Jacobians of Curves

Lecture notes by Ferdinand Wagner

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1. Introduction

1.1. A note on limits and their derived functors

Let $X_\bullet\colon\ \dots \xrightarrow{p_{i+1}} X_i \xrightarrow{p_i} \dots \xrightarrow{p_2} X_1 \xrightarrow{p_1} X_0$ be a diagram of abelian groups or R-modules. As usual, we may view $X_\bullet$ as a functor $X_\bullet\colon (\mathbb{N},\geqslant) \to \mathbf{Ab}$ or $\mathbf{Mod}(R)$, where the category $(\mathbb{N},\geqslant)$ has the nonnegative integers as objects and an arrow $j \to i$ iff $j \geqslant i$. Let

$$d\colon \prod_{i=0}^{\infty} X_i \longrightarrow \prod_{i=0}^{\infty} X_i\ ,\quad d\,(x_i)_{i=0}^{\infty} = (p_{i+1}(x_{i+1}) - x_i)_{i=0}^{\infty}\ .$$

Then we put

$$\varprojlim_{i\in\mathbb{N}} X_i = \ker d \quad \text{and} \quad \varprojlim_{i\in\mathbb{N}}{}^1 X_i = \operatorname{coker} d\ .$$

Remark 1. It is easy to see that $\varprojlim X_i$ equals the usual category-theoretical limit (that's how you construct it). It can also be shown that $\varprojlim^1$ is the first right-derived functor of $\varprojlim$, and that its higher derived functors vanish.

Fact 1. *Let $0 \to X'_\bullet \to X_\bullet \to X''_\bullet \to 0$ be a short exact sequence of diagrams of the above type. Then there is a canonical exact sequence*

$$0 \longrightarrow \varprojlim_{i\in\mathbb{N}} X'_i \longrightarrow \varprojlim_{i\in\mathbb{N}} X_i \longrightarrow \varprojlim_{i\in\mathbb{N}} X''_i \longrightarrow \varprojlim_{i\in\mathbb{N}}{}^1 X'_i \longrightarrow \varprojlim_{i\in\mathbb{N}}{}^1 X_i \longrightarrow \varprojlim_{i\in\mathbb{N}}{}^1 X''_i \longrightarrow 0\ .$$

Proof. Since products preserve exact sequences in $\mathbf{Ab}$ or $\mathbf{Mod}(R)$, we get a diagram

$$\begin{tikzcd}
0 \arrow[r] & \prod_{i=0}^{\infty} X'_i \arrow[r] \arrow[d, "d'"'] & \prod_{i=0}^{\infty} X_i \arrow[r] \arrow[d, "d"'] & \prod_{i=0}^{\infty} X''_i \arrow[r] \arrow[d, "d''"'] & 0 \\
0 \arrow[r] & \prod_{i=0}^{\infty} X'_i \arrow[r] & \prod_{i=0}^{\infty} X_i \arrow[r] & \prod_{i=0}^{\infty} X''_i \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Then the snake lemma finishes the job. □

Fact 2. *Let $X_\bullet$ have the property that for every $i \in \mathbb{N}$ there is a $j \geqslant i$ such that the composition $p_{j,i}\colon X_j \xrightarrow{p_j} X_{j-1} \xrightarrow{p_{j-1}} \dots \xrightarrow{p_{i+1}} X_i$ vanishes. Then*

$$\varprojlim_{i\in\mathbb{N}} X_i = \varprojlim_{i\in\mathbb{N}}{}^1 X_i = 0\ .$$

Proof. If $x = (x_i)_{i=0}^{\infty} \in \varprojlim X_i$, then $x_i = p_{j,i}(x_j)$ for all $j \geqslant i$ by construction, hence $x_i = 0$ for all $i \in \mathbb{N}$. Moreover, let

$$s\colon \prod_{i=0}^{\infty} X_i \longrightarrow \prod_{i=0}^{\infty} X_i \ , \quad s(x)_i = \sum_{j \geqslant i} p_{j,i}(x_j) \ .$$

By assumption s is well-defined. Then

$$d(s(x))_i = p_{i+1}\Bigg(\sum_{j \geqslant i+1} p_{j,i+1}(x_j)\Bigg) - \sum_{j \geqslant i} p_{j,i}(x_j) = -p_{i,i}(x_i) = -x_i \ .$$

Hence $-s$ is a right-inverse of d, so $\varprojlim^1 X_i = \operatorname{coker} d$ vanishes as well. □

Fact 3. *Let $X_\bullet$ have the* **Mittag-Leffler property** *that for every $i \in \mathbb{N}$ there is a $j \geqslant i$ such that for all $k \geqslant j$ the images of $p_{j,i}$ and $p_{k,i}$ in X_i coincide. Then $\varprojlim^1 X_i = 0$.*

Proof. Let's first deal with the special case that each $p_i\colon X_i \to X_{i-1}$ is surjective. Let $x = (x_i)_{i=0}^{\infty} \in \prod_{i=0}^{\infty} X_i$. For every $i \in \mathbb{N}$ we may select $x_j^{(i)} \in X_j$ for all $j \geqslant i$ in such a way that $x_i^{(i)} = x_i$ and $p_{j+1}(x_{j+1}^{(i)}) = x_j^{(i)}$. Then $s(x)$ defined by

$$s(x)_i = \sum_{k=0}^{i-1} x_i^{(k)}$$

is a preimage of x under d, so $\varprojlim^1 X_i = \operatorname{coker} d = 0$ in this case.

Now let $X_\bullet$ be arbitrary with the Mittag-Leffler property. Let $Y_i = \bigcap_{j \geqslant i} p_{j,i}(X_j) \subseteq X_i$. Then $\varprojlim^1 Y_i = 0$ by the special case we just treated, and $\varprojlim^1 X_i/Y_i = 0$ by Fact 2. Since $\varprojlim^1 X_i$ is sandwiched between these two in the exact sequence from Fact 1, this shows $\varprojlim^1 X_i = 0$, as required. □

1.2. The theorem about formal functions

Let $f\colon X \to Y = \operatorname{Spec} A$ be a morphism of quasi-compact schemes. Let $I \subseteq A$ be any ideal. Consider

$$i_n\colon X_n = X \times_Y \operatorname{Spec}(A/I^n) \longrightarrow X \ ,$$

which is a base change of the closed immersion $Y_n = \operatorname{Spec}(A/I^n) \hookrightarrow \operatorname{Spec} A$, hence indeed a closed immersion itself. Also, if f is proper, then so is $X_n \to Y_n$ because properness is another *proper*ty (tee-hee) that is stable under base change (by [AG2, Remark 2.4.1]).

Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules and $\mathcal{F}|_{X_n} = i_n^*\mathcal{F}$ its restriction to X_n (this notation is slightly abusive, but convenient). We put $\mathcal{F}_n = i_{n,*}\mathcal{F}|_{X_n}$. It's easy to check (e.g. affine-locally) that $\mathcal{F}_n \cong \mathcal{F}/I^n\mathcal{F}$. Since i_n is a closed immersion and thus affine, we have an isomorphism $H^p(X, \mathcal{F}_n) \cong H^p(X_n, \mathcal{F}|_{X_n})$ for all $p \geqslant 0$ by [AG2, Corollary 1.5.1]. Together with the canonical projection $\mathcal{F}_{n+1} \cong \mathcal{F}/I^{n+1}\mathcal{F} \to \mathcal{F}/I^n\mathcal{F} \cong \mathcal{F}_n$ this gives canonical morphisms $H^p(X_{n+1}, \mathcal{F}|_{X_{n+1}}) \to H^p(X_n, \mathcal{F}|_{X_n})$ for all $n \in \mathbb{N}$.

