

Jacobians of Curves

Lecture notes by Ferdinand Wagner

Wintersemester 2018/19

This text consists of notes on the lecture Jacobians of Curves, taught at the University of Bonn by Professor Jens Franke in the winter term (Wintersemester) 2018/19.

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1. Introduction and preparations

1.1. A note on limits and their derived functors

Let $X_\bullet: \dots \xrightarrow{p_{i+1}} X_i \xrightarrow{p_i} \dots \xrightarrow{p_2} X_1 \xrightarrow{p_1} X_0$ be a diagram of abelian groups or R -modules. As usual, we may view X_\bullet as a functor $X_\bullet: (\mathbb{N}, \geq) \rightarrow \mathbf{Ab}$ or $\mathbf{Mod}(R)$, where the category (\mathbb{N}, \geq) has the nonnegative integers as objects and an arrow $j \rightarrow i$ iff $j \geq i$. Let

$$d: \prod_{i=0}^{\infty} X_i \longrightarrow \prod_{i=0}^{\infty} X_i, \quad d(x_i)_{i=0}^{\infty} = (p_{i+1}(x_{i+1}) - x_i)_{i=0}^{\infty}.$$

Then we put

$$\varprojlim_{i \in \mathbb{N}} X_i = \ker d \quad \text{and} \quad \varprojlim^1_{i \in \mathbb{N}} X_i = \operatorname{coker} d.$$

Remark 1. It is easy to see that $\varprojlim_{i \in \mathbb{N}} X_i$ equals the usual category-theoretical limit (that's how you construct it). It can also be shown that $\varprojlim^1_{i \in \mathbb{N}}$ is the first right-derived functor of $\varprojlim_{i \in \mathbb{N}}$, and that its higher derived functors vanish.

Fact 1. Let $0 \rightarrow X'_\bullet \rightarrow X_\bullet \rightarrow X''_\bullet \rightarrow 0$ be a short exact sequence of diagrams of the above type. Then there is a canonical exact sequence

$$0 \longrightarrow \varprojlim_{i \in \mathbb{N}} X'_i \longrightarrow \varprojlim_{i \in \mathbb{N}} X_i \longrightarrow \varprojlim_{i \in \mathbb{N}} X''_i \longrightarrow \varprojlim^1_{i \in \mathbb{N}} X'_i \longrightarrow \varprojlim^1_{i \in \mathbb{N}} X_i \longrightarrow \varprojlim^1_{i \in \mathbb{N}} X''_i \longrightarrow 0.$$

Proof. Since products preserve exact sequences in \mathbf{Ab} or $\mathbf{Mod}(R)$, we get a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \prod_{i=0}^{\infty} X'_i & \longrightarrow & \prod_{i=0}^{\infty} X_i & \longrightarrow & \prod_{i=0}^{\infty} X''_i \longrightarrow 0 \\ & & d' \downarrow & & d \downarrow & & d'' \downarrow \\ 0 & \longrightarrow & \prod_{i=0}^{\infty} X'_i & \longrightarrow & \prod_{i=0}^{\infty} X_i & \longrightarrow & \prod_{i=0}^{\infty} X''_i \longrightarrow 0 \end{array}$$

with exact rows. Then the snake lemma finishes the job. □

Fact 2. Let X_\bullet have the property that for every $i \in \mathbb{N}$ there is a $j \geq i$ such that the composition $p_{j,i}: X_j \xrightarrow{p_j} X_{j-1} \xrightarrow{p_{j-1}} \dots \xrightarrow{p_{i+1}} X_i$ vanishes. Then

$$\varprojlim_{i \in \mathbb{N}} X_i = \varprojlim^1_{i \in \mathbb{N}} X_i = 0.$$

Proof. If $x = (x_i)_{i=0}^\infty \in \varprojlim X_i$, then $x_i = p_{j,i}(x_j)$ for all $j \geq i$ by construction, hence $x_i = 0$ for all $i \in \mathbb{N}$. Moreover, let

$$s: \prod_{i=0}^\infty X_i \longrightarrow \prod_{i=0}^\infty X_i, \quad s(x)_i = \sum_{j \geq i} p_{j,i}(x_j).$$

By assumption s is well-defined. Then

$$d(s(x))_i = p_{i+1} \left(\sum_{j \geq i+1} p_{j,i+1}(x_j) \right) - \sum_{j \geq i} p_{j,i}(x_j) = -p_{i,i}(x_i) = -x_i.$$

Hence $-s$ is a right-inverse of d , so $\varprojlim^1 X_i = \text{coker } d$ vanishes as well. \square

Fact 3. *Let X_\bullet have the Mittag-Leffler property that for every $i \in \mathbb{N}$ there is a $j \geq i$ such that for all $k \geq j$ the images of $p_{j,i}$ and $p_{k,i}$ in X_i coincide. Then $\varprojlim^1 X_i = 0$.*

Proof. Let's first deal with the special case that each $p_i: X_i \rightarrow X_{i-1}$ is surjective. Let $x = (x_i)_{i=0}^\infty \in \prod_{i=0}^\infty X_i$. For every $i \in \mathbb{N}$ we may select $x_j^{(i)} \in X_j$ for all $j \geq i$ in such a way that $x_i^{(i)} = x_i$ and $p_{j+1}(x_{j+1}^{(i)}) = x_j^{(i)}$. Then $s(x)$ defined by

$$s(x)_i = \sum_{k=0}^{i-1} x_i^{(k)}$$

is a preimage of x under d , so $\varprojlim^1 X_i = \text{coker } d = 0$ in this case.

Now let X_\bullet be arbitrary with the Mittag-Leffler property. Let $Y_i = \bigcap_{j \geq i} p_{j,i}(X_j) \subseteq X_i$. Then $\varprojlim^1 Y_i = 0$ by the special case we just treated, and $\varprojlim^1 X_i/Y_i = 0$ by Fact 2. Since $\varprojlim^1 X_i$ is sandwiched between these two in the exact sequence from Fact 1, this shows $\varprojlim^1 X_i = 0$, as required. \square

1.2. The theorem about formal functions

Let $f: X \rightarrow Y = \text{Spec } A$ be a morphism of quasi-compact schemes. Let $I \subseteq A$ be any ideal. Consider

$$i_n: X_n = X \times_Y \text{Spec}(A/I^n) \longrightarrow X,$$

which is a base change of the closed immersion $Y_n = \text{Spec}(A/I^n) \hookrightarrow \text{Spec } A$, hence indeed a closed immersion itself. Also, if f is proper, then so is $X_n \rightarrow Y_n$ because properness is another *property* (tee-hee) that is stable under base change (by [AG2, Remark 2.4.1]).

Let \mathcal{F} be a quasi-coherent sheaf of \mathcal{O}_X -modules and $\mathcal{F}|_{X_n} = i_n^* \mathcal{F}$ its restriction to X_n (this notation is slightly abusive, but convenient). We put $\mathcal{F}_n = i_{n,*} \mathcal{F}|_{X_n}$. It's easy to check (e.g. affine-locally) that $\mathcal{F}_n \cong \mathcal{F}/I^n \mathcal{F}$. Since i_n is a closed immersion and thus affine, we have an isomorphism $H^p(X, \mathcal{F}_n) \cong H^p(X_n, \mathcal{F}|_{X_n})$ for all $p \geq 0$ by [AG2, Corollary 1.5.1]. Together with

the canonical projection $\mathcal{F}_{n+1} \cong \mathcal{F}/I^{n+1}\mathcal{F} \rightarrow \mathcal{F}/I^n\mathcal{F} \cong \mathcal{F}_n$ this gives canonical morphisms $H^p(X_{n+1}, \mathcal{F}|_{X_{n+1}}) \rightarrow H^p(X_n, \mathcal{F}|_{X_n})$ for all $n \in \mathbb{N}$.

The canonical morphism $\mathcal{F} \rightarrow i_{n,*}i_n^*\mathcal{F} = \mathcal{F}_n$ induces a morphism

$$H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}_n) \cong H^p(X_n, \mathcal{F}|_{X_n}) \quad (1)$$

for all $p \geq 0$ (the isomorphism on the right-hand side comes from the fact that i_n is a closed immersion, hence affine, and we can apply [AG2, Corollary 1.6.1]). This is a morphism of A -modules, but $H^p(X_n, \mathcal{F}|_{X_n})$ is actually an A/I^n -module, so (1) factors over

$$H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F}) \longrightarrow H^p(X_n, \mathcal{F}|_{X_n}) .$$

This is compatible with the canonical morphisms $H^p(X_{n+1}, \mathcal{F}|_{X_{n+1}}) \rightarrow H^p(X_n, \mathcal{F}|_{X_n})$ (you can just check that on an affine Čech covers). Passing to the limit gives a morphism

$$H^p(X, \mathcal{F})^\wedge \longrightarrow \varprojlim_{n \in \mathbb{N}} H^p(X_n, \mathcal{F}|_{X_n}) , \quad (2)$$

where \wedge denotes the I -adic completion.

Theorem 1 (Grothendieck–Zariski). *When $f: X \rightarrow Y = \operatorname{Spec} A$ is proper (in which case X is automatically a quasi-compact scheme), A is noetherian and \mathcal{F} is a coherent sheaf of \mathcal{O}_X -modules, then (2) is an isomorphism*

$$H^p(X, \mathcal{F})^\wedge \xrightarrow{\sim} \varprojlim_{n \in \mathbb{N}} H^p(X_n, \mathcal{F}|_{X_n}) .$$

Proof. The following proof is essentially the one from [EGAIII, (4.1.7)]. Professor Franke also pointed out that the idea is pretty similar to the proof of the Artin–Rees lemma. Let $I \subseteq A$ be the ideal under consideration and let $R = \bigoplus_{n \geq 0} I^n$ be the Rees algebra associated to I . Then

$$K^p = \bigoplus_{n \geq 0} H^p(X, I^n \mathcal{F})$$

is a module over R as $i \in I^m$ (considered as the m^{th} homogeneous component of R) maps $I^n \mathcal{F}$ to $I^{n+m} \mathcal{F}$.

Claim 1. K^p is a finitely generated R -module for all $p \geq 0$.

