Laplace Transform: Forced Systems

November 3, 2016

Introduction

Goals

In this assignment, you will

- Use the Laplace transform to characterize the output of systems with driven inputs.

Note that there are two problems to turn in. The second one is long. You should spend any QEA time you have left after you are done with the reading and problems on your project.

Introduction

Up to now, you have used the Laplace transform to analyze undriven systems. These are systems which start with certain initial conditions but do not have "input" signals. The output of those systems can be viewed as the natural behavior of the systems.

Here, we shall consider systems with input signals. Such systems are called driven, or forced systems, because the external input drives the system. The resulting output is some combination of the driving function, and the natural behavior of the system.

For instance, consider a system governed by the following differential equation

$$\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + y(t) = x(t)\,, \tag{1}$$

where $y(t)$ is the output signal and $x(t)$ is the input/driving/forcing signal.

Readings and Videos

- Please do the following reading on solving IVPs, most of which are driven systems. Paul's Online Math Notes: Solving IVPs with the Laplace Transform. Note that these notes refer to the unit step function as the Heaviside function. You can refer to this link if you want to read more about the Heaviside function, but all you need to know is the notation: Paul's Online Math Notes: Step Functions.

- Please watch the following video which steps through solving driven problem using the Laplace transform. Example of initial value problem via Laplace transform.

Problems

1. Consider a system described by the following differential equation

$$\frac{d}{dt}y(t) + 2y(t) = x(t) \tag{2}$$

where $x(t) = u(t)$ is a unit step function, with the step at $t = 0$. Suppose that $y(0) = 0$. Find $y(t)$ for $t \geq 0$.

2. Consider the simplified model for a car suspension (similar to the problem you encountered at the end of Module 1) shown in Figure 1. We will treat the input as $x(t)$ which is the height of the bottom

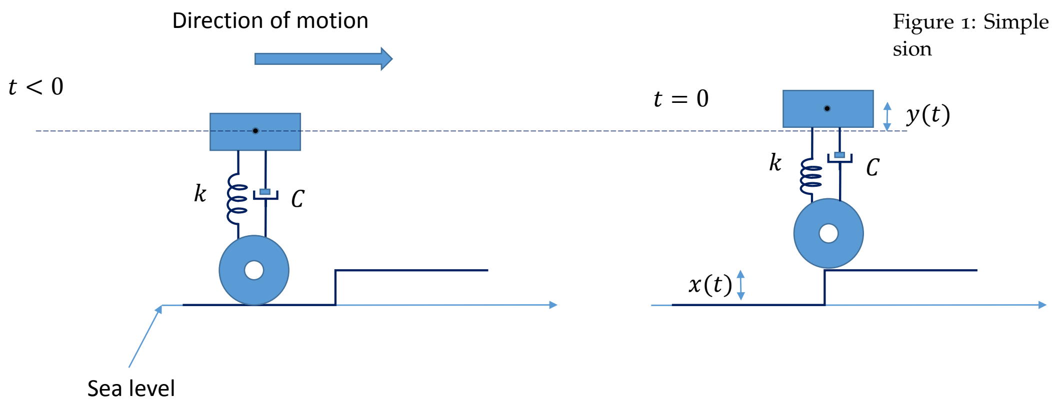

Figure 1: Simple Model for Car Suspension

of the wheel above sea level. The output $y(t)$ is the displacement of a reference point on the body of the car from the dashed line (which is where the reference point is at rest).

The differential equation relating $y(t)$ and $x(t)$ is

$$m\frac{d^2}{dt^2}y(t) + c\frac{d}{dt}y(t) + ky(t) = c\frac{d}{dt}x(t) + kx(t). \tag{3}$$

The constants m, k and c refer to the mass, spring constant, and damping constant of the suspension system.

Suppose that the car moves from left to right on a flat surface, starting at some time in the past and encounters an abrupt bump at time $t = 0$. We model the input as the following step function

$$x(t) = u(t). \tag{4}$$

Videos to set up parts (a), (b), and (e) below are available at the following links. Please feel free to watch them before you attempt the problems, but you don't have to.

- Problem 2 (a)
- Problem 2 (b)
- Problem 2 (e)

(a) Assuming $y(0^-) = 0, x(0^-) = 0$, and $y'(0^-) = 0$, show that

$$Y(s) = \frac{cs + k}{s(ms^2 + cs + k)} \tag{5}$$

(b) Let a_1 and a_2 be the two roots of the polynomial $ms^2 + cs + k$.
If $a_1 \neq a_2$, we can use partial-fraction expansion to write $Y(s)$ as

$$\begin{aligned} Y(s) &= \frac{cs+k}{s(s-a_1)(s-a_2)} \\ &= \frac{k}{a_1 a_2 s} + \frac{a_1 c + k}{a_1(a_1 - a_2)(s-a_1)} + \frac{a_2 c + k}{a_2(a_2-a_1)(s-a_2)} \end{aligned} \tag{6}$$

If $a_1 = a_2$, we can write the following

$$\begin{aligned} Y(s) &= \frac{cs+k}{s(s-a_1)^2} \\ &= \frac{k}{a_1^2 s} - \frac{k}{a_1^2(s-a_1)} + \frac{a_1 c + k}{a_1(s-a_1)^2} \end{aligned} \tag{7}$$

Qualitatively describe how the response of the car differs if a_1 and a_2 are purely real, and if a_1 and a_2 have imaginary components.

(c) Write an expression for $y(t)$ for $t > 0$ in terms of m, c, k, a_1 and a_2.

(d) Suppose that $m = 10^3$ kg, $c = 10^2$ Ns/m and $k = 10^4$ N/m. Plot $y(t)$ from time $t = -1$ to $t = 2$.

(e) What should the relationship between m, c and k be such that the system does not oscillate. How would you change c and/or k to ensure that the car does not oscillate?

(f) Find nominal values of c and/or k so that the car does not oscillate. On the same axes as the plot in part 2d, plot $y(t)$ for these values to verify your answer.

(g) Find values of c and k so that $a_1 = a_2$. On the same axes as the plot in parts 2d and 2f, plot $y(t)$ for these values of c, and k.

(h) Qualitatively describe how your three graphs differ.

Please spend any additional time you have left working on your project.