Basins of Attraction Lab

MATH 310

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Abstract. This lab investigates how different initial guesses converge to specific roots of each function, revealing numerical patterns of convergence. We examine the basins of attraction for various polynomial functions, both real-valued and complex-valued, by using Newton's method. We aim to visualize how different initial guesses converge to specific roots of functions.

1 Introduction

Basins of attraction refer to a set of initial points in the domain of a function that, under the action of a numerical method, converge to a particular root. The convergence is not always guaranteed and is often sensitive to our initial guesses. Thus, the visualization of basins of attraction indicates where the given start points will converge and whether different initial guesses lead to the same root.

This report presents the findings from a series of computational graphics designed to visualize the basins of attraction for various continuous differentiable functions, both real and complex. Using Newton's method through Python implementations with a predefined tolerance of the actual root.

2 Method

2.1 Newton's Method

We apply Newton's method for root-finding with the given functions, that is, finding x such that $f(x) = 0$. Newton's method begins with an initial guess x_0 and then approximates a function by its tangent line. Taking the zero of the tangent line (intersection of x-axis) as the next guess x_1. Repeating this step until catching the root. The recursive step is defined by a formula performed below:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ where } f'(x) \text{ is the first derivative of } f(x) \; .$$

The sequence of x_n indicates the path how to converge to the root, while, we need to ensure Newton's method can be executed by checking if the function is continuous and differentiable, and the first derivative of the function, $f'(x)$, at the guess is not zero. Applying tolerance value 10^{-16} for the method to assign an acceptable error level.

2.2 Computational Implementation

This lab develops a Python Script to implement Newton's method, using pacakges of NumPy and Matplotlib. NumPy is used for numerical computations and Matplotlib library is used for plotting basins.

Hypothesis: We assume that the basins of attraction will be shown around the root, where initial guesses close to its approximated root.

3 Results

3.1 Real-Valued Functions

Quadratic Function (I)

The first function, $f(x) = (x-4)(x+1)$, is a simple quadratic with two distinct real roots shown in figure 1. This function serves as a baseline for observing convergence patterns of basins by applying Newton's method.