Assuming this for the moment, recall that $\mathcal{F}_n \cong \mathcal{F}/I^n \mathcal{F}$ and $H^p(X, \mathcal{F}|_{X_n}) \cong H^p(X, \mathcal{F}_n)$, so the long exact cohomology sequence associated to $0 \rightarrow I^n \mathcal{F} \rightarrow \mathcal{F} \rightarrow \mathcal{F}_n \rightarrow 0$ appears as

$$H^p(X, I^n \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}) \longrightarrow H^p(X_n, \mathcal{F}|_{X_n}) \longrightarrow H^{p+1}(X, I^n \mathcal{F}) . \quad (3)$$

As pointed out after (1), $H^p(X, \mathcal{F}) \rightarrow H^p(X_n, \mathcal{F}|_{X_n})$ factors over $H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F})$, hence we can turn equation (3) into an exact sequence

$$0 \longrightarrow U_n \longrightarrow H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F}) \longrightarrow H^p(X_n, \mathcal{F}|_{X_n}) \longrightarrow V_n \longrightarrow 0 , \quad (4)$$

where U_n is a suitable quotient of $H^p(X, I^n \mathcal{F})$ and $V_n \subseteq H^{p+1}(X, I^n \mathcal{F})$ some submodule. This makes $U = \bigoplus_{n \geq 0} U_n$ a quotient of K^p and $V = \bigoplus_{n \geq 0} V_n$ an R -submodule of K^{p+1} .

Claim 2. We have $\varprojlim U_n = \varprojlim^1 U_n = 0$ and $\varprojlim V_n = \varprojlim^1 V_n = 0$.

Before we prove this (and Claim 1), let's see how Theorem 1 follows from it. Let W_n be the image of $H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F})$ in $H^p(X_n, \mathcal{F}|_{X_n})$. We may split (4) into two short exact sequences $0 \rightarrow U_n \rightarrow H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F}) \rightarrow W_n \rightarrow 0$ and $0 \rightarrow W_n \rightarrow H^p(X, \mathcal{F}|_{X_n}) \rightarrow V_n \rightarrow 0$. Applying Fact 1.1.1 to the first one gives $H^p(X, \mathcal{F})^\wedge \cong \varprojlim^1 W_n$. Then the six-term exact sequence associated to the second proves $\varprojlim W_n \cong \varprojlim H^p(X_n, \mathcal{F}|_{X_n})$ and we are done.

It remains to show the two claims. Note that the Rees algebra R is noetherian. Indeed, I is finitely generated as an ideal in the noetherian ring A , hence R is of finite type over A . Let's also make the following convention: Whenever we write $I^k U_n$ or $I^k V_n$ in the following, this means multiplication as A -modules and the result is contained in U_n resp. V_n again, whereas $R_k U_n$ or $R_k V_n$ means multiplication by the k^{th} homogeneous component of R (which equals I^k as well), so the result is contained in U_{k+n} resp. V_{k+n} .

Proof of Claim 2. Note that U is finitely generated over R , since it is a quotient of the finitely generated R -module K^p . Fix a finite set of generators and let d_0 the maximal non-zero homogeneous components occurring in this set. Then $U_n = R_n U_0 + R_{n-1} U_1 + \dots + R_{n-d_0} U_{d_0}$ for all $n \geq d_0$. In particular, $U_{k+n} = R_k U_n$ for all $n \geq d_0$. Thus, for every $n \geq d_0$ the image of $U_{2n} = R_n U_n$ in U_n is contained in $I^n U_n$. But $U_n \subseteq H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F})$, so $I^n U_n$ vanishes. Therefore, the property from Fact 1.1.2 is fulfilled for all $n \geq d_0$. But then it clearly holds for all $n \geq 0$ as well, so Fact 1.1.2 is applicable.

Similarly, V is finitely generated as a submodule of K^{p+1} , which is finitely generated over the noetherian ring R by Claim 1. By the same argument as above we find a d_1 such that $V_n = R_n V_0 + R_{n-1} V_1 + \dots + R_{n-d_1} V_{d_1}$ for all $n \geq d_1$. In particular, we have $V_{k+n} = R_k V_n$ for all $n \geq d_1$. Thus, for $n \geq d_1$ the image of V_{2n} in V_n is contained in $I^n V_n$. But $I^n V_n$ vanishes again, since V_n is the image of $H^p(X_n, \mathcal{F}|_{X_n})$, which is a A/I^n -module. As above, we can apply Fact 1.1.2. This shows Claim 2.

Proof of Claim 1. Let $v: \tilde{Y} = \text{Spec } R \rightarrow Y$ correspond to $A \hookrightarrow R$ and let $\xi: \tilde{X} = X \times_Y \tilde{Y} \rightarrow X$ be its base change by f . Note that ξ is affine as a base change of the affine morphism v (we use [AG1, Corollary 2.5.1] here). We claim

$$\xi_* \xi^* \mathcal{F} \cong \bigoplus_{n \geq 0} I^n \mathcal{F}.$$

Indeed, this is easily checked affine-locally (where ξ^* is given by tensoring with R); we leave the details to the reader. Also $H^p(\tilde{X}, \xi^* \mathcal{F}) \cong H^p(X, \xi_* \xi^* \mathcal{F})$ as ξ is affine. This shows

$$H^p(\tilde{X}, \xi^* \mathcal{F}) \cong H^p(X, \xi_* \xi^* \mathcal{F}) \cong H^p\left(X, \bigoplus_{n \geq 0} I^n \mathcal{F}\right) \cong \bigoplus_{n \geq 0} H^p(X, I^n \mathcal{F}) = K^p.$$

Note that direct sums usually *don't* commute with cohomology, but here they do, because X is quasi-compact and $\bigoplus_{n \geq 0} I^n \mathcal{F}$ is quasi-coherent (for which we need quasi-compactness as well), so we may compute $H^p(X, \bigoplus_{n \geq 0} I^n \mathcal{F})$ via finite affine Čech covers. In this case, the products in the Čech complex are all finite, hence commute with the direct sum, which is what we needed.

Now $\tilde{f}: \tilde{X} \rightarrow \tilde{Y} = \operatorname{Spec} R$ is proper (as a base change of the proper morphism f), hence the right-hand side is a finitely generated R -module by our finiteness results for the cohomology of proper morphisms (cf. [AG2, Theorem 5]). We win. \square

Remark 1. Note that in the lecture Franke used $\mathcal{K}_n \cong \mathcal{J}^n \mathcal{F}$ instead of $I^n \mathcal{F}$, where $\mathcal{J} = f^{-1} \mathcal{I}$ is the inverse image (in the sense of Definition 1). But $\mathcal{J}^n \mathcal{F} \cong I^n \mathcal{F}$ – which is not that surprising, since the I^n -action on \mathcal{F} is given via the algebraic component $\mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ of f , so $I^n \mathcal{F} = \mathcal{J}^n \mathcal{F}$ is pretty obvious from the construction of f^{-1} described in the proof of Lemma 1 below. I prefer the notation $I^n \mathcal{F}$ – in particular, this is how Grothendieck denotes it in [EGAIII, (4.1.7)], so I believe it's my right to do so as well. Nevertheless, Lemma 1 is *perhaps worthwhile to know* (if you get what I mean), so we will include it now.

Definition 1. Let $f: X \rightarrow Y$ be any morphism of preschemes and $\mathcal{J} \subseteq \mathcal{O}_Y$ a sheaf of ideals on Y . Then define $f^{-1} \mathcal{J}$ to be the image of $f^* \mathcal{J} \rightarrow \mathcal{O}_X$ (which is obtained as the composition of the pull-back of $\mathcal{J} \rightarrow \mathcal{O}_Y$ with the isomorphism $f^* \mathcal{O}_Y \cong \mathcal{O}_X$).

Lemma 1. *Let $f: X \rightarrow Y$ be any morphism of preschemes and $\mathcal{J} \subseteq \mathcal{O}_Y$ quasi-coherent.*

- (a) $f^{-1} \mathcal{J} \subseteq \mathcal{O}_X$ is quasi-coherent.
- (b) Let Y_0 and X_0 be the closed subpreschemes of Y and X defined by \mathcal{J} , $f^{-1} \mathcal{J}$ respectively. Then $X_0 \cong X \times_Y Y_0$.
- (c) For all $n \geq 0$ we have $f^{-1}(\mathcal{J}^n) \cong (f^{-1} \mathcal{J})^n$.

Sketch of a proof. The question is easily seen to be local on both X and Y . So let's consider the affine situation where $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, and $\mathcal{J} = \tilde{J}$ for some ideal $J \subseteq A$. Let $\varphi: A \rightarrow B$ be the morphism of rings corresponding to f . Then $f^{-1} \mathcal{J} = \tilde{I}$ where I is the image of $B \otimes_A J \rightarrow B$ sending $b \otimes j \mapsto b \cdot \varphi(j)$. All three assertions are then easily checked. \square

Remark 2. Recall that for a morphism $f: X \rightarrow Y$ of preschemes and a point $y \in Y$ the **fibre** $f^{-1}\{y\}$ of f at y is defined as the prescheme $f^{-1}\{y\} = X \times_Y \operatorname{Spec} \mathfrak{K}(y)$. This makes sense, since $f^{-1}\{y\}$ is indeed – topologically – the preimage of y , as proved in [AG1, Corollary 1.3.3]. Moreover, $\operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n \rightarrow Y$ is immersive for all $n \geq 1$ and has image $\{y\}$ as well. So [AG1, Corollary 1.3.3] is applicable again and shows that $X_n = X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n)$ has $f^{-1}\{y\}$ as underlying topological space too (but, of course, the prescheme structure differs in general). We may thus think of X_n as the n^{th} infinitesimal thickening of $f^{-1}\{y\}$.

Using this, Theorem 1 can be restated as follows.

Theorem 1a. *Let $f: X \rightarrow Y$ be a proper morphism between locally noetherian¹ preschemes. Let \mathcal{F} be a coherent \mathcal{O}_X -module. For every $y \in Y$ let $X_n = X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n)$ be the n^{th} infinitesimal thickening of $f^{-1}\{y\}$. Then there is an isomorphism*

$$(R^p f_* \mathcal{F})_y^\wedge \xrightarrow{\sim} \varprojlim_{n \in \mathbb{N}} H^p(X_n, \mathcal{F}|_{X_n}),$$

where $^\wedge$ denotes the $\mathfrak{m}_{Y,y}$ -adic completion.

¹Franke only assumes Y to be locally noetherian, but f being of (locally) finite type implies that X is locally noetherian as well by Hilbert's Basissatz. This happens multiple times throughout the text.

