Module 2, B-set 3: Angular Momentum and Mass Moment of Inertia

Just roll with it

November 3, 2016

Learning Goals

By the end of this building block, you should have...

- conceptual understanding of conservation of angular momentum
- insight into the physical meaning of the mass moment of inertia of a solid body
- insight into how a body's mass distribution effects its rotation about an axis
- the skill of determining the mass properties of body in SolidWorks

Overview and Orientation

For this B-set, you'll first have a chance to finish up your pendulum simulations. We ask you to generate plots to illustrate the trajectories of the pendula. You should clearly explain (in a caption or accompanying text) what you are showing with your plots. Then, you will be doing some experiments and calculations that will give you some insight into angular momentum conservation, mass moment of inertia, and how the two are connected. Here you'll do more with the concept that the mass moment of inertia tensor can be written as a scalar matrix.

WARNING: This is a full-week long B-set. It is LONGER than a usual B-set. DO NOT WAIT UNTIL WEDNESDAY TO START IT.

Complete Pendula Simulations (2.5 hours)

1. Complete your simulations of the Simple, Springy, and Spinny-Springy Pendulums.

   (a) Choose and create the 1-to-n figures that you need to illustrate the trajectories of each pendulum for an example case. For each figure, write a caption that clearly explains what the figure is showing. After reviewing your figures, a reader should have a clear understanding of the behavior of the pendulum. Test your material out on someone who is not in QEA.

(b) Provide some evidence (*e.g.*, plots with captions) that validate your simulations.

Complete the In-class Angular Momentum Material (1 hour)

Complete the Inclass 3 material on angular momentum and mass moment of inertia. Do not do the Euler Angler Rate Equation material at this time. You'll do that next class.

Another Look at the Simple Pendulum (30 min)

2. Derive the equation of motion of a simple pendulum in terms of θ using the moment (torque)-angular momemtum relationship

$$\sum \vec{M}_o = \left.\frac{d}{dt}\right|_{o_1} \vec{H}_o \tag{1}$$

where point O is taken to be the pivot.

Conservation of Angular Momentum (2 hours)

In the classroom, there is a rotating platform with a stool on it. We ask that you use it for the following exercises, being very careful both not to damage the bearings of the platform (keep your weight well centered) and to not damage yourself!

3. SKATER'S SPIN Sit on the stool, and start the platform rotating. Experiment with moving your arms in close to your body and stretching them out to your sides.

(a) What happens? Can you explain this qualitatively in terms of angular momentum conservation and moment of inertia?

(b) Quantitatively achieve an approximation of the change in angular velocity for this experiment by estimating the moment of inertia of your body about the axis both with your arms by your sides and with your arms stretched out. It is ok for these purposes to make drastic approximations to the geometry of your body (think rods and cylinders).

(c) By either having your mobile device taking data and holding it in your lap while you do this exercise, or by having a friend take video of the experiment and analyzing it (you can use loggerpro software downloadable from IT), you should be able to approximate the angular velocities with your arms out and with your arms in. Compare the change in angular velocity to that which you would predict with your approximations for the moments of inertia. How close did you get? Are you within order of magnitude? Within factor of two?

4. Also in the classroom is a bicycle wheel on an axle. Have a partner start this spinning, and hand it to you with the wheel axis vertical while you are sitting stationary on the rotating platform. Flip the wheel completely over, such that the axle is once again vertical, but the wheel is now spinning the other way.

   (a) What happens? Can you again explain this in terms of conservation of angular momentum (remember angular momentum is a vector!)?

   (b) Approximate the angular momentum of the bicycle wheel when it is spinning (you may assume all the mass is concentrated at the rim). Use this (and your approximation for the moment of inertia of your body that you calculated above) to approximate how fast you will be spinning after flipping the wheel over. Again, compare your calculation to observation from measurements with your mobile device or simple video analysis. How close did you get?

