A nonlinear user element (UEL) implementation for hyperelastic materials in Abaqus/Standard

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Analyses of large deformation of hyperelastic materials require a different finite element formulation than the small deformation analyses. This document briefly presents one of the approaches to develop finite element formulations for large deformation and its implementation in Abaqus/Standard using its user element *i.e.*, `UEL(...)` subroutine interface. Here, a second Piola-Kirchhoff stress-based total Lagrangian formulation has been adopted to implement using UEL. Some excerpts have been directly adopted or reprinted from Datta, 2024a.

Interested readers should consult textbooks on nonlinear solid mechanics and continuum mechanics (Bonet & Wood, 2008; Bower, 2009; Holzapfel, 2000), nonlinear finite element analyses (Belytschko et al., 2014; de Borst et al., 2012; Reddy, 2015; Wriggers, 2008; Zienkiewicz et al., 2014), textbooks and tutorials on Fortran programming, and Abaqus documentation (Dassault Systèmes, 2024).

1 Summary of total Lagrangian finite element framework

1.1 PK-I stress formulation

In referential (or material) configuration, the governing partial differential equation for stress equilibrium as well as the boundary conditions are given by,

$$
\begin{aligned}
\operatorname{Div}\mathbf{P} + \rho_R \mathbf{B} &= 0 && \text{in } \Omega_0, \\
\mathbf{u} &= \mathbf{g} && \text{on } \Gamma_g, \\
\mathbf{P}\cdot\mathbf{N} &= \mathbf{T} && \text{on } \Gamma_T.
\end{aligned}
\tag{1.1}
$$

where, $\partial\Omega_0 = \Gamma_g \cup \Gamma_T$ is the boundary of the referential domain Ω_0, and Γ_g and Γ_T are two complementary subsurfaces. Here, $\mathbf{P}$ is the first Piola-Kirchhoff (PK-I) stress, $\mathbf{B}$ is the body force per unit mass, ρ_{R} is the mass density in the reference state, and $\mathbf{T}$ is the Piola-Kichhoff traction.

The global weak form in the reference configuration is given by,

$$\mathcal{W}_{\mathbf{u}}(\mathbf{u}) = -\int\limits_{\Omega_0} \mathbf{P} : \mathrm{Grad}(\mathbf{W})\ dV + \int\limits_{\Omega_0} \rho_{\mathrm{R}} \mathbf{B} \cdot \mathbf{W}\ dV + \int\limits_{\Gamma_T} \mathbf{T} \cdot \mathbf{W}\ dS = 0, \tag{1.2}$$

where, $\mathbf{W}$ is the weight function that vanishes on $\Gamma_g \subset \partial\Omega_0$.

Using standard Galerkin discretization, where, $[\mathbf{N}_{\mathbf{u}}^a]$ is the nodal matrix and $[\mathbf{N}_{\mathbf{u}}]$ is the element matrix of the interpolation functions, and for three-dimensional cases, they are given by,

$$\mathbf{N}_{\mathbf{u}}^a = \begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix}, \quad \mathbf{N}_{\mathbf{u}} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \cdots & N_{\mathrm{nnode}} & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \cdots & 0 & N_{\mathrm{nnode}} & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \cdots & 0 & 0 & N_{\mathrm{nnode}} \end{bmatrix}. \tag{1.3}$$

The corresponding vector representation of nodal degrees of freedom of an element, $\mathbf{u}_e$ is given by,

$$\mathbf{u}_e = \begin{bmatrix} u_1^1 & u_2^1 & u_3^1 & u_1^2 & u_2^2 & u_3^2 & \cdots & \cdots & \cdots & u_1^{\mathrm{nnode}} & u_2^{\mathrm{nnode}} & u_3^{\mathrm{nnode}} \end{bmatrix}^{\top}_{\mathrm{ndim}*\mathrm{nnode}\times 1}. \tag{1.4}$$

The first Piola-Kirchhoff stress is not symmetric ($\mathbf{P} \neq \mathbf{P}^{\top}$). Thus, I will have,

$$\mathbf{P} : \mathrm{Grad}\,\mathbf{W} = \mathrm{Grad}\,(\mathbf{N}_{\mathbf{u}}) : \mathbf{P}\mathbf{W}_e = \mathbf{G}_{\mathbf{u}}^{\top}\mathbf{P}\mathbf{W}_e. \tag{1.5}$$

Here, the non-symmetric gradient matrix, $[\mathbf{G}_{\mathbf{u}}]$, is also contains repetitive sub-matrices in the following form,

$$\mathbf{G}_{\mathbf{u}} = \begin{bmatrix} \mathbf{G}_{\mathbf{u}}^1 & \mathbf{G}_{\mathbf{u}}^2 & \mathbf{G}_{\mathbf{u}}^3 & \cdots & \cdots & \mathbf{G}_{\mathbf{u}}^{\mathrm{nnode}} \end{bmatrix}_{\mathrm{ndim}^2\times\mathrm{ndim}*\mathrm{nnode}}, \tag{1.6}$$

where the sub-matrix, $[\mathbf{G}_{\mathbf{u}}^a]$, for two-dimensional and three-dimensional cases are given by,

$$\mathbf{G}_{\mathbf{u}}^a = \begin{bmatrix} N_{,1}^a & 0 \\ 0 & N_{,1}^a \\ N_{,2}^a & 0 \\ 0 & N_{,2}^a \end{bmatrix}_{\mathrm{ndim}^2\times\mathrm{ndim}}, \qquad \mathbf{G}_{\mathbf{u}}^a = \begin{bmatrix} N_{,1}^a & 0 & 0 \\ 0 & N_{,1}^a & 0 \\ 0 & 0 & N_{,1}^a \\ N_{,2}^a & 0 & 0 \\ 0 & N_{,2}^a & 0 \\ 0 & 0 & N_{,2}^a \\ N_{,3}^a & 0 & 0 \\ 0 & N_{,3}^a & 0 \\ 0 & 0 & N_{,3}^a \end{bmatrix}_{\mathrm{ndim}^2\times\mathrm{ndim}}. \tag{1.7}$$

Using this definition, I can write the element-level residual in the reference configuration as,

$$\mathbf{R}_{\mathbf{u}}^e(\mathbf{u}_e) = -\int\limits_{\Omega_0^e} \mathbf{G}_{\mathbf{u}}^{\top}\mathbf{P}(\mathbf{u}_e)\ dV + \int\limits_{\Omega_0^e} \rho_{\mathrm{R}} \mathbf{N}_{\mathbf{u}}^{\top}\mathbf{B}^e\ dV + \int\limits_{\Gamma_T^e} \mathbf{N}_{\mathbf{u}}^{\top}\mathbf{T}^e\ dS = 0. \tag{1.8}$$

For two-dimensional plane strain and plane stress cases and three-dimensional cases, the first Piola-Kirchhoff stress vector is defined as follows,

$$
\begin{aligned}
\mathbf{P} &= \begin{bmatrix} P_{11} & P_{21} & P_{12} & P_{22} \end{bmatrix}^{\top}_{\text{nstress}\times 1}, \\
\text{and} \quad \mathbf{P} &= \begin{bmatrix} P_{11} & P_{21} & P_{31} & P_{12} & P_{22} & P_{32} & P_{13} & P_{23} & P_{33} \end{bmatrix}^{\top}_{\text{nstress}\times 1}.
\end{aligned}
\tag{1.9}
$$