Proof. We may assume that $Y = \operatorname{Spec} A$ is affine, and that A is a noetherian ring. Indeed, replacing Y by an affine neighbourhood $U \cong \operatorname{Spec} A$ and X by $f^{-1}(U)$ doesn't change $(R^p f_* \mathcal{F})_y$ (because the construction of $R^p f_* \mathcal{F}$ is base-local) and also X_n is preserved since $f^{-1}\{y\}$ is already contained in $f^{-1}(U)$ (by [AG2, postnote]).

In this case, $R^p f_* \mathcal{F} = H^p(X, \mathcal{F})^\sim$ by [AG2, Proposition 1.5.1(d)]. Let $\mathfrak{p} \in \operatorname{Spec} A$ be the prime ideal associated to y . Then $R^p f_* \mathcal{F} \cong H^p(X, \mathcal{F})_{\mathfrak{p}}$ and $\mathcal{O}_{Y,y} \cong A_{\mathfrak{p}}$ is flat over A . Let $\mathfrak{m} = \mathfrak{p}A_{\mathfrak{p}} \cong \mathfrak{m}_{Y,y}$ be its maximal ideal. We denote $\pi: \operatorname{Spec} A_{\mathfrak{p}} \rightarrow \operatorname{Spec} A$. Applying [AG2, Fact 4.1.1] to π gives

$$H^p(X \times_Y \operatorname{Spec} A_{\mathfrak{p}}, \pi^* \mathcal{F}) \cong H^p(X, \mathcal{F})_{\mathfrak{p}} \cong (R^p f_* \mathcal{F})_y.$$

Also

$$\begin{aligned} (X \times_Y \operatorname{Spec} A_{\mathfrak{p}}) \times_{\operatorname{Spec} A_{\mathfrak{p}}} \operatorname{Spec}(A_{\mathfrak{p}}/\mathfrak{m}^n) &\cong X \times_Y \left(\operatorname{Spec} A_{\mathfrak{p}} \times_{\operatorname{Spec} A_{\mathfrak{p}}} \operatorname{Spec}(A_{\mathfrak{p}}/\mathfrak{m}^n) \right) \\ &\cong X \times_Y \operatorname{Spec}(A_{\mathfrak{p}}/\mathfrak{m}^n) \\ &\cong X_n \end{aligned}$$

by a bit abstract nonsense. Now Theorem 1 may be applied to $X \times_Y \operatorname{Spec} A_{\mathfrak{p}} \rightarrow \operatorname{Spec} A_{\mathfrak{p}}$ (the base change of f) and the assertion follows. \square

1.3. Application to Zariski's main theorem

1.3.1. A lot of (not necessarily main) theorems by Zariski

Out there in the real world, there are multiple *main theorems* of Zariski around, and usually they're only loosely related. Professor Franke recommends Mumford's *The red book of varieties and schemes* for a discussion of various such version.

Corollary 1. *Let $f: X \rightarrow Y$ be any proper morphism between locally noetherian preschemes and let $d = \sup_{y \in Y} \dim(f^{-1}\{y\})$. If \mathcal{F} is a coherent \mathcal{O}_X -module and $p > d$, then $R^p f_* \mathcal{F} = 0$.*

Proof. Since $R^p f_* \mathcal{F}$ is coherent (this is [AG2, Theorem 5]), $(R^p f_* \mathcal{F})_y$ is a finitely generated $\mathcal{O}_{Y,y}$ -module, hence it vanishes iff its $\mathfrak{m}_{Y,y}$ -adic completion vanishes by Fact A.1.1(b). But $X_n = X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n)$ has underlying space $f^{-1}\{y\}$ (as explained in Remark 1.2.2), hence $H^p(X_n, \mathcal{F}|_{X_n}) = 0$ when $p > d$ by Grothendieck's theorem on cohomological dimension (cf. [AG2, Proposition 1.4.1]). The assertion now follows from Theorem 1a. \square

Definition 1. A morphism $f: X \rightarrow Y$ of finite type is called **quasi-finite at $x \in X$** if x is discrete in its fibre, i.e., if $\{x\}$ is an open and closed subset of $f^{-1}\{y\}$ where $y = f(x)$. We call f **quasi-finite** if it is quasi-finite at every $x \in X$.

The following fact wasn't mentioned in the lecture, but it's *definitely* (in particular, not only *perhaps*) *worthwhile to know!*

Fact 1. *Let $f: X \rightarrow Y$ be a morphism of finite type. Let $x \in X$ be open in its fibre $f^{-1}\{y\}$, where $y = f(x)$. Then f is already quasi-finite at x .*

Proof. Choose an affine open neighbourhood $y \in U \cong \operatorname{Spec} A$. Then $f^{-1}\{y\}$ is contained in $f^{-1}(U)$, so we may w.l.o.g. assume that $Y = \operatorname{Spec} A$ is affine. Put $k = \mathfrak{K}(y)$. Since X may be covered by affine open subsets $\operatorname{Spec} R$, where R is of finite type over A , we may cover the fibre product $f^{-1}\{y\} = X \times_Y \operatorname{Spec} k$ by affine open subsets $\operatorname{Spec}(R \otimes_A k)$, in which $R \otimes_A k$ is a k -algebra of finite type, hence a Jacobson ring. This proves that $f^{-1}\{y\}$ is a Jacobson prescheme as in [AG1, Definition 2.4.2(c)]. But then x is a closed point of the open subset $\{x\} \subseteq f^{-1}\{y\}$, hence also a closed point of $f^{-1}\{y\}$ by [AG1, Fact 2.4.1(c)]. \square

Fact 2. (a) *Any finite morphism is quasi-finite.*

(b) *If k is a field, a morphism $f: X \rightarrow \operatorname{Spec} k$ of finite type is quasi-finite iff it is finite.*

(c) *Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be morphisms of finite type such that g is quasi-finite at $y = f(x)$ for some $x \in X$. Then gf is quasi-finite at x iff f is quasi-finite at x .*

Proof. Maybe that's my bad, but the proof of this is actually annoyingly laborious. We begin with part (a). Let $f: X \rightarrow Y$ be a finite morphism, $x \in X$ and $y = f(x)$. Then the morphism

$$f^{-1}\{y\} = X \times_Y \operatorname{Spec} \mathfrak{K}(y) \longrightarrow \operatorname{Spec} \mathfrak{K}(y)$$

is finite again, as a base changes of finite morphisms are finite again (cf. [AG1, Corollary 1.5.1]). Letting $k = \mathfrak{K}(y)$ this puts us in the situation from (b), so it's sufficient to prove (b).

In the case of (b) we have $f^{-1}\{y\} = X$, so what we need to show is that X carries the discrete topology if f is finite. We know that $X \cong \operatorname{Spec} R$ where R is some finite-dimensional k -algebra (using finiteness of f). For $x \in X$ let \mathfrak{p} be the corresponding prime ideal of R . Then R/\mathfrak{p} is a domain and a finite-dimensional k -vector space, hence a finite field extension of k by Hilbert's Nullstellensatz. This means that \mathfrak{p} is a maximal ideal of R . Consequently, all points of X are closed, so it suffices to show that X has finitely many points. Let $\{x_1, \dots, x_n\}$ be any finite subset of X and $\{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$ the corresponding maximal ideals of R . For every i , we may choose an element $\alpha_i \in \mathfrak{m}_i$ which is not contained in any \mathfrak{m}_j for $j \neq i$ (e.g. by the prime avoidance lemma, cf. [AG1, Lemma 2.5.1]). Put $\beta_i = \prod_{j \neq i} \alpha_j$ (so that $\beta_i \in \mathfrak{m}_j$ for all $j \neq i$ but $\beta_i \notin \mathfrak{m}_i$). We claim that β_1, \dots, β_n are k -linearly independent. Indeed, if $\lambda_1 \beta_1 + \dots + \lambda_n \beta_n = 0$ for some coefficients $\lambda_1, \dots, \lambda_n \in k$, then reducing modulo \mathfrak{m}_i gives $\lambda_i \beta_i = 0$ in $R/\mathfrak{m}_i = \mathfrak{K}(\mathfrak{m}_i)$. But $\beta_i \neq 0$ in $\mathfrak{K}(\mathfrak{m}_i)$, so $\lambda_i = 0$ for all $i = 1, \dots, n$. This proves $\dim_k R \geq n$. But R is finite-dimensional over k , hence X must have finitely many points, as claimed.

Conversely, assume that $f: X \rightarrow \operatorname{Spec} k$ is quasi-finite. Then X is discrete, so it must have finitely many points. Indeed, f being of finite type implies it is quasi-compact (by definition), so X is quasi-compact because $\operatorname{Spec} k$ is, and any discrete quasi-compact space is finite. Let $X = \{x_1, \dots, x_n\}$. Every point $x_i \in X$ together with the restriction $\mathcal{O}_X|_{\{x_i\}}$ of the structure sheaf is a prescheme again, hence affine (because $x_i \in \{x_i\}$ must have an affine neighbourhood). Let $\{x_i\} \cong \operatorname{Spec} R_i$. Then

$$X \cong \coprod_{i=1}^n \operatorname{Spec} R_i \cong \operatorname{Spec} \left(\bigoplus_{i=1}^n R_i \right)$$

is affine. This shows that f is affine, but finiteness is yet to prove. Clearly, it suffices that each R_i is a finite-dimensional k -vector space. Note that R_i has precisely one prime ideal \mathfrak{m}_i (corresponding to x_i), which is then automatically maximal. Since f is of finite type, R_i has finite type over k . In particular R_i is noetherian and we may choose generators r_1, \dots, r_m of \mathfrak{m}_s . Since \mathfrak{m}_i is the only prime ideal of R_s , we have $\mathfrak{m}_i = \text{nil } R_i$. Consequently, there is an $N \in \mathbb{N}$ such that $r_\ell^N = 0$ for all ℓ . Moreover, R_i/\mathfrak{m}_i is a field extension of finite type over k , hence a finite field extension by Hilbert's Nullstellensatz. Let $\beta_1, \dots, \beta_d \in R_i$ be elements whose images modulo \mathfrak{m}_i form a k -basis of R_i/\mathfrak{m}_i . Then it is straightforward to check that R is generated as a k -vector space by the elements

$$\beta_j \cdot r_1^{e_1} r_2^{e_2} \cdots r_n^{e_n} \quad \text{where } 0 \leq e_\ell < N \text{ for all } \ell.$$

This shows $\dim_k R < \infty$, hence f is finite.