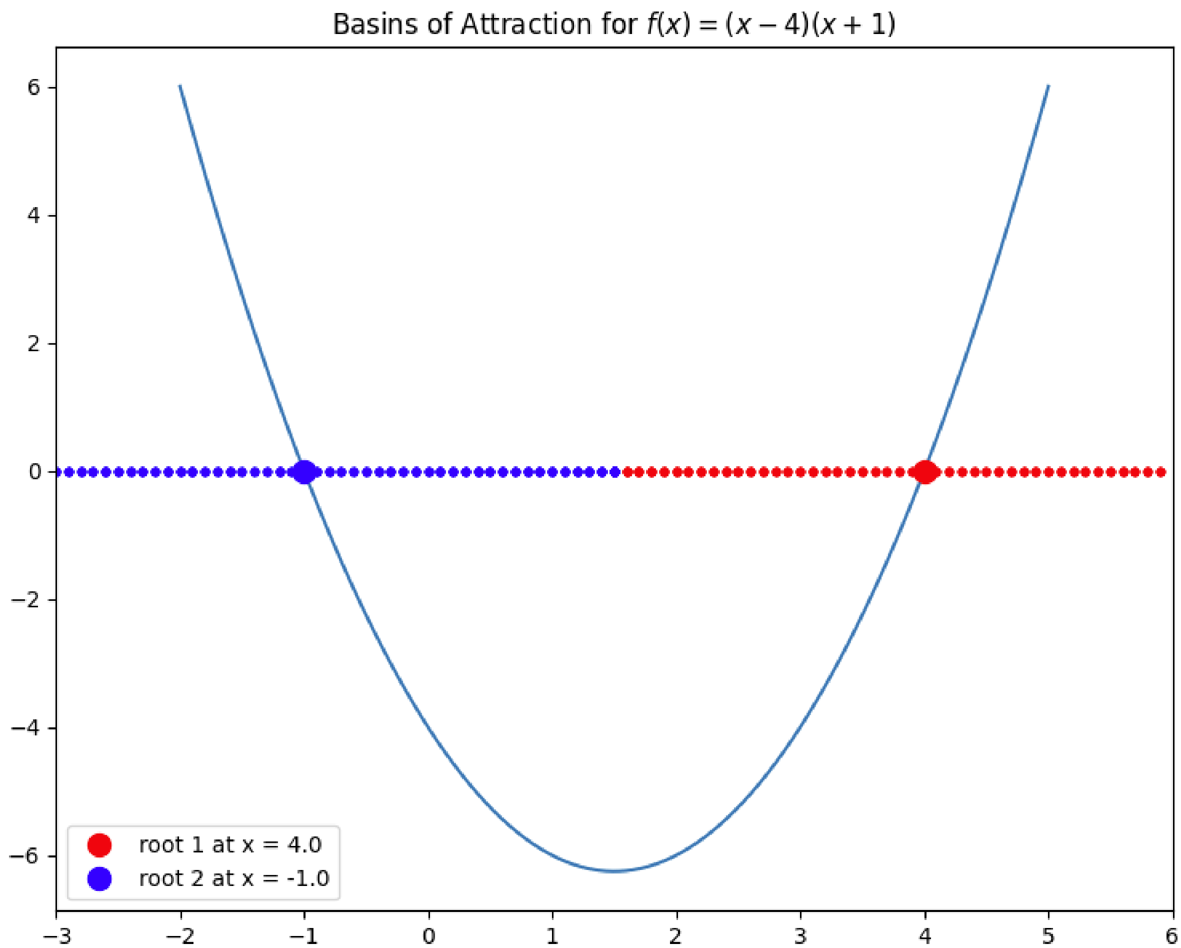

Figure 1: Basins of attraction for a quadratic function $f(x) = (x-4)(x+1)$ with roots at $x = 4$ and $x = -1$. The basins of attraction are clearly demarcated. Red starting points converge to the root at $x = 4$ and the blue initial guesses converge to the root at $x = -1$ within the interval of $[-3, 6]$. The boundary between the two basins lies in the middle between the zeros.

Quadratic Function (II)

The second quadratic function, $g(x) = (x-1)(x+3)$, also has two real roots shown in figure 2. This function provides a comparative case to the first quadratic to assess the consistency of the converging performance.

