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by

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APPROVAL

This is to certify that the Graduate Committee of

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ABSTRACT

Give a concise summary of your thesis. What is your research topic? why it is important and interesting? What problem you have tried to solve, how and what is your contribution.

DEDICATION

This thesis is dedicated to my parents, who has taught me, encouraged me and supported me in my life. Thanks for all your patience, love and unconditional support.

ACKNOWLEDGMENTS

Take this opportunity to thank your advisor, your thesis committee, research collaborators and anyone who helped you in the process of thesis accomplishment.

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CHAPTER 1

INTRODUCTION

Introduction is a very important part of your thesis. You can explain the background, give the notations and basic definitions used in later chapters, or review the recent development of your research topic. In addition, you can summary the contents of following chapters in your thesis and give a clear overview of what and how you have done in your thesis topic.

1.1 Sample Section

This is a sample section.

The study of algebraic structures using its associate graphs is a very exciting field which generates many fascinating results, conjectures and questions [**?**]. There are various ways to associate graphs to algebraic objects such as groups and rings. For instance, the prime graph defined in[**?**], the conjugacy class graph defined in[**?**], the non-commuting graph defined in[**?**], and the nonzero divisor graph defined in[**?**]

1.1.1 Sample Sub Section

This is a sample equation.

$$\frac{\partial}{\partial t}\int\int_{system}(V_{A_r}+V_{A_s})dxdy=0 \tag{1.1}$$

$$\frac{\partial}{\partial t}\int\int_{system}V_{B}dxdy=0 \tag{1.2}$$

CHAPTER 2

OPTION PRICING

An option is a financial contract which gives the holder of the option the right to purchase or sell a prescribed asset at a prescribed time in the future known as the expiry date at a prescribed amount which the exercise or strike price[15]. The most common kinds of prescribed assets which are traded on financial markets are stocks,bonds,currency and commodities.An option is a derivative product because it is traded on an underlying asset.The holder of a call option makes profit if the price of the underlying asset rises on the market whereas the holder of a put option does so when the price of the underlying asset falls on the financial market.The two primary uses of option are for hedging and speculation[15]. There are numerous kinds of options which are traded on financial markets.Vanilla options are options which do not possess any special features or characteristics.Examples are the European and American options.Exotic options possess special features.Examples include Asian options,Barrier options,Basket options.In this We consider The European and American options.

2.1 The European Options

The European option is an option which can only be exercised at its maturity time.The exact or analytical formula for estimating a fair price for European options exist. In 1973 Black and Scholes by making a set of explicit assumptions including the risk-neutrality of the underlying asset price showed that the value European call option satisfies a backward -in-time lognormal partial differential equation of diffusion type which has come to be known as the Black-Scholes equation[1]. Let the $V(S,t)$ be the price of an option which is function of both asset price and time.This option satisfies the following Black-Scholes equation.

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0, \quad S > \overline{S}(t),\ 0 \le t < T. \tag{2.3}$$

where r is the risk-free interest interest rate,σ is the volatility of the asset price,S is the asset price.The Final condition is given by[15]

$$V(S,t) = \begin{cases} max\{E - X, 0\} & \text{for a put option} \\ max\{S - E, 0\} & \text{for a call option} \end{cases}$$

where E is the strike price.

The Boundary condition of the European call option is given as follows:

$$C(S,t) \sim S \quad asS \to \infty, \quad C(0,t) = 0. \tag{2.4}$$

where $C(S,t)$ is the value of the European call option satisfying 2.3. The Boundary condition at of the European put option is given as follows: where $P(S,t)$ is the value of the European put option satisfying equation 2.3 for a time dependent interest rate. Equation 2.3 can be transformed exponentially by making the substitution $S = e^y$ to

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial y^2} + (r - \frac{1}{2}\sigma^2 \frac{\partial U}{\partial y}) - rU = 0 \tag{2.5}$$

$$U(y,T) = \begin{cases} max\{E - e^y, 0\} & \text{for a put option} \\ max\{e^y - E, 0\} & \text{for a call option} \end{cases}$$

The analytical solution of the Black- Scholes partial differential equation 2.3 with corresponding final and initial conditions 2.2 and **??** with a constant volatility and interest rate for the European call option is given as[15]

$$C(S,t) = SN(d_1) - E\exp^{-r(T-t)} N(d_2) \tag{2.6}$$

where $N(.)$ is the cumulative distribution function for the standardized normal random variable given by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp^{-\frac{1}{2}y^2 dy} \tag{2.7}$$

The corresponding analytical solution of the European put option is given by

$$P(S,t) = E\exp^{-r(T-t)} N(-d_2) - SN(-d_1) \tag{2.8}$$

where

$$d_1 = \frac{\log(\frac{S}{E} + (r + \frac{1}{2}\sigma^2)(T-t))}{\sigma\sqrt{T-t}}$$
$$d_2 = \frac{\log(\frac{S}{E} + (r - \frac{1}{2}\sigma^2)(T-t))}{\sigma\sqrt{T-t}}$$

2.2 The American Options

The American option can be exercised at any time prior to expiry.The American option is complicated because at at each time t not only is one interested in the value of the option but also for each asset price S,whether it should be exercised or not.This creates a free boundary problem[15].At each time t there is a particular value of S which lies in the boundary between two regions:one where early exercise is optimal to the other where one should hold on to the option.The optimal exercise prices$_f(t)$ which in general depends on time is not known *priori* unlike the case of European options.The American option valuation can be uniquely specified by a set of constraints among which are the option value must be greater than or equal to the payoff function, the option value must be continuous function of S,replacing the Black-Scholes equation by an inequality and lastly making the derivative of the option with respect to the asset price(option delta) continuous. The value $V(S,t)$ of the American option satisfies the following inequality

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \leq 0 \tag{2.9}$$

The Final condition is at expiry time T given by[15]

$$V(S,t) = \begin{cases} max\{E - X, 0\} & \text{for a put option} \\ max\{S - E, 0\} & \text{for a call option} \end{cases}$$

where E is the strike price.

In the region$0 \leq S \leq S_f(t)$ where early exercise is optimal,the value of the American put option satisfies the following inequality

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP < 0 \tag{2.10}$$

and

$$P = E - S \tag{2.11}$$

In the other region,$S_f(t) < S < \infty$,early exercise is not optimal and the value of the American put option satisfies the Black-Scholes equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0 \tag{2.12}$$

and

$$P > E - S \tag{2.13}$$

The boundary condition at$S_f(t) = S$ are that P and its slope(delta) are continuous.

$$P(S_f(t), t) = max(E - S_f(t), 0) \tag{2.14}$$

$$\frac{\partial P}{\partial S}(S_f(t), t) = -1 \tag{2.15}$$

The boundary condition 2.14 determines the option value at the free boundary,whereas 2.15known as the *smooth pasting condition* determines the location of the free boundary and simultaneously maximizes the benefit to the holder whiles avoiding arbitrage. The value $C(S,t)$ of the American Call option satisfies the corresponding equality in the holding region $0 \leq S \leq S_f(t)$

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{2.16}$$

in the other region where early exercise is optimal $S_f(t) < S < \infty$,the value $C(S,t)$ of the American call option satisfies

