COMBINATORIAL STRUCTURES FOR KNOT THEORY

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1. Basic definitions

Definition 1.1. We let Λ denote the group freely generated by elements ρ and χ subject only to relations $\rho^4 = \chi^2 = 1$. We put $\lambda = \rho^{-1}\chi$ and $\pi = \rho^2\chi$. We define $c\colon \Lambda \to \{\pm 1\}$ by $c(\rho) = c(\chi) = -1$ (so $\ker(c)$ is the smallest normal subgroup containing λ).

Definition 1.2. A *combinatorial link universe* (or just *universe*) is a finite set A with an action of Λ such that

(a) The subgroups $\langle\chi\rangle \simeq C_2$ and $\langle\rho\rangle \simeq C_4$ both act freely on A.
(b) For each $a \in A$, the stabiliser group is contained in $\ker(c)$.

Remark 1.3. It actually follows from (b) that χ has no fixed points, so half of axiom (a) is redundant. It also follows from (b) that ρ can have no fixed points, but ρ^2 could have fixed points. Thus, we could replace axiom (a) by the assumption that ρ^2 has no fixed points.

Example 1.4. The smallest nonempty example is the set $\mathbb{Z}/4$ with $\rho(i) = 1 + i$ and $\chi(i) = 1 - i$. This is called a *loop*.

Remark 1.5. Suppose we start with a geometric link universe $U_0 \subset \mathbb{R}^2$, with crossing set X_0. Suppose also that there are no isolated circles. A *directed arc* for U_0 is pair (u, m), where u a connected component of $U_0 \setminus X_0$, and m is an orientation of u. We define A to be the set of directed arcs. We define $\chi\colon A \to A$ by $\chi(u, m) = (u, -m)$. Next, we can use m to define the starting point $x = \sigma(u, m) \in X_0$ of (u, m). Moving anticlockwise around x, let v be the next component of $U_0 \setminus X_0$ that we encounter, and let n be the unique orientation of v such that the part of v that we encounter is the starting end of v relative to n. We then put $\rho(u, m) = (v, n)$. This gives a combinatorial link universe. To see that axiom (b) is satisfied, note that the sum of the arcs defines a mod two homology class u on U_0. For $x \in \mathbb{R}^2 \setminus U_0$, define $\mu(x) = 1$ if u becomes trivial in $H_1(\mathbb{R}^2 \setminus \{x\}; \mathbb{Z}/2)$, and $\mu(x) = -1$ otherwise. Then define $\theta\colon A \to \{\pm 1\}$ by taking $\theta(a)$ to be the value of μ on the left hand side of a (with respect to the given orientation). We find that $\theta\rho = \theta\chi = -\theta$, so $\theta\alpha = c(\alpha)\theta$ for all $\alpha \in \Lambda$, and axiom (b) follows from this.

If we want to handle a geometric universe that contains isolated circles, we can just add a small twist to each such circle.

Definition 1.6. Given a universe A:

- The *blocks* are the Λ-orbits in A. Each block can be considered as a universe.
- The *components* are the $\langle\chi, \rho^2\rangle$-orbits in A.
- The *crossings* are the $\langle\rho\rangle$-orbits in A.
- The *edges* are the $\langle\chi\rangle$-orbits in A.
- The *faces* are the $\langle\lambda\rangle$-orbits in A.
- A *chequer function* is a map $\theta\colon A \to \{\pm 1\}$ with $\theta\chi = \theta\rho = -\theta$.
- A *height function* is a map $\omega\colon A \to \{\pm 1\}$ with $\omega\rho = -\omega$.
- An *orientation* is a map $\delta\colon A \to \{\pm 1\}$ with $\delta\chi = \delta\rho^2 = -\delta$.

Remark 1.7. If a universe has n crossings, we see that it has $4n$ arcs and $2n$ edges.

Remark 1.8. Axiom (b) in Definition 1.2 is equivalent to the statement that each block admits a chequer function, and thus that the whole universe admits a chequer function.

Remark 1.9. Let Δ be the subgroup of Λ generated by ρ^2 and χ, so the element $\pi = \rho^2\chi$ lies in Δ. Note that $\chi\pi = \pi^{-1}\chi$ and $\rho^2\pi = \pi^{-1}\rho^2$. Every element of Δ can be written as π^m or $\pi^m\chi$ for some $m \in \mathbb{Z}$. The elements π^m have infinite order, but the elements $\pi^m\chi$ are involutions. We have $\pi^{2k}\chi = \pi^k\chi\pi^{-k}$, but $\pi^{2k+1}\chi = \pi^k\rho^2\pi^{-k}$, so every involution is conjugate to π or ρ^2.

