $N = 101$ and FD for $N = 2001$

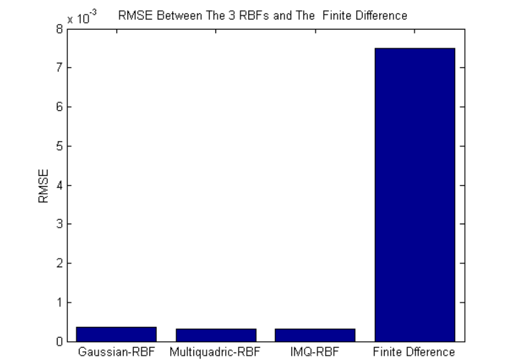


Figure: Comparison of RMSE between the 3 RBFs for $N = 101$ and FD for $N = 2001$

Conclusion

- We evaluated the value of American options using both radial basis functions interpolation approach and finite difference approximation scheme.
- We implemented numerical solutions from three different RBFs namely the Gaussisn,Multiquadric and Inverse-Multiquadric RBFs.
- The three RBFs implemented were considerably efficient and accurate even at a less number of nodes than the finite difference approximation scheme implemented.

Reference

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2. A.Q.M. Khaliq, D.A. Voss and S.H.K. Kazmi, *A linearly implicit predictor-corrector scheme for pricing American options using a penalty method approach*, Journal of Banking and Finance,30 (2006) 489-502.
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Thank you for your attention.Any questions?