The canonical morphism $\mathcal{F} \to i_{n,*}i_n^*\mathcal{F} = \mathcal{F}_n$ induces a morphism

$$H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}_n) \cong H^p(X_n, \mathcal{F}|_{X_n}) \tag{1}$$

for all $p \geqslant 0$ (the isomorphism on the right-hand side comes from the fact that i_n is a closed immersion, hence affine, and we can apply [AG2, Corollary 1.6.1]). This is a morphism of A-modules, but $H^p(X_n, \mathcal{F}|_{X_n})$ is actually an A/I^n-module, so (1) factors over

$$H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F}) \longrightarrow H^p(X_n, \mathcal{F}|_{X_n}) \ .$$

This is compatible with the canonical morphisms $H^p(X_{n+1}, \mathcal{F}|_{X_{n+1}}) \to H^p(X_n, \mathcal{F}|_{X_n})$ (you can just check that on an affine Čech covers). Passing to the limit gives a morphism

$$H^p(X, \mathcal{F})^\frown \longrightarrow \varprojlim_{n \in \mathbb{N}} H^p(X_n, \mathcal{F}|_{X_n}) \ , \tag{2}$$

where $^\frown$ denotes the I-adic completion.

Theorem 1 (Grothendieck–Zariski). *When $f\colon X \to Y = \operatorname{Spec} A$ is proper (in which case X is automatically a quasi-compact scheme), A is noetherian and $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules, then (2) is an isomorphism*

$$H^p(X, \mathcal{F})^\frown \xrightarrow{\sim} \varprojlim_{n \in \mathbb{N}} H^p(X_n, \mathcal{F}|_{X_n}) \ .$$

Proof. The following proof is essentially the one from [EGAIII, (4.1.7)]. Professor Franke also pointed out that the idea is pretty similar to the proof of the Artin–Rees lemma. Let $I \subseteq A$ be the ideal under consideration and let $R = \bigoplus_{n \geqslant 0} I^n$ be the Rees algebra associated to I. Then

$$K^p = \bigoplus_{n \geqslant 0} H^p(X, I^n\mathcal{F})$$

is a module over R as $i \in I^m$ (considered as the m^{th} homogeneous component of R) maps $I^n\mathcal{F}$ to $I^{n+m}\mathcal{F}$.

Claim 1. K^p is a finitely generated R-module for all $p \geqslant 0$.

Assuming this for the moment, recall that $\mathcal{F}_n \cong \mathcal{F}/I^n\mathcal{F}$ and $H^p(X, \mathcal{F}|_{X_n}) \cong H^p(X, \mathcal{F}_n)$, so the long exact cohomology sequence associated to $0 \to I^n\mathcal{F} \to \mathcal{F} \to \mathcal{F}_n \to 0$ appears as

$$H^p(X, I^n\mathcal{F}) \longrightarrow H^p(X, \mathcal{F}) \longrightarrow H^p(X_n, \mathcal{F}|_{X_n}) \longrightarrow H^{p+1}(X, I^n\mathcal{F}) \ . \tag{3}$$

As pointed out after (1), $H^p(X, \mathcal{F}) \to H^p(X_n, \mathcal{F}|_{X_n})$ factors over $H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F})$, hence we can turn equation (3) into an exact sequence

$$0 \longrightarrow U_n \longrightarrow H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F}) \longrightarrow H^p(X_n, \mathcal{F}|_{X_n}) \longrightarrow V_n \longrightarrow 0 \ , \tag{4}$$

where U_n is a suitable quotient of $H^p(X, I^n\mathcal{F})$ and $V_n \subseteq H^{p+1}(X, I^n\mathcal{F})$ some submodule. This makes $U = \bigoplus_{n \geqslant 0} U_n$ a quotient of K^p and $V = \bigoplus_{n \geqslant 0} V_n$ an R-submodule of K^{p+1}.

Claim 2. We have $\varprojlim U_n = \varprojlim^1 U_n = 0$ and $\varprojlim V_n = \varprojlim^1 V_n = 0$.

Before we prove this (and Claim 1), let's see how Theorem 1 follows from it. Let W_n be the image of $H^p(X,\mathcal{F})/I^nH^p(X,\mathcal{F})$ in $H^p(X_n,\mathcal{F}|_{X_n})$. We may split (4) into two short exact sequences $0 \to U_n \to H^p(X,\mathcal{F})/I^nH^p(X,\mathcal{F}) \to W_n \to 0$ and $0 \to W_n \to H^p(X,\mathcal{F}|_{X_n}) \to V_n \to 0$. Applying Fact 1.1.1 to the first one gives $H^p(X,\mathcal{F})^\frown \cong \varprojlim^1 W_n$. Then the six-term exact sequence associated to the second proves $\varprojlim W_n \cong \varprojlim H^p(X_n\mathcal{F}|_{X_n})$ and we are done.

It remains to show the two claims. Note that the Rees algebra R is noetherian. Indeed, I is finitely generated as an ideal in the noetherian ring A, hence R is of finite type over A. Let's also make the following convention: Whenever we write I^kU_n or I^kV_n in the following, this means multiplication as A-modules and the result is contained in U_n resp. V_n again, whereas R_kU_n or R_kV_n means multiplication by the k^{th} homogeneous component of R (which equals I^k as well), so the result is contained on U_{k+n} resp. V_{k+n}.

Proof of Claim 2. Note that U is finitely generated over R, since it is a quotient of the finitely generated R-module K^p. Fix a finite set of generators and let d_0 the maximal non-zero homogeneous components occurring in this set. Then $U_n = R_nU_0 + R_{n-1}U_1 + \ldots + R_{n-d_0}U_{d_0}$ for all $n \geqslant d_0$. In particular, $U_{k+n} = R_kU_n$ for all $n \geqslant d_0$. Thus, for every $n \geqslant d_0$ the image of $U_{2n} = R_nU_n$ in U_n is contained in I^nU_n. But $U_n \subseteq H^p(X,\mathcal{F})/I^nH^p(X,\mathcal{F})$, so I^nU_n vanishes. Therefore, the property from Fact 1.1.2 is fulfilled for all $n \geqslant d_0$. But then it clearly holds for all $n \geqslant 0$ as well, so Fact 1.1.2 is applicable.

Similarly, V is finitely generated as a submodule of K^{p+1}, which is finitely generated over the noetherian ring R by Claim 1. By the same argument as above we find a d_1 such that $V_n = R_nV_0 + R_{n-1}V_1 + \ldots + R_{n-d_1}V_{d_1}$ for all $n \geqslant d_1$. In particular, we have $V_{k+n} = R_kV_n$ for all $n \geqslant d_1$. Thus, for $n \geqslant d_1$ the image of V_{2n} in V_n is contained in I^nV_n. But I^nV_n vanishes again, since V_n is the image of $H^p(X_n,\mathcal{F}|_{X_n})$, which is a A/I^n-module. As above, we can apply Fact 1.1.2. This shows Claim 2.