Since the first Piola-Kirchhoff stress, $\mathbf{P}$, is a nonlinear function of the displacement field, $\mathbf{u}_e$, it needs to be linearized following the standard Newton-Raphson procedure in which the element tangent matrix, $\mathbf{k}^e_{\mathbf{uu}}$, is given by,

$$
\begin{aligned}
\mathbf{k}^e_{\mathbf{uu}} = -\frac{\partial \mathbf{R}^e_{\mathbf{u}}}{\partial \mathbf{u}_e} &= \int_{\Omega^e_0} \mathbf{G}^{\top}_{\mathbf{u}} \frac{\partial \mathbf{P}}{\partial \mathbf{F}} \frac{\partial \mathbf{F}}{\partial \mathbf{u}_e}\ dV, \\
&= \int_{\Omega^e_0} \mathbf{G}^{\top}_{\mathbf{u}} \frac{\partial \mathbf{P}}{\partial \mathbf{F}} \frac{\partial \mathbf{N}_{\mathbf{u}}}{\partial \mathbf{X}}\ dV, \\
&= \int_{\Omega^e_0} \mathbf{G}^{\top}_{\mathbf{u}} \frac{\partial \mathbf{P}}{\partial \mathbf{F}} \mathbf{G}_{\mathbf{u}}\ dV, \\
&= \int_{\Omega^e_0} \mathbf{G}^{\top}_{\mathbf{u}} \mathbf{A} \mathbf{G}_{\mathbf{u}}\ dV.
\end{aligned}
\tag{1.10}
$$

In finite element literature, $\mathbf{A}$ is defined as the matrix form of the first elasticity tensor, $\mathbb{A} = \frac{\partial \mathbf{P}}{\partial \mathbf{F}}$. The dimension of the element stiffness (tangent) matrix is $[\mathbf{k}^e_{\mathbf{uu}}]_{\text{ndim}*\text{nnode}\times\text{ndim}*\text{nnode}}$.

1.2 PK-II stress formulation

By substituting the relation $\mathbf{P} = \mathbf{FS}$ into the weak form (in the previous section) for an arbitrary test function, I can have,

$$
\mathcal{W}_{\mathbf{u}}(\mathbf{u}) = -\int_{\Omega_0} \mathbf{FS} : \text{Grad}(\mathbf{W})\ dV + \int_{\Omega_0} \rho_{\text{R}} \mathbf{B} \cdot \mathbf{W}\ dV + \int_{\Gamma_T} \mathbf{T} \cdot \mathbf{W}\ dS = 0,
\tag{1.11}
$$

where, $\mathbf{S} = \mathbf{F}^{-1}\mathbf{P}$ is the second Piola-Kirchhoff stress and $\mathbf{F} = \text{Grad}(\mathbf{X})$ is the deformation gradient. Using the standard Galerkin discretization, the discretized element residual vector, $\mathbf{R}^a_{\mathbf{u}}$, corresponds to any degree of freedom (DOFs) within the element can be written as,

$$
\begin{aligned}
\mathbf{R}^a_{\mathbf{u}}(\mathbf{u}_e) &= -\int_{\Omega^e_0} \mathbf{FS} : \text{Grad}(\mathbf{N}^a_{\mathbf{u}})\ dV + \int_{\Omega^e_0} \rho_{\text{R}} \mathbf{N}^{a\top}_{\mathbf{u}} \mathbf{B}^e\ dV + \int_{\Gamma^e_T} \mathbf{N}^{a\top}_{\mathbf{u}} \mathbf{T}^e\ dS = 0, \\
\Rightarrow \mathbf{R}^a_{\mathbf{u}}(u_i) &= -\int_{\Omega^e_0} \frac{\partial N^a}{\partial X_I} F_{iJ} S_{JI}\ dV + \int_{\Omega^e_0} \rho_{\text{R}} N^a B^e_i\ dV + \int_{\Gamma^e_T} N^a T^e_i\ dS = 0.
\end{aligned}
\tag{1.12}
$$

The Voigt vector representation of the second Piola-Kirchhoff stress, $\mathbf{S}$, for three-dimensional and two-dimensional plane strain and plane stress cases, are given by,

$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{22} & S_{33} & S_{23} & S_{13} & S_{12} \end{bmatrix}^\top, \qquad \text{and} \qquad \mathbf{S} = \begin{bmatrix} S_{11} & S_{22} & S_{12} \end{bmatrix}^\top. \tag{1.13}$$

Following the standard procedure of the Newton-Raphson method, the component of the element tangent matrix is given by,

$$\begin{aligned} \mathbf{k}^{ab}_{\mathbf{uu}} &= -\frac{\mathbf{R}^a_{\mathbf{u}}}{\partial \mathbf{u}^b}, \\ \Rightarrow k^{ab}_{u_i u_k} &= \int_{\Omega^e_0} \frac{\partial N^a}{\partial X_J} S_{JL} \delta_{ik} \frac{\partial N^b}{\partial X_L}\, dV + \int_{\Omega^e_0} \frac{\partial N^a}{\partial X_J} \left(F_{iI} \mathbb{C}_{IJKL} F_{kK} \right) \frac{\partial N^b}{\partial X_L}\, dV. \end{aligned} \tag{1.14}$$

where,

$$\mathbb{C}_{IJKL} = 2\frac{\partial S_{JI}}{\partial C_{KL}} = \frac{\partial S_{JI}}{\partial E_{KL}} \tag{1.15}$$

is defined as the material tangent tensor (or second elasticity tensor).

The matrix form of the element tangent matrix $[\mathbf{k}^e_{\mathbf{uu}}]$ and the element residual vector, $\mathbf{R}^e_{\mathbf{u}}$, are given by,

$$\begin{aligned} \mathbf{k}^e_{\mathbf{uu}} &= \int_{\Omega^e_0} \left[\mathbf{G}^\top_{\mathbf{u}} \Sigma_{\mathbf{S}} \mathbf{G}_{\mathbf{u}} + (\mathbf{B}_{\mathbf{u}} \boldsymbol{\Sigma}_{\mathbf{F}})^\top \mathbf{D}_{\mathbb{C}} (\mathbf{B}_{\mathbf{u}} \Sigma_{\mathbf{F}}) \right] dV, \\ \mathbf{R}^e_{\mathbf{u}} &= -\int_{\Omega^e_0} (\mathbf{B}_{\mathbf{u}} \Sigma_{\mathbf{F}})^\top \mathbf{S}\, dV + \int_{\Omega^e_0} \rho_{\mathrm{R}} \mathbf{N}^\top_{\mathbf{u}} \mathbf{B}^e\, dV + \int_{\Gamma^e_T} \mathbf{N}^\top_{\mathbf{u}} \mathbf{T}^e\, dS. \end{aligned} \tag{1.16}$$

Here, $\mathbf{S}$ is the second Piola-Kirchhoff stress in Voigt vector form, $\Sigma_{\mathbf{F}}$ and $\Sigma_{\mathbf{S}}$ are expanded matrix forms of the deformation gradient, and PK-II stress tensor, respectively, $\mathbf{B}_{\mathbf{u}}$ is the symmetric gradient matrix (in the context of small deformation it is also known as strain-displacement matrix), $[\mathbf{G}_{\mathbf{u}}]$ is the non-symmetric gradient matrix, and $\mathbf{D}_{\mathbb{C}}$ is the Voigt matrix form of the material tangent.