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC < 0 \tag{2.17}$$

and

$$P = E - S \tag{2.18}$$

2.3 Penalty Method

We introduce a penalty term into the Black-Scholes equation, to obtain a parabolic nonlinear partial differential equation of the form[9]

$$\frac{\partial P_\epsilon}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P_\epsilon}{\partial S^2} + rS\frac{\partial P_\epsilon}{\partial S} - rP_\epsilon + \frac{{}_\epsilon C}{P_\epsilon + \epsilon - q(S)} = 0, \quad 0 \leq S \leq S_\infty,\ 0 \leq t < T. \tag{2.19}$$

Here $0 < \epsilon \ll 1$ is a small regularization parameter, $C \geq rE$ is a positive constant, and $q(S) = E - S$ is the barrier function. The value S_∞ is a (relatively very large) price for which the option is worthless. The following are terminal and boundary conditions which accompany the transformation

$$P_\epsilon(S,T) = \max(E-S,0) \tag{2.20}$$

$$P_\epsilon(0,t) = E, \tag{2.21}$$

$$P_\epsilon(S_\infty,t) = 0. \tag{2.22}$$

Lemma 2.1 *Assume that the weight coefficients* $w_j > 0$ *for* $j = 1, \cdots, p$ *and* $\mathbf{C}_{11}$ *is invertible. Then there is a solution* $\hat{\beta}$ *for the weighted elastic net such that*

$$\mathrm{sgn}(\hat{\beta}) = \mathrm{sgn}(\beta^*)$$

if and only if the following conditions hold:

$$|\mathbf{x}_j^T \mathbf{X}_S[(\mathbf{C}_{11} + \lambda_2 \mathbf{W}^2)^{-1}(\mathbf{C}_{11}\beta_S^* + \frac{\mathbf{X}_S^T \epsilon}{n} - \lambda_1 \mathbf{b}) - \beta_S^*] - \frac{\mathbf{x}_j^T \epsilon}{n}| \le \lambda_1 w_j, \text{ for } j \in S^c, \quad (*)$$

and

$$\mathrm{sgn}((\mathbf{C}_{11} + \lambda_2 \mathbf{W}^2)^{-1}(\mathbf{C}_{11}\beta_S^* + \frac{\mathbf{X}_S^T \epsilon}{n} - \lambda_1 \mathbf{b})) = \mathrm{sgn}(\beta_S^*). \quad (**)$$

Proof. Recall that $\mathbf{y} = \mathbf{X}\beta^* + \epsilon$, $\mathbf{W} = \mathrm{diag}[w_1, \cdots, w_p]$, and $\mathbf{b} = \mathbf{W}_S \mathrm{sgn}(\beta_S^*)$.

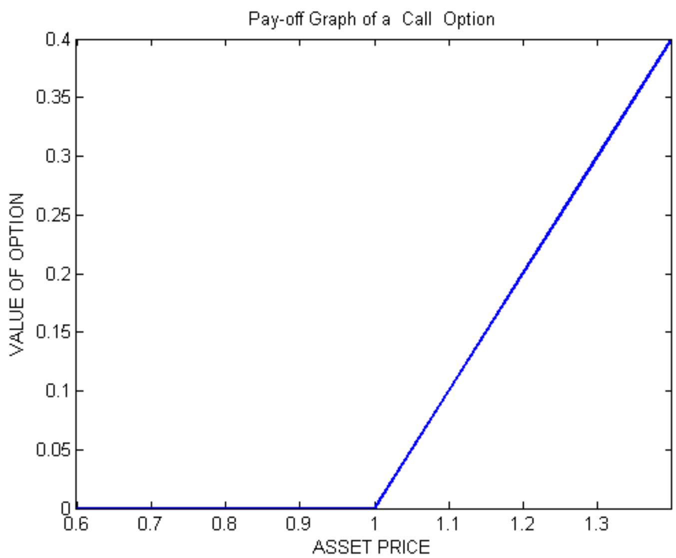

Figure 1: The pay-off of the European Call Option

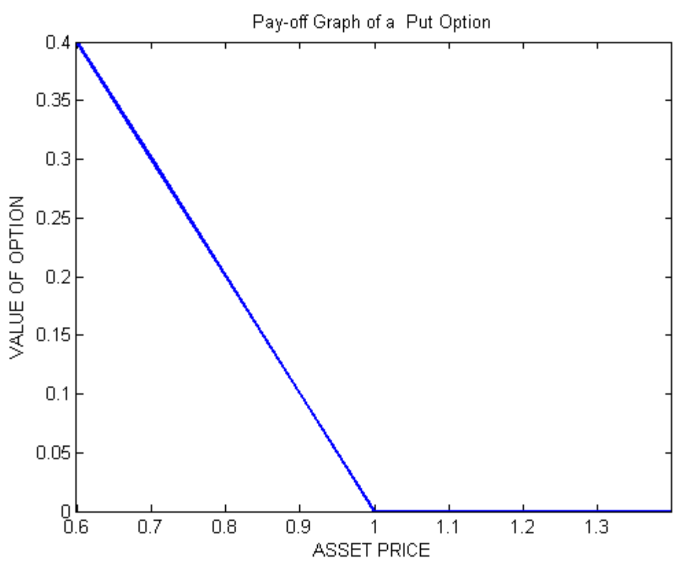

Figure 2: The pay-off of the European Put Option

CHAPTER 3

FINITE DIFFERENCE METHODS

3.1 Finite Difference Approximations

The method of Finite Difference Approximation which is based on Taylor series expansions of functions near the point of interest will be used to discretize the Black-Scholes partial differential equation[15]

$$\frac{\partial P}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial^2}+rS\frac{\partial P}{\partial S}-rP+\frac{{}_{\epsilon}C}{P_{\epsilon}+\epsilon-q(s)} \tag{3.23}$$

The first order partial derivative $\frac{\partial P}{\partial S}$ is approximated by central differencing with spatial step size h=$\frac{S_f-S_0}{N}$ and time step size$k$=$\frac{T_f-T_0}{M}$ as follows.We apply the ETD-BE scheme performing a Backward-Euler approximation on $\frac{\partial P}{\partial t}$ treat the penalty$Q$ term explicitly.

$$\frac{\partial P^2}{\partial S^2}=\frac{P_{i+1,j}-2P_{i,j}+P_{i-1,j}}{h^2}$$

$$\frac{\partial P}{\partial S}=\frac{P_{i+1,j}-P_{i-1,j}}{2h}$$

Using the approximations above ,the Black-Scholes partial differential equation is then applied to mesh points (nk,mk) ,$n=1,2...N-1$,at the time level $t=mk$,$m=1,2,...,M$.At each n we have

$$\frac{P_{i,j}-P_{i,j-1}}{k}=\frac{1}{2}\sigma^2 S^2\left[\frac{P_{i+1,j}-2P_{i,j}+P_{i-1,j}}{h^2}\right]+rs\left[\frac{P_{i+1,j}-P_{i-1,j}}{2h}\right]-rP_{i,j}+\frac{{}_{\epsilon}C}{P_{\epsilon}+\epsilon-q(s)}$$

$$P_{i,j}-P_{i,j-1}=\frac{1}{2}\frac{\sigma^2S^2k}{h^2}\left[P_{i+1,j}-2P_{i,j}+P_{i-1,j}\right]+\frac{krS}{2h}\left[P_{i+1,j}-P_{i-1,j}\right]-rkP_{i,j}+\frac{{}_{\epsilon}Ck}{P_{\epsilon}+\epsilon-q(s)}$$