Part (c). Since g is quasi-finite at y , the subset $\{y\} \subseteq g^{-1}\{g(y)\}$ is open and closed, hence $f^{-1}\{y\} \subseteq (gf)^{-1}\{g(y)\}$ is open and closed. This means that $\{x\}$ is open and closed in the fibre $(gf)^{-1}\{g(y)\}$ iff it is open and closed in $f^{-1}\{y\}$ and we win. \square

Theorem 2 (Grothendieck's version of Zariski's main theorem). (a) *Let $f: X \rightarrow Y$ be a quasi-finite proper morphism between locally noetherian preschemes. Then f is finite.*

(b) *Let $f: X \rightarrow Y$ be a quasi-finite and separated morphism between noetherian preschemes. Then there exists a factorization $X \xrightarrow{j} \overline{X} \xrightarrow{g} Y$ of f where j is an open immersion and g is finite.*

(c) *If $f: X \rightarrow Y$ is any morphism of finite type between locally noetherian preschemes, then*

$$U = \{x \in X \mid f \text{ is quasi-finite at } x\}$$

is open in X , and the restriction $f|_U$ is quasi-finite (by definition).

Proof. Part (a). We may assume that $Y = \text{Spec } A$ is affine (indeed, all involved properties are base-local). Let $\mathcal{J} \subseteq \mathcal{O}_X$ be a sheaf of ideals, then \mathcal{J} is coherent as X is locally noetherian. Since f is quasi-finite, all fibres carry the discrete topology. In particular, they are zero-dimensional and Corollary 1 shows that $R^1 f_* \mathcal{J} = 0$. Then also $0 = R^1 f_* \mathcal{J}(Y) = H^1(X, \mathcal{J})$ (using [AG2, Proposition 1.5.1(d)]), hence X is affine by Serre's affinity criterion. This shows that f is affine. Moreover, $f_* \mathcal{O}_X$ is a coherent \mathcal{O}_Y -module by [AG2, Theorem 5], hence f is finite.

Part (b) is hard, see the discussion on page 12. We only prove a special case there, which, however, is sufficient to prove (c). But before we can do this, we need to prove some more theorems of Zariski. \square

Theorem 3 (Zariski's connectedness theorem). *Let $f: X \rightarrow Y$ be a proper morphism between locally noetherian schemes, whose algebraic component $f^*: \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ is an isomorphism.*

(a) *The fibres $f^{-1}\{y\}$ are connected for all $y \in Y$.*

(b) *The set*

$$U = \{x \in X \mid \{x\} = f^{-1}\{f(x)\}\} = \{x \in X \mid f \text{ is quasi-finite at } x\}$$

is open in X , and the restriction $f|_U$ is quasi-finite (by definition).

Proof. Part (a). Assume $f^{-1}\{y\}$ is not connected, say, $f^{-1}\{y\} = U_1 \cup U_2$ for disjoint non-empty open subsets $U_1, U_2 \subseteq f^{-1}\{y\}$. Since all infinitesimal thickenings $X_n = X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n)$ have underlying topological space $f^{-1}\{y\}$, there is a unique $\varepsilon_n \in \mathcal{O}_{X_n}(X_n) = H^0(X_n, \mathcal{O}_{X_n})$ such that $\varepsilon_n|_{U_1} = 0$ and $\varepsilon_n|_{U_2} = 1$. The sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ clearly defines an element ε of

$$\varprojlim_{n \in \mathbb{N}} H^0(X_n, \mathcal{O}_{X_n}) \cong (f_* \mathcal{O}_X)_{\hat{y}} \cong \hat{\mathcal{O}}_{Y,y}.$$

The left isomorphism here is due to Theorem 1a and the fact that $\mathcal{O}_{X_n} = \mathcal{O}_X|_{X_n}$, and the right one holds by assumption. Hence $\hat{\mathcal{O}}_{Y,y}$ is a local ring (by Corollary A.1.5) with an idempotent $\varepsilon \neq 0, 1$. Then $1 - \varepsilon \neq 0, 1$ is another non-trivial idempotent. Both ε and $1 - \varepsilon$ can't be units in $\hat{\mathcal{O}}_{Y,y}$, otherwise $\varepsilon^2 = \varepsilon$ implies $\varepsilon = 1$ (and similar for $1 - \varepsilon$). But then they are elements of the maximal ideal \mathfrak{m} , so $\varepsilon + (1 - \varepsilon) = 1$ is an element of \mathfrak{m} as well, contradiction!

Part (b). By (a), any point $x \in X$ is open and closed in its fibre iff $f^{-1}\{f(x)\} = \{x\}$. Therefore the two definitions of U indeed coincide.

We must show that U is open. This is a local question with respect to Y , hence we may assume that $Y = \operatorname{Spec} A$ is affine. Let $x \in U$ and $V \subseteq X$ an affine open neighbourhood of x . Put $Z = X \setminus V$. Then $Z \subseteq X$ is closed and disjoint from $f^{-1}\{f(x)\} = \{x\}$. As f is proper, $Z' = f(Z) \subseteq Y$ is closed, and $y = f(x) \notin Z'$. There's an $\alpha \in A$ such that $y \notin V(\alpha)$ and $V(\alpha) \supseteq Z'$. Let $Y_1 = Y \setminus V(\alpha)$. Note that $Y_1 \cong \operatorname{Spec} A_\alpha$ is affine and $x \in X_1 = f^{-1}(Y_1) \subseteq V$. Then $X_1 = X \setminus V(f^*\alpha) = V \setminus V(f^*\alpha)$ is affine as well, so the restriction $f|_{X_1}: X_1 \rightarrow Y_1$ of f is affine and proper. But every affine proper morphism is finite (because $f|_{X_1,*} \mathcal{O}_{X_1}$ is a coherent \mathcal{O}_{Y_1} -module by [AG2, Theorem 5]), so $f|_{X_1}$ is, in particular, quasi-finite by Fact 2 and $U \cap X_1 = X_1$. This proves that U is open. \square

Remark 1. On first glance, the argument from Theorem 3(b) might look like it proves that every proper morphism is affine, but what it actually shows is the following: If $f: X \rightarrow Y$ is a proper morphism such that for each $x \in X$ the fibre $f^{-1}\{f(x)\}$ is contained in some affine subset $V \subseteq X$, then f is already affine (and hence finite).

Remark 2. Recall that a prescheme X is called **normal** if it is integral and all local rings $\mathcal{O}_{X,x}$ (which are domains if X is integral) are normal (cf. [AG1, Definition 2.4.5]). This is the case iff $\mathcal{O}_X(U)$ is a normal domain for all affine $U \subseteq X$, cf. the discussion in [AG1, Remark 2.5.1].

Corollary 2 (Zariski's birationality theorem). *Let $f: X \rightarrow Y$ be a proper morphism between locally noetherian preschemes, where Y is normal. Suppose that f is **birational** in the sense that there is a dense open subset $U \subseteq Y$ such that the restriction $f|_{f^{-1}(U)}: f^{-1}(U) \xrightarrow{\sim} U$ is an isomorphism and $f^{-1}(U)$ is dense in X . Then all assertions from Theorem 3 apply to f . In particular, f has connected fibres.*

Proof. First note that U is irreducible as an open subset of the irreducible space Y (irreducibility of Y is implied by Y being normal). Hence X is irreducible because it has the dense irreducible subset $f^{-1}(U) \cong U$. Let $\operatorname{Spec} A \cong V \subseteq Y$ be an affine open subset, where A is a domain. Then $f^{-1}(V)$ is open in X , hence dense in X and thus irreducible. Since U is dense in Y , the intersection $U \cap V$ is non-empty, hence $f^{-1}(U \cap V) \subseteq f^{-1}(V)$ is a non-empty open subset and thereby dense again. This shows that we can actually reduce to the case $Y = \operatorname{Spec} A$ (all the

other involved properties are clearly base-local). Moreover, we may assume that X is integral. Indeed, the assertions from Theorem 3 are purely topological, so we may replace X by its reduction $X^{\text{red}} = V(\text{nil}(\mathcal{O}_X))$ to obtain an X which is irreducible and reduced (hence integral) and has the same underlying topological space as the original one.

Claim 1. The ring $B = \mathcal{O}_X(X)$ is a domain in the above situation, and A and B have the same field of quotients K . Moreover, we have $A \subseteq B$ as subrings of K .

Believing this for the moment, the proof can be finished as follows. Since B is finitely generated as an A -module (because $f_*\mathcal{O}_X = \tilde{B}$ is coherent by [AG2, Theorem 5]), it is integral over A . But A is integrally closed in K , hence $A \subseteq B$ implies $A = B$. We conclude $f_*\mathcal{O}_X \cong \mathcal{O}_Y$, as needed.

Unfortunately, the proof of Claim 1 wasn't discussed in the lecture, but I think it should have been. Since X and Y are irreducible, they have unique generic points η_X and η_Y . As η_Y is dense in Y , we have $\eta_Y \in U$ and similarly $\eta_X \in f^{-1}(U)$. Hence $f(\eta_X) = \eta_Y$ and the induced morphism $\mathcal{O}_{Y,\eta_Y} \xrightarrow{\sim} \mathcal{O}_{X,\eta_X}$ is an isomorphism by the birationality assumption. Moreover, η_Y corresponds to $0 \in \text{Spec } A$, hence $\mathcal{O}_{Y,\eta_Y} \cong K$ is the quotient field of A . So we should prove that \mathcal{O}_{X,η_X} is the quotient field of $B = \mathcal{O}_X(X)$ as well.

It's clear that B is a domain because X is integral. Since $U \subseteq \text{Spec } A$ is open, we find an affine open subset $V = \text{Spec } A \setminus V(\alpha) \subseteq U$. Then $f^{-1}(V) = X \setminus V(f^*\alpha) = f^{-1}(U) \setminus V(f^*\alpha) \cong V$ is affine again by birationality of f . We know that X is quasi-compact and separated since so are f and $\text{Spec } A$. In particular, [AG1, Proposition 1.5.1(c)] is applicable to \mathcal{O}_X and gives $\mathcal{O}_X(f^{-1}(V)) \cong \mathcal{O}_X(X)_{f^*\alpha}$, so these two rings have the same quotient field. But $\mathcal{O}_X(f^{-1}(V)) \cong \mathcal{O}_Y(V) \cong A_\alpha$ has quotient field K , so we win.