Remark 1. Based on the PK-I and PK-II stress-based total Lagrangian finite element formulations, the following relation holds between the first and second elasticity tensors,

$$\mathbb{A}_{iJkL} = S_{JL}\delta_{ik} + F_{iI}\mathbb{C}_{IJKL}F_{kK}, \qquad \text{where,} \quad \mathbb{A}_{iJkL} = \frac{\partial P_{iJ}}{\partial F_{kL}} \text{ and } \mathbb{C}_{IJKL} = 2\frac{\partial S_{IJ}}{\partial C_{KL}}.$$

Similar to the interpolation function matrix, $[\mathbf{N}_{\mathbf{u}}]$, the symmetric gradient matrix (also known as strain-displacement matrix), $[\mathbf{B}_{\mathbf{u}}]$, is also composed of repetitive sub-matrices, $[\mathbf{B}^a_{\mathbf{u}}]$. For three-dimensional cases, the matrix form of $[\mathbf{B}_{\mathbf{u}}]$ and $[\mathbf{B}^a_{\mathbf{u}}]$ are given by,

$$\mathbf{B_u} = \begin{bmatrix} \mathbf{B_u^1} & \mathbf{B_u^2} & \mathbf{B_u^3} & \cdots & \cdots & \mathbf{B_u^{nnode}} \end{bmatrix}_{\text{nstress}\times\text{ndim}*\text{nnode}}, \text{ where, } \mathbf{B_u^a} = \begin{bmatrix} N^a_{,1} & 0 & 0 \\ 0 & N^a_{,2} & 0 \\ 0 & 0 & N^a_{,3} \\ 0 & N^a_{,3} & N^a_{,2} \\ N^a_{,3} & 0 & N^a_{,1} \\ N^a_{,2} & N^a_{,1} & 0 \end{bmatrix}_{\text{nstress}\times\text{ndim}}. \tag{1.17}$$

For two-dimensional plane stress/ strain and axisymmetric cases, $[\mathbf{B_u^a}]$ will have different forms and can be found in finite element textbooks (Fish & Belytschko, 2007; Hughes, 1987; Zienkiewicz et al., 2013, 2014).

$[\Sigma_\mathbf{F}]_{\text{ndim}*\text{nnode}\times\text{ndim}*\text{nnode}}$ is a square banded diagonal matrix of dimension $\text{ndim} * \text{nnode} \times \text{ndim} * \text{nnode}$, and for a three-dimensional case, it appears as,

$$\Sigma_\mathbf{F} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ F_{31} & F_{32} & F_{33} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & & \cdots & & & & & \\ & & & & \cdots & & & & & \\ & & & & \cdots & & & & & \\ & & & & \cdots & & & & & \\ & & & & \cdots & & & & & \\ & & & & \cdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & F_{11} & F_{12} & F_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & F_{21} & F_{22} & F_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & F_{31} & F_{32} & F_{33} \end{bmatrix}_{\text{ndim}*\text{nnode}\times\text{ndim}*\text{nnode}}. \tag{1.18}$$

For a two-dimensional plane-strain case, $\Sigma_\mathbf{F}$ can be reduced by eliminating every third row and column. $[\Sigma_\mathbf{S}]$ is the expanded PK-II stress matrix, and for a two-dimensional and a three-dimensional case, it is given by,

$$\Sigma_\mathbf{S} = \begin{bmatrix} \mathbf{S}_{11} & 0 & \mathbf{S}_{12} & 0 \\ 0 & \mathbf{S}_{11} & 0 & \mathbf{S}_{12} \\ \mathbf{S}_{12} & 0 & \mathbf{S}_{22} & 0 \\ 0 & \mathbf{S}_{12} & 0 & \mathbf{S}_{22} \end{bmatrix}, \quad \Sigma_\mathbf{S} = \begin{bmatrix} \mathbf{S}_{11} & 0 & 0 & \mathbf{S}_{12} & 0 & 0 & \mathbf{S}_{13} & 0 & 0 \\ 0 & \mathbf{S}_{11} & 0 & 0 & \mathbf{S}_{12} & 0 & 0 & \mathbf{S}_{13} & 0 \\ 0 & 0 & \mathbf{S}_{11} & 0 & 0 & \mathbf{S}_{12} & 0 & 0 & \mathbf{S}_{13} \\ \mathbf{S}_{12} & 0 & 0 & \mathbf{S}_{22} & 0 & 0 & \mathbf{S}_{23} & 0 & 0 \\ 0 & \mathbf{S}_{12} & 0 & 0 & \mathbf{S}_{22} & 0 & 0 & \mathbf{S}_{23} & 0 \\ 0 & 0 & \mathbf{S}_{12} & 0 & 0 & \mathbf{S}_{22} & 0 & 0 & \mathbf{S}_{23} \\ \mathbf{S}_{13} & 0 & 0 & \mathbf{S}_{23} & 0 & 0 & \mathbf{S}_{33} & 0 & 0 \\ 0 & \mathbf{S}_{13} & 0 & 0 & \mathbf{S}_{23} & 0 & 0 & \mathbf{S}_{33} & 0 \\ 0 & 0 & \mathbf{S}_{13} & 0 & 0 & \mathbf{S}_{23} & 0 & 0 & \mathbf{S}_{33} \end{bmatrix}. \tag{1.19}$$

$[\Sigma_\mathbf{S}]$ has a dimension of $[\Sigma_\mathbf{S}]_{\text{ndim}^2\times\text{ndim}^2}$. It is also possible to represent $\mathbf{G_u^a}$ and consequently $\mathbf{G_u}$ and $\Sigma_\mathbf{S}$ matrices in alternative matrix forms which will essentially give the same result (de Borst et al., 2012; Reddy, 2015).

Based on the finite element formulation presented in this document, a simplified pseudo-code or procedure for developing the Abaqus/Standard UEL subroutine is in **Procedure** 1.