Note that Δ acts on A in such a way that ρ^2 and χ are free involutions. As all other involutions in Δ are conjugate to χ or ρ^2, they also act freely. Thus, the stabiliser of each point is of the form $\langle\pi^m\rangle$ for some m. Moreover, we have $c(\pi) = -1$ so m must be even, say $m = 2k$. Note also that $\langle\pi^{2k}\rangle$ is normal in Δ, so we get a free action of $\Delta/\langle\pi^{2k}\rangle$ on the orbit. In particular, the size of each orbit is divisible by 4.

Definition 1.10. Let A be a universe, and let $X = A/\langle\rho\rangle$ be the set of crossings. Let $\sigma\colon A \to X$ be the natural projection, and put $\tau = \sigma\chi$. If $a \in A$ with $\sigma(a) = x$ and $\tau(a) = y$, we will say that a is an arc from x to y.

Definition 1.11. The *mirror image* of A is the same set with the same χ, but with ρ replaced by ρ^{-1}.

Remark 1.12. Note that the map $\lambda = \rho^{-1}\chi$ is conjuagate (via χ) to the inverse of $\rho\chi$. This means that the $\langle\rho^{-1}\chi\rangle$-orbits biject with the $\langle\rho\chi\rangle$-orbits, so a universe and its mirror image have the same number of faces.

To do:

- Basic examples: twist knots, torus knots, closures of braids, chains of links.

2. Geometric realisation

Definition 2.1. The *geometric realisation* of A is

$$|A| = ((A \times \Delta_2)/\sim),$$

where we take $\Delta_2 = \{(x, y, z) \in [0, 1]^2 \mid x + y + z = 1\}$, and the equivalence relation is generated by $(a; x, y, 0) \sim (\chi(a), y, x, 0)$ and $(a; 0, x, z) \sim (\lambda(a); x, 0, z)$.

Proposition 2.2. *The space $|A|$ is a closed surface, and the set of chequer functions bijects naturally with the set of orientations. There is a natural CW structure, with a two-cell for each face, a one-cell for each edge, and a zero-cell for each crossing. Thus, if there are m faces and n crossings then the Euler characteristic is $m - n$.*

Proof. Put $P = A \times \Delta_2$, and let $q\colon P \to |A|$ be the natural projection map. Note that P is compact Hausdorff, and our equivalence relation is easily seen to give a closed subspace of $P \times P$, so the quotient space $|A|$ is again compact Hausdorff. We say that a subset $T \subseteq P$ is *saturated* if $T = q^{-1}(q(T))$, or equivalently $t \in T$ whenever $t \sim t'$ for some $t' \in T$. We will cover P by saturated open sets as defined below.

Fix an arc $a \in A$.

- We put
  $$\widetilde{U}_a = \{(\rho^k(a); x, y, z) \in P \mid k \in \mathbb{Z}/4,\ x > 1/2\} \amalg \{(\chi\rho^k(a); x, y, z) \in P \mid k \in \mathbb{Z}/4,\ y > 1/2\}.$$
  Note here that the elements $\rho^i(a)$ are all distinct, and the elements $\chi\rho^k(a)$ are all distinct, but it can happen in exceptional cases that $\chi\rho^k(a) = \rho^m(a)$. Nonetheless, the conditions on x and y ensure that the above union is disjoint even in those cases.
- We put
  $$\widetilde{V}_a = \{(\chi^j(a); x, y, z) \in P \mid j \in \mathbb{Z}/2,\ xy > 0\}.$$
- Now let d be the smallest positive integer such that $\lambda^d(a) = a$, and put
  $$\widetilde{W}_a = \{(\lambda^k(a); x, y, z) \in P \mid k \in \mathbb{Z}/d,\ z > 0\}.$$

It is straightforward to check that each of these sets is open in P and is saturated, so the images $U_a = q(\widetilde{U}_a)$, $V_a = q(\widetilde{V}_a)$ and $W_a = q(\widetilde{W}_a)$ are open in $|A|$.