Let $\beta = \frac{1}{2}\frac{\sigma^2 S^2 k}{h^2}$, $\frac{krS}{2h}$ and $Q = \frac{{}_\epsilon Ck}{P_\epsilon + \epsilon - q(s)}$

$$P_{i,j} - P_{i,j-1} = \beta P_{i+1,j} - 2\beta P_{i,j} + \beta P_{i-1,j} + \alpha P_{i+1,j} - \alpha P_{i-1,j} - rkp_{i,j} + Q$$

$$P_{i,j} + 2\beta P_{i,j} + rkP_{i,j} - \beta P_{i+1,j} - \alpha P_{i-1,j} - \beta P_{i-1,j} = P_{i,j-1} + Q$$

$$(1 + 2\beta + rk)P_{i,j} - (\alpha + \beta)P_{i+1,j} + (\alpha - \beta)P_{i-1,j} = P_{i,j-1} + Q$$

This leads to the following tridiagonal system

$$A = \begin{pmatrix} 1+2\beta+rk & -(\alpha+\beta) & ... & 0 \\ \alpha-\beta & 1+2\beta+rk & & \\ & \ddots & \ddots & \\ & & & \\ 0 & ...... & \alpha-\beta & 1+2\beta+rk-(\alpha+\beta) \\ 0 & ... & \alpha-\beta & 1+2\beta+rk \end{pmatrix}.$$

The boundary condition vector $\mathbf{b}_j$ is given by

$$\mathbf{b}_j = [(\alpha - \beta)P_1, 0, ...0, -(\alpha + \beta)P_{N,}]^T$$

$$AV_{i,j} + b_j = V_{i,j-1} + Q$$

The eigenvalues of the matrix A can be shown to

$$\begin{aligned} \lambda_s &= 1 + 2\beta + rk + 2\sqrt{-(\alpha+\beta)(\alpha-\beta)}\cos\tfrac{S\pi}{N+1} \\ S &= 1 \cdots N-1 \end{aligned}$$

$$\lambda_s = 1 + 2\beta + rk + 2\sqrt{\beta^2 - \alpha^2}\cos\frac{S\pi}{N+1}$$

If $0 < \alpha \leq \beta$ then the eigenvalues are real numbers satisfying

$$-(1 + 2\beta + rk + 2\sqrt{\beta^2 - \alpha^2}) \leq \lambda_s \leq 1 + 2\beta + rk + 2\sqrt{\beta^2 - \alpha^2}$$

If $\beta \leq \alpha$,then the eigenvalues are complex numbers satisfying

$$\begin{aligned}\lambda_s &= 1 + 2\beta + rk + 2\sqrt{-(-\beta^2 - \alpha^2)} \\ &= 1 + 2\beta + rk + 2\sqrt{\alpha^2 - \beta^2} i \\ &= 1 + 2\beta + rk + 2i\gamma_j\end{aligned}$$

where

$$-\sqrt{\alpha^2 - \beta^2} \leq \gamma_j \leq \sqrt{\alpha^2 - \beta^2}$$

The eigenvalues of $\mathbf{A}$ are essential in determining the stability of the numerical scheme above.The system is stable if $max|\lambda_s| \leq 1$ for $S = 1...N - 1$

4 Time Stepping Scheme

We Consider the following nonlinear parabolic initial-boundary value problem:

$$\begin{aligned}u_t + Au &= F(t, u) && \text{in } \Omega, \quad t \in (0, T\,], \\ u &= v && \text{on } \partial\Omega, \quad t \in (0, T\,], \\ u(\cdot, 0) &= u_0 && \text{in } \Omega,\end{aligned} \tag{4.1}$$

where ω is a bounded domain in $\mathbb{R}^d$ with Lipschitz continuous boundary, A represents a uniformly elliptic operator, and F is a sufficiently smooth, nonlinear reaction term. One should have in mind the following type of differential operator:

$$A := -\sum_{j,k=1}^{d} \frac{\partial}{\partial x_j}\left(a_{j,k}(x)\frac{\partial}{\partial x_k}\right) + \sum_{j=1}^{d} b_j(x)\frac{\partial}{\partial x_j} + b_0(x), \tag{4.2}$$

where the coefficients $a_{j,k}$ and b_j are C^∞ (or sufficiently smooth) functions on $\overline{\Omega}$, $a_{j,k} = a_{k,j}$, $b_0 \geq 0$, and for some $c_0 > 0$

$$\sum_{j,k=1}^{d} a_{j,k}(\cdot)\xi_j\xi_k \geq c_0|\xi|^2, \quad \text{on } \overline{\Omega}, \quad \text{for all } \xi \in \mathbb{R}^{\mathrm{d}}. \tag{4.3}$$

However, we shall use A and F based on an abstract formulation for convenience of the development of the numerical scheme and its analysis. The initial value problem (16) is reset to be posed in a Hilbert space X, as follows. Consider now A to be a linear, self-adjoint, positive definite, closed operator with a compact inverse, defined on a dense domain $D(A) \subset X$. The operator A could represent any of $\{A_h\}_{0<h\leq h_0}$, obtained through spatial discretization, and X We assume the resolvent set $\rho(A)$ satisfies, for some $\alpha \in (0, \frac{\pi}{2})$, $\rho(A) \supset \overline{\Sigma}_\alpha$, where $\Sigma_\alpha := z \in := \alpha < |\arg(z)| \leq \pi, z \neq 0$. Also, assume there exists $M \geq 1$ such that

$$\|(zI - A)^{-1}\| \geq M|z|^{-1},\ z \in \Sigma_\alpha. \tag{4.4}$$

It follows that $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ which is the solution operator for (16), and $|e^{-tA}| \leq C$ for $t \geq 0$. Also, we assume that $F(t, u(t))$ is Lipschitz on $[0, T] \times X$, i.e. it satisfies the following assumption:

Assumption1 *$F : [0, T] \times X \to X$ and U be an open subset of $[0, T] \times X$. For every $(t, x) \in U$ there exists a neighborhood $V \subset U$ and a real number L_T such that*

$$\|F(t_1, x_1) - F(t_2, x_2)\| \leq L_T(|t_1 - t_2| + \|x_1 - x_2\|X) \tag{4.5}$$

for all $(t_i, x_i) \in V$. Using the standard representation:

$$E(t) := e^{-tA} = \frac{1}{2\pi i}\int_\Gamma e^{-tz}(zI - A)^{-1}dz,$$

where

$$\Gamma := z \in : \arg(z) = \theta$$

, oriented so that $\mathrm{Im}(z)$ decreases, for any

$\theta \in (\alpha, \frac{\pi}{2})$ and the Duhamel principle, the exact solution can be written as

$$u(t) = E(t)v + \int_0^t E(t - s)F(s, u(s))ds. \tag{4.6}$$

Let $0 < k \leq k_0$, for some k_0, and $t_n = nk$, $0 \leq n \leq N$. Replacing t by $t + k$, using basic properties of E and by the change of variable $s - t = k\tau$, we arrive at the following recurrence formula for the exact solution:

$$u(t_{n+1}) = e^{-kA}u(t_n) + k\int_0^1 e^{-kA(1-\tau)}F(t_n + \tau k, u(t_n + \tau k))\, d\tau. \tag{4.7}$$