Procedure 1: A large deformation total Lagrangian (PK-II) UEL subroutine implementation in Abaqus/Standard

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Input : `PROPS, COORDS, JELEM, JTYPE, NNODE, NDOFEL, TIME, DTIME, U, DU, V, A, PREDEF, JDLTYP, NDLOAD, MDLOAD, DDLMAG, ALMAG`
Output : `AMATRX, RHS, PNEWDT, ENERGY, SVARS`

Get $\texttt{nInt} \leftarrow \texttt{PROPS}$ and $\texttt{ndim}, \texttt{nstress} \leftarrow \texttt{JTYPE}$
Initialize: $\{\mathbf{N}^a_{\mathbf{u}}, \mathbf{B}^a_{\mathbf{u}}, \mathbf{G}^a_{\mathbf{u}}, \mathbf{N}_{\mathbf{u}}, \mathbf{B}_{\mathbf{u}}, \mathbf{G}_{\mathbf{u}}, \Sigma_{\mathbf{F}}, \Sigma_{\mathbf{S}}, \mathbf{k}^e_{\mathbf{uu}}, \mathbf{R}^e_{\mathbf{u}}\} = 0$
Get nodal displacement vectors of the element: $\mathbf{u}_e, \Delta\mathbf{u}_e$
Reshape nodal coordinate and displacement vector in matrix form: $\overline{\mathbf{x}_e}, [\overline{\mathbf{u}}_e], \overline{\Delta\mathbf{u}_e}$
Get $w_{\text{int}}, \boldsymbol{\xi}_{\text{int}} \leftarrow$ SUBROUTINE $\texttt{gaussQuadrtr}(\texttt{ndim}, \texttt{nNode})$
for $k = 1$ **to** nInt **do**
  Get $\mathbf{N}, \frac{\partial \mathbf{N_u}}{\partial \boldsymbol{\xi}} \leftarrow$ SUBROUTINE $\texttt{interpFunc}(\texttt{ndim}, \texttt{nNode})$
  Calculate: $\frac{\partial \mathbf{X}}{\partial \boldsymbol{\xi}} = \overline{\mathbf{x}_e} \frac{\partial \mathbf{N_u}}{\partial \boldsymbol{\xi}}$ `// map to the reference configuration to calculate` $\mathbf{F}$
  Calculate: $\frac{\partial \mathbf{N_u}}{\partial \mathbf{X}} = \frac{\partial \mathbf{N_u}}{\partial \boldsymbol{\xi}} \left(\frac{\partial \mathbf{X}}{\partial \boldsymbol{\xi}}\right)^{-1}$ and J_ξ
  Calculate deformation gradient: $\mathbf{F} = \mathbb{1} + [\overline{\mathbf{u}}_e] \frac{\partial \mathbf{N_u}}{\partial \mathbf{X}}$
  for $i = 1$ **to** nNode **do**
    Form $\mathbf{N}^a_{\mathbf{u}}(\texttt{ndim}, \texttt{NDOFEL})$, $\mathbf{B}^a_{\mathbf{u}}(\texttt{nstress}, \texttt{ndim})$, and $\mathbf{G}^a_{\mathbf{u}}(\texttt{ndim}^2, \texttt{ndim})$ matrices
    $\mathbf{N_u}(1:\texttt{ndim}, \texttt{ndim}*(\texttt{i}-1)+1:\texttt{ndim}*\texttt{i}) = \mathbf{N}^a_{\mathbf{u}}(1:\texttt{ndim}, 1:\texttt{ndim})$
    $\mathbf{B_u}(1:\texttt{nstress}, \texttt{ndim}*(\texttt{i}-1)+1:\texttt{ndim}*\texttt{i}) = \mathbf{B}^a_{\mathbf{u}}(1:\texttt{nstress}, 1:\texttt{ndim})$
    $\mathbf{G_u}(1:\texttt{ndim}^2, \texttt{ndim}*(\texttt{i}-1)+1:\texttt{ndim}*\texttt{i}) = \mathbf{G}^a_{\mathbf{u}}(1:\texttt{ndim}^2, 1:\texttt{ndim})$
  end
  `// end of nodal point loop`
  Get $\mathbf{S}, \mathbf{e}, \boldsymbol{\sigma}, \mathbf{D}_{\mathbb{C}} \leftarrow$ SUBROUTINE $\texttt{umatHyperelastic}(\texttt{PROPS}, \mathbf{F})$ `// UMAT returns the constitutive response`
  Form $\Sigma_{\mathbf{F}}(\texttt{ndim}*(\texttt{i}-1)+1:\texttt{ndim}*\texttt{i}, \texttt{ndim}*(\texttt{j}-1)+1:\texttt{ndim}*\texttt{j}) = \mathbf{F}$ `// double nested for loop over nNode with i=j`
  Form $\Sigma_{\mathbf{S}}(\texttt{ndim}*(\texttt{i}-1)+1:\texttt{ndim}*\texttt{i}, \texttt{ndim}*(\texttt{j}-1)+1:\texttt{ndim}*\texttt{j}) = S_{ij}\mathbb{1}$ `// double nested for loop over ndim`
  Calculate: $\mathbf{k}^e_{\mathbf{uu}} = \mathbf{k}^e_{\mathbf{uu}} + w_{\text{int}}(k)\det(J_\xi)\left(\mathbf{G}^\top_{\mathbf{u}}\Sigma_{\mathbf{S}}\mathbf{G_u} + (\mathbf{B_u}\Sigma_{\mathbf{F}})^\top \mathbf{D}_{\mathbb{C}}(\mathbf{B_u}\Sigma_{\mathbf{F}})\right)$
  Calculate: $\mathbf{R}^e_{\mathbf{u}} = \mathbf{R}^e_{\mathbf{u}} - w_{\text{int}}(k)\det(J_\xi)\left((\mathbf{B_u}\Sigma_{\mathbf{F}})^\top \mathbf{S} - \rho_{\text{R}}\mathbf{N}^\top_{\mathbf{u}}\mathbf{b}\right)$
end
`// end of integration point loop`
Assign $\texttt{AMATRX} = \mathbf{k}^e_{\mathbf{uu}}$ and $\texttt{RHS} = \mathbf{R}^e_{\mathbf{u}}$

The current implementation includes three-dimensional elements and two-dimensional plane-strain elements. Since Abaqus does not natively support viewing element output from the user element, a layer of additional dummy elements with `UVARM(...)` user subroutine from

Abaqus was used to view the results. The same set of elements can be used to apply the body force and traction-type boundary conditions on the user elements.

Remark 2. Formulation and implementation of two-dimensional plane stress and axisymmetric cases require some additional effort which has not been shown here. For plane stress cases, the same element matrix operators can be used as the plane strain, however, the constitutive calculation is different. Constitutive calculation requires enforcing the plane stress condition $\sigma_{33} = 0$ to obtain the third axial component of the deformation gradient, F_{33}, and consequently modifying the tangent moduli and calculate the components of the stress tensor (Zienkiewicz et al., 2014). On the other hand, for axisymmetric cases, the constitutive calculation is the same as three-dimensional cases but the element matrix operators, matrix form of the material/ spatial tangent tensor, and the stress vector are needed to be modified to account for the third axial component of the stress and strain tensor.

2 F-bar method to alleviate volumetric locking

de Souza Neto et al., 1996 proposed a simple and easily implementable approach called the F-bar method for fully-integrated bilinear quadrilateral and trilinear hexahedral elements. In this approach, the deformation gradient is decomposed into deviatoric and volumetric parts, and evaluated at the centroid of the element as well as at the integration points. By combining the volumetric deformation gradient from the centroid and the deviatoric deformation gradient from the integration points, a modified deformation gradient, $\overline{\mathbf{F}}$ is constructed. All the constitutive response is evaluated using this modified deformation gradient, and consequently, the residual and element tangent matrix are also modified.

