

Jacobians of Curves

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1. Introduction and preparations

1.1. A note on limits and their derived functors

Let $X_\bullet: \dots \xrightarrow{p_{i+1}} X_i \xrightarrow{p_i} \dots \xrightarrow{p_2} X_1 \xrightarrow{p_1} X_0$ be a diagram of abelian groups or R -modules. As usual, we may view X_\bullet as a functor $X_\bullet: (\mathbb{N}, \geq) \rightarrow \mathbf{Ab}$ or $\mathbf{Mod}(R)$, where the category (\mathbb{N}, \geq) has the nonnegative integers as objects and an arrow $j \rightarrow i$ iff $j \geq i$. Let

$$d: \prod_{i=0}^{\infty} X_i \longrightarrow \prod_{i=0}^{\infty} X_i, \quad d(x_i)_{i=0}^{\infty} = (p_{i+1}(x_{i+1}) - x_i)_{i=0}^{\infty}.$$

Then we put

$$\varprojlim_{i \in \mathbb{N}} X_i = \ker d \quad \text{and} \quad \varprojlim^1_{i \in \mathbb{N}} X_i = \operatorname{coker} d.$$

Remark 1. It is easy to see that $\varprojlim_{i \in \mathbb{N}} X_i$ equals the usual category-theoretical limit (that's how you construct it). It can also be shown that $\varprojlim^1_{i \in \mathbb{N}}$ is the first right-derived functor of $\varprojlim_{i \in \mathbb{N}}$, and that its higher derived functors vanish.

Fact 1. Let $0 \rightarrow X'_\bullet \rightarrow X_\bullet \rightarrow X''_\bullet \rightarrow 0$ be a short exact sequence of diagrams of the above type. Then there is a canonical exact sequence

$$0 \longrightarrow \varprojlim_{i \in \mathbb{N}} X'_i \longrightarrow \varprojlim_{i \in \mathbb{N}} X_i \longrightarrow \varprojlim_{i \in \mathbb{N}} X''_i \longrightarrow \varprojlim^1_{i \in \mathbb{N}} X'_i \longrightarrow \varprojlim^1_{i \in \mathbb{N}} X_i \longrightarrow \varprojlim^1_{i \in \mathbb{N}} X''_i \longrightarrow 0.$$

Proof. Since products preserve exact sequences in \mathbf{Ab} or $\mathbf{Mod}(R)$, we get a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \prod_{i=0}^{\infty} X'_i & \longrightarrow & \prod_{i=0}^{\infty} X_i & \longrightarrow & \prod_{i=0}^{\infty} X''_i \longrightarrow 0 \\ & & d' \downarrow & & d \downarrow & & d'' \downarrow \\ 0 & \longrightarrow & \prod_{i=0}^{\infty} X'_i & \longrightarrow & \prod_{i=0}^{\infty} X_i & \longrightarrow & \prod_{i=0}^{\infty} X''_i \longrightarrow 0 \end{array}$$

with exact rows. Then the snake lemma finishes the job. □

Fact 2. Let X_\bullet have the property that for every $i \in \mathbb{N}$ there is a $j \geq i$ such that the composition $p_{j,i}: X_j \xrightarrow{p_j} X_{j-1} \xrightarrow{p_{j-1}} \dots \xrightarrow{p_{i+1}} X_i$ vanishes. Then

$$\varprojlim_{i \in \mathbb{N}} X_i = \varprojlim^1_{i \in \mathbb{N}} X_i = 0.$$

Proof. If $x = (x_i)_{i=0}^\infty \in \varprojlim X_i$, then $x_i = p_{j,i}(x_j)$ for all $j \geq i$ by construction, hence $x_i = 0$ for all $i \in \mathbb{N}$. Moreover, let

$$s: \prod_{i=0}^\infty X_i \longrightarrow \prod_{i=0}^\infty X_i, \quad s(x)_i = \sum_{j \geq i} p_{j,i}(x_j).$$

By assumption s is well-defined. Then

$$d(s(x))_i = p_{i+1} \left(\sum_{j \geq i+1} p_{j,i+1}(x_j) \right) - \sum_{j \geq i} p_{j,i}(x_j) = -p_{i,i}(x_i) = -x_i.$$

Hence $-s$ is a right-inverse of d , so $\varprojlim^1 X_i = \text{coker } d$ vanishes as well. \square

Fact 3. *Let X_\bullet have the Mittag-Leffler property that for every $i \in \mathbb{N}$ there is a $j \geq i$ such that for all $k \geq j$ the images of $p_{j,i}$ and $p_{k,i}$ in X_i coincide. Then $\varprojlim^1 X_i = 0$.*

Proof. Let's first deal with the special case that each $p_i: X_i \rightarrow X_{i-1}$ is surjective. Let $x = (x_i)_{i=0}^\infty \in \prod_{i=0}^\infty X_i$. For every $i \in \mathbb{N}$ we may select $x_j^{(i)} \in X_j$ for all $j \geq i$ in such a way that $x_i^{(i)} = x_i$ and $p_{j+1}(x_{j+1}^{(i)}) = x_j^{(i)}$. Then $s(x)$ defined by

$$s(x)_i = \sum_{k=0}^{i-1} x_i^{(k)}$$

is a preimage of x under d , so $\varprojlim^1 X_i = \text{coker } d = 0$ in this case.

