Epidemiological Modelling of Coronavirus Infection in Quarantined Population

A Outline of Project for MATH371 Wi2020

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2020

Outline

Motivation and Objective

- Motivation: COVID-19

The Models

- SIRD
- SIR with Quarantine

Assumptions

Methods of Estimation

- Least-square (Plateaued)
- Time Scale Matching

Result

Objective

- to explore the SIRD model and its stability points
- to estimate parameters of the SIR model given COVID-19 epidemiological data from Wuhan of China, Italy, and Canada
- to explore SIRD models that embed age compartments, and quarantining of population

COVID-19

COVID-19 is caused by a novel virus named SARS-CoV-2 that found its first confirmed case in December 2020. By March 2020, $780,000$ cases of COVID-19 have been confirmed around the world, and it is still developing.

SIRD Model

The fraction of individuals susceptible to contract COVID-19 in a population , $\frac{S}{N} = s$, fraction of infected individuals, $\frac{I}{N} = i$, fraction of removed individuals, $\frac{R}{N} = r$ and fraction of dead, $\frac{D}{N} = d$ change in dynamic described in the following planar system:

$$
\begin{aligned}
\dot{s} &= -\beta s i & s(0) &= s_0 \\
\dot{i} &= \beta s i - (\alpha + \mu) i & i(0) &= i_0 \\
\dot{r} &= \alpha i & r(0) &= r_0 \\
\dot{d} &= \mu i & d(0) &= 0
\end{aligned}
\tag{1}
$$

where:

β = infection rate; $\frac{1}{\beta}$ = average days to be infected

α = recovery rate; $\frac{1}{\alpha}$ = average days to recover

μ = death rate

R_0 = basic reproduction number, the avg number infected by 1 person

Stability Analysis of the SIRD

Taking out $d = 1 - s - i - d$, the planar system in (1) can be rewritten as

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$$

where $\mathbf{x}^T = [s \;\; i \;\; r]$
at steady state (ss), $(\dot{x}) = 0 \implies$

$$\mathbf{x} = \hat{\mathbf{x}} = \begin{pmatrix} s_{ss} \\ 0 \\ r \end{pmatrix}$$

with Jacobian matrix

$$J\hat{\mathbf{x}} = \begin{pmatrix} 0 & \beta s_{ss} & 0 \\ 0 & \beta s_{ss} - (\alpha + \mu) & 0 \\ 0 & \alpha & 0 \end{pmatrix}$$

and eigenvalues $\lambda_1, \lambda_2 = 0, \lambda_3 = bs - (\alpha + \mu) < 0$ since $s_{ss} < s_{I_{max}}$

Phase Plane Analysis

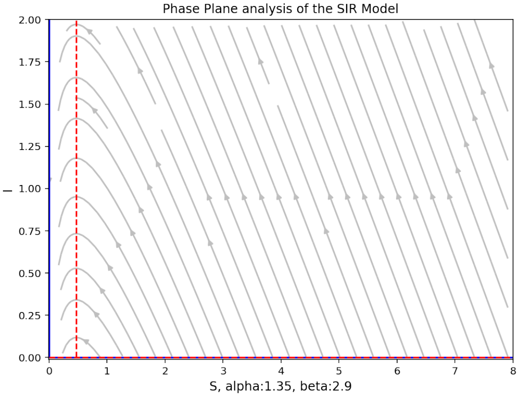

Figure: Phase plane plot obtained from Jay Newby 2020 lecture materials

The phase plane plot reveals that $i = 0$, $s = 0$ is a stable point.

SIR with Quarantine

The SIR model with quarantine adds a compartment $\dot{q} = \frac{Q}{N}$, where Q represents the quarantined infected individuals. Infected individuals are quarantined at rate γ, recover at rate α, and perish at rate μ.

$$
\begin{aligned}
\dot{s} &= -\beta s i & s(0) &= s_0 \\
\dot{i} &= \beta s i - (\alpha + \mu) i & i(0) &= i_0 \\
\dot{r} &= \alpha i & r(0) &= r_0 \\
\dot{q} &= \gamma i - \mu q - \alpha q & q(0) &= 0 \\
\dot{d} &= \mu i & d(0) &= 0
\end{aligned}
\tag{2}
$$

Assumptions

The SIR family of models assume:

- instantaneous infection, recovery of individuals
- no delay in any effect, including death

which is not captured in the publicly available data collected by the health authority.

In addition, while the solutions to the SIR model is unique, the optimal parameters found by equation (3) takes different values given different initial guess fed to the solver.