The fact that $A \subseteq B$ as subrings of K follows from the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \mathcal{O}_{Y,\eta_Y} & \xrightarrow{\sim} & \mathcal{O}_{X,\eta_X} \end{array}$$

in which every arrow except the top one is injective, hence $A \rightarrow B$ is injective as well. \square

Fact 3. Every proper morphism $f: X \rightarrow Y$ between locally noetherian preschemes can be factorized via the **Stein factorization** as

$$f: X \xrightarrow{\tilde{f}} \mathbf{Spec}_Y(f_*\mathcal{O}_X) \xrightarrow{g} Y.$$

In this composition, g is finite and the assumptions of Theorem 3 hold for \tilde{f} .

Sketch of a proof. It's pretty obvious that this factorization exists (to construct \tilde{f} , use the adjunction from [AG2, Proposition 1.6.2(b)]). To show that \tilde{f} and g have the required properties,

we look at things locally and assume that $Y = \operatorname{Spec} A$ is affine (and A is noetherian). Then the factorization looks like

$$f: X \xrightarrow{\tilde{f}} \operatorname{Spec} \mathcal{O}_X(X) \xrightarrow{g} \operatorname{Spec} A ,$$

so g is affine. Moreover, $\mathcal{O}_X(X)$ is a finitely generated A -module because $f_*\mathcal{O}_X$ is a coherent \mathcal{O}_Y -module (by [AG2, Theorem 5], as usual), so g is actually finite. Also proving that $\tilde{f}_*\mathcal{O}_X = \mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(X)}$ is straightforward, so it remains to show that \tilde{f} is proper. But g is finite, hence separated, and $g\tilde{f} = f$ is proper, so \tilde{f} is proper as well by [AG2, Proposition 2.4.1]. \square

1.3.2. Discussion of Theorem 2(b) and (c)

Proof of Theorem 2(c). Let's assume that $f: X \rightarrow Y$ factors over

$$f: X \xrightarrow{j} \overline{X} \xrightarrow{\bar{f}} Y , \tag{1}$$

where j is an open immersion and \bar{f} is proper. Then

$$\{x \in X \mid f \text{ is quasi-finite at } x\} = X \cap \left\{x \in \overline{X} \mid \bar{f} \text{ is quasi-finite at } x\right\} . \tag{2}$$

Indeed, a point $x \in X$ is open in $\bar{f}^{-1}\{y\}$ (where $y = f(x)$) iff it is open in the open subset $f^{-1}\{y\} = X \cap \bar{f}^{-1}\{y\} \subseteq \bar{f}^{-1}\{y\}$. In view of Fact 1 this shows (2). We thus have reduced (c) (under the assumption that \bar{f} exists) to the case of proper morphisms.

If $f: X \rightarrow Y$ is proper, then consider its Stein factorization. Since g is finite, it's quasi-finite as well by Fact 2(a). So Fact 2(c) shows that

$$\{x \in X \mid f \text{ is quasi-finite at } x\} = \{x \in X \mid \tilde{f} \text{ is quasi-finite at } x\} .$$

But the right-hand side is open in X by Fact 3 and Theorem 3(b) and we're happy!

Note that such an \bar{f} always exists when X and Y are affine. Indeed, if X has finite type over Y and both are affine, we get a closed embedding $X \hookrightarrow \mathbb{A}_Y^n$ for some $n \in \mathbb{N}$. Together with the open embedding $\mathbb{A}_Y^n \hookrightarrow \mathbb{P}_Y^n$ this makes X a closed subscheme of an open subscheme of \mathbb{P}_Y^n . But then X is also an open subscheme of some closed subscheme $\overline{X} \subseteq \mathbb{P}_Y^n$. This gives a factorization

$$f: X \xrightarrow{j} \overline{X} \xrightarrow{\bar{f}} Y$$

in which $\bar{f}: \overline{X} \hookrightarrow \mathbb{P}_Y^n \rightarrow Y$ is (strongly) projective, hence proper by [AG2, Proposition 2.4.2]. But (c) is completely local on both X and Y (thanks to Fact 1), so by checking the affine case we have actually covered all of (c). \square

The rest of this subsection will be used to sketch a proof of Theorem 2(b). We begin with the observation that Theorem 2(b) becomes much easier when things are proper.

Lemma 1. *Let $f: X \rightarrow Y$ be a quasi-finite separated morphism between noetherian schemes, and assume that f factors as in (1). Then Theorem 2(b) holds for f .*

Remark 3. It can be shown that such a factorization (1) always exists for morphisms of finite type between noetherian schemes, for which Professor Franke refers to notes of *Brian Conrad* or *Paul Vojta*, although he isn't sure whether using their results to prove Theorem 2(b) doesn't involve any circular reasoning.

Proof of Lemma 1. Let $\bar{f} = g \circ \tilde{f}$ be the Stein factorization of \bar{f} . Put $\bar{Y} = \mathbf{Spec}_Y \bar{f}_* \mathcal{O}_{\bar{X}}$ for convenience. Since $g: \bar{Y} \rightarrow Y$ is finite by Fact 3 (for which we need properness of \bar{f}), we're done if we show that the composition $\tilde{f}|_X: X \hookrightarrow \bar{X} \rightarrow \bar{Y}$ is an open embedding. Since $f = g\tilde{f}|_X$ and g are quasi-finite (g is even finite), Fact 2(c) shows that $\tilde{f}|_X$ is quasi-finite as well.

In particular $\tilde{f}|_X$ is injective, and for all $x \in X$ we have $\tilde{f}^{-1}\{\tilde{f}(x)\} = \{x\}$, because $\mathcal{O}_{\bar{Y}} \cong \tilde{f}_* \mathcal{O}_{\bar{X}}$ (by Fact 3), so the argument from Theorem 3(b) can be applied. If V is any open neighbourhood of $x \in X$, then $\tilde{f}(\bar{X} \setminus V) \subseteq \bar{Y}$ is closed because \tilde{f} is proper (by Fact 3), hence closed. Moreover, $\tilde{f}(\bar{X} \setminus V)$ doesn't contain $\tilde{f}(x)$ as $\tilde{f}^{-1}\{\tilde{f}(x)\} = \{x\}$. Thus, the complement U of $\tilde{f}(\bar{X} \setminus V)$ in \bar{Y} is open and $\tilde{f}^{-1}(U) \subseteq V$ is an open neighbourhood of x . This shows that $\tilde{f}|_X$ is actually an open map! Together with $\mathcal{O}_{\bar{Y}} \cong \tilde{f}_* \mathcal{O}_{\bar{X}}$ we see that $\tilde{f}|_X$ is an open embedding, as claimed. \square

Definition 2. A morphism $f: X \rightarrow Y$ of preschemes is called **quasi-affine**, if it satisfies the following equivalent conditions:

- (a) For all affine open $U \subseteq Y$, $f^{-1}(U)$ is a quasi-affine scheme (that is, a quasi-compact open subscheme of an affine scheme).
- (b) Y can be covered by affine open U such that $f^{-1}(U)$ is quasi-affine.
- (c) f factors as $X \xrightarrow{j} \bar{X} \xrightarrow{\bar{f}} Y$, where j is an open embedding and \bar{f} is affine.
- (d) The canonical morphism $X \rightarrow \mathbf{Spec}_Y(f_* \mathcal{O}_X)$ in the Stein factorization is an open immersion.

For a proof of equivalence, check out [Stacks, Tag 01SJ]. Note that Franke apparently had a different proof (for the case of quasi-affine morphisms of finite type) in mind than the Stacks Project guys, but I have no idea how this was going to work (whereas the Stacks Project proof is pretty clear to me and works without restrictions on f). Nevertheless, the following lemma he suggested is perhaps worthwhile to know.

Lemma 2. Let X be a noetherian prescheme, \mathcal{M} a quasi-coherent \mathcal{O}_X -module, and $U \subseteq X$ an open subset. Let $\mathcal{N} \subseteq \mathcal{M}|_U$ be a coherent submodule. Then there is a coherent $\tilde{\mathcal{N}} \subseteq \mathcal{M}$ such that $\tilde{\mathcal{N}}|_U = \mathcal{N}$. In particular, \mathcal{M} is the union of its coherent submodules.

Proof. We proceed by noetherian induction. Because Y is noetherian, there is a \subseteq -maximal open subset U to which \mathcal{N} extends (and, with slight abuse of notation, we denote some fixed extension to U by \mathcal{N} as well). Assuming $U \neq X$, we will derive a contradiction. Pick $x \in X \setminus U$ and let $V \cong \mathbf{Spec} A$ be an affine open neighbourhood of x .

Claim. There is a coherent $\mathcal{Q} \subseteq \mathcal{M}|_V$ such that $\mathcal{Q}|_{U \cap V} \supseteq \mathcal{N}|_{U \cap V}$.

Indeed, $V \cong \operatorname{Spec} A$ is a noetherian topological space (because Y is noetherian), so the open subset $U \cap V$ is quasi-compact. This means that we can cover it by finitely many affine open subsets $V_i \cong \operatorname{Spec} A_{f_i}$ for $i = 1, \dots, n$. For every i we know that $\mathcal{N}(V_i) \subseteq \mathcal{M}(V_i) \cong \mathcal{M}(V)_{f_i}$ is a finitely generated A_{f_i} -module (because \mathcal{N} is coherent), so we can choose finitely many $\mu_{i,j} \in \mathcal{M}(V)$, $j = 1, \dots, k_i$, whose images in $\mathcal{N}(V_i)$ form a set of generators. Let $\mathcal{Q} \subseteq \mathcal{M}|_V$ be the subsheaf generated by $\{\mu_{i,j} \mid i \leq n, j \leq k_i\} \in \mathcal{M}(V)$. Then \mathcal{Q} has the required property, proving the claim.