This is the basis for deriving ETD schemes. *The ETD-BE Scheme.* Denoting the semidiscrete approximation to $u(t_n)$ by u_n (note that only the time-variable is discretized) and $F(t_n, u_n)$ by F_n, the simplest approximation to the integral is to impose that F is constant for $t \in [t_n, t_{n+1}]$, i.e. $F \approx F_n$. This yields (from (22))

$$u(t_{n+1}) \approx e^{-kA}u(t_n) + e^{-kA}k\int_0^1 e^{kA\tau}\, d\tau\, F_n = e^{-kA}u(t_n) - A^{-1}\left(e^{-kA} - I\right) F_n \tag{4.8}$$

This semidiscrete scheme is not useful until the matrix exponential is discretized efficiently. Noting that

$$\begin{aligned}
-A^{-1}\big(e^{-kA} - I\big) &= -A^{-1}\big(\,(I + kA)^{-1} - I\big) \\
&= -A^{-1}\big(I - (I + kA)\,\big)(I + kA)^{-1} \\
&= k\,(I + kA)^{-1} \\
&= kR_{0,1}\big(kA\big),
\end{aligned} \tag{4.9}$$

we arrive at the following fully discrete first order scheme, where v now denotes the fully discrete solution. This is the same as a standard first order linearly implicit scheme. The **ETD-BE** scheme is as follows:

$$u_{n+1} = R_{0,1}(kA)u_n + kR_{0,1}\big(kA\big)\, F(t_n, u_n). \tag{4.10}$$

$$(I + kA)u_{n+1} = u_n + kF(t_n, u_n) \tag{4.11}$$

This is an exponential time differencing version of the Backward Euler scheme, which can be more efficient than backward Euler for nonlinear problems. We will use this scheme as an initial damping scheme for problems with irregular data. For nonlinear systems it has the advantage over backward Euler in being explicit as far as the nonlinear part is concerned, thus eliminating the need for extra time consuming nonlinear solvers at each step, for instance, a modified Newton's method.

4.1 Stability Analysis

Let us denote U^n the theoretical solution of the finite difference scheme(25) and let V^n be the numerical solution.

Setting $E^n = U^n - V^n$,we have the following relationship:

$(I + kA)E^{n+1} = E^n + k(F(U^n) - F(V^n))$

In the following we suppose that f is a smooth function satisfying the relationship:

$$|f'(u)| \leq L, for\, u \,\epsilon\, T \tag{4.12}$$

where T is an interval of $\Re$ containing the solution of scheme(25).

The mean value theorem involves

$$F(U^n) - F(V^n) = F'(W^n)E^n$$

with

$$F'(W^n) = diag(f'(w_1^n)...f'(w_{N-1}^n))$$

Hence

$$(I + kA)E^{n+1} = kA + kF'(W^n)E^n$$

Setting $R^n = (kA)^{-1}kA + kF'(W^n)$,the positivity of the matrix $(I + kA)^{-1}$ implies the positivity of the matrix R_n.

with the previous notations,we have $|E^{n+1}| \leq R_n|E^n|$ whose elements are $|E_1^n|$, for

$i = 1...N-1$ and Let $|R_n|$ be the matrix with elements $|Rn_{ij}|$,for $i, j = 1...N-1$. From the positivity of the matrix R_n the following relation follows:

$$|E^{n+1}| \leq R_n|E^n|$$

From condition(26) we deduce $F'(W^n) \leq L \cdot I$ and therefore:

$$R_n \leq (I + kA)^{-1}(1 + Lk)I$$

setting

$$R = (I + kA)^{-1}(1 + Lk)I$$

we have$|E^n| \leq R_n|E^0|$ with $|E^0| = |U^0 - V^0|$.

Let $\rho(R)$ be the spectral radius of the matrix R.if $\rho(R) < 1$ then

$$\lim_{n \to +\infty} mathbbR^n = 0$$

Therefore,if

$$\rho(R) < 1, \lim_{n \to +\infty} |E^n| = 0$$

and scheme (25) is numerically stable.

4.2 Algorithm

We use **ETD-BE** method for time stepping which leads to the following equation

$$[\Phi - kR]c^n = [\Phi + kR]c^{n+1} + kQ^{n+1} \tag{4.13}$$

The terminal condition serves as an initial condition for the ODE system. After collocation at the points x_i, $i = 1, \ldots, N$, the coefficients $c_j(T)$ are given as the solution of the linear system

$$\Phi c(T) = \mathbf{P}$$

where Φ is as above, and $\mathbf{P} \; = [P_\epsilon(x_1, T), \ldots, P_\epsilon(x_N, T)]^T$.

Since radial basis functions do not satisfy the boundary conditions automatically, they are satisfied by adding specific equations to enforce them at each time step.

An algorithm for **ETD-BE** method is as follows:

1. Choose a time step k .
2. Assemble the matrices Φ and R.
3. Compute the matrices $R_1 = \Phi - kR$ and $R_2 = \Phi + kR$.
4. Factor the matrices Φ and R_1.
5. Initialize the solution vector $\boldsymbol{P}$ via $P(x_i, T) = \max(E - x_i, 0)$, $i = 1, ..., N$.
6. For each time step
   (a) Update the coefficients by solving $\Phi c = P$.
   (b) Compute $b = R_2 c$ the vector Q_c
   (c) Find the next coefficients by solving the linear system $R_1 c = b + kQ_c$.
   (d) Update the solution vector P via $P(x_i, t)$=Φc,$i = 2, ..., N - 1$.
   (e) Enforce the boundary conditions $P(x_1, t)$=E and $P(X_N, t)$=0.

Each time step involves the solution of two linear systems. From the theory of radial basis function interpolation it is well known that the matrix Φ is invertible for any choice of (distinct) collocation points (=centers) x_i.

Of course, this does not ensure satisfaction of the positivity constraint. However, the plots resulting from our numerical experiments indicate that this constraint is indeed satisfied for our choices of parameters.