5. In the classroom are a few tops. Play with them and make observations of their motion.

   (a) Draw a free body diagram of a spinning top when its axis is completely vertical. What direction is the angular momentum vector of the spinning top?

   (b) Now draw a free body diagram of a spinning top when its rotation axis is tilted by an angle ϕ with respect to the vertical. What direction is the angular momentum vector?

   (c) What is the torque on the top due to gravity and taken about the pivot point sitting on the desk? (Assume the top has mass m and that the center of mass is a distance h from the point of the top). What direction is this torque vector?

   (d) As we discussed in class, we can write the rotational equation of motion as $\vec{\tau} = \frac{d\vec{L}}{dt}$ or, in words, the torque (moment) equals the time rate of change of the angular momentum vector. If this is true, what is the direction of the *change in the angular momentum* of the top?

   (e) Draw a picture of this showing the tilted top, the direction of the angular momentum vector and the direction of $d\vec{L}$, which is the vector change in the angular momentum. If the angular momentum changes by $d\vec{L}$ in the time interval dt, indicate in your picture where the new angular momentum vector will be at that new time dt later.

(f) We can repeat this for each infinitesimal time interval dt. What is the net resulting motion of the change in angular momentum that results from the gravitational torque???

(g) EXTRA We can use simple geometry as a shortcut to find an expression for the frequency of this rotation about the vertical direction (termed precession) which results from the gravitational torque. Use your expression for the torque to find $\frac{d\vec{L}}{dt}$. In the same interval dt that the axis of the top will moves from $\vec{L}$ to $\vec{L} + \vec{d}L$ the angular positions moves through an angle $d\theta$ which defines the angular velocity $\frac{d\theta}{dt}$. Can you use this to find the angular velocity of the precession in terms of the original angular momentum of the top?

Mystery Cube(sat) (3 hours)

In class we started to learn about the moment of inertia tensor of a three dimensional object. In this exercise we will be investigating a 'mystery cube' using just the on-diagonal terms of this tensor.

We will be using a rotation table to measure the moment of inertia of objects around various axes, and use that information to deduce things about the mass distribution in our object.

6. In order to use the rotation table, you need to understand the equations that govern its motion. The rotation table consists of a rotational platform. Around the axis of this platform are spindles of various different radii, around which we can wind a string. This string then spins out over a pulley and is connected to a hanging mass. When we drop the mass, it pulls the string over the pulley, exerting a force out the outside radius of the spindle around which the string is wrapped. Keeping in mind that the tension in a string is constant throughout the string:

(a) Draw a free body diagram for the hanging mass, and use this to write the equation of motion for the acceleration of the mass when it is falling.

(b) Draw a free body diagram for the rotation table and use this to write down an equation of motion for the angular acceleration of the rotating platform when there is tension on the string.

(c) The tension in the string is constant, so we can use that to combine the two equations into one, but that is not enough: we need to relate the linear acceleration of the falling mass to the rotational acceleration of the rotating table. Assuming the string does not stretch, what is this relation?

(d) Use the combined equations of motion to find an expression for the moment of inertia of the platform (and anything on it) in terms of the mass hanging on the string and its linear acceleration (which we will measure).

7. Ok, now we have what we need to do measurements around a given axis (the axis of the rotation table) of any object! Go up to AC 426 where a whole bunch of rotation tables have been set up by our fabulous ninjas. The measurement of the linear acceleration of the falling mass is done via a relatively antiquated DAQ systems made by Vernier. You will need to download the Vernier Loggerpro software from the IT website before you begin: it is reasonably small and quick to download. If you plug in the LabPro unit to your computer before you start the software, it may auto-recognize the photogate. If it does not, go to the Experiment » Set Up Sensors » Show All Interfaces, and drag the photogate icon to Dig/Sonic1. To set up for the rotation table experiment, we can use a shortcut to avoid a lot of random calculations: go to File » Open » Probes & Sensors » Photogates » Pulley and press Connect. This will set everything up to measure the linear acceleration of the string going over the pulley, which is the same as the linear acceleration of the falling mass.