Let $j: U \cap V \hookrightarrow V$ be the obvious inclusion and put $\overline{\mathcal{N}} = \mathcal{Q} \cap j_* \mathcal{N}|_{U \cap V}$. This guy is coherent (because subsheaves of a coherent one are coherent again on locally noetherian preschemes) and satisfies $\overline{\mathcal{N}}|_{U \cap V} = \mathcal{N}|_{U \cap V}$. But then \mathcal{N} can be extended to a coherent sheaf $\tilde{\mathcal{N}}$ on $U \cup V$ via $\tilde{\mathcal{N}}|_U = \mathcal{N}$ and $\mathcal{N}|_V = \mathcal{K}$. This contradicts maximality of U . \square

Lemma 3. *If $f: X \rightarrow Y$ is a quasi-finite and quasi-affine morphism between noetherian preschemes. Then Theorem 2(b) holds for f .*

Proof. Note that quasi-affine morphisms are automatically separated. We will show that $g: \operatorname{Spec}_Y(f_* \mathcal{O}_X) \rightarrow Y$ is proper, which shows that f factors as in (1). Then Theorem 2(b) holds by Lemma 1. To show properness of g , we define

$$\mathcal{R} = \bigoplus_{n \geq 0} f_* \mathcal{O}_X(n),$$

viewed as a graded quasi-coherent \mathcal{O}_Y -algebra. Note that $\operatorname{Spec}_Y(f_* \mathcal{O}_X) \cong \operatorname{Proj}_Y(\mathcal{R})$ as Y -preschemes (indeed, locally this reduces to [AG1, Example 2.6.2]). Moreover, \mathcal{R} is locally of finite type over \mathcal{O}_Y . In fact, the 0th homogeneous component $\mathcal{R}_0 \cong f_* \mathcal{O}_X$ is locally of finite type over \mathcal{O}_Y (because f is of finite type), and \mathcal{R} is generated by \mathcal{R}_0 and the element $1 \in \mathcal{R}_1(Y)$ (which acts as a shift). Therefore, $\operatorname{Proj}_Y(\mathcal{R}) \rightarrow Y$ is proper by [AG2, Proposition 2.4.2], whence we're done. \square

So we see that to prove Theorem 2(b) it is sufficient to show that any quasi-finite and separated morphism between noetherian preschemes is quasi-affine. This is a base-local question again (since quasi-affinity is base-local again by Definition 2), so we may assume that $Y = \operatorname{Spec} A$ for some noetherian ring A .

Professor Franke sketches a proof following [SGA1, Exposé VIII.6], where the technique of *faithfully flat descent* is used to reduce the question to the case of complete local rings.

So we may assume $Y = \operatorname{Spec} A$, where A is a noetherian complete local ring (with respect to its maximal ideal \mathfrak{m}) and that Zariski's main theorem is true for morphisms $f': X' \rightarrow Y'$, where in addition to the other assumptions we have $\dim Y' < \dim Y$. Let s be the unique closed point of A (given by \mathfrak{m}).

Lemma 4. *If B is any finite A -algebra, then $\operatorname{Spec} B$ has a decomposition $\operatorname{Spec} B = \bigcup_{i=1}^n U_i$, where the U_i are disjoint open subsets such that the only closed point s of $\operatorname{Spec} A$ has precisely one preimage in each U_i .*

Proof. By Corollary A.1.7, B has finitely many maximal ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ and these are precisely the prime ideals over \mathfrak{m} . For every \mathfrak{q}_i we will construct an idempotent $e_i \in B$ such that $e_i \in \mathfrak{q}_i$ but $e_i \notin \mathfrak{q}_j$ for $j \neq i$. Then $\mathfrak{q}_i \in V(e_i)$ but $\mathfrak{q}_j \notin V(e_i)$ for $j \neq i$, and $V(e_i)$ is an open and closed subset of B (because e_i is an idempotent), so we see that the \mathfrak{q}_i are contained in distinct connected components of $\text{Spec } B$. But B is noetherian, so $\text{Spec } B$ has finitely many connected components (as pointed out in [AG1, Lemma 2.4.2]). Then every connected component is open and we can construct the required U_i as unions of suitable connected components of $\text{Spec } B$.

Let $\bar{\mathfrak{q}}_i = \mathfrak{q}_i/\mathfrak{m}$ be the prime ideals of $B/\mathfrak{m}B$ (which are automatically maximal). Then the intersection $\bigcap_{j=1}^n \bar{\mathfrak{q}}_j$ is the nilradical $\text{nil}(B/\mathfrak{m}B)$, so by the Chinese remainder theorem we find an element $\bar{e}_i \in (B/\mathfrak{m}B)/\text{nil}(B/\mathfrak{m}B)$ such that $\bar{e}_i \equiv 0 \pmod{\bar{\mathfrak{q}}_i}$ and $\bar{e}_i \equiv 1 \pmod{\bar{\mathfrak{q}}_j}$ for $j \neq i$. Note that $\bar{e}_i^2 = \bar{e}_i$. Since $B/\mathfrak{m}B$ is noetherian, there is an $N \in \mathbb{N}$ such that $\text{nil}(B/\mathfrak{m}B)^N = 0$. Hence $B/\mathfrak{m}B$ is $\text{nil}(B/\mathfrak{m}B)$ -adically complete, which means we can lift \bar{e}_i to an idempotent $\tilde{e}_i \in B/\mathfrak{m}B$ by Hensel's lemma (cf. Proposition A.1.1). Now B is $\mathfrak{m}B$ -adically complete by Proposition A.1.4, so using Hensel's lemma once again we can lift \tilde{e}_i to an idempotent $e_i \in B$ with the required properties. We're done. \square

Let $f: X \rightarrow Y = \text{Spec } A$ be quasi-finite and separated, and let $x \in f^{-1}\{s\}$. Because the fibre $f^{-1}\{s\}$ is discrete, there is an affine open $U \subseteq X$ such that $U \cap f^{-1}\{s\} = \{x\}$. Zariski's main theorem applies to the affine morphism $f|_U: U \rightarrow \text{Spec } A$, which therefore factors as

$$U \xhookrightarrow{j} \text{Spec } B \longrightarrow Y$$

where B is as above and j is an open embedding. By Lemma 4 (and shrinking U if necessary) we may assume that B has only one closed point $j(x)$, i.e. is local itself. But then the only open subset of $\text{Spec } B$ containing the closed point $j(x)$ is $\text{Spec } B$ itself, hence j is an isomorphism. We have thus found an affine open neighbourhood $U \cong \text{Spec } B$ of x which is finite over Y . Then $U \rightarrow X$ is proper (as $U \rightarrow Y$ is finite, hence proper, and f is separated, so [AG2, Proposition 2.4.1] applies), so U is also closed. Putting $U = U_1$ and $X \setminus U = X_1$, what we proved is that $X = X_1 \amalg U_1$ is the disjoint union of its open subsets X_1 and U_1 .

Iterating this for the remaining preimages² of s in X provides a decomposition

$$X = X' \amalg \coprod_{i=1}^n U_i$$

into disjoint open subsets, where $\coprod_{i=1}^n U_i \cong \coprod_{i=1}^n \text{Spec } \mathcal{O}_X(U_i) \cong \text{Spec } (\oplus_{i=1}^n \mathcal{O}_X(U_i))$ is finite over Y and $f' = f|_{X'}: X' \rightarrow Y$ has image in $Y' = Y \setminus \{s\}$, hence the induction assumption applies to $f': X' \rightarrow Y'$. This means that f' can be written as $f': X' \hookrightarrow \bar{X}' \rightarrow Y'$, where X' is an open subscheme of \bar{X}' , which in turn is finite over Y' . Still we aren't done yet, as we need something finite over Y rather than Y' .

Here it is time for our final lemma.

²There are finitely many of them. Indeed, $f^{-1}\{s\} \subseteq X$ is closed (as the preimage of the closed point s), hence quasi-compact (because X is noetherian, hence quasi-compact). But the fibre $f^{-1}\{s\}$ is supposed to be discrete, so it must be finite.

Lemma 5. *Let A be a noetherian ring and $S \subseteq \operatorname{Spec} A$ an open subscheme. If $f: X \rightarrow S$ is a finite morphism, then there is a finite A -algebra B such that X is an open subscheme of $\operatorname{Spec} B$ and the diagram*

$$\begin{array}{ccc} \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A \\ \uparrow & & \uparrow \\ X & \xrightarrow{f} & S \end{array}$$

commutes.

Proof. Since $\operatorname{Spec} A$ is a noetherian space, its open subset S is quasi-compact. We thus find a finite cover $S = \bigcup_{i=1}^n S_i$, where $S_i \cong \operatorname{Spec} A_{\alpha_i}$ (for some $f_i \in A$) are affine open subschemes. Putting $X_i = f^{-1}(S_i)$ we get an affine open cover $X = \bigcup_{i=1}^n X_i$, where $X_i \cong \operatorname{Spec} B_i$ for some finite A_{α_i} -algebra B_i by finiteness of f .

Note that X is quasi-compact because S and f are. Moreover, S is separated as an open subscheme of a separated prescheme, hence X is separated as well, as finite morphisms are separated. If we denote the image of α_i under the restriction $A \rightarrow \mathcal{O}_S(S)$ by α_i as well, we see that $S_i = S \setminus V(\alpha_i)$, so $A_{\alpha_i} \cong \mathcal{O}_S(S \setminus V(\alpha_i)) \cong \mathcal{O}_S(S)_{\alpha_i}$. Note that this works because S is quasi-compact and separated, so [AG1, Proposition 1.5.1(c)] applies. By the same argument applied to X , we get $B_i \cong \mathcal{O}_X(X \setminus V(f^*\alpha_i)) \cong \mathcal{O}_X(X)_{f^*\alpha_i}$.

Note that $\mathcal{O}_X(X)$ becomes an A -algebra via $A \rightarrow \mathcal{O}_S(S) \rightarrow \mathcal{O}_X(X)$. For every i let $x_{i,j} \in B_i$, $j = 1, \dots, k_i$ be generators of B_i as an A_{α_i} -module. Multiplying by a suitable power of $f^*\alpha_i$ we can assume that the $x_{i,j}$ are from $\mathcal{O}_X(X)$ and integral over A (rather than just integral over A_{α_i}). Now let Λ be the image of A in $\mathcal{O}_X(X)$ and put

$$B = \Lambda[x_{i,j} \mid i \leq n, j \leq k_i] .$$

Then B is finite over A and contains the $f^*\alpha_i$ (because these guys are already in Λ). Moreover, we have $B_{f^*\alpha_i} = \mathcal{O}_X(X)_{f^*\alpha_i} \cong B_i$, because B already contains a set of generators of B_i over A_{α_i} . So $X = \bigcup_{i=1}^n X_i$ is a union of open subschemes of $\operatorname{Spec} B$, hence an open subscheme itself and $\operatorname{Spec} B \rightarrow \operatorname{Spec} A$ clearly has the required properties. \square

Hence Zariski's main theorem may likely be proved if we were able to reduce the case of general Y to the case of noetherian complete local rings.

