Introduction to Resurgence

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Consider this integral:

$$Z(g) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2 - \frac{g}{4}x^4} \, \mathrm{d}x \tag{1}$$

zero-dimensional QFT whose action is $S = \frac{1}{2}x^2 + \frac{g}{4}x^4$.
Perturbative series for g is

$$\begin{aligned} Z(g) &= \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{g}{4}\right)^n x^{4n} \, \mathrm{d}x && (2) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{g}{4}\right)^n \int_{-\infty}^{\infty} x^{4n} e^{-\frac{1}{2}x^2} \, \mathrm{d}x && (3) \\ &= \sum_{n=0}^{\infty} \sqrt{2}\frac{\Gamma(2n+\frac{1}{2})}{\Gamma(n+1)}(-g)^n && (4) \\ &\cong \sum n!(-g)^n && (5) \end{aligned}$$

Because of the factrial growth, the radius of convergence is 0!

Asymptotic series

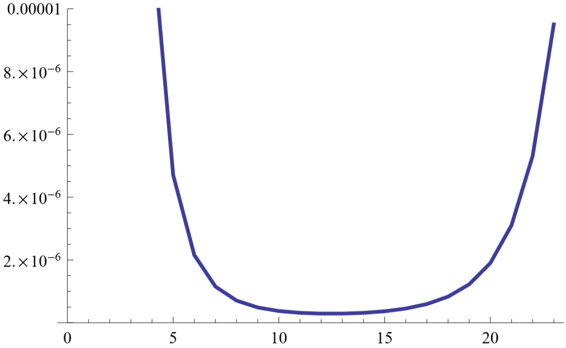


Figure: vertical axis:$|Z_{true}(g) - Z_{pert.}(g)|$, horizontal axis:$N$

When the series is $\sum_k A^{-k} k! g^k$, the critical N is $N_* = \left|\frac{A}{g}\right|$.

Perturbative series is ill-defined!
$\rightarrow$Resurgence gives a method to recover the series.

Resurgence method

$$S(g) = \sum_{n=0}^{\infty} a_n g^n \tag{6}$$

Borel transform is defined as

$$B[S](z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n = \sum_{n=0}^{\infty} b_n z^n \tag{7}$$

Borel summation is defined as

$$\tilde{S}(g) = \frac{1}{g} \int_0^{\infty} e^{-z/g} B[S](z) \, \mathrm{d}z \tag{8}$$

In fact, this procedure comes from this identity:

$$1 = \int_0^{\infty} e^{-t} \frac{t^n}{n!} \, \mathrm{d}t \tag{9}$$

quartic oscillator

Let's consider this Euclidian Action:

$$S_E = \int_0^\beta \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{1}{4}gx^4\right) \mathrm{d}\tau \tag{10}$$

The partition function is $Z = \operatorname{tr} e^{-\beta H} = \int \mathcal{D}x e^{-S_E}$. When we calculate it perturbatively for g and using this formula:

$$E_0 = \lim_{\beta\to\infty} -\frac{1}{\beta} \log Z(\beta) \tag{11}$$

It gives

$$E_0 = \frac{1}{2} + \sum_{k=1}^{\infty} a_k g^k \tag{12}$$

$$a_k = (-1)^{k+1}\sqrt{\frac{6}{\pi^3}}\left(\frac{3}{4}\right)^k \Gamma\left(k+\frac{1}{2}\right) \tag{13}$$

Indeed, it is factorial divergence!→Let's apply resurgence

Using Borel summation,

$$E_0 = \frac{1}{2} + \sqrt{\frac{6}{\pi^3}} \frac{3}{4} g \int_0^\infty \sqrt{x} \frac{e^{-x}}{1 + \frac{3}{4} g x} \, \mathrm{d}x \tag{14}$$

This is finite for $g \geqq 0$. Therefore,

Remark

1. Resurgence cures divergent series and gives a meaningful result.
2. This result is valid even if $g >> 1$!(e.g. $E_0 \simeq 1.14$ for $g = 100$)

And this result is consistent with other calculations.
(See: S.Graffi, et.al, PLB 32, 631 and E.Delabaere, et.al, JMP 38, 6126)

Singularities in Borel summation

In the case of $S(g) = \sum n! g^n$, Borel sum is

$$\tilde{S}(g) = \frac{1}{g} \int_0^\infty \frac{e^{-z/g}}{1-z} \, \mathrm{d}z \tag{15}$$

It has a singularity($z = 1$) in the integral.

$$\tilde{S}_\pm(g) = \frac{1}{g} \int_0^{\infty \pm i\theta} \frac{e^{-z/g}}{1-z} \, \mathrm{d}z \tag{16}$$

The difference of two integrals is

$$\begin{aligned} \tilde{S}_+(g) - \tilde{S}_-(g) &= -\frac{1}{2\pi i} \mathrm{Res}_{z=1} \left(\frac{1}{g} \frac{e^{-z/g}}{1-z} \right) && (17) \\ &= -\frac{1}{2\pi i g} e^{-\frac{1}{g}} && (18) \end{aligned}$$

This term looks nonperturbative for g. Physical meaning...?