To construct a modified deformation gradient, $\overline{\mathbf{F}}$, at the integration point of interest, the volumetric part from the centroid, $\mathbf{F}_0^{\text{vol}}$, is combined with the deviatoric part at the integration point, $\mathbf{F}^{\text{dev}}$, as below,

$$\overline{\mathbf{F}} = \mathbf{F}^{\text{dev}}\mathbf{F}_0^{\text{vol}} = \left(\frac{\det \mathbf{F}_0}{\det \mathbf{F}}\right)^{1/3} \mathbf{F}. \tag{2.1}$$

For two-dimensional plane strain cases, $\overline{\mathbf{F}}$, needs to be computed as,

$$\overline{\mathbf{F}} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$[\overline{\mathbf{F}}_{\text{pe}}]$}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \tag{2.2}$$

where, $[\overline{\mathbf{F}}_{\text{pe}}]_{2\times 2}$ is defined as the following

$$\overline{\mathbf{F}}_{\text{pe}} = \mathbf{F}^{\text{dev}}\mathbf{F}_0^{\text{vol}} = \left(\frac{\det \mathbf{F}_0}{\det \mathbf{F}}\right)^{1/2} \mathbf{F}_{\text{pe}}. \tag{2.3}$$

2.1 PK-I stress-based total Lagrangian F-bar formulation

Based on the definition of the modified deformation gradient, for three-dimensional cases, I can write the element residual as,

$$\mathbf{R}^e_{\mathbf{u}}(\mathbf{u}_e) = -\int\limits_{\Omega^e_0} \mathbf{G}^\top_{\mathbf{u}} \left(\frac{\det \mathbf{F}_0}{\det \mathbf{F}}\right)^{-2/3} \mathbf{P}\ dV + \int\limits_{\Omega^e_0} \rho_{\mathrm{R}} \mathbf{N}^\top_{\mathbf{u}} \mathbf{B}\ dV + \int\limits_{\Gamma^e_T} \mathbf{N}^\top_{\mathbf{u}} \mathbf{T}_e\ dS, \tag{2.4}$$

The matrix form of the element tangent matrix can be written as,

$$\mathbf{k}^e_{\mathbf{uu}} = \int\limits_{\Omega^e_0} \left(\frac{\det \mathbf{F}_0}{\det \mathbf{F}}\right)^{-1/3} \mathbf{G}^\top_{\mathbf{u}} \mathbf{A} \mathbf{G}_{\mathbf{u}}\ dV + \int\limits_{\Omega^e_0} \left(\frac{\det \mathbf{F}_0}{\det \mathbf{F}}\right)^{-2/3} \left(\mathbf{G}^\top_{\mathbf{u}} \mathbf{Q}_0 \mathbf{G}^0_{\mathbf{u}} - \mathbf{G}^\top_{\mathbf{u}} \mathbf{Q} \mathbf{G}_{\mathbf{u}}\right)\ dV \tag{2.5}$$

where, $\mathbf{A}$, $\mathbf{Q}_0$, and $\mathbf{Q}$ are the matrix form of the following fourth-order tensors

$$\begin{aligned}
\mathbb{A} &= \frac{\partial \mathbf{P}}{\partial \mathbf{F}} && \Rightarrow \mathbb{A}_{iJkL} = \frac{\partial \overline{P}_{iJ}}{\partial \overline{F}_{kL}}, \\
\mathbb{Q} &= \frac{1}{3}\ \mathbb{A} : \left(\overline{\mathbf{F}} \otimes \mathbf{F}^{-\top}\right) - \frac{2}{3}\ \mathbf{P} \otimes \mathbf{F}^{-\top} && \Rightarrow \mathbb{Q}_{iJkL} = \frac{1}{3} \mathbb{A}_{iJmN} \overline{F}_{mN} F^{-1}_{kL} - \frac{2}{3} P_{iJ} F^{-1}_{kL}, \\
\mathbb{Q}_0 &= \frac{1}{3}\ \mathbb{A} : \left(\overline{\mathbf{F}} \otimes \mathbf{F}_0^{-\top}\right) - \frac{2}{3}\ \mathbf{P} \otimes \mathbf{F}_0^{-\top} && \Rightarrow \mathbb{Q}_{0iJkL} = \frac{1}{3} \mathbb{A}_{iJmN} \overline{F}_{mN} F^{-1}_{0,kL} - \frac{2}{3} P_{iJ} F^{-1}_{0,kL}
\end{aligned} \tag{2.6}$$

Similarly, for the two-dimensional plane strain cases, the element residual can be written as,

$$\mathbf{R}^e_{\mathbf{u}} = -\int\limits_{\Omega^e_0} \mathbf{G}^\top_{\mathbf{u}} \left(\frac{\det \mathbf{F}_0}{\det \mathbf{F}}\right)^{-1/2} \mathbf{P}\ dV + \int\limits_{\Omega^e_0} \rho_{\mathrm{R}} \mathbf{N}^\top_{\mathbf{u}} \mathbf{B}\ dV + \int\limits_{\Gamma^e_T} \mathbf{N}^\top_{\mathbf{u}} \mathbf{T}_e\ dS \tag{2.7}$$

The matrix form of the element tangent matrix can be written as,

$$\mathbf{k}^e_{\mathbf{uu}} = \int\limits_{\Omega^e_0} \mathbf{G}^\top_{\mathbf{u}} \mathbf{A} \mathbf{G}_{\mathbf{u}}\ dV + \int\limits_{\Omega^e_0} \left(\frac{\det \mathbf{F}_0}{\det \mathbf{F}}\right)^{1/2} \left(\mathbf{G}^\top_{\mathbf{u}} \mathbf{Q}_0 \mathbf{G}_{\mathbf{u}}{}^0 - \mathbf{G}^\top_{\mathbf{u}} \mathbf{Q} \mathbf{G}_{\mathbf{u}}\right)\ dV \tag{2.8}$$

where, $\mathbf{A}$, $\mathbf{Q}_0$, and $\mathbf{Q}$ are the matrix form of the following fourth-order tensors

$$\begin{aligned}
\mathbb{A} &= \frac{\partial \mathbf{P}}{\partial \mathbf{F}} && \Rightarrow \mathbb{A}_{iJkL} = \frac{\partial \overline{P}_{iJ}}{\partial \overline{F}_{kL}}, \\
\mathbb{Q} &= \frac{1}{2}\ \mathbb{A} : \left(\overline{\mathbf{F}} \otimes \mathbf{F}^{-\top}\right) - \frac{1}{2}\ \mathbf{P} \otimes \mathbf{F}^{-\top} && \Rightarrow \mathbb{Q}_{iJkL} = \frac{1}{2} \mathbb{A}_{iJmN} \overline{F}_{mN} F^{-1}_{kL} - \frac{1}{2} P_{iJ} F^{-1}_{kL}, \\
\mathbb{Q}_0 &= \frac{1}{2}\ \mathbb{A} : \left(\overline{\mathbf{F}} \otimes \mathbf{F}_0^{-\top}\right) - \frac{2}{3}\ \mathbf{P} \otimes \mathbf{F}_0^{-\top} && \Rightarrow \mathbb{Q}_{0iJkL} = \frac{1}{2} \mathbb{A}_{iJmN} \overline{F}_{mN} F^{-1}_{0,kL} - \frac{1}{2} P_{iJ} F^{-1}_{0,kL}
\end{aligned} \tag{2.9}$$