(a) First, to test your system, measure the moment of inertia of the solid ABS plastic gray platform. (You should get something around 0.0091 kgm^2).

(b) Next there are a few uniform cylindrical objects in the lab. You can measure the total mass and the dimesions of these objects. Assume they are of uniform density and calculate the moment of inertia about the cylindrical axis. Then measure this using the rotation table and compare with your calculation.

(c) Now I would like you to calculate and measure the moment of inertia of a textbook about an axis through the center of the book. You can accurately approximate the textbook as a rectangular prism. The integrals for the moment of inertia for this are substantially uglier to calculate, so you will probably want to set this up in Matlab using integral2 (or integral3). This should be just like some of the calculations you did for integrating moment for the boat project last semester, except using r^2 instead of r. Make sure to compare your calculation and your experimental measurement to each other! How close did you get?

(d) Lastly! The fun thing about a Cubesat is that it is definitely not uniform in its mass distribution. To get you used to this idea and how this affects things, we would like you to explore one of the 'Mystery Cubes' which have been prepared for you. These are foam cubes which each have an additional mass secured inside in an unknown position (you may treat this additional mass as a point mass). Your job is to figure out WHERE that additional mass is located through measurement. Things to consider:

i. Hold the cube in one hand and try to balance it: can you feel the anomaly in the mass distribution? (What does it mean to balance something?) Can you approximate the location of the added mass from this information?

ii. Try tossing the cube with a partner (try not to damage it!!! your job gets much harder if the cube is no longer a cube). Can you see differences in the way it spins in the air based on the difference in mass distribution (what would you expect?).

iii. Try measuring the moment around the axes centered on each face of the cube using the rotating platform. (You will want to substitute the cardboard platform for the ABS platform. Why??? Make sure you measure the I of the cardboard platform, too. Also, you may need the double-sided tape to secure the cube.). Can you start to get quantitative information about where the extra mass is located? Don't forget that you will want to take into account the moment of inertia of the platform and of the foam cube in your calculations!

iv. Does this give you enough information to localize the mass? What if I told you how much the additional mass weighs? If I didn't give you that information, What additional information can you get to further localize the added mass?

(e) Postscript: During these exercises, you were at times spinning objects around axes which did not pass through their center of mass. What happened when you did this? Why did this happen?

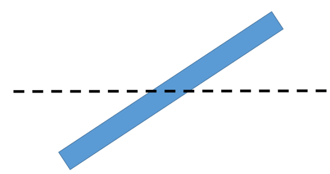
Properties of Mass Moment of Inertia (4 hours)

We know you love volume integrals, but for complicated objects, Solidworks can calculate the mass moment of inertia properties of an object for you. Click on the Mass Properties tool (under the Tools tab) to display the part's mass moment of inertia. This command will display the following mass moment of inertia properties of your object:

- The principal moments and principal axes of the object (we'll define these later) taken at the center of mass.
- The moments of inertia of the object taken aligned with the Solidworks reference coordinate system but around the center of mass.
- The moments of inertia of the object aligned with the Solidworks reference coordinate system and with the origin at the reference coordinate system origin.

The following exercise is made to help you think about the moment of inertia tensor and its dependence on coordinate system.

8. Create a thin rod with a square or rectangular cross section in SolidWorks and have SolidWorks calculate the moment of inertia properties.
   (a) First, look at the principal axes. What is special about these directions (how do they relate to the shape of the object)? What do rotations about these axes look like?
   (b) Look at the principal moments of inertia. What do you notice about these numbers? What does it mean for one moment of inertia to be much smaller than the other two? Can you understand this in terms of the general definition of moment of inertia in terms of mr^2?
   (c) Look at the moments of inertia in the two other coordinate systems. What do you notice? Depending on how you created your rod with respect to Solidworks' default reference coordinate system these may or may not be different from each other. What are the values of the off-diagonal terms?
   (d) Now, create a new reference coordinate system, by going to the features menu and selecting reference geometry and choosing coordinate system from the menu. Highlight the origin selection box and click on one of the corners of your rod, then select the x axis box and click on one of the diagonally-located corners of your rod (ie, you want to create a reference coordinate system which is NOT aligned with the axis of the rod). Calculate the moment of inertia properties of the rod with respect to this new coordinate system by using the 'report coordinate values relative to' selection in mass properties. What has changed about your principal axes? What has changed about your moments of inertia with respect to this new coordinate system?