CHAPTER 5

RBF-MESHFREE METHODS

CHAPTER 6

DISCRETIZATION AND ALGORITHMS

CHAPTER 7

NUMERICAL METHODS AND STABILITY ANALYSIS

7.1 Stability

Let error at the n^{th} time level be defined by

$$e^n = V^n_{exact} - V^n_{app} \tag{7.14}$$

Where V^n_{exact} and V^n_{app} is the exact solution and approximate solution obtained by the numerical process 2.3 and 2.3

$$e^n = \mathbf{H}e^{n=1} \tag{7.15}$$

where $\mathbf{H}$ is the amplification matrix which is given by

$$\mathbf{H} = \Phi^{-1}[\Phi + \theta\Delta t\mathbf{R}]^{-1}\Phi \tag{7.16}$$

The numerical scheme is stable if the spectral radius of $\mathbf{B}$, $\rho(\mathbf{B}) \leq 1$ We substitute the value of $\mathbf{B}$ into equation 7.15 to obtain

$$[\Phi - (1-\theta)\Delta tM]\Phi^{-1}e^n = [\Phi + \theta\Delta t\mathbf{R}]\Phi^{-1}e^{n+1} \tag{7.17}$$

This implies

$$[I - (1-\theta)\Delta tM)]e^n = [I + \theta\Delta tM]e^{n+1} \tag{7.18}$$

where $M = \mathbf{R}\Phi^{-1}$ and I is an $N \times N$ identity matrix. The Numerical scheme is stable if all eigenvalues of the matrix $[\Phi - (1-\theta)\Delta tM]^{-1}[I + \theta\Delta tM]$ are less than one.

$$\left|\frac{1 + \theta\Delta t\lambda_M}{\Phi - (1-\theta)\Delta t\lambda_M}\right| \leq 1 \tag{7.19}$$

where λ_M represents the eigenvalues of the matrix M

The value of θ determines whether the system is Explicit,Crank-Nicolson on Implicit Euler method.In the first case,when $\theta = 1$ the system reduces to the explicit Euler method whose stability condition becomes

$$|1 = \Delta t \lambda_M| \leq 1 \tag{7.20}$$

The explicit Euler method will be stable if

$$\Delta t \geq \frac{-2}{\lambda_M} \quad and \quad \lambda_M \leq 0. \tag{7.21}$$

In the second case When $\theta = 0$.we obtain the implicit Euler method which is unconditionally stable because $\lambda_M \leq 0$ the errors decay to zero. Lastly when $\theta = 0.5$ we obtain the Crank-Nicolson method.The inequality 7.19 still holds as long as $\lambda_M \leq 0$.The Crank-Nicolson is in this case unconditionally stable.

CHAPTER 8

NUMERICAL EXPERIMENTS AND RESULTS

We compare the results of RBF methods to those obtained earlier by [7, 8] using finite differences [2].

8.1 Example 1:One Asset

First we verify our single asset code which will be used to calculate the boundary conditions for the two-asset basket option.

The effects of the penalty parameter ϵ were studied more extensively in [7]. There it was observed that, for the finite difference method, the error introduced by the penalty term was roughly on the order of ϵ. Since we are interested in comparing the meshfree formulation with the finite difference framework we consider only the case $\epsilon = 0.01$ here. The other parameters for our single asset American put problem are: $r = 0.1$, $\sigma = 0.2$, $D = 0$, $E = 1$, $T = 1$, $t_0 = 0$, $S_0 = 0$, and $S_\infty = 2$. We used the a constant time step of $k = 0.01$ and different values of nodes which are presented in the tables given below.

S	FD 1001	RBF 41	RBF 81	RBF 101
0.6	0.4000037	4.000079284231482e-01	4.000137274388471e-01	4.000140155227794e-0
0.7	0.3001161	3.002190386780568e-01	3.002244584184045e-001	3.002248173328785e-0
0.8	0.2020397	2.026026872289588e-01	2.026217219929730e-01	2.026237471581654e-0
0.9	0.1169591	1.176250974697884e-01	1.176927181568679e-01	1.177005653395421e-0
1.0	0.0602833	5.952849811933896e-02	5.963331031716317e-02	5.964562793120096e-0
1.1	0.0293272	2.739145951270390e-02	2.748615753723662e-02	2.749734464846737e-0
1.2	0.0140864	1.210898358038832e-02	1.216998804034032e-02	1.217722986470464e-0
1.3	0.0070408	5.620025402083054e-03	5.650899377080217e-03	5.654603136082172e-0
1.4	0.0038609	3.034489511209617e-03	3.047863317966305e-03	3.049553028430248e-0
RMSE		3.800531000028380e-04	3.679472322155780e-04	3.665413983887557e-0
CPU time		7.800049999999814e-02	3.276021000000000e-01	9.516060999999993e-0
COND number		8.540711140948943e+03	9.371517158130049e+03	9.477187078272415e+0

Table 1: Values of American option at $t = 0$ using Gaussian-RBF with $c = 1.5$.

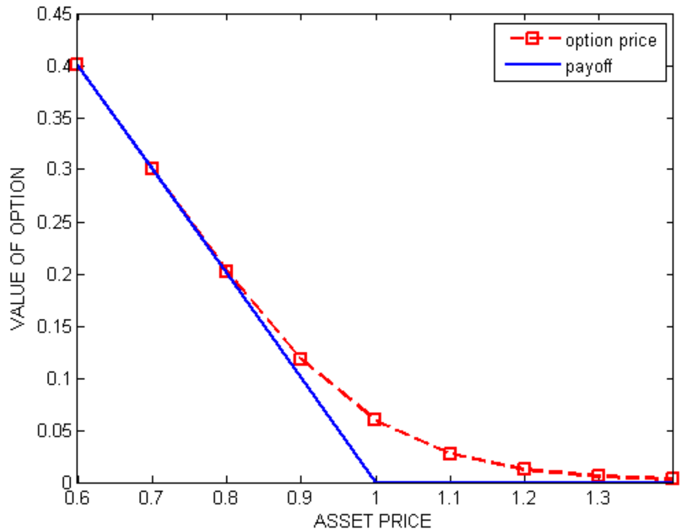

Figure 3: Graph of Gaussian-RBF at $N = 101$ nodes

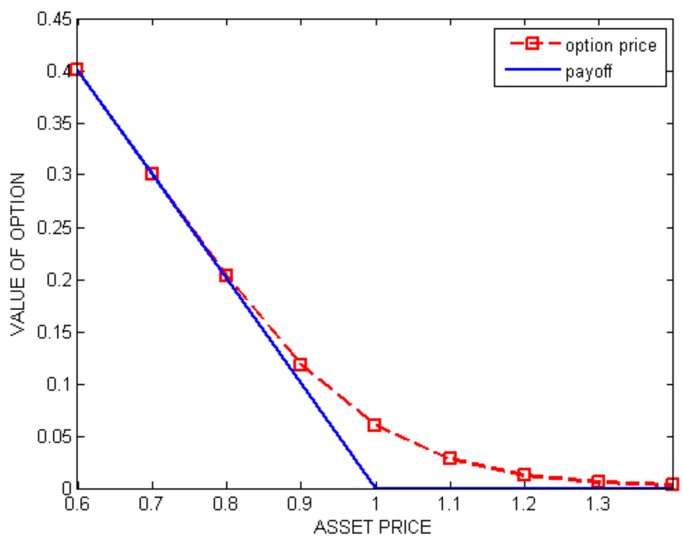

Figure 4: Graph of Multiquadric-RBF at $N = 101$ nodes

S	FD 1001	RBF 41	RBF 81	RBF 101
0.6	0.4000037	4.000136025947318e-01	4.000143544675322e-01	4.000144691188708e-0
0.7	0.3001161	3.002281861141670e-01	3.002328054598784e-01	3.002333928317463e-0
0.8	0.2020397	2.026990698419577e-01	2.027118053149325e-01	2.027132241451394e-0
0.9	0.1169591	1.180054049292558e-01	1.180378873827233e-01	1.180415024971369e-0
1.0	0.0602833	6.003360140092645e-02	6.011839322106172e-02	6.012808397308078e-0
1.1	0.0293272	2.777378229860611e-02	2.787819507876477e-02	2.789129963720338e-0
1.2	0.0140864	1.230516651466951e-02	1.240253754684245e-02	1.241824783091772e-0
1.3	0.0070408	5.682600710233188e-03	5.790463354382935e-03	5.817249701233540e-0
1.4	0.0038609	3.019767309641319e-03	3.204100755105077e-03	3.263693145839186e-0
RMSE		3.468133973660428e-04	3.255796482865938e-04	3.212911530746730e-0
CPU time		1.404009000000031e-01	2.652017000000058e-01	5.460034999999976e-0
COND number		1.350297755352479e+04	5.276831666999462e+04	8.204955512679130e+0

Table 2: Values of American option at $t = 0$ using Multiquadric-RBF with $c = 1.0$.