Proof of Claim 1. Let $\upsilon\colon \widetilde{Y} = \operatorname{Spec} R \to Y$ correspond to $A \hookrightarrow R$ and let $\xi\colon \widetilde{X} = X \times_Y \widetilde{Y} \to X$ be its base change by f. Note that ξ is affine as a base change of the affine morphism υ (we use [AG1, Corollary 2.5.1] here). We claim

$$\xi_*\xi^*\mathcal{F} \cong \bigoplus_{n\geqslant 0} I^n\mathcal{F} .$$

Indeed, this is easily checked affine-locally (where ξ^* is given by tensoring with R); we leave the details to the reader. Also $H^p(\widetilde{X},\xi^*\mathcal{F}) \cong H^p(X,\xi_*\xi^*\mathcal{F})$ as ξ is affine. This shows

$$H^p(\widetilde{X},\xi^*\mathcal{F}) \cong H^p(X,\xi_*\xi^*\mathcal{F}) \cong H^p\Big(X,\bigoplus_{n\geqslant 0} I^n\mathcal{F}\Big) \cong \bigoplus_{n\geqslant 0} H^p(X,I^n\mathcal{F}) = K^p .$$

Note that direct sums usually *don't* commute with cohomology, but here they do, because X is quasi-compact and $\bigoplus_{n\geqslant 0} I^n\mathcal{F}$ is quasi-coherent (for which we need quasi-compactness as well), so we may compute $H^p\left(X,\bigoplus_{n\geqslant 0} I^n\mathcal{F}\right)$ via finite affine Čech covers. In this case, the products in the Čech complex are all finite, hence commute with the direct sum, which is what we needed.

Now $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y} = \operatorname{Spec} R$ is proper (as a base change of the proper morphism f), hence the right-hand side is a finitely generated R-module by our finiteness results for the cohomology of proper morphisms (cf. [AG2, Theorem 5]). We win. $\square$

Remark 1. Note that in the lecture Franke used $\mathcal{K}_n \cong \mathcal{J}^n\mathcal{F}$ instead of $I^n\mathcal{F}$, where $\mathcal{J} = f^{-1}\mathcal{I}$ is the inverse image (in the sense of Definition 1). But $\mathcal{J}^n\mathcal{F} \cong I^n\mathcal{F}$ – which is not that surprising, since the I^n-action on $\mathcal{F}$ is given via the algebraic component $\mathcal{O}_Y \to f_*\mathcal{O}_X$ of f, so $I^n\mathcal{F} = \mathcal{J}^n\mathcal{F}$ is pretty obvious from the construction of f^{-1} described in the proof of Lemma 1 below. I prefer the notation $I^n\mathcal{F}$ – in particular, this is how Grothendieck denotes it in [EGAIII, (4.1.7)], so I believe it's my right to do so as well. Nevertheless, Lemma 1 is *perhaps worthwhile to know* (if you get what I mean), so we will include it now.

Definition 1. Let $f\colon X \to Y$ be any morphism of preschemes and $\mathcal{J} \subseteq \mathcal{O}_Y$ a sheaf of ideals on Y. Then define $f^{-1}\mathcal{J}$ to be the image of $f^*\mathcal{J} \to \mathcal{O}_X$ (which is obtained as the composition of the pull-back of $\mathcal{J} \to \mathcal{O}_Y$ with the isomorphism $f^*\mathcal{O}_Y \cong \mathcal{O}_X$).

Lemma 1. *Let $f\colon X \to Y$ be any morphism of preschemes and $\mathcal{J} \subseteq \mathcal{O}_Y$ quasi-coherent.*

(a) *$f^{-1}\mathcal{J} \subseteq \mathcal{O}_X$ is quasi-coherent.*

(b) *Let Y_0 and X_0 be the closed subpreschemes of Y and X defined by $\mathcal{J}$, $f^{-1}\mathcal{J}$ respectively. Then $X_0 \cong X \times_Y Y_0$.*

(c) *For all $n \geqslant 0$ we have $f^{-1}(\mathcal{J}^n) \cong (f^{-1}\mathcal{J})^n$.*

Sketch of a proof. The question is easily seen to be local on both X and Y. So let's consider the affine situation where $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, and $\mathcal{J} = \widetilde{J}$ for some ideal $J \subseteq A$. Let $\varphi\colon A \to B$ be the morphism of rings corresponding to f. Then $f^{-1}\mathcal{J} = \widetilde{I}$ where I is the image of $B \otimes_A J \to B$ sending $b \otimes j \mapsto b \cdot \varphi(j)$. All three assertions are then easily checked. $\square$

Remark 2. Recall that for a morphism $f\colon X \to Y$ of preschemes and a point $y \in Y$ the **fibre** $f^{-1}\{y\}$ of f at y is defined as the prescheme $f^{-1}\{y\} = X \times_Y \operatorname{Spec} \mathfrak{K}(y)$. This makes sense, since $f^{-1}\{y\}$ is indeed – topologically – the preimage of y, as proved in [AG1, Corollary 1.3.3]. Moreover, $\operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n \to Y$ is immersive for all $n \geqslant 1$ and has image $\{y\}$ as well. So [AG1, Corollary 1.3.3] is applicable again and shows that $X_n = X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n)$ has $f^{-1}\{y\}$ as underlying topological space too (but, of course, the prescheme structure differs in general). We may thus think of X_n as the n^{th} infinitesimal thickening of $f^{-1}\{y\}$.

Using this, Theorem 1 can be restated as follows.

Theorem 1a. *Let $f\colon X \to Y$ be a proper morphism between locally noetherian[1] preschemes. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. For every $y \in Y$ let $X_n = X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n)$ be the n^{th} infinitesimal thickening of $f^{-1}\{y\}$. Then there is an isomorphism*

$$(R^p f_*\mathcal{F})^\wedge_y \xrightarrow{\sim} \varprojlim_{n \in \mathbb{N}} H^p(X_n, \mathcal{F}|_{X_n}) \,,$$

where $^\wedge$ denotes the $\mathfrak{m}_{Y,y}$-adic completion.

[1] Franke only assumes Y to be locally noetherian, but f being of (locally) finite type implies that X is locally noetherian as well by Hilbert's Basissatz. This happens multiple times throughout the text.

Proof. We may assume that $Y = \operatorname{Spec} A$ is affine, and that A is a noetherian ring. Indeed, replacing Y by an affine neighbourhood $U \cong \operatorname{Spec} A$ and X by $f^{-1}(U)$ doesn't change $(R^p f_* \mathcal{F})_y$ (because the construction of $R^p f_* \mathcal{F}$ is base-local) and also X_n is preserved since $f^{-1}\{y\}$ is already contained in $f^{-1}(U)$ (by [AG2, postnote]).

In this case, $R^p f_* \mathcal{F} = H^p(X, \mathcal{F})^\sim$ by [AG2, Proposition 1.5.1(d)]. Let $\mathfrak{p} \in \operatorname{Spec} A$ be the prime ideal associated to y. Then $R^p f_* \mathcal{F} \cong H^p(X, \mathcal{F})_\mathfrak{p}$ and $\mathcal{O}_{Y,y} \cong A_\mathfrak{p}$ is flat over A. Let $\mathfrak{m} = \mathfrak{p} A_\mathfrak{p} \cong \mathfrak{m}_{Y,y}$ be its maximal ideal. We denote $\pi \colon \operatorname{Spec} A_\mathfrak{p} \to \operatorname{Spec} A$. Applying [AG2, Fact 4.1.1] to π gives

$$H^p(X \times_Y \operatorname{Spec} A_\mathfrak{p}, \pi^* \mathcal{F}) \cong H^p(X, \mathcal{F})_\mathfrak{p} \cong (R^p f_* \mathcal{F})_y \ .$$

Also

$$\begin{aligned}(X \times_Y \operatorname{Spec} A_\mathfrak{p}) \times_{\operatorname{Spec} A_\mathfrak{p}} \operatorname{Spec}(A_\mathfrak{p}/\mathfrak{m}^n) &\cong X \times_Y \left(\operatorname{Spec} A_\mathfrak{p} \times_{\operatorname{Spec}_{A_\mathfrak{p}}} \operatorname{Spec}(A_\mathfrak{p}/\mathfrak{m}^n)\right) \\ &\cong X \times_Y \operatorname{Spec}(A_\mathfrak{p}/\mathfrak{m}^n) \\ &\cong X_n\end{aligned}$$

by a bit abstract nonsense. Now Theorem 1 may be applied to $X \times_Y \operatorname{Spec} A_\mathfrak{p} \to \operatorname{Spec} A_\mathfrak{p}$ (the base change of f) and the assertion follows. □

1.3. Application to Zariski's main theorem

Out there in the real world, there are multiple *main theorems* of Zariski around, and usually they're only loosely related. Professor Franke recommends Mumford's *The red book of varieties and schemes* for a discussion of various such version.