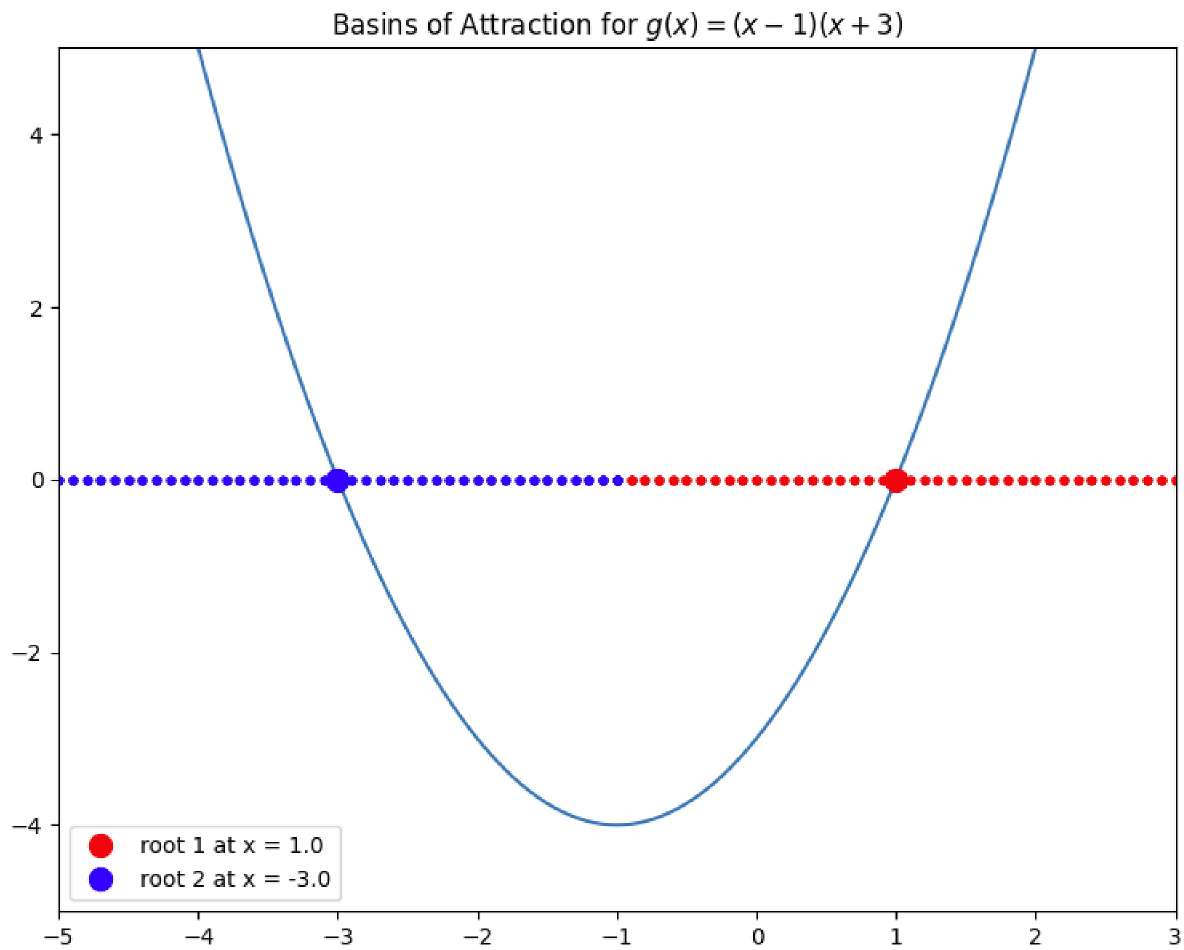

Figure 2: Basins of attraction for a quadratic function $g(x) = (x-1)(x+3)$ with roots at $x = 1$ and $x = -3$. Similar pattern is shown compared the first function. The root at $x = -3$ attracts red colored starting points from the right of the midpoint, and the root at $x = 1$ attracts initial guesses from the left, with the blue color.

Cubic Function

The cubic function, $h(x) = (x-4)(x-1)(x+3)$, reveals a more complex basins of attraction graph shown in figure 3. However, different from the patterns in the first two functions, several initial guesses close to the local minimum or maximum will converge to an unexpected far-away root. For instance, some starting points around the local maximum, shown in the left side of the graph, converge to the root at $x = 4$, where is the most right root.