2. Flat morphisms

Definition 1. A morphism $f: X \rightarrow Y$ of preschemes is called **flat** iff it has the following equivalent properties.

- (a) If $U \subseteq X$ and $V \subseteq Y$ are affine opens such that $f(U) \subseteq V$, then $\mathcal{O}_X(U)$ is flat as an $\mathcal{O}_Y(V)$ -module.
- (b) It is possible to cover X by affine open subsets for which such V may be found.
- (c) For any $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,y}$ -module, where $y = f(x)$.

We call f **faithfully flat** if it is flat and surjective.

Sketch of a proof of equivalence. The implication $(a) \Rightarrow (b)$ is trivial. If you think about it, showing $(b) \Rightarrow (c) \Rightarrow (a)$ comes down to the following fact from commutative algebra: An A -algebra B is flat if and only if $B_{\mathfrak{q}}$ is a flat $A_{\mathfrak{p}}$ algebra for all primes $\mathfrak{p} \in \text{Spec } A$ and $\mathfrak{q} \in \text{Spec } B$ above \mathfrak{p} .

Indeed, if B is flat over A , then so is $B_{\mathfrak{q}}$ (by exactness of localization), hence $B_{\mathfrak{q}}$ is also flat over $A_{\mathfrak{p}}$. Conversely, if $B_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$, then also over A because $A_{\mathfrak{p}}$ is flat over A (exactness of localization again). But then the localizations of B at any prime are flat over A , hence so is B itself (cf. [Hom, Fact 1.2.6(d)]). \square

2.1. Flat base change and cohomology

Let A be a ring, B an A -algebra and $f: X \rightarrow \text{Spec } A$ a quasi-compact and separated morphism. Let

$$\tilde{f}: \tilde{X} = X \times_{\text{Spec } A} \text{Spec } B \longrightarrow \text{Spec } B$$

its base change along $\text{Spec } B \rightarrow \text{Spec } A$. Also let $\pi: \tilde{X} \rightarrow X$ be projection to the other fibre product factor. We want to investigate the relation between the cohomology of X and \tilde{X} . For a quasi-coherent \mathcal{O}_X -module \mathcal{F} and an affine Čech cover $\mathcal{U}: X = \bigcup_{i \in I} U_i$ of X , the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$ calculates $H^\bullet(X, \mathcal{F})$. Pulling back \mathcal{U} gives a Čech cover

$$\pi^{-1}\mathcal{U}: \tilde{X} = \bigcup_{i \in I} U_i \times_{\text{Spec } A} \text{Spec } B,$$

of \tilde{X} , whose components $U_i \times_{\text{Spec } A} \text{Spec } B$ are affine again (as fibre products of affine schemes), hence $\pi^{-1}\mathcal{U}$ may be used to compute $H^\bullet(\tilde{X}, \pi^*\mathcal{F})$. Explicitly, we obtain

$$\check{C}^\bullet(\pi^{-1}\mathcal{U}, \pi^*\mathcal{F}) \cong \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A B. \quad (1)$$

This gives a morphism

$$H^\bullet(X, \mathcal{F}) \otimes_A B \longrightarrow H^\bullet(\tilde{X}, \pi^* \mathcal{F}) , \tag{2}$$

which clearly is an isomorphism if B is flat over A .

A. Appendix

A.1. Some prerequisites about completions

We briefly recall the most important facts about completions. An excellent introduction to this subject can be found in [AM94, Section 10].

Definition 1. Let A be a ring (commutative with 1) and I an ideal in A . Let M be an A -module.

- (a) The **I -adic topology** on M is the unique topology such that $\{I^n\}_{n \in \mathbb{N}}$ is a fundamental system of neighbourhoods of 0 and M (with its additive structure) becomes a topological group in this topology.
- (b) The **completion** of M with respect to the I -adic topology is

$$\hat{M} = \varprojlim_{n \in \mathbb{N}} M/I^n M.$$

Note that \hat{A} is a ring again. We call M **complete** in the I -adic topology if the canonical morphism $M \rightarrow \hat{M}$ is an isomorphism.

Remark 1. M with its I -adic topology is *pseudo-metrizable* via $d(x, y) = e^{-\sup\{n \mid x-y \in I^n\}}$. It is easy to check that \hat{M} is also the completion of M in the analytical sense, i.e. the set of Cauchy sequences modulo the zero sequences.

Example 1. If $I^n = 0$ for some $n \in \mathbb{N}$, then any A -module is complete in the I -adic topology.

Example 2. If $A = \mathbb{Z}$ and $I = p\mathbb{Z}$ for some prime p , then $\hat{A} = \mathbb{Z}_p$ is the ring of p -adic integers.

Proposition 1 (Hensel's lemma). *Suppose the ring A is complete in the I -adic topology. Let $P \in A[T]$ be a polynomial and $a_0 \in A$ such that $P(a_0) \equiv 0 \pmod{I}$ and $P'(a_0)$ is a unit in A/I . Then there is a unique $a \in A$ such that $a \equiv a_0 \pmod{I}$ and $P(a) = 0$.*

Proof. Step 1. Consider the special case $I^2 = 0$. For $\delta \in I$ we have $P(a_0 + \delta) = P(a_0) + \delta P'(a_0)$ since all terms of order δ^2 or higher vanish in the binomial expansion. Now $P'(a_0)$ being a unit in A/I gives a unique $\delta \in I$ such that $a = a_0 + \delta$ satisfies $P(a) = 0$.

Step 2. Suppose that $I^{2^n} = 0$ for some $n \in \mathbb{N}$. Using induction on n (with the base case being precisely Step 1) we may assume that Hensel's lemma holds for $A/I^{2^{n-1}}$. In particular, there is a unique a_{n-1} such that $P(a_{n-1}) \equiv 0 \pmod{I^{2^{n-1}}}$ and $a_{n-1} \equiv a_0 \pmod{I}$. Moreover, $P'(a_{n-1})$ is invertible in $A/I^{2^{n-1}}$. Indeed, this follows from Hensel's lemma applied to $A/I^{2^{n-1}}$ (for which it

holds by induction hypothesis) and the polynomial $Q = P'(a_{n-1})T - 1$. The derivative $Q'(a_{n-1})$ equals $P'(a_{n-1})$ which is invertible in A/I since $P'(a_{n-1}) \equiv P'(a_0) \pmod{I}$, so Hensel's lemma is indeed applicable. Now replacing I by $I^{2^{n-1}}$ and a_0 by a_{n-1} reduces the situation to Step 1, proving the inductive step.

Step 3. Now let I be arbitrary. By Step 2 there is for every $n \in \mathbb{N}$ a unique $a_n \in A/I^{2^n}$ such that $P(a_n) \equiv 0 \pmod{I^{2^n}}$ and $a_n \equiv a_0 \pmod{I}$. Then $a_n \equiv a_{n-1} \pmod{I^{2^{n-1}}}$ is forced by uniqueness. Hence $a = (a_n)_{n \in \mathbb{N}}$ defines an element of

$$\varprojlim_{n \in \mathbb{N}} A/I^{2^n} = \varprojlim_{n \in \mathbb{N}} A/I^n = \widehat{A},$$

providing the desired element $a \in \widehat{A}$. \square

Corollary 1. *Let A be complete in the I -adic topology.*

- (a) *If $a \in A$ becomes a unit in A/I , then already $a \in A^\times$.*
- (b) *For every idempotent $\pi \in A/I$ there is a unique idempotent in A whose image modulo I is π . Therefore, $\operatorname{Spec} A$ and $\operatorname{Spec} A/I$ have the same connected components.*
- (c) *I is contained in the Jacobson radical $\operatorname{rad} A$.*

Proof. Part (a) follows from Proposition 1 applied to $P = aT - 1$ (whose derivative a is a unit in A/I by assumption, so this is fine). For (b) we use the polynomial $P = T^2 - T$. Again, $P'(\pi) = 2\pi - 1$ is a unit in A/I since $(2\pi - 1)^2 = 4\pi^2 - 4\pi + 1 = 1$ in A/I . To prove (c) recall the characterization

$$\operatorname{rad} A = \{x \in A \mid 1 - ax \in A^\times \text{ for all } a \in A\}.$$

If $x \in I$, then $1 - ax$ is a unit in A/I , hence also in A by (a). \square

Proposition 2. *Let A be noetherian and $N \subseteq M$ finitely generated A -modules. Then the I -adic topology on N coincides with the induced topology by the I -adic topology on M .*

Sketch of a proof. By the Artin–Rees lemma (cf. [Alg2, Proposition 3.4.1]) there exists a number $c \in \mathbb{N}$ such that $N \cap I^{n+c}M \subseteq I^nN$. From this, the assertion is easily deduced. \square

Fact 1. (a) *The canonical morphism $\widehat{M} = \varprojlim M/I^nM \rightarrow M/IM$ is surjective.*

- (b) *If M is finitely generated and I is contained in the Jacobson radical of A , then $\widehat{M} = 0$ implies $M = 0$.*

Proof. For (a), note that the composition $M \rightarrow \widehat{M} \rightarrow M/IM$ equals the projection $M \rightarrow M/IM$ by definition of the limit. Since the latter is surjective, so is $\widehat{M} \rightarrow M/IM$. In particular, part (a) shows that $\widehat{M} = 0$ implies $M = IM$. In the situation of (b) this is equivalent to $M = 0$ by Nakayama's lemma (which – as we all know – Professor Franke also likes to attribute to Azumaya and Krull, even though he regards Krull as a noob compared to Grothendieck). \square

Corollary 2. *If A is noetherian, then the functor $M \mapsto \widehat{M}$ is exact on the category of finitely generated A -modules.*

Proof. Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be a short exact sequence of finitely generated A -modules. Then $M' + I^n M$ is the kernel of $M \rightarrow M''/I^n M''$. Using $(M' + I^n M)/I^n M \cong M'/(M' \cap I^n M)$ we get short exact sequences

$$0 \longrightarrow M'/(M' \cap I^n M) \longrightarrow M/I^n M \longrightarrow M''/I^n M'' \longrightarrow 0 \quad (*)$$

for every $n \in \mathbb{N}$. Since $M'/(M' \cap I^n M)$ is sandwiched between $M'/I^n M'$ and $M'/I^{n+c} M'$ for some $c \in \mathbb{N}$ by the Artin–Rees lemma, it's easy to see that

$$\varprojlim_{n \in \mathbb{N}} M'/(M' \cap I^n M) = \varprojlim_{n \in \mathbb{N}} M'/I^{n+c} M' = \varprojlim_{n \in \mathbb{N}} M'/I^n M' = \widehat{M}'.$$