Corollary 1. *Let $f \colon X \to Y$ be any proper morphism between locally noetherian preschemes and let $d = \sup_{y \in Y} \dim\left(f^{-1}\{y\}\right)$. If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module and $p > d$, then $R^p f_* \mathcal{F} = 0$.*

Proof. Since $R^p f_* \mathcal{F}$ is coherent (this is [AG2, Theorem 5]), $(R^p f_* \mathcal{F})_y$ is a finitely generated $\mathcal{O}_{Y,y}$-module, hence it vanishes iff it $\mathfrak{m}_{Y,y}$-adic completion vanishes by Fact A.1.1(b). But $X_n = X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n)$ has underlying space $f^{-1}\{y\}$ (as explained in Remark 1.2.2), hence $H^p(X_n, \mathcal{F}|_{X_n}) = 0$ when $p > d$ by Grothendieck's theorem on cohomological dimension (cf. [AG2, Proposition 1.4.1]). The assertion now follows from Theorem 1a. □

Definition 1. A morphism $f \colon X \to Y$ of finite type is called **quasi-finite at $x \in X$** if x is discrete in its fibre, i.e., if $\{x\}$ is an open and closed subset of $f^{-1}\{y\}$ where $y = f(x)$. We call f **quasi-finite** if it is quasi-finite at every $x \in X$.

The following fact wasn't mentioned in the lecture, but it's *definitely* (in particular, not only *perhaps*) *worthwhile to know*!

Fact 1. *Let $f \colon X \to Y$ be a morphism of finite type. Let $x \in X$ be open in its fibre $f^{-1}\{y\}$, where $y = f(x)$. Then f is already quasi-finite at x.*

Proof. Choose an affine open neighbourhood $y \in U \cong \operatorname{Spec} A$. Then $f^{-1}\{y\}$ is contained in $f^{-1}(U)$, so we may w.l.o.g. assume that $Y = \operatorname{Spec} A$ is affine. Put $k = \mathfrak{K}(y)$. Since X may be covered by affine open subsets $\operatorname{Spec} R$, where R is of finite type over A, we may cover the fibre product $f^{-1}\{y\} = X \times_Y \operatorname{Spec} k$ by affine open subsets $\operatorname{Spec}(R \otimes_A k)$, in which $R \otimes_A k$ is a k-algebra of finite type, hence a Jacobson ring. This proves that $f^{-1}\{y\}$ is a Jacobson prescheme as in [AG1, Definition 2.4.2(c)]. But then x is a closed point of the open subset $\{x\} \subseteq f^{-1}\{y\}$, hence also a closed point of $f^{-1}\{y\}$ by [AG1, Fact 2.4.1(c)]. $\square$

Fact 2. (a) *Any finite morphism is quasi-finite.*

(b) *If k is a field, a morphism $f\colon X \to \operatorname{Spec} k$ of finite type is quasi-finite iff it is finite.*

(c) *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of finite type such that g is quasi-finite at $y = f(x)$ for some $x \in X$. Then gf is quasi-finite at x iff f is quasi-finite at x.*

Proof. Maybe that's my bad, but the proof of this is actually annoyingly laborious. We begin with part (a). Let $f\colon X \to Y$ be a finite morphism, $x \in X$ and $y = f(x)$. Then the morphism

$$f^{-1}\{y\} = X \times_Y \operatorname{Spec} \mathfrak{K}(y) \longrightarrow \operatorname{Spec} \mathfrak{K}(y)$$

is finite again, as a base changes of finite morphisms are finite again (cf. [AG1, Corollary 1.5.1]). Letting $k = \mathfrak{K}(y)$ this puts us in the situation from (b), so it's sufficient to prove (b).

In the case of (b) we have $f^{-1}\{y\} = X$, so what we need to show is that X carries the discrete topology if f is finite. We know that $X \cong \operatorname{Spec} R$ where R is some finite-dimensional k-algebra (using finiteness of f). For $x \in X$ let $\mathfrak{p}$ be the corresponding prime ideal of R. Then $R/\mathfrak{p}$ is a domain and a finite-dimensional k-vector space, hence a finite field extension of k by Hilbert's Nullstellensatz. This means that $\mathfrak{p}$ is a maximal ideal of R. Consequently, all points of X are closed, so it suffices to show that X has finitely many points. Let $\{x_1, \dots, x_n\}$ be any finite subset of X and $\{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$ the corresponding maximal ideals of R. For every i, we may choose an element $\alpha_i \in \mathfrak{m}_i$ which is not contained in any $\mathfrak{m}_j$ for $j \neq i$ (e.g. by the prime avoidance lemma, cf. [Alg1, Lemma 2.5.1]). Put $\beta_i = \prod_{j \neq i} \alpha_j$ (so that $\beta_i \in \mathfrak{m}_j$ for all $j \neq i$ but $\beta_i \notin \mathfrak{m}_i$). We claim that $\beta_1, \dots, \beta_n$ are k-linearly independent. Indeed, if $\lambda_1\beta_1 + \dots + \lambda_n\beta_n = 0$ for some coefficients $\lambda_1, \dots, \lambda_n \in k$, then reducing modulo $\mathfrak{m}_i$ gives $\lambda_i\beta_i = 0$ in $R/\mathfrak{m}_i = \mathfrak{K}(\mathfrak{m}_i)$. But $\beta_i \neq 0$ in $\mathfrak{K}(\mathfrak{m}_i)$, so $\lambda_i = 0$ for all $i = 1, \dots, n$. This proves $\dim_k R \geqslant n$. But R is finite-dimensional over k, hence X must have finitely many points, as claimed.

Conversely, assume that $f\colon X \to \operatorname{Spec} k$ is quasi-finite. Then X is discrete, so it must have finitely many points. Indeed, f being of finite type implies it is quasi-compact (by definition), so X is quasi-compact because $\operatorname{Spec} k$ is, and any discrete quasi-compact space is finite. Let $X = \{x_1, \dots, x_n\}$. Every point $x_i \in X$ together with the restriction $\mathcal{O}_X|_{\{x_i\}}$ of the structure sheaf is a prescheme again, hence affine (because $x_i \in \{x_i\}$ must have an affine neighbourhood). Let $\{x_i\} \cong \operatorname{Spec} R_i$. Then

$$X \cong \coprod_{i=1}^n \operatorname{Spec} R_i \cong \operatorname{Spec}\left(\bigoplus_{i=1}^n R_i\right)$$

is affine. This shows that f is affine, but finiteness is yet to prove. Clearly, it suffices that each R_i is a finite-dimensional k-vector space. Note that R_i has precisely one prime ideal $\mathfrak{m}_i$

(corresponding to x_i), which is then automatically maximal. Since f is of finite type, R_i has finite type over k. In particular R_i is noetherian and we may choose generators $r_1, \dots, r_m$ of $\mathfrak{m}_s$. Since $\mathfrak{m}_i$ is the only prime ideal of R_s, we have $\mathfrak{m}_i = \operatorname{nil} R_i$. Consquently, there is an $N \in \mathbb{N}$ such that $r_\ell^N = 0$ for all ℓ. Moreover, $R_i/\mathfrak{m}_i$ is a field extension of finite type over k, hence a finite field extension by Hilbert's Nullstellensatz. Let $\beta_1, \dots, \beta_d \in R_i$ be elements whose images modulo $\mathfrak{m}_i$ form a k-basis of $R_i/\mathfrak{m}_i$. Then it is straightforward to check that R is generated as a k-vector space by the elements

$$\beta_j \cdot r_1^{e_1} r_2^{e_2} \cdots r_n^{e_n} \quad \text{where } 0 \leqslant e_\ell < N \text{ for all } \ell\,.$$

This shows $\dim_k R < \infty$, hence f is finite.