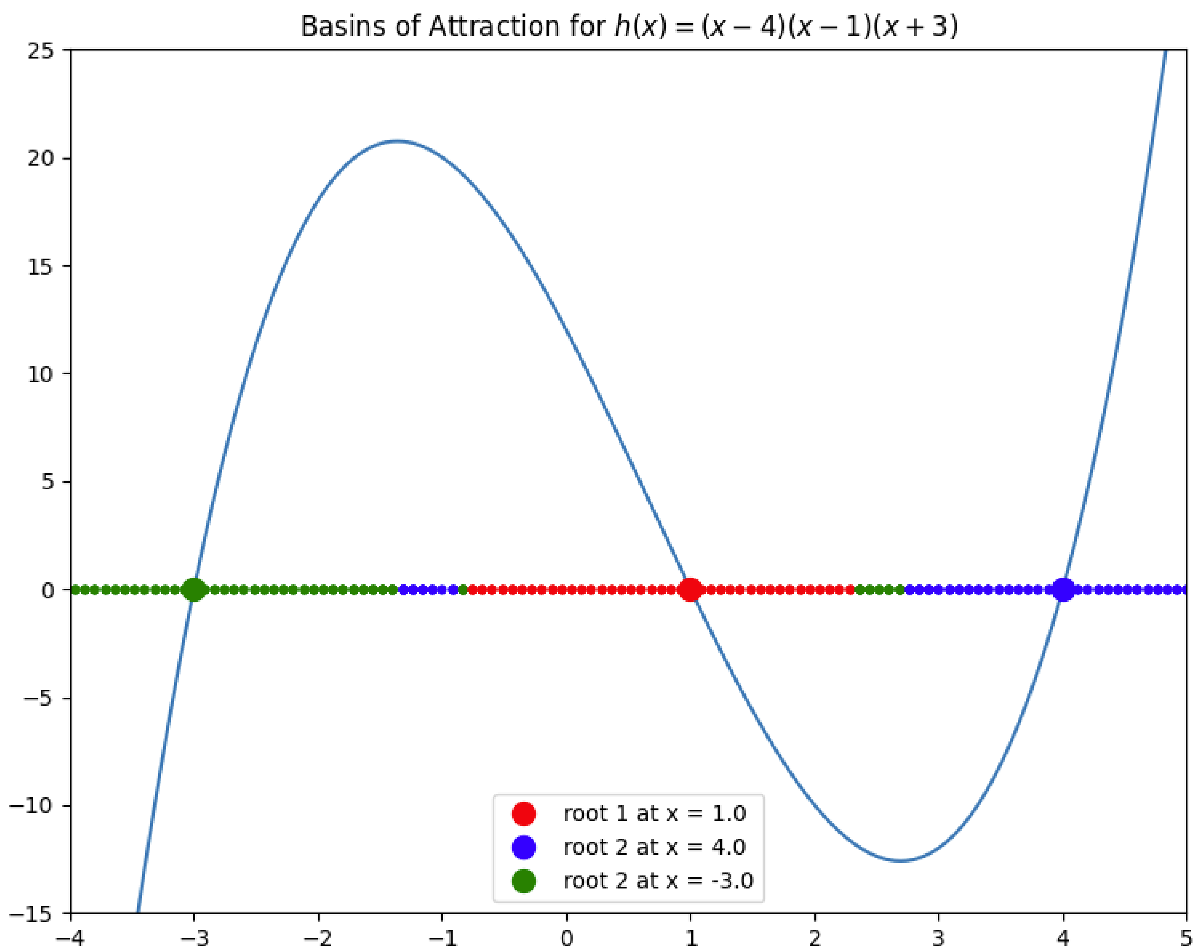

Figure 3: Basins of attraction for a Cubic function $h(x) = (x-4)(x-1)(x+3)$ with roots at $x = 1$, $x = 4$ and $x = -3$.This function reveals its basins of attraction with three distinct regions. Red points converge to the root at $x = 1$, blue ones converge to $x = 4$, and green ones converge to $x = -3$.

Quartic Function (Own Case)

A quartic function, $f(x) = (x-2)(x-1)(x+1) - 4$, is selected to explore basins of attraction in scenarios with higher polynomial degrees and multiple real roots, shown in figure 4. In this plot, the root at $x = 0$ appears to attract initial guesses from a domain within two minima symmetrically. The roots at $x \approx 2.236$ and $x \approx -2.236$ also have their basins nearby. However, when we offer initial guesses close to one local minimum, for example, the blue dot at $x \approx -1.8$, it leads to a convergence to the root at $x \approx 2.236$ instead of the other two zeros, which are apparently closer to the initial guess.