Now let X_\bullet be arbitrary with the Mittag-Leffler property. Let $Y_i = \bigcap_{j \geq i} p_{j,i}(X_j) \subseteq X_i$. Then $\varprojlim^1 Y_i = 0$ by the special case we just treated, and $\varprojlim^1 X_i/Y_i = 0$ by Fact 2. Since $\varprojlim^1 X_i$ is sandwiched between these two in the exact sequence from Fact 1, this shows $\varprojlim^1 X_i = 0$, as required. \square

1.2. The theorem about formal functions

Let $f: X \rightarrow Y = \text{Spec } A$ be a morphism of quasi-compact schemes. Let $I \subseteq A$ be any ideal. Consider

$$i_n: X_n = X \times_Y \text{Spec}(A/I^n) \longrightarrow X,$$

which is a base change of the closed immersion $Y_n = \text{Spec}(A/I^n) \hookrightarrow \text{Spec } A$, hence indeed a closed immersion itself. Also, if f is proper, then so is $X_n \rightarrow Y_n$ because properness is another *property* (tee-hee) that is stable under base change (by [AG2, Remark 2.4.1]).

Let \mathcal{F} be a quasi-coherent sheaf of \mathcal{O}_X -modules and $\mathcal{F}|_{X_n} = i_n^* \mathcal{F}$ its restriction to X_n (this notation is slightly abusive, but convenient). We put $\mathcal{F}_n = i_{n,*} \mathcal{F}|_{X_n}$. It's easy to check (e.g. affine-locally) that $\mathcal{F}_n \cong \mathcal{F}/I^n \mathcal{F}$. Since i_n is a closed immersion and thus affine, we have an isomorphism $H^p(X, \mathcal{F}_n) \cong H^p(X_n, \mathcal{F}|_{X_n})$ for all $p \geq 0$ by [AG2, Corollary 1.5.1]. Together with

the canonical projection $\mathcal{F}_{n+1} \cong \mathcal{F}/I^{n+1}\mathcal{F} \rightarrow \mathcal{F}/I^n\mathcal{F} \cong \mathcal{F}_n$ this gives canonical morphisms $H^p(X_{n+1}, \mathcal{F}|_{X_{n+1}}) \rightarrow H^p(X_n, \mathcal{F}|_{X_n})$ for all $n \in \mathbb{N}$.

The canonical morphism $\mathcal{F} \rightarrow i_{n,*}i_n^*\mathcal{F} = \mathcal{F}_n$ induces a morphism

$$H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}_n) \cong H^p(X_n, \mathcal{F}|_{X_n}) \quad (1)$$

for all $p \geq 0$ (the isomorphism on the right-hand side comes from the fact that i_n is a closed immersion, hence affine, and we can apply [AG2, Corollary 1.6.1]). This is a morphism of A -modules, but $H^p(X_n, \mathcal{F}|_{X_n})$ is actually an A/I^n -module, so (1) factors over

$$H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F}) \longrightarrow H^p(X_n, \mathcal{F}|_{X_n}) .$$

This is compatible with the canonical morphisms $H^p(X_{n+1}, \mathcal{F}|_{X_{n+1}}) \rightarrow H^p(X_n, \mathcal{F}|_{X_n})$ (you can just check that on an affine Čech covers). Passing to the limit gives a morphism

$$H^p(X, \mathcal{F})^\wedge \longrightarrow \varprojlim_{n \geq 1} H^p(X_n, \mathcal{F}|_{X_n}) , \quad (2)$$

where \wedge denotes the I -adic completion.

Theorem 1 (Grothendieck–Zariski). *When $f: X \rightarrow Y = \operatorname{Spec} A$ is proper (in which case X is automatically a quasi-compact scheme), A is noetherian and \mathcal{F} is a coherent sheaf of \mathcal{O}_X -modules, then (2) is an isomorphism*

$$H^p(X, \mathcal{F})^\wedge \xrightarrow{\sim} \varprojlim_{n \geq 1} H^p(X_n, \mathcal{F}|_{X_n}) .$$

Proof. The following proof is essentially the one from [EGAIII, (4.1.7)]. Professor Franke also pointed out that the idea is pretty similar to the proof of the Artin–Rees lemma. Let $I \subseteq A$ be the ideal under consideration and let $R = \bigoplus_{n \geq 0} I^n$ be the Rees algebra associated to I . Then

$$K^p = \bigoplus_{n \geq 0} H^p(X, I^n \mathcal{F})$$

is a module over R as $i \in I^m$ (considered as the m^{th} homogeneous component of R) maps $I^n \mathcal{F}$ to $I^{n+m} \mathcal{F}$.

Claim 1. K^p is a finitely generated R -module for all $p \geq 0$.