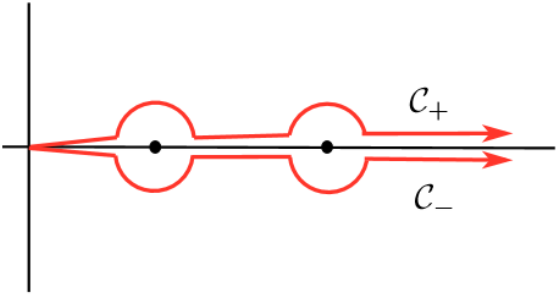


Figure: $\tilde{S}_+(g) - \tilde{S}_-(g)$ picks up information of poles(Borel singularities)

(from Marcos Marino arXiv:1206.6272v2[hep-th])

Unstable vacuum

Let's consider cubic potential:

$$S_E = \int_0^\beta \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + gx^3 \right) \mathrm{d}\tau \tag{19}$$

Perturbatively, it gives

$$E_0 = \frac{1}{2} + \sum_{k=1}^{\infty} a_k g^{2k} \tag{20}$$

$$a_k = -\sqrt{\frac{\pi}{2}}\sqrt{60}\frac{60^k}{2^{3k}}\Gamma\left(k + \frac{3}{2}\right) \tag{21}$$

Using Borel summation,

$$\tilde{E}_0 = \frac{1}{2} - \sqrt{\frac{\pi}{2}}\sqrt{60}\frac{15}{2}g^2 \int_0^\infty \left(x^{\frac{3}{2}} \frac{e^{-x}}{1 - \frac{15}{2}g^2 x} \right) \mathrm{d}x \tag{22}$$

Borel singularity and nonperturbative effect

$$\tilde{E}_{0+} - \tilde{E}_{0-} = \frac{1}{\sqrt{\pi} i} \frac{1}{g} e^{-\frac{2}{15 g^2}} \tag{23}$$

As a result,

$$\tilde{E}_0 = \tilde{E}_{0\pm} \mp \frac{1}{\sqrt{\pi} i} \frac{1}{g} e^{-\frac{2}{15 g^2}} \tag{24}$$

When we calculate E_0 with using 1-loop calculation

$$\operatorname{Im} E_0 \propto \frac{1}{g} e^{-\frac{2}{15 g^2}} \tag{25}$$

Remark

As you know, the imaginary part of energy means tunneling probability($\operatorname{Im} E_0 = -\frac{\Gamma}{2}$), which is a nonperturbative effect. We can recover this effect from perturvative series!

double-well potential

Let's consider this Euclidian Action:

$$S_E = \int_0^\beta \left(\frac{1}{2}\dot{x}^2 + \frac{x^2}{2}(1 - \sqrt{g}x)^2 \right) \quad (26)$$

The potential is a double-well type. Again when we calculate E_0 with perturbation, (for large k)

$$E_0 = \sum_{k=0}^{\infty} a_k g^k \quad (27)$$

$$a_k \simeq -\frac{3^{k+1}}{\pi}\Gamma(k+2) \quad (28)$$

After Using Borel summation, it gives

$$\tilde{E}_0 = -\frac{1}{\pi}\frac{1}{g}\int_0^\infty e^{-x}\frac{1}{1-3gx}\,\mathrm{d}x \quad (29)$$

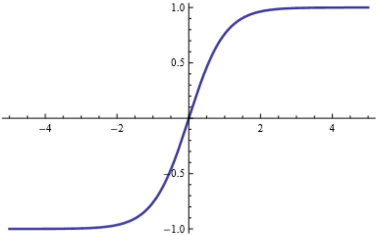
$\mathrm{Im}\,\tilde{E}_0 \propto 1/g e^{-\frac{1}{3g}}$. What is the meaning of this term...?

Instanton-Antiinstanton(bion)

Classical solution of this action is

- obvious vacuum(not time dependent): $x(\tau) = 0, \frac{1}{\sqrt{g}}$
- instanton solution(time dependent):

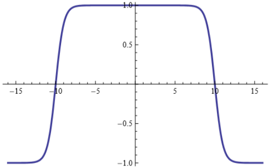
$$x(\tau) = \frac{1}{2\sqrt{g}} \tanh\left(\frac{\tau - \tau_0}{2}\right)$$



But this solution does not satisfy periodic boundary condition!

bion

$$x_{bion}(\tau) = \frac{1}{2\sqrt{g}}\left(\tanh\left(\frac{\tau-\tau_0}{2}\right) - \tanh\left(\frac{\tau-\tau_1}{2}\right)\right) \qquad (30)$$



If we substitute $x_{bion}(\tau)$ into EoM

$$\left.\frac{\delta S}{\delta x}\right|_{x=x_{bion}} = O\left(e^{-(\tau_1-\tau_0)}\right) \qquad (31)$$

if $\beta \to \infty$, we can set $(\tau_1 - \tau_0) \to \infty$. Therefore it can be a solution.

multi-bion solutions($0 \to \sqrt{\frac{1}{g}} \to 0 ...$) also can satisfy the EoM in $\beta \to \infty$. I calculate the contribution with WKB, it gives

$$E_0 \simeq \frac{1}{2} - \frac{1}{g} e^{-\frac{1}{3g}} \tag{32}$$

Real but nonperturbative correction. And this term looks relevant to the term from Borel singularity!

Relationship between Perturbative and nonperturbative

Actually, if we can know all Borel singularities, we can reconstruct perturbative series.

$$S(g) = \sum a_k g^k \tag{33}$$

$$a_k = \frac{1}{2\pi i} \int_0^\infty \frac{1}{z^{k+1}} \sum \frac{1}{g} \oint_{poles} e^{-z/g} B[S](z)\, \mathrm{d}z \tag{34}$$

In short, pertutabative series and nonperturbative effect are relevant each other! (what is the meaning of complex ones...?)

Summary

- Resurgence gives a method to recover an exact value from asymptotic series.
- Borel singularity is relevant to nonperturbative effect(instanton, bion)
- If we can know a distribution of Borel singularities, it means we calculate a perturbative series in all order and vice versa.

Bibliography

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