2.2 PK-II stress-based total Lagrangian F-bar formulation

For three-dimensional cases, using the definition of the second Piola-Kirchhoff stress, $\mathbf{S}$, the matrix form of element residual vector, $\mathbf{R}^e_{\mathbf{u}}(\mathbf{u}_e)$, is given by,

$$\mathbf{R}^e_{\mathbf{u}} = -\int\limits_{\Omega^e_0} \left(\frac{\det \mathbf{F}_0}{\det \mathbf{F}}\right)^{-2/3} (\mathbf{B}_{\mathbf{u}}\Sigma_{\mathbf{F}})^\top \mathbf{S}\ dV + \int\limits_{\Omega^e_0} \rho_{\mathrm{R}} \mathbf{N}^\top_{\mathbf{u}} \mathbf{B}\ dV + \int\limits_{\Gamma^e_T} \mathbf{N}^\top_{\mathbf{u}} \mathbf{T}_e\ dS \tag{2.10}$$

The matrix form of the element tangent matrix is given by,

$$\begin{aligned}\mathbf{k}^e_{\mathbf{uu}} = &\int\limits_{\Omega^e_0} \left(\frac{\det \mathbf{F}_0}{\det \mathbf{F}}\right)^{-1/3} \Big(\mathbf{G}^\top_{\mathbf{u}} \Sigma_{\mathbf{S}} \mathbf{G}_{\mathbf{u}} + (\mathbf{B}_{\mathbf{u}}\boldsymbol{\Sigma}_{\mathbf{F}})^\top \mathbf{D}_{\mathbb{C}} (\mathbf{B}_{\mathbf{u}}\Sigma_{\mathbf{F}})\Big)\ dV \\ &+ \int\limits_{\Omega^e_0} \left[\left(\frac{\det \mathbf{F}_0}{\det \mathbf{F}}\right)^{-2/3} \Big(\mathbf{G}^\top_{\mathbf{u}} \mathbf{Q}^0_{\mathrm{R}} \mathbf{G}^0_{\mathbf{u}} - \mathbf{G}^\top_{\mathbf{u}} \mathbf{Q}_{\mathrm{R}} \mathbf{G}_{\mathbf{u}}\Big)\right]\ dV\end{aligned} \tag{2.11}$$

where, $\mathbf{Q}^0_{\mathrm{R}}$ and $\mathbf{Q}_{\mathrm{R}}$ are the matrix form of the following fourth-order tensors

$$\begin{aligned}(\mathbb{Q}^0_{\mathrm{R}})_{iJkL} &= \frac{1}{3}\overline{F}_{iI}\left(2\frac{\partial \overline{S}_{IJ}}{\partial \overline{C}_{MN}}\right)\overline{C}_{MN} F^{-\top}_{0,kL} - \frac{1}{3}\overline{P}_{iJ} F^{-\top}_{0,kL}, \\ (\mathbb{Q}_{\mathrm{R}})_{iJkL} &= \frac{1}{3}\overline{F}_{iI}\left(2\frac{\partial \overline{S}_{IJ}}{\partial \overline{C}_{MN}}\right)\overline{C}_{MN} F^{-\top}_{kL} - \frac{1}{3}\overline{P}_{iJ} F^{-\top}_{kL}.\end{aligned} \tag{2.12}$$

Using the definition of the second Piola-Kirchhoff stress, for plane-strain case, the matrix form of element residual vector, $\mathbf{R}^e_{\mathbf{u}}(\mathbf{u}_e)$, can be written as,

$$\mathbf{R}^e_{\mathbf{u}} = -\int\limits_{\Omega^e_0} \left(\frac{\det \mathbf{F}_0}{\det \mathbf{F}}\right)^{-1/2} (\mathbf{B}_{\mathbf{u}}\Sigma_{\mathbf{F}})^\top \mathbf{S}\ dV + \int\limits_{\Omega^e_0} \rho_{\mathrm{R}} \mathbf{N}^\top_{\mathbf{u}} \mathbf{B}\ dV + \int\limits_{\Gamma^e_T} \mathbf{N}^\top_{\mathbf{u}} \mathbf{T}_e\ dS \tag{2.13}$$

The matrix form of the element tangent matrix is given by,

$$\begin{aligned}\mathbf{k}^e_{\mathbf{uu}} = &\int\limits_{\Omega^e_0} \Big[\mathbf{G}^\top_{\mathbf{u}} \Sigma_{\mathbf{S}} \mathbf{G}_{\mathbf{u}} + (\mathbf{B}_{\mathbf{u}}\boldsymbol{\Sigma}_{\mathbf{F}})^\top \mathbf{D}_{\mathbb{C}} (\mathbf{B}_{\mathbf{u}}\Sigma_{\mathbf{F}})\Big]\ dV \\ &+ \int\limits_{\Omega^e_0} \left[\left(\frac{\det \mathbf{F}_0}{\det \mathbf{F}}\right)^{-1/2} \mathbf{G}^\top_{\mathbf{u}} \Big(\mathbf{Q}^0_{\mathrm{R}} \mathbf{G}^0_{\mathbf{u}} - \mathbf{Q}_{\mathrm{R}} \mathbf{G}_{\mathbf{u}}\Big)\right]\ dV\end{aligned} \tag{2.14}$$

where, $\mathbf{Q}^0_{\mathrm{R}}$ and $\mathbf{Q}_{\mathrm{R}}$ are the matrix form of the following fourth-order tensors

$$\begin{aligned}(\mathbb{Q}^0_{\mathrm{R}})_{iJkL} &= \frac{1}{2}\overline{F}_{iI}\left(2\frac{\partial \overline{S}_{IJ}}{\partial \overline{C}_{MN}}\right)\overline{C}_{MN} F^{-\top}_{0,kL}, \\ (\mathbb{Q}_{\mathrm{R}})_{iJkL} &= \frac{1}{2}\overline{F}_{iI}\left(2\frac{\partial \overline{S}_{IJ}}{\partial \overline{C}_{MN}}\right)\overline{C}_{MN} F^{-\top}_{kL}.\end{aligned} \tag{2.15}$$

Remark 3. While the F-bar approach proposed by de Souza Neto et al., 1996 is simple and easy to implement in an existing finite element framework, the resultant tangent matrix is no longer symmetric because of the additional term added in there. This requires using an unsymmetric solver which may increase the computation time by a few folds in case the material tangent is symmetric. Alternative to the F-bar formulation is to use the $\overline{\mathbf{B}}$-bar formulation for fully-integrated first-order quadrilateral and hexahedral elements (default in Abaqus/ Standard) or mixed/ hybrid $\mathbf{u} - p$ element.

3 Constitutive models for hyperelastic materials

In this implementation, two types of constitutive models have been included to represent the mechanical behavior of hyperelastic materials: (a) the Neo-Hookean model and (b) the Arruda-Boyce model. For both models, the coupled strain energy density representation has been adopted to model quasi-incompressible behaviors. The built-in hyperelastic material models available in Abaqus/Standard are represented by an uncoupled strain energy density formulation in which the deformation gradient is also split into deviatoric and volumetric parts. Users can specify the bulk modulus of the materials to be a few orders of magnitude larger than the shear modulus to enforce quasi-incompressibility. However, it should be noted that standard displacement based elements will behave poorly under certain deformation states because of *volumetric locking*, and users are recommended to use F-bar elements in such case.