S	FD 1001	RBF 41	RBF 81	RBF 101
0.6	0.4000037	4.000081846731239e-01	3.999944677251256e-01	3.999848597023653e-0
0.7	0.3001161	3.002194386244130e-01	3.002110833603666e-01	3.002039340350107e-0
0.8	0.2020397	2.026110427716545e-01	2.026254392415477e-01	2.026245530783953e-0
0.9	0.1169591	1.176619409499281e-01	1.177464819756828e-01	1.177650983203513e-0
1.0	0.0602833	5.958535731817690e-02	5.973980242505235e-02	5.978150361976092e-0
1.1	0.0293272	2.745184583462177e-02	2.761876322846362e-02	2.767204819099156e-0
1.2	0.0140864	1.216693881001230e-02	1.231781400468144e-02	1.237667347724274e-0
1.3	0.0070408	5.680432372339327e-03	5.835817190838141e-03	5.909043172578131e-0
1.4	0.0038609	3.110299805121982e-03	3.332465656092971e-03	3.447089842679579e-0
RMSE		3.673986998156583e-04	3.341736920608062e-04	3.213851282534414e-0
CPU time		2.808017999999990e-01	1.060806800000002	1.388408899999998
COND number		1.511671497888172e+03	1.890239112015076e+03	2.011469801137463e+0

Table 3: Values of American option at $t = 0$ using Inverse-Multiquadric-RBF with

$c = 1$.

8.2 Finite Difference Results

c	RMSE	CPU Time
0.5	6.180467527014752e-002	1.404009000000031e-001
0.7	5.943695978989005e-002	1.248008000000027e-001
0.9	1.048115147988156e-003	2.028012999999902e-001
1.1	3.763686518227007e-004	9.360060000000203e-002
1.3	3.799290451810216e-004	1.404009000000031e-001
1.5	3.799938827710493e-004	1.092007000000024e-001
1.7	3.799855151911536e-004	9.360060000000203e-002
1.9	3.800289462433457e-004	9.360059999997361e-002
2.1	3.800742178954350e-004	1.716011000000037e-001
2.3	3.801378742389429e-004	9.360060000000203e-002
2.5	3.803864448512753e-004	1.092007000000024e-001
2.7	3.813691028102942e-004	9.360060000000203e-002
2.9	3.849603077913177e-004	1.092007000000024e-001
3.1	3.978068233174443e-004	1.560010000000034e-001
3.3	4.454215456274659e-004	1.092007000000024e-001
3.5	6.300820110166411e-004	1.560010000000034e-001

Table 4: Comparison of different c valuesfor Gaussian-RBF and their RMSE .

c	RMSE	CPU Time
0.2	8.206261648864639e-00	1.232407899999998e+000
0.4	2.148128533777387e-003	1.185607599999997e+000
0.8	3.060974866471791e-004	1.060806799999995e+000
1.0	3.212911530746730e-004	8.736055999999906e-001
1.3	3.466796105517317e-004	8.736056000000190e-001
1.5	3.516081867974326e-004	8.268052999999895e-001
1.8	3.542657007260861e-004	9.984063999999933e-001
2.1	3.554989239753822e-004	1.294808300000000e+000
2.5	3.566160605759570e-004	1.123207199999996e+000
3.0	3.576356069306907e-004	8.892056999999909e-001
3.5	3.583680279546284e-004	9.984063999999933e-001
4.0	3.588897897535030e-004	7.644048999999882e-001

Table 5: Comparison of different c values for MQ-RBF and their RMSE .

