COMBINATORIAL TANGLES

N. P. STRICKLAND

We want to define a category (or 2-category) of combinatorial tangles. The objects will be t-sets, as defined below.

Definition 0.1. A *t-set* is a finite, totally ordered set A equipped with a map $\epsilon_A \colon A \to \{\pm 1\}$. Given a t-set A, we put

$$\begin{aligned} A_+ &= \{a \in A \mid \epsilon(a) = 1\} \\ A_- &= \{a \in A \mid \epsilon(a) = -1\} \\ \|A\| &= \sum_i \epsilon_i = |A_+| - |A_-|. \end{aligned}$$

We write $\overline{A} = \{\overline{a} \mid a \in A\}$ ordered by $\overline{a} < \overline{b}$ iff $b < a$, and with $\epsilon_{\overline{A}}(\overline{a}) = -\epsilon_A(a)$. Given another t-set B, we define $A + B$ to be the set $A \amalg B$, ordered so that A precedes B, and with $\epsilon_{A \amalg B}$ given by ϵ_A on A and by ϵ_B on B.

Next, we define a groupoid $T(A)$ as follows (whose objects are called *combinatorial tangles* or just *tangles*). An object of $T(A)$ is a tuple $(X, P, \omega, \alpha, \beta, \phi)$ where

- X is a finite set (whose elements are called *crossings*).
- P is a finite set (whose elements are called *arcs*).
- ω is a map from P to $\{0, 1\}$. Arcs u with $\omega(u) = 0$ are called *closed*, and those with $\omega(u) = 1$ are called *open*. We write PC and PO for the subsets of closed and open arcs.
- α is a map from $\mathbb{Z}/4 \times X$ to PO.
- β is a map from A to PO.
- ϕ is a map from X to $\{\pm 1\}$.
- α and β together give a bijection $A_- \amalg \{3, 4\} \times X \to PO$, with inverse ϕ_0 say.
- α and β together give a bijection $A_+ \amalg \{0, 1\} \times X \to PO$, with inverse ϕ_1 say.

A morphism from $(X, P, \omega, \alpha, \beta, \phi)$ to $(X', P', \omega', \alpha', \beta', \phi')$ is a pair of bijections $X \to X'$ and $P \to P'$ that are compatible with the other structure.

The interpretation is supposed to be as follows. We identify A with the set $\{(0, i, 0) \in \mathbb{R}^3 \mid 1 \leq i \leq |A|\}$ and we suppose that we have an oriented tangle in the half-space $x \leq 0$ with boundary A. We suppose that the orientation is arranged so that the arcs run from negative points (ie $(0, i, 0)$ with $\epsilon_i = -1$) to positive points. We suppose that the tangle projects nicely to give a tangle diagram in the xy plane with crossing set X. We let PO denote the set of open arcs (homeomorphic to $[0, 1]$) in this diagram, and we let PC denote the set of freely floating circles. We then put $P = PO \amalg PC$, and define ω in the obvious way. Next, given $x \in X$ we let $\alpha(0, x), \dots, \alpha(3, x)$ be the four arcs that touch x, numbered so that in anticlockwise order around x we have the incoming end of $\alpha(0, x)$, the incoming end of $\alpha(1, x)$, the outgoing end of $\alpha(2, x)$ and the outgoing end of $\alpha(3, x)$. Also, for $a \in A$ we let $\beta(a)$ denote the unique arc that touches a. It is reasonably clear that the axioms are satisfied. We say that a combinatorial tangle is *geometric* if it arises as above.

Reidermeister moves for combinatorial tangles are as follows. Consider a tangle $T = (X, P, \omega, \alpha, \beta, \phi)$.

(1) Suppose there is a crossing x with $\alpha(2, x) = \alpha(3, x)$. We construct a new tangle $T' = (X', P', \omega', \alpha', \beta', \phi')$ as follows:
   - $X' = X \setminus \{x\}$
   - P' is the quotient of P in which all the elements $\alpha(i, x)$ are identified together to give a new element, which we call p.
   - ω' is the same as ω, except that $\omega'(p) = 0$ if $\alpha(0, x) = \alpha(1, x)$, and $\omega'(p) = 1$ otherwise.