We now define maps $\widetilde{f}_a$, $\widetilde{g}_a$ and $\widetilde{h}_a$ from $\widetilde{U}_a$, $\widetilde{V}_a$ and $\widetilde{W}_a$ to $\mathbb{C}$. These will use the map $\phi_{\alpha\beta}\colon \Delta_2 \to \mathbb{C}$ given by

$$\phi_{\alpha\beta}(x, y, z) = (x + y)\exp\left(2\pi i\frac{x\alpha + y\beta}{x + y}\right).$$

It is straightforward to check that this is continuous even at $(0,0,1)$, and if $\beta-\alpha<2\pi$ it gives a homeomorphism from Δ_2 to $\{re^{2\pi it} \mid 0\leq r\leq 1,\ \alpha\leq t\leq\beta\}$. We put

$$\widetilde{f}_a(\rho^k(a);x,y,z)=2\phi_{2k/8,(2k+1)/8}(y,z,x) \qquad \widetilde{f}_a(\chi\rho^k(a);x,y,z)=2\phi_{2k/8,(2k-1)/8}(x,z,y)$$
$$\widetilde{g}_a(a;x,y,z)=x-y+z\sqrt{1-(x-y)^2}i \qquad \widetilde{g}_a(\chi(a);x,y,z)=y-x-z\sqrt{1-(x-y)^2}i$$
$$\widetilde{h}_a(\lambda^k(a);x,y,z)=\phi_{k/d,(k+1)/d}(x,y,z).$$

Note that $\widetilde{f}_a$ sends the equivalent points $(\rho^k(a);x,y,0)$ and $(\chi\rho^k(a);y,x,0)$ to $2i^k y$, and it also sends the equivalent points $(\rho^k(a);x,0,z)$ and $(\lambda^{-1}\rho^k(a);0,x,z)=(\chi\rho^{k+1}(a);0,x,z)$ to $2i^{k+1/2}z$. These are the only relations that occur in $\widetilde{U}_a$, so there is an induced map $f_a\colon U_a\to\mathbb{C}$. Similarly, $\widetilde{g}_a$ respects the relation $(a;x,y,0)\sim(\chi(a);y,x,0)$ and so induces a map $g_a\colon V_a\to\mathbb{C}$. Moreover, $\widetilde{h}_a$ respects the relations $(\lambda^k(a);0,x,z)\sim(\lambda^{k+1}(a);x,0,z)$ and so induces $h_a\colon W_a\to\mathbb{C}$. It is not hard to see that in each case, we get a homeomorphism with the open unit disc.

All claims in the proposition are now reasonably clear. □

Definition 2.3. We say that A is *spherical* if it admits a chequer function, and for each block B, the space $|B|$ has euler characteristic two (and so is homeomorphic to S^2).

3. Reidemeister moves

Definition 3.1. Let A be a universe, and let $a\in A$ be an arc with $\lambda(a)=a$ (so $\chi(a)=\rho(a)$) but $\chi\rho^2(a)\neq\rho^3(a)$. We define

$$A^*=A\setminus\{\rho^k(a)\mid k\in\mathbb{Z}/4\},$$

and we let ρ^* denote the restriction of ρ to A^*. Note that χ exchanges a and $\rho(a)$, so it preserves $A\setminus\{a,\rho(a)\}=A^*\amalg\{\rho^2(a),\rho^3(a)\}$. By assumption, χ does not exchange $\rho^2(a)$ and $\rho^3(a)$. It follows that the arcs $b=\chi\rho^2(a)$ and $c=\chi\rho^3(a)$ lie in A^* and are distinct, and that χ preserves $A^*\setminus\{b,c\}$. We define χ^* to be the unique involution on A^* that exchanges b and c but otherwise agrees with χ. Note that if θ is a chequer function on A then $\theta(b)+\theta(c)=0$ so $\theta|_{A^*}$ is a chequer function on A^*. Thus, A^* is again a universe. We define $R_1(A,a)=A^*$, and we call this operation *Reidemeister move 1.*

Remark 3.2. The condition $\chi\rho^2(a)\neq\rho^3(a)$ means that we do not allow a Reidemeister move that would convert a figure eight to an untwisted loop.