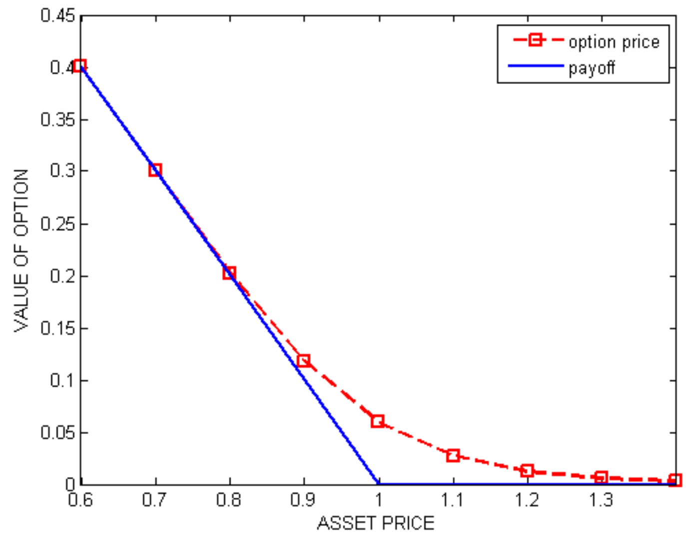

Figure 5: Graph of Inverse Multiquadric-RBF at $N = 101$ nodes

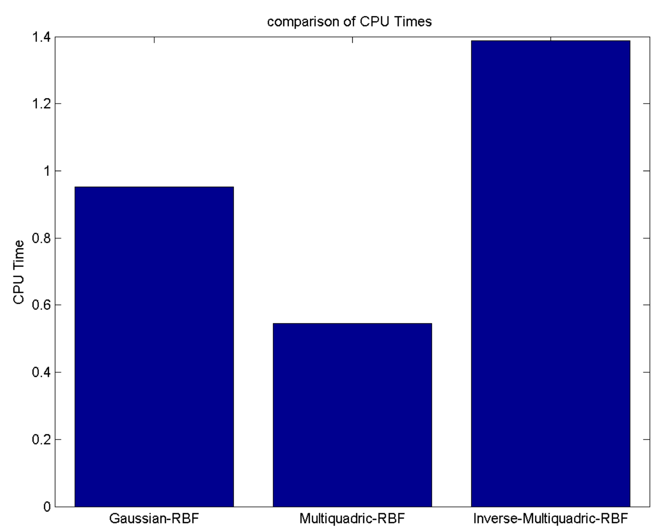

Figure 6: Comparison CPU times of RBF's at $N = 101$ nodes

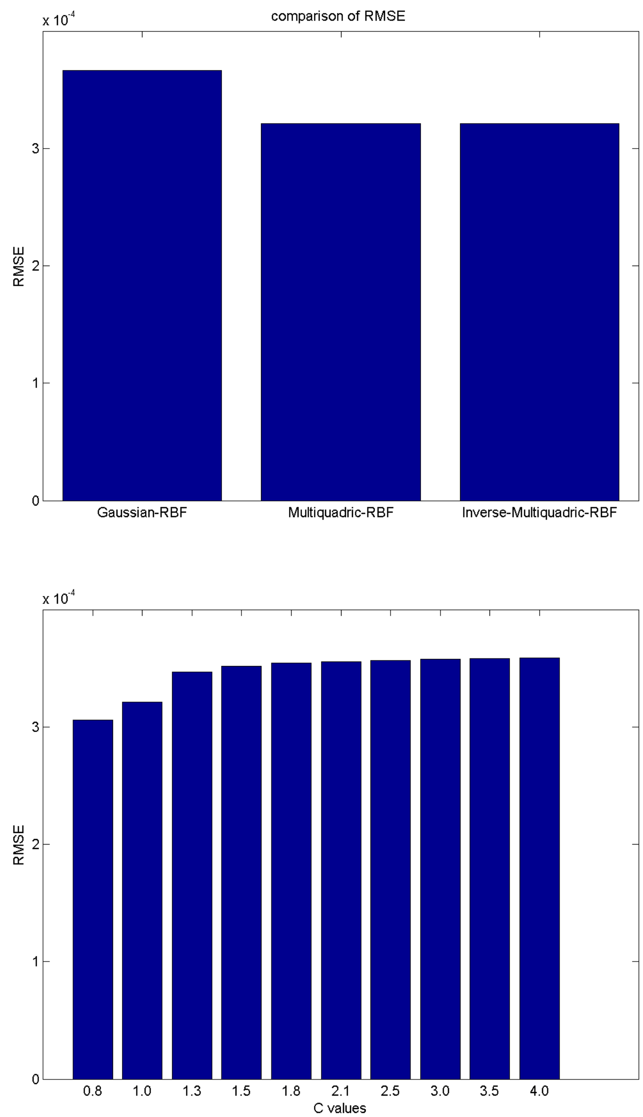

Figure 7: Comparison RMSE of RBF's at $N = 101$ nodes(above) and RMSE for different c values using MQ-RBF

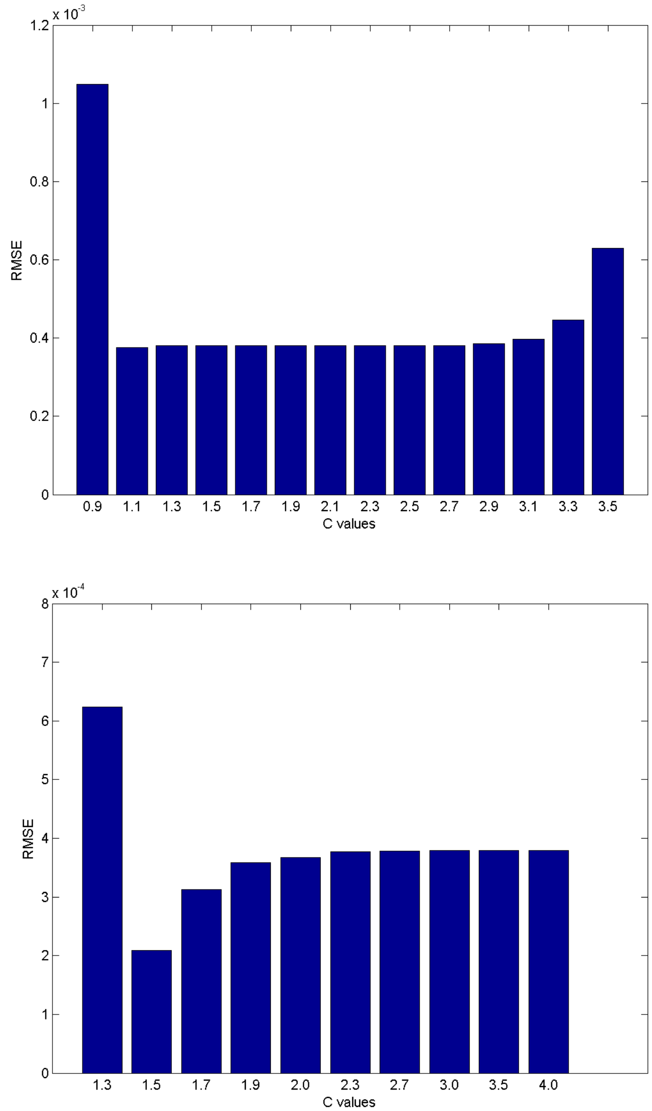

Figure 8: Comparison RMSE of Different c values for Gaussian-RBF(above) and IMQ-RBF at $N = 101$ nodes

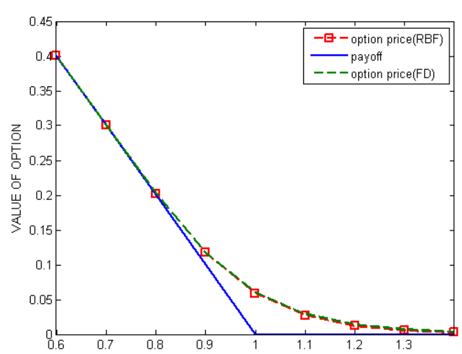

Figure 9: Comparison between RBF and Finite Difference Solutions

c	RMSE	CPU Time
0.5	4.951360842189655e-002	6.708043000000004e-001
0.7	1.532664011401649e-002	4.992032000000002e-001
0.9	7.794887198969099e-003	5.148032999999970e-001
1.1	2.578845380361731e-003	7.020045000000010e-001
1.3	6.232380475217872e-004	3.744024000000010e-001
1.5	2.089107130800181e-004	4.212027000000020e-001
1.7	3.126186527403213e-004	4.836031000000034e-001
1.9	3.582142527202816e-004	3.432022000000004e-001
2.0	3.800742178954350e-004	5.460034999999976e-001
2.3	3.673986998156583e-004	4.992031999999966e-001
2.7	3.768248122380777e-004	6.240039999999993e-001
3.0	3.787147145926920e-004	5.148032999999970e-001
3.5	3.790322985167134e-004	5.148032999999970e-001
4.0	3.793193743515548e-004	2.340015000000051e-001

Table 6: Comparison of different c values for IMQ-RBF and their RMSE .

S	Option Value	FD1001
0.6	0.4001500	0.4000037
0.7	0.3002437	0.3001161
0.8	0.2049304	0.2020397
0.9	0.1288876	0.1169591
1.0	0.0746978	0.0602833
1.1	0.0399911	0.0293272
1.2	0.0199422	0.0140864
1.3	0.0093622	0.0070408
1.4	0.0041843	0.0038609
CPUTIME	7.2540	
RMSE	0.0075	

Table 7: Finite Difference solution at $N = 2001, \epsilon = 10^{-4}, \Delta t = 0.001$.