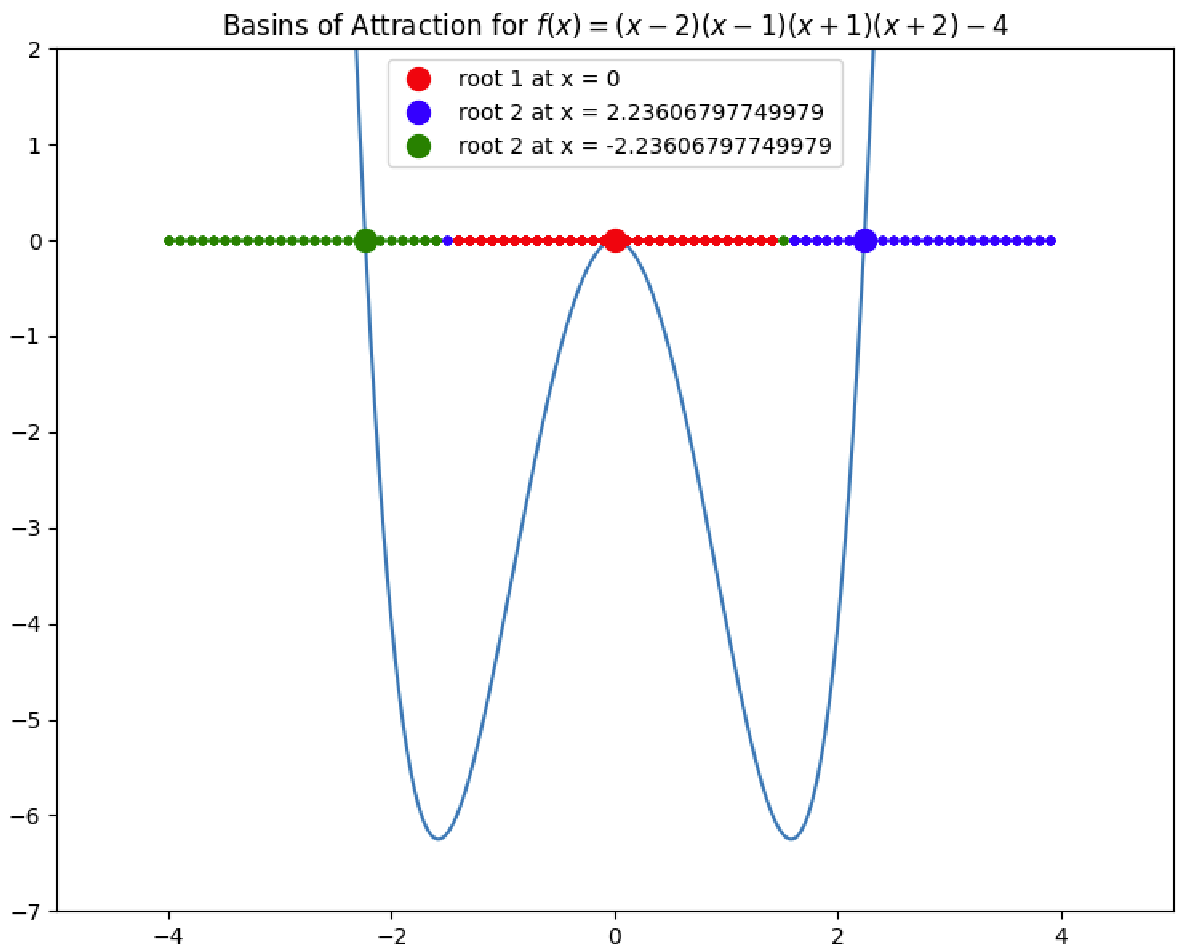

Figure 4: Basins of attraction for a quadratic function $f(x) = (x-2)(x-1)(x+1) - 4$ with roots at $x = 0$, $x \approx 2.236$, and $x \approx -2.236$. Red starting points converge to the middle root at $x = 0$, blue points converge to the right root at $x \approx 2.236$, and green points converge to the left zero at $x \approx -2.236$.

This visualization serves a valuable understanding that there will be clear separations between basins around each root, but a sensitivity of initial guesses is clarified. Specifically, when we choose a starting point near the boundaries of basins, it will slightly deviate and then lead to convergence to a far-away root, which is unexpected.

3.2 Complex-Valued Function

The visualization for a complex function $f(z) = z^3 - 1$, where z is a complex variable, reveals a symmetrical distribution of basins in the complex plane shown in figure 5. The symmetrical shape centered at the origin reflects that starting points are arbitrarily converge to different roots based on which side of the boundary they stands.