Definition 3.3. Let A be a universe, and let $a\in A$ be an arc with $\lambda^2(a)=a$, or equivalently $\chi\rho(a)=\rho^3\chi(a)$. This means that among the arcs $b_{ijk}=\chi^i\rho^j\chi^k(a)$ we have the coincidences $b_{110}=b_{031}$ and $b_{131}=b_{010}$, as well as the automatic identities $b_{101}=b_{000}$ and $b_{100}=b_{001}$. Suppose that there are no further coincidences, so we have 12 distinct arcs of the form b_{ijk}. We put

$$A^*=A\setminus\{b_{0jk}\mid j\in\mathbb{Z}/4,\ k\in\mathbb{Z}/2\},$$

and note that this is preserved by ρ. We define ρ^* to be the restriction of ρ to A^*. Next, we observe that the set

$$A'=A\setminus\{b_{ijk}\mid i,k\in\mathbb{Z}/2,\ j\in\mathbb{Z}/4\}$$

is preserved by χ, and that A^* consists of A' together with the four distinct points b_{111}, b_{120}, b_{121} and b_{130}. We let χ^* denote the unique involution on A^* that agrees with χ on A' and satisfies $\chi^*(b_{111})=b_{130}$ and $\chi^*(b_{120})=b_{121}$. One can check that any chequer function on A restricts to give a chequer function on A^*, so A^* is again a universe. We define $R_2(A,a)=A^*$, and we call this operation *Reidemeister move 2.*

Remark 3.4. Our auxiliary condition forbids all cases in which there are some monogons involved, but that should be harmless, because we can create or destroy monogons using Reidemeister move 1. Our auxiliary condition also forbids cases where Reidemeister move 2 would create a floating circle. This should again be harmless, because we can first add a twist to the circle, and then remove it.

Definition 3.5. Let A be a universe, and let $a\in A$ be an arc such that $\lambda^3(a)=a$. We will define a new universe A^*, with the same underlying set as A and the same action of χ, but with a different map ρ^* in place of ρ. To define this, let B be the subset of A consisting of the elements $b_{ijk}=\rho^{2i}\chi^j\lambda^k(a)$, for $i,j\in\mathbb{Z}/2$

and $k \in \mathbb{Z}/3$. We will assume that there are no coincidences among these elements, so $|B| = 12$. Using $\rho = \chi\lambda^{-1}$ we see that

$$\rho(b_{00i}) = b_{0,1,i-1} \qquad \rho(b_{01i}) = b_{1,0,i+1}$$
$$\rho(b_{10i}) = b_{1,1,i-1} \qquad \rho(b_{11i}) = b_{0,0,i+1}.$$

Thus, the set B is closed under ρ. We define ρ^* to be the same as ρ on $A \setminus B$, and we put

$$\rho^*(b_{00i}) = b_{1,1,i+1} \qquad \rho^*(b_{01i}) = b_{0,0,i-1}$$
$$\rho^*(b_{10i}) = b_{0,1,i+1} \qquad \rho^*(b_{11i}) = b_{1,0,i-1}.$$

One can check that $(\rho^*)^2(b_{ijk}) = b_{i+2,j,k} = \rho^2(b_{ijk})$. As the map $(\rho^*)^2 = \rho^2$ is a free involution, it follow that ρ^* gives a free action of C_4 on A^*. If θ is any chequer function for A, we see that $\theta(b_{ijk}) = (-1)^j\theta(a)$, and it follows that $\theta\rho^* = -\theta$, so θ is also a chequer function for A^*. Thus, A^* is again a universe. We define $R_2(A, a) = A^*$, and we call this operation *Reidemeister move 3*.

Remark 3.6. The auxiliary condition $|B| = 12$ forbids the case where $\lambda(a) = a$. I suspect that all other kinds of coincidences are automatically excluded by the definition of a universe, but I have not checked that completely.

Definition 3.7. Universes A and A' are *Reidemeister equivalent* if they are related by a series of Reidemeister moves. More explicitly, there should exist a sequence $B_0, \dots, B_r$ of universes where

- $B_0 \simeq A$ and $B_r \simeq A'$;
- For all i, either B_{i+1} is obtained (up to isomorphism) by applying a Reidemeister move to B_i, or B_i is obtained by applying a Reidemeister move to B_{i+1}.

To do:

- Effect of Reidemeister moves on crossings and orientations.