S	Option Value	FD1001
0.6	0.4015118	0.4000037
0.7	0.3028086	0.3001161
0.8	0.2100498	0.2020397
0.9	0.1334426	0.1169591
1.0	0.0778183	0.0602833
1.1	0.0419259	0.0293272
1.2	0.0211123	0.0140864
1.3	0.0100855	0.0070408
1.4	0.0046513	0.0038609
CPUTIME	1.0920	
RMSE	0.0098	

Table 8: Finite Difference solution at $N = 2001, \epsilon = 10^{-3}, \Delta t = 0.01$.

S	Option Value	FD1001
0.6	0.4134769	0.4000037
0.7	0.3191524	0.3001161
0.8	0.2295821	0.2020397
0.9	0.1513232	0.1169591
1.0	0.0913208	0.0602833
1.1	0.0513920	0.0293272
1.2	0.0277877	0.0140864
1.3	0.0149302	0.0038609
1.4	0.0082378	0.0038609
CPUTIME	0.312002	
RMSE	0.0216	

Table 9: Finite Difference solution at $N = 2001, \epsilon = 10^{-2}, \Delta t = 0.1$.

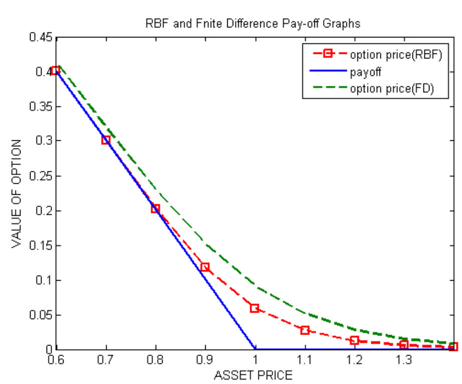

Figure 10: RBF N=101, k=0.01 and FD N=2001,k=0.01

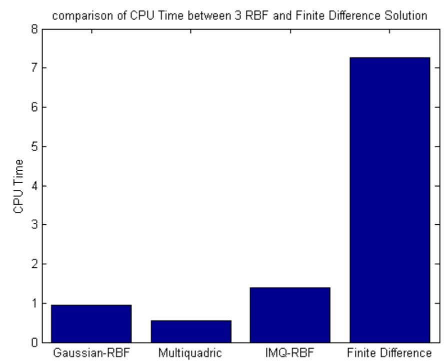

Figure 11: Comparison of CPU Times between the 3 RBFs for N = 101 and FD for N=2001

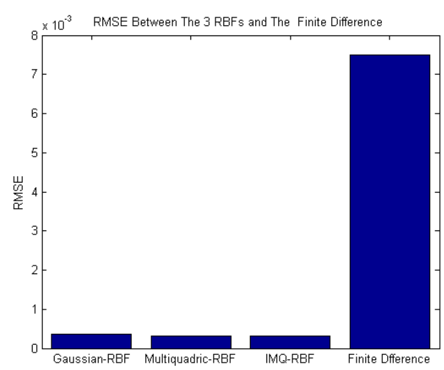

Figure 12: Comparison of RMSE between the 3 RBFs for N = 101 and FD for N=2001

CHAPTER 9

CONCLUSION

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APPENDICES

APPENDIX A

SAMPLE APPENDIX

A.1 Long Table

Here We show how to create long table -table which run on more than one page.

Table A.1: Long Table

	A Column A				Column B		Column C				
C	A1	A2	A3	A4	B1	B2	C1	C2	C3	C4	C5
418773365010	0	0	0	1	1	1	0	0	1	0	0
419196187410	0	0	0	1	1	1	0	0	1	0	0
419222133330	0	0	0	1	1	1	0	0	1	0	0
419465256210	0	0	0	1	1	1	0	0	1	0	0
419491202130	0	0	0	1	1	1	0	0	1	0	0
419914024530	0	0	0	1	1	1	0	0	1	0	0
593129947410	0	0	0	1	1	1	0	0	0	0	0
593552769810	0	0	0	1	1	1	0	0	0	0	0
593578715730	0	0	0	1	1	1	0	0	0	0	0
593821838610	0	0	0	1	1	1	0	0	0	0	0
593847784530	0	0	0	1	1	1	0	0	0	0	0
594270606930	0	0	0	1	1	1	0	0	0	0	0
656532346470	0	0	0	1	1	1	1	0	1	1	0
656955168870	0	0	0	1	1	1	1	0	1	1	0
656981114790	0	0	0	1	1	1	1	0	1	1	0
657224237670	0	0	0	1	1	1	1	0	1	1	0
657250183590	0	0	0	1	1	1	1	0	1	1	0
657673005990	0	0	0	1	1	1	1	0	1	1	0
799769020050	0	0	0	1	1	1	0	1	1	0	0
800191842450	0	0	0	1	1	1	0	1	1	0	0
800217788370	0	0	0	1	1	1	0	1	1	0	0
800460911250	0	0	0	1	1	1	0	1	1	0	0
800486857170	0	0	0	1	1	1	0	1	1	0	0
800909679570	0	0	0	1	1	1	0	1	1	0	0
830888928870	0	0	0	1	1	1	1	0	0	1	0
831311751270	0	0	0	1	1	1	1	0	0	1	0
831337697190	0	0	0	1	1	1	1	0	0	1	0
831580820070	0	0	0	1	1	1	1	0	0	1	0
831606765990	0	0	0	1	1	1	1	0	0	1	0
832029588390	0	0	0	1	1	1	1	0	0	1	0
955811146290	1	1	1	1	1	1	0	1	1	0	0

Table A.1:

	A Column A				Column B		Column C				
C	A1	A2	A3	A4	B1	B2	C1	C2	C3	C4	C5
956284899570	1	1	1	1	1	1	0	1	1	0	0
956553968370	1	1	1	1	1	1	0	1	1	0	0
956976790770	1	1	1	1	1	1	0	1	1	0	0
957002736690	1	1	1	1	1	1	0	1	1	0	0
974125602450	0	0	0	1	1	1	0	1	1	1	1
974548424850	0	0	0	1	1	1	0	1	1	1	1
974574370770	0	0	0	1	1	1	0	1	1	1	1
974817493650	0	0	0	1	1	1	0	1	1	1	1
974843439570	0	0	0	1	1	1	0	1	1	1	1
975266261970	0	0	0	1	1	1	0	1	1	1	1
1037950823910	0	0	0	1	1	1	1	1	1	1	0
1037976769830	0	0	0	1	1	1	1	1	1	1	0
1038219892710	0	0	0	1	1	1	1	1	1	1	0
1038245838630	0	0	0	1	1	1	1	1	1	1	0
1038668661030	0	0	0	1	1	1	1	1	1	1	0
1116199694610	0	0	0	1	1	1	0	0	1	0	1
1116622517010	0	0	0	1	1	1	0	0	1	0	1
1116648462930	0	0	0	1	1	1	0	0	1	0	1
1116891585810	0	0	0	1	1	1	0	0	1	0	1
1116917531730	0	0	0	1	1	1	0	0	1	0	1
1117340354130	0	0	0	1	1	1	0	0	1	0	1
1130167728690	1	1	1	1	1	1	0	1	0	0	0
1130641481970	1	1	1	1	1	1	0	1	0	0	0

A.2 Sample Section

Insert Text Here.

APPENDIX B

SAMPLE APPENDIX 2

B.1 Section Heading

Insert text here.

B.1.1 Subsection Heading