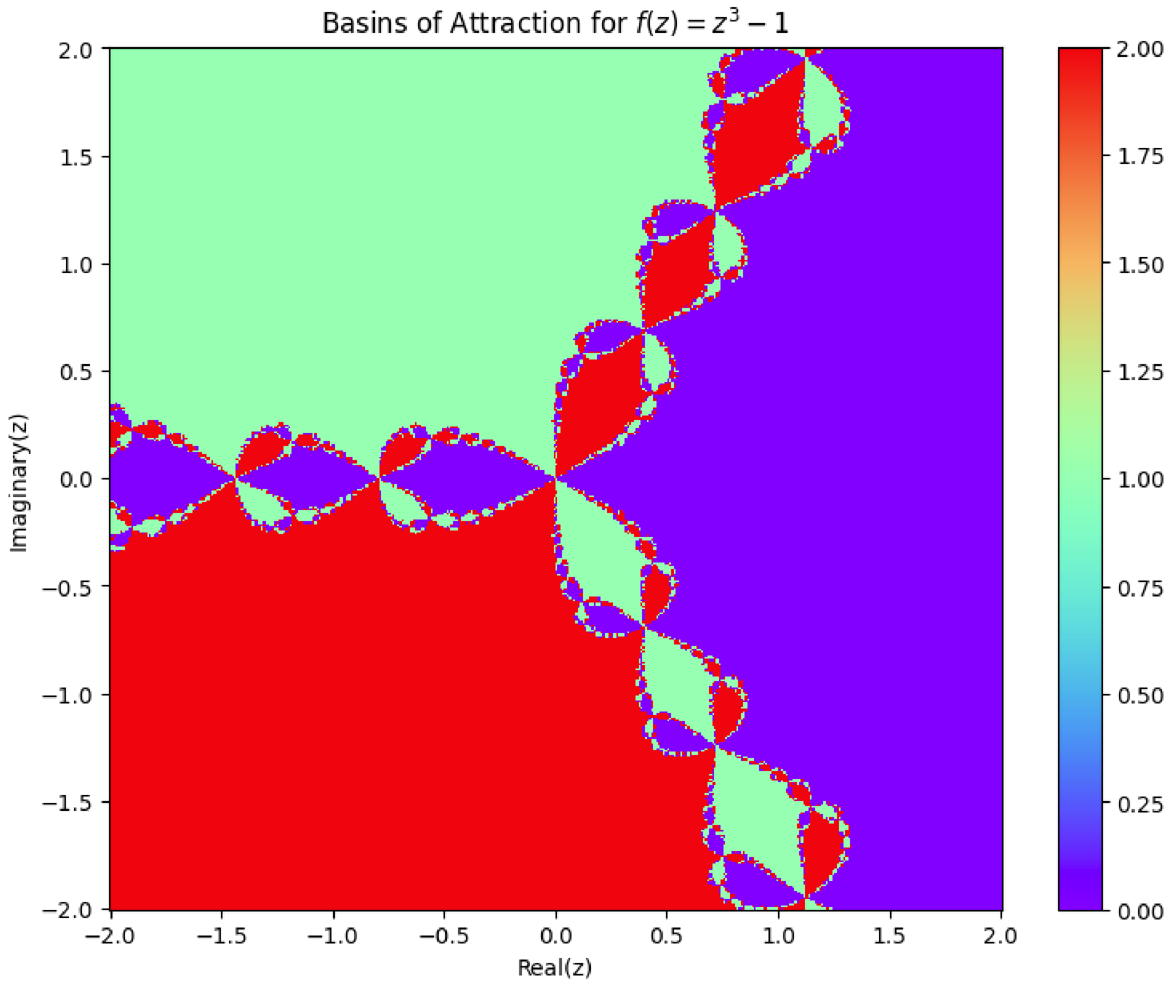

Figure 5: Basins of attraction for the function $f(z) = z^3 - 1$, where z is a complex variable. Three roots are $z = 1$ in red, $z = \frac{1}{2}(-1 + \sqrt{3}i)$ in purple, and $z = \frac{1}{2}(-1 - \sqrt{3}i)$ in green. Colors from red to purple, reveals three roots, and the intersected parts formed a "flower" clarify boundaries between basins. Since the machine precision, the graph illustrate the sensitivity of the initial guesses which affects root-convergence.

3.3 Own Case: Complex-Valued Function

Similar the previous real-valued function case, we take $f(z) = z^4 - 1$, where z is a complex variable, to observe a higher-dimensional plot shown in figure 6. Similar to the previous complex-valued function but with one more degree. Four basins of attraction intersect at the origin and reveal how slight deviation of initial guesses will result in unexpected root-convergence.