Minimisation Problem (Plateaued)

To estimate parameters from data from regions that **have reached** a plateau-ing trend in the number of confirmed cases (infected), we solve the following problem:

$$
\begin{aligned}
\underset{\alpha,\beta=1,\mu,N}{\text{minimize}} \quad & \left(\frac{I_{max}-\widehat{I}_{max}}{I_{max}}\right)^2 + \frac{1}{T}\sum_{t=1}^{T}\left(\frac{R_t-\widehat{R_t}}{R_{max}}\right)^2 + \left(\frac{D_t-\widehat{D_t}}{D_{max}}\right)^2 \\
\text{subject to} \quad & 0 \leq \frac{\alpha+\mu}{\beta} \leq \left(\frac{N-(I+R+D)}{N}\right)\Bigg|_{I=I_{max}}
\end{aligned}
\tag{3}
$$

For Non-Plateaued Data

For data from regions that have not reached the plateauing stage of the data, we solve the following minimisation problem for each compartment i, r, d.
I.e. assuming $s = 1$, for i:

$$\begin{aligned} &\underset{i_0, k_0}{\text{minimize}} && E = \sum_{t=1}^{T} (i_t - \widehat{i_t})^2 \\ &\text{subject to} && \hat{i} = i_0 e^{k_0 i} \\ &\text{where } k = \beta - \alpha - \mu \end{aligned} \tag{4}$$

For Non-Plateaued Data

From Eq(4) we get

$$\begin{aligned}\hat{i} &= i_0 e^{k_0 t} \\ \hat{r} &= r_0 e^{k_0 t} \\ \hat{d} &= d_0 e^{k_0 t}\end{aligned} \tag{5}$$

In addition,

$$r_0 = \frac{i_0 \alpha}{k_0}, \quad d_0 = \frac{i_0 \mu}{k_0}$$

$$\implies \alpha = \frac{r_0 k_0}{i_0}, \quad \mu = \frac{d_0 k_0}{i_0}$$

$$\implies \beta = k_0 + \frac{k_0}{i_0}(d_0 + r_0)$$

Time Scale of Estimated Data

Using Euler's method, a differential form $\dot{s} = -\beta si$ can be expressed as

$$s_{t+1} = s_t + dt(-\beta s_t i_t)$$

Which we consider as

$$s_{t+1} = s_t + \beta dt(-s_t i_t)$$

Thus, a change in β leads to a change in the timescale. In Eq(3) we assumed $\beta = 1$.

The Time-scale constant

The estimated data has time scale t'. It is scaled to match the timescale used by the data in t time scale. We obtain

$$k = \frac{t_m}{t'_m} \tag{6}$$

where t_m is the time at which $i(t)$ is maximum and t'_m is the time when $\hat{i}(t')$ is maximum.

Eq (6) implies $t' = \frac{t}{k}$. Thus, to match timescales, $\hat{s}(t') = \hat{s}(\frac{t}{k})$. In addition, since $\beta = 1$ in Eq(3), the estimated parameters are scaled i.e.

$$\alpha = \frac{\hat{\alpha}}{k}, \beta = \frac{1}{k}, \mu = \frac{\hat{\mu}}{k}, N = \hat{N}$$

.

Result

The estimated parameters depends on the initial conditions fed to the optimization algorithm.

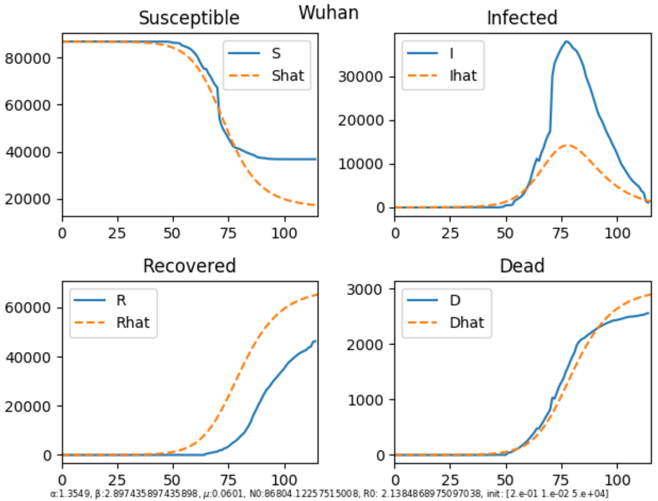

Figure: Estimated s, i, r, d from Wuhan epidemiological data. $R_0 = 2.138$

Using initial guess $N_0 = 5x10^4$, the estimated $\alpha \approx 1.35, \beta \approx 2.9, \mu \approx 0.06, N \approx 9x10^4$

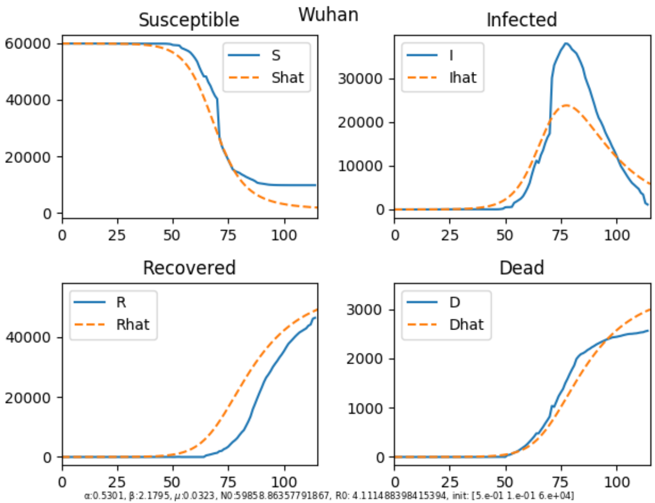

Figure: Estimated s, i, r, d from Wuhan epidemiological data. $R_0 = 4.111$

Using initial guess $N_0 = 6x10^4$, the estimated

$\alpha \approx 0.53, \beta \approx 2.18, \mu \approx 0.03, N \approx 6x10^4$

Appendix

- The code and data used for this project are published here: Github/NatashaTing/covid19-modelling.