- α' and β' are obtained in the obvious way by composing α and β with the quotient map $P \to P'$.
- ϕ' is the restriction of ϕ.

The construction of T' from T is the *first Reidermeister move.*

(2) Suppose instead that we have crossings x and y with $\alpha(2,x) = \alpha(1,y)$ and $\alpha(3,x) = \alpha(0,y)$ and $\phi(x) + \phi(y) = 0$. In this case we construct T' as follows:
- $X' = X \setminus \{x, y\}$
- P' is the quotient of P in which the elements $\alpha(0,x), \alpha(2,x) = \alpha(1,y)$ and $\alpha(3,y)$ are identified together to give p, and the elements $\alpha(1,x), \alpha(3,x) = \alpha(0,y)$ and $\alpha(2,y)$ are identified together to give q.
- ω' is the same as ω, except that $\omega'(p) = 0$ iff $\alpha(3,y) = \alpha(0,x)$, and $\omega'(q) = 0$ iff $\alpha(2,y) = \alpha(1,x)$.
- α' and β' are obtained in the obvious way by composing α and β with the quotient map $P \to P'$.
- ϕ' is the restriction of ϕ.

The construction of T' from T is the *second Reidermeister move.*

(3) Finally, suppose we have $x, y, z \in X$ with

$$\alpha(3,x) = \alpha(1,y) \qquad \alpha(2,x) = \alpha(1,z) \qquad \alpha(2,y) = \alpha(0,z).$$

We then put $T' = (X, P, \omega, \alpha', \beta, \phi)$, where α' is the same as α except

$$\begin{array}{llll} \alpha'(0,x) = \alpha(0,z) & \alpha'(1,x) = \alpha(1,x) & \alpha'(2,x) = \alpha(2,z) & \alpha'(3,x) = \alpha(3,x) \\ \alpha'(0,y) = \alpha(1,z) & \alpha'(1,y) = \alpha(1,y) & \alpha'(2,y) = \alpha(3,z) & \alpha'(3,y) = \alpha(3,y) \\ \alpha'(0,z) = \alpha(0,y) & \alpha'(1,z) = \alpha(0,x) & \alpha'(2,z) = \alpha(0,y) & \alpha'(3,z) = \alpha(2,x). \end{array}$$

This construction is the *third Reidemeister move.*

Now suppose we have two sign sequences A_0 and A_1. We define $T(A_0, A_1)$ to be $T(\overline{A_0} + A_1)$.

Now suppose we have a t-set A. We put $1_A = (\emptyset, A, 1, \emptyset, \beta, \emptyset)$, where $\beta\colon \overline{A} + A \to P = A$ is given by $\beta(\overline{a}) = \beta(a) = a$.

Now suppose we have

$$T_0 = (X_0, P_0, \omega_0, \alpha_0, \beta_0, \phi_0) \in T(A_0, A_1) = T(\overline{A_0} + A_1)$$
$$T_1 = (X_1, P_1, \omega_1, \alpha_1, \beta_1, \phi_1) \in T(A_1, A_2) = T(\overline{A_1} + A_2).$$

We put $T_1 \circ T_0 = (X, P, \omega, \alpha, \beta, \phi)$, where
- $X = X_0 \amalg X_1$
- P is the largest quotient of $P_0 \amalg P_1$ in which $\beta_0(a_1)$ is identified with $\beta_1(\overline{a_1})$ for all $a_1 \in A_1$.
- $\alpha\colon \mathbb{Z}/4 \times X \to P$ is obtained by combining α_0 and α_2, and composing with the quotient map $P_0 \amalg P_2 \to P$.
- $\beta\colon \overline{A_0} \amalg A_2 \to P$ is obtained by combining β_0 and β_2, and composing with the quotient map $P_0 \amalg P_2 \to P$.
- ω is 1 on the images of α and β, and 0 elsewhere.
- ϕ is given by ϕ_i on X_i.

References