Hopefully, there are a few things you noticed in this activity:

- The moment of inertia depends on origin.
- The moment of inertia depends on coordinate system.
- The principal axes are aligned with the axes of symmetry of the object (if it has symmetries).
- In the principal axes coordinate system, the moment of inertia matrix is diagonal.
- In coordinate systems not aligned with the principal axes, the off-diagonal elements of the moment of inertia matrix are generally not zero.

9. To understand the significance of the off-diagonal terms in the moment of inertia tensor, consider a rotation of your rod around an axis like that picture to the right. For the purposes of this exercise, we will consider a rotation axis which goes through the center of mass of your rod, but makes an angle of $\theta \neq 90$ with the long axis of the rod. (Note: this is very similar to the problem of the two masses from in class).
   (a) What does this rotation look like? Use a pencil or other object to visualize the rotation about this axis.
   (b) Is this rotation stable? In other words, if you were to start your rod spinning about this axis with no external forces or torques present, would it keep spinning about this axis? Why or why not? If not, what would happen? (If it helps, think about the 'centrifugal force' associated with each part of the rod. What will that centrifugal force tend to do to the rod as it spins?)
   (c) Consider the angular momentum associated with this rotation. What is the direction of the instantaneous angular momentum vector? What is happening to the angular momentum vector as a function of time?
   (d) The angular momentum is related to the rotation and the moment of inertia tensor as

$$\vec{L}_{physics} = \vec{H}_{ME} = \overleftrightarrow{\mathbf{I}}\vec{\omega} \tag{2}$$

   If $\overleftrightarrow{\mathbf{I}}$ is not diagonal in the coordinate system for which $\vec{\omega}$ is along a coordinate direction or equivalently we work in a coordinate system for which $\overleftrightarrow{\mathbf{I}}$ is diagonal and consider an $\vec{\omega}$ which is not along a single coordinate direction, will $\vec{H}$ in general be parallel to $\vec{\omega}$? If $\vec{H}$ is not parallel to $\vec{\omega}$ what happens to the direction of the vector $\vec{H}$ as the object undergoes its rotation?

Now the Fun Part

10. When you were introduced to angular momentum, you learned that the angular analog of $\vec{F} = \frac{d\vec{p}}{dt}$ is that an applied torque (moment) results in a change in angular momentum:

$$\vec{\tau}_{physics} = \frac{d\vec{L}}{dt} \tag{3}$$

$$\vec{m}_{ME} = \frac{d\vec{H}}{dt} \tag{4}$$

(These are the same equation: one written in physics notation, one in ME notation). But equations go both ways: if an object is spinning about a non-principal axis, the time dependent angular momentum means that there must be a torque acting on the system to keep the object spinning around this axis. If this torque is removed the system will not keep spinning about the same axis. Use this relation and your Solidworks calculated principal moments of inertia to quantify how much torque you would need to keep your rectangular rod spinning with an angular velocity ω around the axis at an angle θ from the long axis of the rod?

- Note this means that if you use a motor to spin something off of a principal axis, the 'for every action there is an equal and opposite reaction' thing means that there are all sorts of nasty torques on your axle which can eventually cause fatigue failure of the axle.

Diagonalizing the Mass Moment of Inertia

Hopefully by this point, the significance of having off-diagonal elements in the moment of inertia tensor, and the important of knowing the principal axes and moments of an object have been impressed upon you! So, we need a way to figure out what the principal axes of an object are...or to put it another way, in which coordinate system is the moment of inertia tensor diagonal?