Part (c). Since g is quasi-finite at y, the subset $\{y\} \subseteq g^{-1}\{g(y)\}$ is open and closed, hence $f^{-1}\{y\} \subseteq (gf)^{-1}\{g(y)\}$ is open and closed. This means that $\{x\}$ is open and closed in the fibre $(gf)^{-1}\{g(y)\}$ iff it is open and closed in $f^{-1}\{y\}$ and we win. □

Theorem 2 (Grothendieck's version of Zariski's main theorem)**.** (a) *Let $f\colon X \to Y$ be a quasi-finite proper morphism between locally noetherian preschemes. Then f is finite.*

(b) *Let $f\colon X \to Y$ be a quasi-finite morphism between noetherian preschemes. Then there exists a factorization $X \xhookrightarrow{j} \overline{X} \xrightarrow{g} Y$ of f where j is an open immersion and g is finite.*

(c) *If $f\colon X \to Y$ is any morphism of finite type between locally noetherian preschemes, then*

$$U = \{x \in X \mid f \text{ is quasi-finite at } x\}$$

is open in X, and the restriction $f|_U$ is quasi-finite (by definition).

Proof. Part (a). We may assume that $Y = \operatorname{Spec} A$ is affine (indeed, all involved properties are base-local). Let $\mathcal{J} \subseteq \mathcal{O}_X$ be a sheaf of ideals, then $\mathcal{J}$ is coherent as X is locally noetherian. Since f is quasi-finite, all fibres carry the discrete topology. In particular, they are zero-dimensional and Corollary 1 shows that $R^1 f_* \mathcal{J} = 0$. Then also $0 = R^1 f_* \mathcal{J}(Y) = H^1(X, \mathcal{J})$ (using [AG2, Proposition 1.5.1(d)]), hence X is affine by Serre's affinity criterion. This shows that f is affine. Moreover, $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-module by [AG2, Theorem 5], hence f is finite.

Part (b) is hard, see the discussion on page 12. We only prove a special case there, which, however, is sufficient to prove (c). But before we can do this, we need to prove some more theorems of Zariski. □

Theorem 3 (Zariski's connectedness theorem)**.** *Let $f\colon X \to Y$ be a proper morphism between locally noetherian schemes, whose algebraic component $f^*\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.*

(a) *The fibres $f^{-1}\{y\}$ are connected for all $y \in Y$.*

(b) *The set*

$$U = \left\{x \in X \;\middle|\; \{x\} = f^{-1}\{f(x)\}\right\} = \{x \in X \mid f \text{ is quasi-finite at } x\}$$

is open in X, and the restriction $f|_U$ is quasi-finite (by definition).

Proof. Part (a). Assume $f^{-1}\{y\}$ is not connected, say, $f^{-1}\{y\} = U_1 \cup U_2$ for disjoint non-empty open subsets $U_1, U_2 \subseteq f^{-1}\{y\}$. Since all infinitesimal thickenings $X_n = X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n)$ have underlying topological space $f^{-1}\{y\}$, there is a unique $\varepsilon_n \in \mathcal{O}_{X_n}(X_n) = H^0(X_n, \mathcal{O}_{X_n})$ such that $\varepsilon_n|_{U_1} = 0$ and $\varepsilon_n|_{U_2} = 1$. The sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ clearly defines an element ε of

$$\varprojlim_{n\in\mathbb{N}} H^0(X_n, \mathcal{O}_{X_n}) \cong (f_*\mathcal{O}_X)_y^\wedge \cong \widehat{\mathcal{O}}_{Y,y}\ .$$

The left isomorphism here is due to Theorem 1a and the fact that $\mathcal{O}_{X_n} = \mathcal{O}_X|_{X_n}$, and the right one holds by assumption. Hence $\widehat{\mathcal{O}}_{Y,y}$ is a local ring (by Corollary A.1.5) with an idempotent $\varepsilon \neq 0, 1$. Then $1 - \varepsilon \neq 0, 1$ is another non-trivial idempotent. Both ε and $1-\varepsilon$ can't be units in $\widehat{\mathcal{O}}_{Y,y}$, otherwise $\varepsilon^2 = \varepsilon$ implies $\varepsilon = 1$. But then they are elements of the maximal ideal $\mathfrak{m}$, so $\varepsilon + (1-\varepsilon) = 1$ is an element of $\mathfrak{m}$ as well, contradiction!

Part (b). By (a), any point $x \in X$ is open and closed in its fibre iff $f^{-1}\{f(x)\} = \{x\}$. Therefore the two definitions of U indeed coincide.

We must show that U is open. This is a local question with respect to Y, hence we may assume that $Y = \operatorname{Spec} A$ is affine. Let $x \in U$ and $V \subseteq X$ an affine open neighbourhood of x. Put $Z = X \setminus V$. Then $Z \subseteq X$ is closed and disjoint from $f^{-1}\{f(x)\} = \{x\}$. As f is proper, $Z' = f(Z) \subseteq Y$ is closed, and $y = f(x) \notin Z'$. There's an $\alpha \in A$ such that $y \notin V(\alpha)$ and $V(\alpha) \supseteq Z'$. Let $Y_1 = Y \setminus V(\alpha)$. Note that $Y_1 \cong \operatorname{Spec} A_\alpha$ is affine and $x \in X_1 = f^{-1}(Y_1) \subseteq V$. Then $X_1 = X \setminus V(f^*\alpha) = V \setminus V(f^*\alpha)$ is affine as well, so the restriction $f|_{X_1}\colon X_1 \to Y_1$ of f is affine and proper. But every affine proper morphism is finite (because $f|_{X_1,*}\mathcal{O}_{X_1}$ is a coherent $\mathcal{O}_{Y_1}$-module by [AG2, Theorem 5]), so $f|_{X_1}$ is, in particular, quasi-finite by Fact 2 and $U \cap X_1 = X_1$. This proves that U is open. $\square$

Remark 1. On first glance, the argument from Theorem 3(b) might look like it proves that every proper morphism is affine, but what it actually shows is the following: If $f\colon X \to Y$ is a proper morphism such that for each $x \in X$ the fibre $f^{-1}\{f(x)\}$ is contained in some affine subset $V \subseteq X$, then f is already affine (and hence finite).

Remark 2. Recall that a prescheme X is called **normal** if it is integral and all local rings $\mathcal{O}_{X,x}$ (which are domains if X is integral) are normal (cf. [AG1, Definition 2.4.5]). This is the case iff $\mathcal{O}_X(U)$ is a normal domain for all affine $U \subseteq X$, cf. the discussion in [AG1, Remark 2.5.1].