4. The Jones polynomial

Definition 4.1.
- A *link diagram* is a universe equipped with a height function $\omega\colon A \to \{\pm 1\}$.
- An *oriented universe* is a universe equipped with an orientation $\delta\colon A \to \{\pm 1\}$. In this context we put $A_+ = \delta^{-1}\{1\}$ and $A_- = \chi(A_+) = \delta^{-1}\{-1\}$.
- An *oriented link diagram* is a universe equipped with both a height function and an orientation.

Lemma 4.2. *Let A be an oriented universe. Then there is a unique map $\zeta\colon X \to A$ with $\sigma\zeta = 1$ and*

$$\delta\zeta(x) = \delta\rho\zeta(x) = 1, \qquad \delta\rho^2\zeta(x) = \delta\rho^3\zeta(x) = -1.$$

Definition 4.3. Let D be an oriented link diagram. The *sign* of a crossing $x \in X$ is $\omega\zeta(x)$. We also define the *writhe* $w(D) = \sum_{x \in X} \omega\zeta(x)$.

Definition 4.4. Let A be a link universe. A *state* of A is a free involution $\xi\colon A \to A$, such that for all a we have $\sigma(a) \in \{\rho(a), \rho^{-1}(a)\}$. We write $\Sigma(A)$ for the set of states. Given a state ξ, we put $\kappa_0(\xi) = A/\langle \chi, \xi\rangle$, and call this the set of *cut components* for ξ. We put $d(\xi) = |\kappa_0(\xi)|$.

Remark 4.5. Fix a directed arc a_0, and put $x = \sigma(a_0)$ and $a_i = \rho^i(a_0)$, so $\sigma^{-1}\{x\} = \{a_0, a_1, a_2, a_3\}$. Any state ξ must act on this set as a transposition pair, but it cannot exchange a_0 and a_2. Thus, there are precisely two possibilities: the restriction of ξ is either $(a_0\ a_1)(a_2\ a_3)$ or $(a_0\ a_3)(a_1\ a_2)$. Thus, we have $|\Sigma(A)| = 2^{|X|}$.

Definition 4.6. Now suppose we have a link diagram D with height function ω. Given a state ξ and a crossing $x \in X$, define $\phi(\xi, x) \in \{A, B\}$ as follows:

- If $\xi(a) = \rho^{-\omega(a)}(a)$ for all $a \in \sigma^{-1}\{x\}$, then $\phi(\xi, x) = A$.
- If $\xi(a) = \rho^{\omega(a)}(a)$ for all $a \in \sigma^{-1}\{x\}$, then $\phi(\xi, x) = B$.

(One can check that these are the only two possibilities.)

We also define

$$\phi(\xi) = \prod_{x\in X} \phi(\xi, x) \in \mathbb{Z}[A, B]$$

$$\psi(D) = \sum_{\xi} \phi(\xi) C^{d(\xi)} \in \mathbb{Z}[A, B, C].$$

We then let $\psi_0(D)$ denote the image of $\psi(D)$ in the ring

$$\mathbb{Z}[A, B, C]/(AB - 1, ABC + A^2 + B^2) = \mathbb{Z}[A^{\pm 1}].$$

Finally, if D also has an orientation we put $\psi_1(D) = (-A^{-3})^{w(D)}\psi_0(D)$. We call $\psi_1(D)$ the *Jones polynomial* of D.

Remark 4.7. It is traditional to normalise the Jones polynomial so that the Jones polynomial of an unlinked circle is 1; this has the effect that the Jones polynomial converts connected sums (in a sense that we have not defined here) to products. We have instead normalised the Jones polynomial so that $\psi_1(\emptyset) = 1$ and $\psi_1(D \amalg D') = \psi_1(D)\psi_1(D')$.

Theorem 4.8. *If D and D' are Reidemeister equivalent then $\psi_1(D) = \psi_1(D')$.*

To do:

- Discuss the skein relation.

5. Dowker-Thistlethwaite theory

Definition 5.1. Recall that $\Delta = \langle \chi, \rho^2 \rangle = \langle \chi, \pi \rangle$. For $n > 0$ we define $DT_0(n) = \mathbb{Z}/(2n) \times \{\pm 1\}$, with Δ-action by $\chi(i, \epsilon) = (i, -\epsilon)$ and $\pi(i, \epsilon) = (i + \epsilon, \epsilon)$ (so $\rho^2(i, \epsilon) = (i - \epsilon, -\epsilon)$).

