Theorem 1. *"Kinetic Energy" is a well-defined notion. That is, regardless of the path over which a force acts on a particle of mass m traveling at velocity v, the energy (work) required to stop the particle is*

$$W = -\frac{1}{2}mv^2$$

Proof. Let's first look at the energy required to stop the particle using a constant deceleration from velocity $v = v_0$ to $v = v_1$. A deceleration over time span Δt is

$$a = \frac{v_1 - v_0}{\Delta t}$$

Of course, the energy (work) exerted is the product of the force times the distance across which the force acts.:

$$W = F \cdot d$$

where $F = ma$. Thus, so far, we know that

$$W = m\frac{v_1 - v_0}{\Delta t} \cdot d \tag{1}$$

The distance d that the particle travels is given by

$$d = \int_0^{\Delta t} v(t)\, dt$$

where Δt is the time over which the action takes place and $v(t)$ is the (variable) velocity at given time t. Observing constant deceleration,

$$v(t) = v_0 + \frac{v_1 - v_0}{\Delta t}t, \quad t \in [0, \Delta t]$$

So, we get

$$\begin{aligned} d &= \int_0^{\Delta t} v_0 + \frac{v_1 - v_0}{\Delta t}t\, dt \\ &= v_0\Delta t + \frac{1}{2}(v_1 - v_0)\Delta t \\ &= \frac{1}{2}\left(v_0 + v_1\right)\Delta t \end{aligned}$$

Combinging this with equation 1, we get

$$\begin{aligned} W &= m\frac{v_1 - v_0}{\Delta t} \cdot \frac{1}{2}\left(v_0 + v_1\right)\Delta t \\ &= \frac{1}{2}m\left(v_1^2 - v_0^2\right) \end{aligned}$$

One can immediately see that for the case of $v_1 = 0$, this simplifies to

$$W = -\frac{1}{2}mv_0^2$$

Of course, it follows from this that the energy required to move a particle—via transitions of constant acceleration—through velocities

$$v_0, v_1, \ldots, v_n$$

is

$$\begin{aligned} W &= \frac{1}{2}m\left(v_1^2 - v_0^2\right) + \frac{1}{2}m\left(v_2^2 - v_1^2\right) + \cdots + \frac{1}{2}m\left(v_n^2 - v_{n-1}^2\right) \\ &= \frac{1}{2}m\left(v_n^2 - v_0^2\right) \end{aligned}$$

Thus, since any (integrable) path can be approximated arbitrarily well by a piecewise linear function, we get that the work required to stop the particle over any path is as claimed. $\square$