Corollary 2 (Zariski's birationality theorem)**.** *Let $f\colon X \to Y$ be a proper morphism between locally noetherian preschemes, where Y is normal. Suppose that f is* **birational** *in the sense that there is a dense open subset $U \subseteq Y$ such that the restriction $f|_{f^{-1}(U)}\colon f^{-1}(U) \xrightarrow{\sim} U$ is an isomorphism and $f^{-1}(U)$ is dense in X. Then all assertions from Theorem 3 apply to f. In particular, f has connected fibres.*

Proof. First note that U is irreducible as an open subset of the irreducible space Y (irreducibility of Y is implied by Y being normal). Hence X is irreducible because it has the dense irreducible subset $f^{-1}(U) \cong U$. Let $\operatorname{Spec} A \cong V \subseteq Y$ be an affine open subset, where A is a domain. Then $f^{-1}(V)$ is open in X, hence dense in X and thus irreducible. Since U is dense in Y, the intersection $U \cap V$ is non-empty, hence $f^{-1}(U \cap V) \subseteq f^{-1}(V)$ is a non-empty open subset and thereby dense again. This shows that we can actually reduce to the case $Y = \operatorname{Spec} A$ (all the other involved properties are clearly base-local). Moreover, we may assume that X is integral. Indeed, the assertions from Theorem 3 are purely topological, so we may replace X by its reduction $X^{\mathrm{red}} = V(\operatorname{nil}(\mathcal{O}_X))$ to obtain an X which is irreducible and reduced (hence integral) and has the same underlying topological space as the original one.

Claim 1. The ring $B = \mathcal{O}_X(X)$ is a domain in the above situation, and A and B have the same field of quotients K. Moreover, we have $A \subseteq B$ as subrings of K.

Believing this for the moment, the proof can be finished as follows. Since B is finitely generated as an A-module (because $f_*\mathcal{O}_X = \widetilde{B}$ is coherent by [AG2, Theorem 5]), it is integral over A. But A is integrally closed in K, hence $A \subseteq B$ implies $A = B$. We conclude $f_*\mathcal{O}_X \cong \mathcal{O}_Y$, as needed.

Unfortunately, the proof of Claim 1 wasn't discussed in the lecture, but I think it should have been. Since X and Y are irreducible, they have unique generic points η_X and η_Y. As η_Y is dense in Y, we have $\eta_Y \in U$ and similarly $\eta_X \in f^{-1}(U)$. Hence $f(\eta_X) = \eta_Y$ and the induced morphism $\mathcal{O}_{Y,\eta_Y} \xrightarrow{\sim} \mathcal{O}_{X,\eta_X}$ is an isomorphism by the birationality assumption. Moreover, η_Y corresponds to $0 \in \operatorname{Spec} A$, hence $\mathcal{O}_{Y,\eta_Y} \cong K$ is the quotient field of A. So we should prove that $\mathcal{O}_{X,\eta_X}$ is the quotient field of $B = \mathcal{O}_X(X)$ as well.

It's clear that B is a domain because X is integral. Since $U \subseteq \operatorname{Spec} A$ is open, we find an affine open subset $V = \operatorname{Spec} A \setminus V(\alpha) \subseteq U$. Then $f^{-1}(V) = X \setminus V(f^*\alpha) = f^{-1}(U) \setminus V(f^*\alpha) \cong V$ is affine again by birationality of f. We know that X is quasi-compact and separated since so are f and $\operatorname{Spec} A$. In particular, [AG1, Proposition 1.5.1(c)] is applicable to $\mathcal{O}_X$ and gives $\mathcal{O}_X(f^{-1}(V)) \cong \mathcal{O}_X(X)_{f^*\alpha}$, so these two rings have the same quotient field. But $\mathcal{O}_X(f^{-1}(V)) \cong \mathcal{O}_Y(V) \cong A_\alpha$ has quotient field K, so we win.

The fact that $A \subseteq B$ as subrings of K follows from the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\ \mathcal{O}_{Y,\eta_Y} \arrow[r, "\sim"] & \mathcal{O}_{X,\eta_X} \end{tikzcd}$$

in which every arrow except the top one is injective, hence $A \to B$ is injective as well. □

Fact 3. *Every proper morphism $f\colon X \to Y$ between locally noetherian preschemes can be factorized via the* **Stein factorization** *as*

$$f\colon X \xrightarrow{\widetilde{f}} \mathbf{Spec}_Y(f_*\mathcal{O}_X) \xrightarrow{g} Y\ .$$

In this composition, g is finite and the assumptions of Theorem 3 hold for $\widetilde{f}$.

Sketch of a proof. It's pretty obvious that this factorization exists (to construct $\widetilde{f}$, use the adjunction from [AG2, Proposition 1.6.2(b)]). To show that $\widetilde{f}$ and g have the required properties, we look at things locally and assume that $Y = \operatorname{Spec} A$ is affine (and A is noetherian). Then the facorization looks like

$$f\colon X \xrightarrow{\widetilde{f}} \operatorname{Spec} \mathcal{O}_X(X) \xrightarrow{g} \operatorname{Spec} A\ ,$$

so g is affine. Moreover, $\mathcal{O}_X(X)$ is a finitely generated A-module because $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-module (by [AG2, Theorem 5], as usual), so g is actually finite. Also proving that $\widetilde{f}_*\mathcal{O}_X = \mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(X)}$ is straightforward, so it remains to show that $\widetilde{f}$ is proper. But g is finite, hence separated, and $g\widetilde{f} = f$ is proper, so $\widetilde{f}$ is proper as well by [AG2, Proposition 2.4.1]. $\square$

Discussion of Theorem 2(b) and (c). Let's assume that $f\colon X \to Y$ factors over

$$f\colon X \overset{j}{\hookrightarrow} \overline{X} \xrightarrow{\overline{f}} Y\ ,$$

where j is an open immersion and $\overline{f}$ is proper. It can be shown that such a factorization always exists for morphisms of finite type between noetherian schemes, for which Professor Franke refers to notes of *Brian Conrad* or *Paul Vojta*, although he isn't sure whether using their results to prove Theorem 2(b) doesn't involve any circular reasoning.

Be that as it may, if we can factorize f as above, then

$$\{x \in X \mid f \text{ is quasi-finite at } x\} = X \cap \left\{x \in \overline{X} \;\middle|\; \overline{f} \text{ is quasi-finite at } x\right\}\ . \tag{1}$$

Indeed, a point $x \in X$ is open in $\overline{f}^{-1}\{y\}$ (where $y = f(x)$) iff it is open in the open subset $f^{-1}\{y\} = X \cap \overline{f}^{-1}\{y\} \subseteq \overline{f}^{-1}\{y\}$. In view of Fact 1 this shows (1). We thus have reduced (c) (under the assumption that $\overline{f}$ exists) to the case of proper morphisms.

If $f\colon X \to Y$ is proper, then consider its Stein factorization. Since g is finite, it's quasi-finite as well by Fact 2(a). So Fact 2(c) shows that

$$\{x \in X \mid f \text{ is quasi-finite at } x\} = \{x \in X \mid \widetilde{f} \text{ is quasi-finite at } x\}\ .$$

But the right-hand side is open in X by Fact 3 and Theorem 3(b) and we're happy!

Note that such an $\overline{f}$ always exists when X and Y are affine. Indeed, if X has finite type over Y and both are affine, we get a closed embedding $X \hookrightarrow \mathbb{A}^n_Y$ for some $n \in \mathbb{N}$. Together with the open embedding $\mathbb{A}^n_Y \hookrightarrow \mathbb{P}^n_Y$ this makes X a closed subprescheme of an open subprescheme of $\mathbb{P}^n_Y$. But then X is also an open subprescheme of some closed subprescheme $\overline{X} \subseteq \mathbb{P}^n_Y$. This gives a factorization

$$f\colon X \overset{j}{\hookrightarrow} \overline{X} \xrightarrow{\overline{f}} Y$$

in which $\overline{f}\colon \overline{X} \hookrightarrow \mathbb{P}^n_Y \to Y$ is (strongly) projective, hence proper by [AG2, Proposition 2.4.2]. But (c) is completely local on both X and Y (thanks to Fact 1), so by checking the affine case we have actually covered all of (c). $\square$

A. Appendix

A.1. Some prerequisites about completions

We briefly recall the most important facts about completions. An excellent introduction to this subject can be found in [AM94, Section 10].