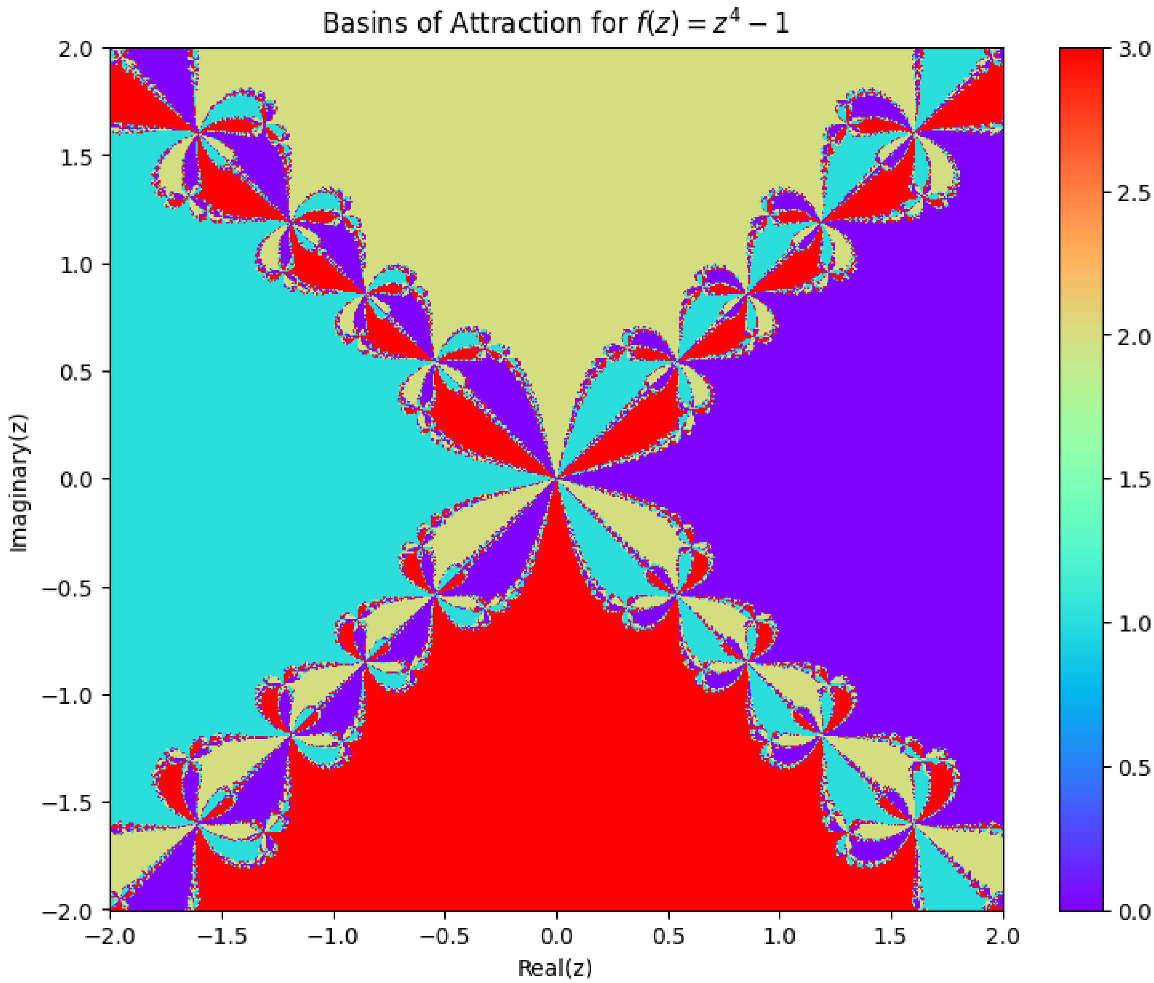

Figure 6: Basins of attraction for the function $f(z) = z^4 - 1$, where z is a complex variable. Roots are $x = 1$ in red, $x = -1$ in brown/yellow, $x = i$ purple, and $x = -i$ in blue. Four-fold symmetrical pattern is visualized to determine the convergence to various roots.

4 Conclusions

Basins of Attraction visualizations effectively capture one property of Newton's method: its powerful convergence properties within the basin of attraction and its potential for divergence or slow convergence near the basin boundaries.

The basins of attraction visualized for quadratic, cubic, and quartic functions almost align with the theoretical expectations for real-valued function where basins tend to be clearly separated. However, the complexity of cubic and quartic polynomials demonstrates that initial guesses close to local extrema will lead the convergence to a far-away root, which reveals a complexity of basin attraction in higher dimensions since the Newton's method leverages tangent lines. That is, taking the local extrema as the middle, if the absolute slope value at the starting point is relatively smaller, the farther zero will be converged.

For complex-valued functions, the results reveal the partitioning nature of the basins' boundaries. Each initial guess with slight deviations can lead to convergence to an unexpected root, shown in a "flower" intersection.

Since we apply tolerance for Newton's method because of the machine precision issues, this is also a considerable reason that causes such deviations of convergence for basins of attraction in higher-dimensional graphs and complex-valued function evaluations.

Bibliography

[1] Sullivan, E.(2022). Section 2.7.1 Basins of Attraction. *Numerical methods: An Inquiry-Based Approach With Python.*

[2] Python 3.12.1, *Numpy, Matplotlib*

[3] Online sources (learning how to plot for complex-valued functions):
Visualizing Complex-valued Functions
How to plot a complex number in Python using Matplotlib ?
Duke University Exercises