Let's state this another way. If we are in the principal axes coordinate system, then the moment of inertia tensor is diagonal, so if we consider the angular momentum resulting from a rotation around a principal axis, it is along the same direction and simply proportional to the angular velocity vector:

$$\overleftrightarrow{\mathbf{I}}\,\vec{\omega} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \omega_x \\ 0 \\ 0 \end{bmatrix} = \lambda_1 \omega_x \hat{i} = \lambda_1 \vec{\omega} = \vec{H} \tag{5}$$

The simple relation

$$\overleftrightarrow{\mathbf{I}}\,\vec{\omega} = \lambda_1 \vec{\omega} \tag{6}$$

should be something you recognize from last semester It is the eigenvector equation! If $\vec{\omega}$ is along an eigenvector of the moment of inertia tensor this will be true, and the principal moment of inertia associated with this direction is the eigenvalue of the tensor associated with that eigenvector. So if we have the moment of inertia tensor for an object in an arbitrary coordinate system, by finding the eigenvalues and eigenvectors of the tensor, we can find the principal axes and principal moments of the object, and hence know about which axes it will stably rotate.

Effect of Center of Mass

If we are going to uniquely separate rotational motion of a body from translational motion, then the rotational motion will be (by definition) taken around the center of mass of the object. However, it is not always convenient to calculate the moment of inertia tensor around the center of mass. If we calculate the moment of inertia tensor with respect to a different origin, we can relate the two via the moment of inertia due to the displacement of the center of mass. The moment of inertia of the center of mass can be regarded as that of a point mass. If the center of mass is located at a vector displacement $\vec{a} = \left[a_x, a_y, a_z\right]$ from the origin we are using to calculate the moment of inertia tensor we can write:

$$\overleftrightarrow{\mathbf{I}} = \overleftrightarrow{\mathbf{I}'} - M \begin{bmatrix} a^2 & a_x a_y & a_x a_z \\ a_x a_y & a^2 & a_y a_z \\ a_x a_z & a_y a_z & a^2 \end{bmatrix} \tag{7}$$

Where M is the total mass of the object, $\overleftrightarrow{\mathbf{I}}$ is the moment of inertia with respect to the center of mass and $\overleftrightarrow{\mathbf{I}'}$ is the moment of inertia with respect to the other origin. This is referred to as the parallel axis theorem. It is reasonably easy to derive directly from the moment of inertia formula, but I will not show the derivation here. It is available in any dynamics textbook.

Putting it All Together

11. Create a rectangular solid body in Solidworks with one vertex located at the origin and edges coincident with the X, Y, and Z directions. Make the sides along each direction a different length, i.e., all cross-sections should be rectangular, not square. Assign the body a material by right-clicking the part at the top of the feature tree in the left side of the window.

    (a) Verify by hand calculations the moments of inertia with respect to the center of mass and with respect to the ’output coordinate system’ (the part’s reference coordinate system). Use the parallel axis theorem for the latter.

(b) Verify the principal mass moments of inertia and the direction of the principal axes. The inertia matrix will already be diagonal for a symmetric body.

(c) Create a new coordinate system (Go to the top menu and follow 'Insert-Reference Geometry-Coordinate System'). Align it in some direction in space. Display the mass properties with respect to the new coordinate system.

(d) Verify the principal moments and axes from the non-diagonal tensor in your new coordinate system by calculating the eigenvalues and eigenvectors of the inertia matrix using MATLAB. (Note Solidworks give you the absolute values of the moment term. You'll need to account for the negative sign for the off diagonal terms).

(e) Modify the block or make a new part that does not have any symmetry. Use Solidworks to compute the inertia matrix with respect to a reference frame centered at the center of mass and parallel to the output coordinate system. Transfer that inertia tensor into MATLAB and determine the corresponding principal moments of inertia and principal axes. Do the results match Solidworks' principal inertia tensor? What do the principal axes of a non-symmetric body capture? Are they in obvious places?