3.1 (Quasi)-compressible Neo-Hookean material

The strain energy density, Ψ, for a quasi-incompressible Neo-Hookean type material is given by,

$$\Psi = \frac{\mu}{2}(I_1 - 3 - 2\ln J) + \frac{\kappa}{2}(\ln J)^2. \tag{3.1}$$

where, $I_1 = \mathrm{tr}(\mathbf{C})$ and $J = \det(\mathbf{F})$, and μ and κ are the material parameters representing the shear modulus and bulk modulus.

The second Piola-Kirchhoff stress, $\mathbf{S}$, and the material tangent, $\mathbb{C}$, are given by,

$$\begin{aligned} \mathbf{S} &= \mu(\mathbb{1} - \mathbf{C}^{-1}) + \kappa(\ln J)\mathbf{C}^{-1}, \\ \mathbb{C}_{IJKL} = 2\frac{\partial S_{IJ}}{C_{KL}} &= \kappa C^{-1}_{IJ} C^{-1}_{KL} + (\mu - \kappa \ln J)\left(C^{-1}_{IK} C^{-1}_{JL} + C^{-1}_{JK} C^{-1}_{IL}\right). \end{aligned} \tag{3.2}$$

3.2 (Quasi)-compressible Arruda-Boyce material

The strain energy density, Ψ, for a quasi-incompressible Arruda-Boyce type model is given by,

$$
\begin{gathered}
\Psi = \mu\left[\lambda_{\mathrm{L}}^2\left(\frac{\lambda_c\beta_c}{\lambda_{\mathrm{L}}} + \ln\frac{\beta_c}{\sinh\beta_c}\right) - \left(\frac{\lambda_{\mathrm{L}}}{3}\right)\ln J\right] + \frac{\kappa}{2}(\ln J)^2, \\
\text{where, } \beta_c = \mathcal{L}^{-1}\left(\frac{\lambda_c}{\lambda_{\mathrm{L}}}\right), \quad \text{and} \quad \lambda_c = \sqrt{\frac{I_1}{3}},
\end{gathered}
\tag{3.3}
$$

The second Piola-Kirchhoff stress, $\mathbf{S}$, is given by,

$$
\mathbf{S} = \mu\left(\frac{\lambda_{\mathrm{L}}}{3\lambda_c}\beta_c\right)\mathbb{1} - \left[\frac{\mu\lambda_{\mathrm{L}}}{3} - \kappa(\ln J)\right]\mathbf{C}^{-1}. \tag{3.4}
$$

Using index notation, the material tangent, $\mathbb{C}$, can be written as,

$$
\begin{aligned}
\mathbb{C}_{IJKL} &= 2\frac{\partial S_{IJ}}{C_{KL}}, \\
&= \frac{\mu}{9\lambda_c^2}\left(\frac{\partial\beta_c}{\partial\left(\frac{\lambda_c}{\lambda_{\mathrm{L}}}\right)} - \frac{\lambda_{\mathrm{L}}}{\lambda_c}\beta_c\right)\delta_{IJ}\delta_{KL} + \kappa C_{IJ}^{-1}C_{KL}^{-1}, \\
&\quad + \left[\frac{\mu\lambda_{\mathrm{L}}}{3} - \kappa(\ln J)\right]\left(C_{IK}^{-1}C_{JL}^{-1} + C_{JK}^{-1}C_{IL}^{-1}\right).
\end{aligned}
\tag{3.5}
$$

In this implementation, I used the approximation provided in Bergström, 1999 to evaluate the inverse Langevin function, $\mathcal{L}^{-1}(x)$, and its derivative, $D\mathcal{L}^{-1}(x)$, appearing in the Arruda-Boyce model.

4 Modeling in Abaqus

If and only if the user element has the same topology as any standard built-in element available in Abaqus, I can build the primary model in Abaqus/CAE, export the input file (`.inp`), and modify it to execute with Abaqus/Standard solver. Building the primary model is a standard exercise, and it is strongly recommended to check the option **Do not use parts and assemblies in input files** from the **model attributes** drop-down menu before exporting the input file of a model. Once the input is available, it must be modified following the instructions in the Abaqus user manual.

The first step is to include the element definition for the user elements in the input file by removing the standard element defined with `*Element` keyword in generating the model.

```
*User Element, Type=< >, Nodes=< >, Coordinates=< >, Properties=<
   >, Iproperties=< >
<list of degrees of freedom>
*Element, type=< >
<list of element nodal connectivity>
```

The second step is to define the properties of the user elements by element sets by removing standard keywords `*Section` and `*Material`.

```
*Uel property, Elset= < >
<list of properties>
```

The third step is optional, but if the user wants to visualize the results in Abaqus/Viewer, a set of overlaying dummy elements needs to be added to the input file.

```
*Element, Type=< >
<list of element nodal connectivity>
*Elset, Elset=elDummy
< >
*Solid section, Elset=elDummy, Material=Dummy
*Material, Name=Dummy
*Elastic
1.e-20
*User output variables
< >
```

The fourth step is to request element-level output in load steps.

```
*Element output, Elset=< >
uvarm
```

Remark 4. If F-bar element formulation is used to run an analysis, the user needs to ensure using `unsymm` keyword in the element definition and `unsymm=yes` in the `step` definition.

5 Source code availability

GitHub repository: https://github.com/bibekanandadatta/Abaqus-UEL-Hyperelasticity.

6 Citation

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Appendix A. Updated Lagrangian finite element framework

In the current configuration, the governing partial differential equation for stress equilibrium as well as the boundary conditions are given by,

$$\begin{aligned} \operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{B} &= 0 \qquad && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g} && \text{on } \Gamma_g, \\ \boldsymbol{\sigma} \cdot \mathbf{n} &= \mathbf{t} && \text{on } \Gamma_t. \end{aligned} \tag{A.1}$$

where, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\rho\mathbf{B}$ is the body force, $\partial\Omega = \Gamma_g \cup \Gamma_t$ is the boundary of the referential domain Ω, and Γ_g and Γ_t are two complementary subsurfaces. For an arbitrary vector test function, $\mathbf{w}$, the corresponding weak form of the governing equation, $\mathcal{W}_{\mathbf{u}}$, is given by,

$$\mathcal{W}_{\mathbf{u}}(\mathbf{u}) = -\int\limits_{\Omega} \boldsymbol{\sigma} : \mathrm{grad}(\mathbf{w}) \; dv + \int\limits_{\Omega} \rho \mathbf{B} \cdot \mathbf{w} \; dv + \int\limits_{\Gamma_t} \mathbf{t} \cdot \mathbf{w} \; ds = 0. \tag{A.2}$$