Moreover, each $M'/(M' + I^{n+1} M) \rightarrow M'/(M' + I^n M)$ is clearly surjective, so Fact 1.1.3 gives

$$\varprojlim_{n \in \mathbb{N}}^1 M'/(M' \cap I^n M) = 0.$$

Thus, taking the limit over $(*)$ gives a short exact sequence $0 \rightarrow \widehat{M}' \rightarrow \widehat{M} \rightarrow \widehat{M}'' \rightarrow 0$ by Fact 1.1.1. We are done. \square

Corollary 3. *Let A be a Noetherian ring.*

- (a) *When M is a finitely generated A -module, then $\widehat{M} \cong M \otimes_A \widehat{A}$.*
- (b) *\widehat{A} is flat as an A -module.*
- (c) *Suppose that I is contained in the Jacobson radical of A . If $\mu: M \rightarrow N$ is a morphism of finitely generated A -modules such that $\widehat{\mu}: \widehat{M} \rightarrow \widehat{N}$ is an isomorphism, then μ is an isomorphism.*

Proof. Part (a). Every finitely generated A -module is finitely presented as well since A is noetherian. So take a representation $M \cong \text{coker}(A^m \rightarrow A^n)$ for some $m, n \in \mathbb{N}$. It's obvious that $(A^n)^\wedge \cong \widehat{A}^n \cong A^n \otimes_A \widehat{A}$. Since both completion and tensor products commute with cokernels, this shows $\widehat{M} \cong M \otimes_A \widehat{A}$ as well.

Part(b). By Corollary 2 and (a), $-\otimes_A \widehat{A}$ is exact on finitely generated A -modules. By [Hom, Proposition 1.2.2] this is sufficient for flatness.

Part (c). Let $K = \ker \mu$ and $Q = \text{coker } \mu$. Since completion is exact on finitely generated A -modules, we get an exact sequence

$$0 \longrightarrow \widehat{K} \longrightarrow \widehat{M} \xrightarrow{\widehat{\mu}} \widehat{N} \longrightarrow \widehat{Q} \longrightarrow 0.$$

But $\widehat{\mu}$ is an isomorphism, so $\widehat{K} = 0$ and $\widehat{Q} = 0$. By Fact 1 this shows $K = 0$ and $Q = 0$. We are done. \square

Corollary 4. *If $J \subseteq A$ is any ideal and M a finitely generated A -module, then $(JM)^\wedge \rightarrow \widehat{M}$ defines an isomorphism $(JM)^\wedge \xrightarrow{\sim} J\widehat{M}$.*

Proof. We may view $(JM)^\wedge$ as a submodule of \widehat{M} since completion preserves injectivity of the inclusion $JM \subseteq M$ by Corollary 2. It's easy to see that $J\widehat{M}$ is contained in $(JM)^\wedge$. To prove the converse, take generators j_1, \dots, j_n of J . Then completion preserves surjectivity of $(j_1, \dots, j_n): M^n \twoheadrightarrow JM$ and we are done. \square

Corollary 5. *If A is a noetherian local ring with maximal ideal \mathfrak{m} , then \widehat{A} is local with maximal ideal $\mathfrak{m}\widehat{A}$.*

Proof. We proved this in [Hom, Corollary 2.2.2]. \square

Proposition 3. *Let A be noetherian and $I \subseteq A$ any ideal, then the I -adic completion \widehat{A} is noetherian again.*

To prove this, we need to prove the evil twin of Hilbert's Basissatz first.

Lemma 1. *If A is noetherian, then so is the power series ring $A[[T]]$.*

Proof. We can (and will) basically copy the proof of Hilbert's Basissatz. Let $J \subseteq A[[T]]$ be any ideal and put $J_n = \left\{ a_n \mid \sum_{k=n}^\infty a_k T^k \in J \right\}$ for $n \geq 0$. Then $(J_n)_{n \in \mathbb{N}}$ form an ascending sequence of ideals in A . Noetherianness of A tells us that this sequence becomes eventually stationary, say, at $n = s$. So we may choose $a^{(i)} = \sum_{k \geq s} a_k T^k \in A[[T]]$ for $i = 1, \dots, N$ such that $a_s^{(1)}, \dots, a_s^{(N)}$ generate J_s . Then $a^{(1)}, \dots, a^{(N)}$ generate $J \cap T^s A[[T]]$. Indeed, given any $b = \sum_{k \geq s} b_k T^k \in J$ we can inductively choose coefficients $r_k^{(1)}, \dots, r_k^{(N)} \in A$ such that $r^{(i)} = \sum_{k \geq 0} r_k^{(i)} T^k$ satisfy $r^{(1)} a^{(1)} + \dots + r^{(N)} a^{(N)} = b$ up to degree T^{s+k} . This works because $J_{k+s} = J_s$ for all $k \geq 0$ is generated by $a_s^{(1)}, \dots, a_s^{(N)}$ again.

Now $A[[T]]/T^s A[[T]]$ is a finitely generated A -module, hence the image of J in it is finitely generated as well, A being noetherian. We thus may choose $a^{(N+1)}, \dots, a^{(N+M)} \in J$ whose images modulo $T^s A[[T]]$ generate the image of J in $A[[T]]/T^s A[[T]]$. Then $a^{(1)}, \dots, a^{(N+M)}$ generate J and our job's done here. \square

Proof of Proposition 3. Let r_1, \dots, r_n be generators of I . Then sending $X_i \mapsto r_i$ defines a surjective morphism $A[[X_1, \dots, X_n]] \twoheadrightarrow \widehat{A}$. Since $A[[X_1, \dots, X_n]]$ is noetherian by Lemma 1 and induction on n , so is its quotient \widehat{A} . \square

Corollary 6. *Suppose that A is a noetherian local ring and $I \subseteq A$ any (proper) ideal. Then $\dim A = \dim \widehat{A}$. In particular, A is regular iff \widehat{A} is regular.*

Proof. Let \mathfrak{m} be the maximal ideal of A . Then $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{A}$ (this equality holds because of Corollary 4) is the maximal ideal of the local ring \widehat{A} as was shown in the proof of [Hom, Corollary 2.2.2]. Since $I \subseteq \mathfrak{m}$, the quotients $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ already have I -torsion, hence

$$\mathfrak{m}^i/\mathfrak{m}^{i+1} \cong \left(\mathfrak{m}^i/\mathfrak{m}^{i+1} \right)^\wedge \cong \widehat{\mathfrak{m}^i}/\widehat{\mathfrak{m}^{i+1}}$$

(the last isomorphism follows from exactness of completion). This shows that the associated graded rings $\text{gr}(A, \mathfrak{m})$ and $\text{gr}(\hat{A}, \hat{\mathfrak{m}})$ agree, hence (A, \mathfrak{m}) and $(\hat{A}, \hat{\mathfrak{m}})$ have the same Hilbert–Samuel polynomials, which shows $\dim A = \dim \hat{A}$ by [Alg2, Theorem 20].

Now A and \hat{A} have the same residue field k and $\mathfrak{m}/\mathfrak{m}^2 \cong \hat{\mathfrak{m}}/\hat{\mathfrak{m}}^2$ by the I -torsion arguments we have seen several times now, so $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim_k \hat{\mathfrak{m}}/\hat{\mathfrak{m}}^2$. Clearly this implies that A is regular iff \hat{A} is. \square

Remark 2. In a similar fashion one can show that a noetherian local ring is Cohen–Macaulay, or Gorenstein, or a complete intersection, iff its I -adic completion is one as well. For example, for Cohen–Macaulayness one would need to show $\text{depth}_A(A) = \text{depth}_{\hat{A}}(\hat{A})$, which follows from the isomorphism $\text{Ext}_A^p(k, A) \cong \text{Ext}_{\hat{A}}^p(k, \hat{A})$ that was described in the proof of [Hom, Proposition 2.4.2].

Proposition 4. *Let A be a noetherian ring which is complete in the I -adic topology and let M be a finitely generated A -module. Then M is I -adically complete.*

Proof. Note that this is clearly fulfilled if $M = A^n$ is a finitely generated free A -module. Now let M be arbitrary. Since A is noetherian, M can be represented as coker $(A^m \rightarrow A^n)$. Because A^m, A^n equal their own completions (as we have just seen) and completion is exact (by Corollary 2), $M = \hat{M}$ holds as well. \square

Corollary 7. *Let A be a noetherian local ring maximal ideal \mathfrak{m} . Let B be a finite A -algebra.*

- (a) *Then B has only finitely many prime ideals over \mathfrak{m} , and all of them are maximal.*
- (b) *If A is, in addition, \mathfrak{m} -adically complete, then all maximal ideals of B lie over \mathfrak{m} . In particular, B is **semi-local** (i.e. has finitely many maximal ideals). The same is true if $A \subseteq B$.*

Proof. Part (a). If $\mathfrak{q} \in \text{Spec } B$ is a prime ideal over \mathfrak{m} , then B/\mathfrak{q} is a finitely generated domain over the residue field $k = A/\mathfrak{m}$, hence a finite field extension of k , so \mathfrak{q} is maximal. Moreover, $B/\mathfrak{m}B$ is a finite-dimensional k -algebra, hence it has only finitely many maximal ideals by the argument from Fact 1.3.2(b).

Part (b). If A is \mathfrak{m} -adically complete, then B is $\mathfrak{m}B$ -adically complete by Proposition 4, so $\mathfrak{m}B$ is contained in the Jacobson radical $\text{rad } B$ by Corollary 1(c). Then all maximal ideals of B lie over \mathfrak{m} .

Now assume $A \subseteq B$. Let $\mathfrak{q} \in \text{Spec } B$ be a maximal ideal and $\mathfrak{p} = \mathfrak{q} \cap A$. Then $A/\mathfrak{p} \subseteq B/\mathfrak{q}$ is an integral ring extension in which B/\mathfrak{q} is a field, hence so is A/\mathfrak{p} , (by [Alg1, Proposition 1.5.1(d)]) which proves $\mathfrak{p} = \mathfrak{m}$. \square

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