Definition 5.2. Let σ be an involution on $\mathbb{Z}/(2n)$ such that $\sigma(i) = 1 - i \pmod 2$ for all i, and let $\phi\colon \mathbb{Z}/(2n) \to \{\pm 1\}$ be a map such that $\phi\sigma = -\phi$. We define $DT(\sigma, \phi)$ to be the set $\mathbb{Z}/(2n) \times \{\pm 1\}$ equipped with the maps $\chi(i, \epsilon) = (i, -\epsilon)$ and

$$\rho(i, 1) = \begin{cases} (\sigma(i), 1) & \text{if } \phi(i) = 1 \\ (\sigma(i) - 1, -1) & \text{if } \phi(i) = -1 \end{cases}$$

$$\rho(i, -1) = \begin{cases} (\sigma(i+1) - 1, -1) & \text{if } \phi(i+1) = 1 \\ (\sigma(i+1), 1) & \text{if } \phi(i+1) = -1. \end{cases}$$

Proposition 5.3. *$D(\sigma, \phi)$ is a universe, with underlying Δ-set $DT_0(n)$.*

Proof. It is clear that χ acts on $DT(\sigma, \phi)$ in the same way that it acts on $DT_0(n)$. A check of cases shows that ρ^2 also acts on $DT(\sigma, \phi)$ in the same way that it acts on $DT_0(n)$ (and the same then follows for $\pi = \rho^2\chi$). It is easy to see that ρ^2 has no fixed points, and that the map $\theta(i, \epsilon) = (-1)^i\epsilon$ is a chequer function. It follows that we have a universe as claimed. □

Proposition 5.4. *Let A be any universe whose underlying Δ-set is $DT_0(n)$; then there is a unique pair (σ, ϕ) such that $A = DT(\sigma, \phi)$.*

Proof. Each ρ-orbit must consist of two distinct ρ^2-orbits, and each ρ^2-orbit contains a unique point whose second component is $+1$. Thus, for each $i \in \mathbb{Z}/(2n)$ there exists a unique element $\sigma(i) \in \mathbb{Z}/(2n) \setminus \{i\}$ such that $(i, 1)$ and $(\sigma(i), 1)$ generate the same ρ-orbit. It is clear from this characterisation that σ is an involution without fixed points. Note that $(i, 1)$ and $(\sigma(i), 1)$ cannot be linked by ρ^2, so they must be linked by $\rho^{\pm 1}$. If $(\sigma(i), 1) = \rho^{\pm 1}(i, 1)$ then we find that $\pi^{i-\sigma(i)}\rho^{\pm 1}$ stabilises $(i, 1)$, so we must have $c(\pi^{i-\sigma(i)}\rho^{\pm 1}) = 1$, so $\sigma(i) = 1 - i \pmod 2$. Now put $\phi(i) = 1$ if $(\sigma(i), 1) = \rho(i, 1)$, and $\phi(i) = -1$ if $(\sigma(i), 1) = \rho^{-1}(i, 1)$. Let ρ_0 denote the map defined using σ and ϕ as in Definition 5.2. This has $\rho_0^2 = \rho^2$, and ρ_0 has the same orbits as ρ, and ρ_0 agrees with ρ on at least one point in each orbit; it follows easily that $\rho = \rho_0$. □

Corollary 5.5. *Let A be a universe with only one component. Then $A \simeq D(\sigma, \phi)$ for some σ and ϕ.*

Proof. Using Remark 1.9 we see that A is Δ-equivariantly isomorphic to $DT_0(n)$ for some n. Given this, the claim follows easily from Proposition 5.4. $\square$

Remark 5.6. Note that $DT(\sigma, -\phi)$ is the mirror image of $DT(\sigma, \phi)$, so it is spherical if and only if $DT(\sigma, \phi)$ is spherical.

Theorem 5.7. *Let σ be an involution on $\mathbb{Z}/(2n)$ with $\sigma(i) = 1 - i \pmod 2$, and put*

$$\Phi = \{\phi \mid DT(\sigma, \phi) \textit{ is spherical } \}.$$

Then, under mild irreducibility conditions, we have either $\Phi = \emptyset$ or $\Phi = \{\phi, -\phi\}$ for some ϕ.

Proof. I think that this is an accurate reformulation of a result proved by Dowker and Thistlethwaite. However, I have not yet digested the proof. $\square$