Definition 1. Let R be a ring (commutative with 1) and I an ideal in R. Let M be an R-module.

(*a*) The **I-adic topology** on M is the unique topology such that $\{I^n\}_{n\in\mathbb{N}}$ is a fundamental system of neighbourhoods of 0 and M (with its additive structure) becomes a topological group in this topology.

(*b*) The **completion** of M with respect to the I-adic topology is

$$\widehat{M} = \varprojlim_{n\in\mathbb{N}} M/I^nM\ .$$

Note that $\widehat{R}$ is a ring again. We call M **complete** in the I-adic topology if the canonical morphism $M \to \widehat{M}$ is an isomorphism.

Remark 1. M with its I-adic topology is *pseudo-metrizable* via $d(x,y) = \mathrm{e}^{-\sup\{n \mid x-y\in I^n\}}$. It is easy to check that $\widehat{M}$ is also the completion of M in the analytical sense, i.e. the set of Cauchy sequences modulo the zero sequences.

Example 1. If $I^n = 0$ for some $n \in \mathbb{N}$, then any R-module is complete in the I-adic topology.

Example 2. If $R = \mathbb{Z}$ and $I = p\mathbb{Z}$ for some prime p, then $\widehat{R} = \mathbb{Z}_p$ is the ring of p-adic integers.

Proposition 1 (Hensel's lemma)**.** *Suppose the ring R is complete in the I-adic topology. Let $P \in R[T]$ be a polynomial and $a_0 \in R$ such that $P(a_0) \equiv 0 \bmod I$ and $P'(a_0)$ is a unit in R/I. Then there is a unique $a \in R$ such that $a \equiv a_0 \bmod I$ and $P(a) = 0$.*

Proof. Step 1. Consider the special case $I^2 = 0$. For $\delta \in I$ we have $P(a_0+\delta) = P(a_0) + \delta P'(a_0)$ since all terms of order δ^2 or higher vanish in the binomial expansion. Now $P'(a_0)$ being a unit in R/I gives a unique $\delta \in I$ such that $a = a_0 + \delta$ satisfies $P(a) = 0$.

Step 2. Suppose that $I^{2^n} = 0$ for some $n \in \mathbb{N}$. Using induction on n (with the base case being precisely Step 1) we may assume that Hensel's lemma holds for $R/I^{2^{n-1}}$. In particular, there is a unique a_{n-1} such that $P(a_{n-1}) \equiv 0 \bmod I^{2^{n-1}}$ and $a_{n-1} \equiv a_0 \bmod I$. Moreover, $P'(a_{n-1})$ is invertible in $R/I^{2^{n-1}}$. Indeed, this follows from Hensel's lemma applied to $R/I^{2^{n-1}}$ (for which it holds by induction hypothesis) and the polynomial $Q = P'(a_{n-1})T - 1$. The derivative $Q'(a_{n-1})$ equals $P'(a_{n-1})$ which is invertible in R/I since $P'(a_{n-1}) \equiv P'(a_0) \bmod I$, so Hensel's lemma is indeed applicable. Now replacing I by $I^{2^{n-1}}$ and a_0 by a_{n-1} reduces the situation to Step 1, proving the inductive step.

Step 3. Now let I be arbitrary. By Step 2 there is for every $n \in \mathbb{N}$ a unique $a_n \in R/I^{2^n}$ such that $P(a_n) \equiv 0 \bmod I^{2^n}$ and $a_n \equiv a_0 \bmod I$. Then $a_n \equiv a_{n-1} \bmod I^{2^{n-1}}$ is forced by uniqueness. Hence $a = (a_n)_{n\in\mathbb{N}}$ defines an element of

$$\varprojlim_{n\in\mathbb{N}} R/I^{2^n} = \varprojlim_{n\in\mathbb{N}} R/I^n = \widehat{R}\,,$$

providing the desired element $a \in \widehat{R}$. $\square$

Corollary 1. *Let R be complete in the I-adic topology.*

(*a*) *If $a \in R$ becomes a unit in R/I, then already $a \in R^\times$.*

(*b*) *For every idempotent $\pi \in R/I$ there is a unique idempotent in R whose image modulo I is π. Therefore,* $\operatorname{Spec} R$ *and* $\operatorname{Spec} R/I$ *have the same connected components.*

(*c*) *I is contained in the Jacobson radical* $\operatorname{rad} R$*.*

Proof. Part (*a*) follows from Proposition 1 applied to $P = aT - 1$ (whose derivative a is a unit in R/I by assumption, so this is fine). For (*b*) we use the polynomial $P = T^2 - T$. Again, $P'(\pi) = 2\pi - 1$ is a unit in R/I since $(2\pi - 1)^2 = 4\pi^2 - 4\pi + 1 = 1$ in R/I. To prove (*c*) recall the characterization

$$\operatorname{rad} R = \left\{x \in R \;\middle|\; 1 - rx \in R^\times \text{ for all } r \in R\right\}\,.$$

If $x \in I$, then $1 - rx$ is a unit in R/I, hence also in R by (*a*). $\square$

Proposition 2. *Let R be noetherian and $N \subseteq M$ finitely generated R-modules. Then the I-adic topology on N coincides with the induced topology by the I-adic topology on M.*

Sketch of a proof. By the Artin–Rees lemma (cf. [Alg2, Proposition 3.4.1]) there exists a number $c \in \mathbb{N}$ such that $N \cap I^{n+c}M \subseteq I^nN$. From this, the assertion is easily deduced. $\square$

Fact 1. (*a*) *The canonical morphism $\widehat{M} = \varprojlim M/I^nM \to M/IM$ is surjective.*

(*b*) *If M is finitely generated and I is contained in the Jacobson radical of R, then $\widehat{M} = 0$ implies $M = 0$.*

Proof. For (*a*), note that the composition $M \to \widehat{M} \to M/IM$ equals the projection $M \to M/IM$ by definition of the limit. Since the latter is surjective, so is $\widehat{M} \to M/IM$. In particular, part (*a*) shows that $\widehat{M} = 0$ implies $M = IM$. In the situation of (*b*) this is equivalent to $M = 0$ by Nakayama's lemma (which – as we all know – Professor Franke also likes to attribute to Azumaya and Krull, even though he regards Krull as a noob compared to Grothendieck). $\square$

Corollary 2. *If R is noetherian, then the functor $M \mapsto \widehat{M}$ is exact on the category of finitely generated R-modules.*

Proof. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely generated R-modules. Then $M' + I^n M$ is the kernel of $M \twoheadrightarrow M''/I^n M''$. Using $(M' + I^n M)/I^n M \cong M'/(M' \cap I^n M)$ we get short exact sequences

$$0 \longrightarrow M'/(M' \cap I^n M) \longrightarrow M/I^n M \longrightarrow M''/I^n M'' \longrightarrow 0 \tag{$*$}$$

for every $n \in \mathbb{N}$. Since $M'/(M' \cap I^n M)$ is sandwiched between $M'/I^n M'$ and $M'/I^{n+c}M'$ for some $c \in \mathbb{N}$ by the Artin–Rees lemma, it's easy to see that

$$\varprojlim_{n \in \mathbb{N}} M'/(M' \cap I^n M) = \varprojlim_{n \in \mathbb{N}} M'/I^{n+c}M' = \varprojlim_{n \in \mathbb{N}} M'/I^n M' = \widehat{M}' \,.$$