By employing the standard Galerkin discretization, the residual for each element, $\mathbf{R}^e_{\mathbf{u}}$, can be written as,

$$\begin{aligned} \mathbf{R}^e_{\mathbf{u}}(\mathbf{u}_e) &= -\int\limits_{\Omega^e} \mathbf{B}^\top_{\mathbf{u}} \boldsymbol{\sigma} \; dv + \int\limits_{\Omega^e} \rho \mathbf{N}^\top_{\mathbf{u}} \mathbf{B}^e \; dv + \int\limits_{\Gamma^e_t} \mathbf{N}^\top_{\mathbf{u}} \mathbf{t}^e \; ds, \\ \Rightarrow R^a_{\mathbf{u}}(u_i) &= -\int\limits_{\Omega^e} \frac{\partial N^a}{\partial x_j} \sigma_{ij} \; dv + \int\limits_{\Omega^e} \rho N^a B^e_i \; dv + \int\limits_{\Gamma^e_t} N^a t^e_i \; ds. \end{aligned} \tag{A.3}$$

Following the standard Newton-Raphson procedure, the element tangent matrix, $\mathbf{k}_{\mathbf{uu}}$, can be written as,

$$\begin{aligned} \mathbf{k}^{ab}_{\mathbf{uu}} &= -\frac{\mathbf{R}^a_{\mathbf{u}}}{\partial \mathbf{u}^b}, \\ \Rightarrow k^{ab}_{u_i u_k} &= \int\limits_{\Omega^e_0} \frac{\partial N^a}{\partial x_j} \sigma_{jl} \delta_{ik} \frac{\partial N^b}{\partial x_l} \; dv + \int\limits_{\Omega^e_0} \frac{\partial N^a}{\partial x_j} \mathbb{c}_{ijkl} \frac{\partial N^b}{\partial x_l} \; dv. \end{aligned} \tag{A.4}$$

where,

$$\mathbb{c}_{ijkl} = J^{-1} F_{iI} F_{jJ} \mathbb{C}_{IJKL} F_{kK} F_{lL} \tag{A.5}$$

is defined as the spatial tangent tensor (or fourth elasticity tensor). Similar to the material tangent tensor, $\mathbb{c}$, also possesses minor symmetry. The relation between the first and fourth elasticity tensor (spatial tangent) is as follows,

$$\mathbb{A}_{iJkL} = J F^{-1}_{Jj} \left(\sigma_{jl} \delta_{ik} + \mathbb{c}_{ijkl} \right) F^{-1}_{Ll}. \tag{A.6}$$

In matrix form, the element tangent matrix, $\mathbf{k}^e_{\mathbf{uu}}$, can be written as,

$$\mathbf{k}^e_{\mathbf{uu}} = \int\limits_{\Omega^e} \left(\mathbf{G}^\top_{\mathbf{u}} \Sigma_{\boldsymbol{\sigma}} \mathbf{G}_{\mathbf{u}} + \mathbf{B}^\top_{\mathbf{u}} \mathbf{D} \mathbf{B}_{\mathbf{u}} \right) \; dv. \tag{A.7}$$

where $\mathbf{D}$ is the Voigt matrix form of the spatial tangent tensor, , and $\Sigma_{\boldsymbol{\sigma}}$ is the expanded matrix form of the Cauchy stress tensors with a similar form of $\Sigma_{\mathbf{S}}$ introduced in the previous section. The non-symmetric gradient matrix, $\mathbf{G}_{\mathbf{u}}$, and strain-displacement matrix, $\mathbf{B}_{\mathbf{u}}$, have the same form as total Lagrangian formulation, however, they are evaluated using the current coordinate instead of the initial coordinate. Compared to the total Lagrangian implementation, in the updated Lagrangian approach coordinates are updated at every iteration by adding the displacement vector, $\mathbf{u}$, to the initial nodal coordinate vector, $\mathbf{X}$, before evaluating the interpolation functions and their derivatives. From the element tangent matrix expression, It is evident that the updated Lagrangian approach requires an additional geometric stiffness term compared to the small deformation model. Thus it is possible to modify the existing small deformation implementation easily when the updated Lagrangian framework is chosen.

Appendix B. F-bar formulation in updated Lagrangian framework

In the updated Lagrangian framework, the element residual remains the same after modifying the deformation gradient, *i.e.*,

$$\mathbf{R}^e_{\mathbf{u}} = -\int\limits_{\Omega^e} \mathbf{B}^\top_{\mathbf{u}} \boldsymbol{\sigma}(\overline{\mathbf{F}})\ dv + \int\limits_{\Omega^e} \rho \mathbf{N}^\top_{\mathbf{u}} \mathbf{B}^e\ dv + \int\limits_{\Gamma^e_t} \mathbf{N}^\top_{\mathbf{u}} \mathbf{t}^e\ ds, \tag{B.1}$$

where, the internal force vector in the residual term is calculated as usual using the stress computed using the modified deformation gradient, $\overline{\mathbf{F}}$. However, modified deformation gradient, $\overline{\mathbf{F}}$, results in a modified element tangent matrix as follows,

$$\mathbf{k}^e_{\mathbf{uu}} = \int\limits_{\Omega^e} \left[\mathbf{G}^\top_{\mathbf{u}} \Sigma_{\boldsymbol{\sigma}} \mathbf{G}_{\mathbf{u}} + \mathbf{B}^\top_{\mathbf{u}} \mathbf{D} \mathbf{B}_{\mathbf{u}} + \mathbf{G}^\top_{\mathbf{u}} \mathbf{q} (\mathbf{G}^0_{\mathbf{u}} - \mathbf{G}_{\mathbf{u}})\right]\ dv \tag{B.2}$$

Here, the first two terms are standard as shown before, and the third term appeared in the tangent matrix because of the modified deformation gradient, $\overline{\mathbf{F}}$. Here, $\mathbf{G}^0_{\mathbf{u}}$ is the non-symmetric gradient matrix evaluated at the centroid of the element, and $\mathbf{q}$ is the matrix form of the following fourth-order tensor,

$$\begin{aligned} \mathbb{q}_{\text{3D}} &= \frac{1}{3}\mathbb{a} : (\mathbb{1} \otimes \mathbb{1}) - \frac{2}{3}\boldsymbol{\sigma} \otimes \mathbb{1} \qquad &\Rightarrow (\mathbb{q}_{\text{3D}})_{ijkl} &= \frac{1}{3}\mathbb{a}_{ijmn}\delta_{mn}\delta_{kl} - \frac{2}{3}\sigma_{ij}\delta_{kl} \\ \mathbb{q}_{\text{pe}} &= \frac{1}{2}\mathbb{a} : (\mathbb{1} \otimes \mathbb{1}) - \frac{1}{2}\boldsymbol{\sigma} \otimes \mathbb{1} \qquad &\Rightarrow (\mathbb{q}_{\text{pe}})_{ijkl} &= \frac{1}{2}\mathbb{a}_{ijmn}\delta_{mn}\delta_{kl} - \frac{1}{2}\sigma_{ij}\delta_{kl} \end{aligned} \tag{B.3}$$

$$\text{where, } \mathbb{a}_{ijkl} = \frac{1}{\det \overline{\mathbf{F}}} \overline{F}_{jJ} \overline{F}_{lL} \mathbb{A}_{iJkL}, \quad \text{and} \quad \mathbb{A}_{iJkL} = \frac{\partial \overline{P}_{iJ}}{\partial \overline{F}_{kL}}. \tag{B.4}$$