Moreover, each $M'/(M' + I^{n+1}M) \to M'/(M' + I^n M)$ is clearly surjective, so Fact 1.1.3 gives

$$\varprojlim_{n \in \mathbb{N}}{}^1 \, M'/(M' \cap I^n M) = 0 \,.$$

Thus, taking the limit over ($*$) gives a short exact sequence $0 \to \widehat{M}' \to \widehat{M} \to \widehat{M}'' \to 0$ by Fact 1.1.1. We are done. □

Corollary 3. *Let R be a Noetherian ring.*

(*a*) *When M is a finitely generated R-module, then $\widehat{M} \cong M \otimes_R \widehat{R}$.*

(*b*) *$\widehat{R}$ is flat as an R-module.*

(*c*) *Suppose that I is contained in the Jacobson radical of R. If $\mu\colon M \to N$ is a morphism of finitely generated R-modules such that $\widehat{\mu}\colon \widehat{M} \to \widehat{N}$ is an isomorphism, then μ is an isomorphism.*

Proof. Part (a). Every finitely generated R-module is finitely presented as well since R is noetherian. So take a representation $M \cong \operatorname{coker}(R^m \to R^n)$ for some $m, n \in \mathbb{N}$. It's obvious that $(R^n)^\wedge \cong \widehat{R}^n \cong R^n \otimes_R \widehat{R}$. Since both completion and tensor products commute with cokernels, this shows $\widehat{M} \cong M \otimes_R \widehat{R}$ as well.

Part(b). By Corollary 2 and (a), $- \otimes_R \widehat{R}$ is exact on finitely generated R-modules. By [Hom, Proposition 1.2.2] this is sufficient for flatness.

Part (c). Let $K = \ker \mu$ and $Q = \operatorname{coker} \mu$. Since completion is exact on finitely generated R-modules, we get an exact sequence

$$0 \longrightarrow \widehat{K} \longrightarrow \widehat{M} \xrightarrow{\widehat{\mu}} \widehat{N} \longrightarrow \widehat{Q} \longrightarrow 0 \,.$$

But $\widehat{\mu}$ is an isomorphism, so $\widehat{K} = 0$ and $\widehat{Q} = 0$. By Fact 1 this shows $K = 0$ and $Q = 0$. We are done. □

Corollary 4. *If $J \subseteq R$ is any ideal and M a finitely generated R-module, then $(JM)^\wedge \to \widehat{M}$ defines an isomorphism $(JM)^\wedge \xrightarrow{\sim} J\widehat{M}$.*

Proof. We may view $(JM)^\frown$ as a submodule of $\widehat{M}$ since completion preserves injectivity of the inclusion $JM \subseteq M$ by Corollary 2. It's easy to see that $J\widehat{M}$ is contained in $(JM)^\frown$. To prove the converse, take generators $j_1, \dots, j_n$ of J. Then completion preserves surjectivity of $(j_1, \dots, j_n)\colon M^n \twoheadrightarrow JM$ and we are done. $\square$

Corollary 5. *If R is a noetherian local ring with maximal ideal $\mathfrak{m}$, then $\widehat{R}$ is local with maximal ideal $\mathfrak{m}\widehat{R}$.*

Proof. We proved this in [Hom, Corollary 2.2.2]. $\square$

Proposition 3. *Let R be noetherian and $I \subseteq R$ any ideal, then the I-adic completion $\widehat{R}$ is noetherian again.*

To prove this, we need to prove the evil twin of Hilbert's Basissatz first.

Lemma 1. *If R is noetherian, then so is the power series ring $R[\![T]\!]$.*

Proof. We can (and will) basically copy the proof of Hilbert's Basissatz. Let $J \subseteq R[\![T]\!]$ be any ideal and put $J_n = \left\{ a_n \;\middle|\; \sum_{k=n}^\infty a_k T^k \in J \right\}$ for $n \geqslant 0$. Then $(J_n)_{n\in\mathbb{N}}$ form an ascending sequence of ideals in R. Noetherianness of R tells us that this sequence becomes eventually stationary, say, at $n = s$. So we may choose $a^{(i)} = \sum_{k=s}^\infty a_k T^k \in R[\![T]\!]$ for $i = 1, \dots, N$ such that $a_s^{(1)}, \dots, a_s^{(N)}$ generate J_s. Then $a^{(1)}, \dots, a^{(N)}$ generate $J \cap T^s R[\![T]\!]$. Indeed, given any $b = \sum_{k=s}^\infty b_k T^k \in J$ we can inductively choose coefficients $r_k^{(1)}, \dots, r_k^{(N)} \in R$ such that $r^{(i)} = \sum_{k=0}^\infty r_k^{(i)} T^k$ satisfy $r^{(1)}a^{(1)} + \ldots + r^{(N)}a^{(N)} = b$ up to degree T^{s+k}. This works because $J_{k+s} = J_s$ for all $k \geqslant 0$ is generated by $a_s^{(1)}, \dots, a_s^{(N)}$ again.

Now $R[\![T]\!]/T^s R[\![T]\!]$ is a finitely generated R-module, hence the image of J in it is finitely generated as well, R being noetherian. We thus may choose $a^{(N+1)}, \dots, a^{(N+M)} \in J$ whose images modulo $T^s R[\![T]\!]$ generate the image of J in $R[\![T]\!]/T^s R[\![T]\!]$. Then $a^{(1)}, \dots, a^{(N+M)}$ generate J and our job's done here. $\square$

Proof of Proposition 3. Let $r_1, \dots, r_n$ be generators of I. Then sending $X_i \mapsto r_i$ defines a surjective morphism $R[\![X_1, \dots, X_n]\!] \twoheadrightarrow \widehat{R}$. Since $R[\![X_1, \dots, X_n]\!]$ is noetherian by Lemma 1 and induction on n, so is its quotient $\widehat{R}$. $\square$

Corollary 6. *Suppose that R is a noetherian local ring and $I \subseteq R$ any (proper) ideal. Then $\dim R = \dim \widehat{R}$. In particular, R is regular iff $\widehat{R}$ is regular.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R. Then $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$ (this equality holds because of Corollary 4) is the maximal ideal of the local ring $\widehat{R}$ as was shown in the proof of [Hom, Corollary 2.2.2]. Since $I \subseteq \mathfrak{m}$, the quotients $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ already have I-torsion, hence

$$\mathfrak{m}^i/\mathfrak{m}^{i+1} \cong \left(\mathfrak{m}^i/\mathfrak{m}^{i+1}\right)^\frown \cong \widehat{\mathfrak{m}}^i/\widehat{\mathfrak{m}}^{i+1}$$

(the last isomorphism follows from exactness of completion). This shows that the associated graded rings $\operatorname{gr}(R, \mathfrak{m})$ and $\operatorname{gr}(\widehat{R}, \widehat{\mathfrak{m}})$ agree, hence $(R, \mathfrak{m})$ and $(\widehat{R}, \widehat{\mathfrak{m}})$ have the same Hilbert–Samuel polynomials, which shows $\dim R = \dim \widehat{R}$ by [Alg2, Theorem 20].

Now R and $\widehat{R}$ have the same residue field k and $\mathfrak{m}/\mathfrak{m}^2 \cong \widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2$ by the I-torsion arguments we have seen several times now, so $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim_k \widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2$. Clearly this implies that R is regular iff $\widehat{R}$ is. □

Remark 2. In a similar fashion one can show that a noetherian local ring is Cohen–Macaulay, or Gorenstein, or a complete intersection, iff its I-adic completion is one as well. For example, for Cohen–Macaulayness one would need to show $\operatorname{depth}_R(R) = \operatorname{depth}_{\widehat{R}}(\widehat{R})$, which follows from the isomorphism $\operatorname{Ext}^p_R(k, R) \cong \operatorname{Ext}^p_{\widehat{R}}(k, \widehat{R})$ that was described in the proof of [Hom, Proposition 2.4.2].

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