Assuming this for the moment, recall that $\mathcal{F}_n \cong \mathcal{F}/I^n \mathcal{F}$ and $H^p(X, \mathcal{F}|_{X_n}) \cong H^p(X, \mathcal{F}_n)$, so the long exact cohomology sequence associated to $0 \rightarrow I^n \mathcal{F} \rightarrow \mathcal{F} \rightarrow \mathcal{F}_n \rightarrow 0$ appears as

$$H^p(X, I^n \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}) \longrightarrow H^p(X_n, \mathcal{F}|_{X_n}) \longrightarrow H^{p+1}(X, I^n \mathcal{F}) . \quad (3)$$

As pointed out after (1), $H^p(X, \mathcal{F}) \rightarrow H^p(X_n, \mathcal{F}|_{X_n})$ factors over $H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F})$, hence we can turn equation (3) into an exact sequence

$$0 \longrightarrow U_n \longrightarrow H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F}) \longrightarrow H^p(X_n, \mathcal{F}|_{X_n}) \longrightarrow V_n \longrightarrow 0 , \quad (4)$$

where U_n is a suitable quotient of $H^p(X, I^n \mathcal{F})$ and $V_n \subseteq H^{p+1}(X, I^n \mathcal{F})$ some submodule. This makes $U = \bigoplus_{n \geq 0} U_n$ a quotient of K^p and $V = \bigoplus_{n \geq 0} V_n$ an R -submodule of K^{p+1} .

Claim 2. We have $\varprojlim U_n = \varprojlim^1 U_n = 0$ and $\varprojlim V_n = \varprojlim^1 V_n = 0$.

Before we prove this (and Claim 1), let's see how Theorem 1 follows from it. Let W_n be the image of $H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F})$ in $H^p(X_n, \mathcal{F}|_{X_n})$. We may split (4) into two short exact sequences $0 \rightarrow U_n \rightarrow H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F}) \rightarrow W_n \rightarrow 0$ and $0 \rightarrow W_n \rightarrow H^p(X, \mathcal{F}|_{X_n}) \rightarrow V_n \rightarrow 0$. Applying Fact 1.1.1 to the first one gives $H^p(X, \mathcal{F})^\wedge \cong \varprojlim^1 W_n$. Then the six-term exact sequence associated to the second proves $\varprojlim W_n \cong \varprojlim H^p(X_n, \mathcal{F}|_{X_n})$ and we are done.

It remains to show the two claims. Note that the Rees algebra R is noetherian. Indeed, I is finitely generated as an ideal in the noetherian ring A , hence R is of finite type over A . Let's also make the following convention: Whenever we write $I^k U_n$ or $I^k V_n$ in the following, this means multiplication as A -modules and the result is contained in U_n resp. V_n again, whereas $R_k U_n$ or $R_k V_n$ means multiplication by the k^{th} homogeneous component of R (which equals I^k as well), so the result is contained on U_{k+n} resp. V_{k+n} .

Proof of Claim 2. Note that U is finitely generated over R , since it is a quotient of the finitely generated R -module K^p . Fix a finite set of generators and let d_0 the maximal non-zero homogeneous components occurring in this set. Then $U_n = R_n U_0 + R_{n-1} U_1 + \dots + R_{n-d_0} U_{d_0}$ for all $n \geq d_0$. In particular, $U_{k+n} = R_k U_n$ for all $n \geq d_0$. Thus, for every $n \geq d_0$ the image of $U_{2n} = R_n U_n$ in U_n is contained in $I^n U_n$. But $U_n \subseteq H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F})$, so $I^n U_n$ vanishes. Therefore, the property from Fact 1.1.2 is fulfilled for all $n \geq d_0$. But then it clearly holds for all $n \geq 0$ as well, so Fact 1.1.2 is applicable.

Similarly, V is finitely generated as a submodule of K^{p+1} , which is finitely generated over the noetherian ring R by Claim 1. By the same argument as above we find a d_1 such that $V_n = R_n V_0 + R_{n-1} V_1 + \dots + R_{n-d_1} V_{d_1}$ for all $n \geq d_1$. In particular, we have $V_{k+n} = R_k V_n$ for all $n \geq d_1$. Thus, for $n \geq d_1$ the image of V_{2n} in V_n is contained in $I^n V_n$. But $I^n V_n$ vanishes again, since V_n is the image of $H^p(X_n, \mathcal{F}|_{X_n})$, which is a A/I^n -module. As above, we can apply Fact 1.1.2. This shows Claim 2.

Proof of Claim 1. Let $v: \tilde{Y} = \text{Spec } R \rightarrow Y$ correspond to $A \hookrightarrow R$ and let $\xi: \tilde{X} = X \times_Y \tilde{Y} \rightarrow X$ be its base change by f . Note that ξ is affine as a base change of the affine morphism v (we use [AG1, Corollary 2.5.1] here). We claim

$$\xi_* \xi^* \mathcal{F} \cong \bigoplus_{n \geq 0} I^n \mathcal{F}.$$

Indeed, this is easily checked affine-locally (where ξ^* is given by tensoring with R); we leave the details to the reader. Also $H^p(\tilde{X}, \xi^* \mathcal{F}) \cong H^p(X, \xi_* \xi^* \mathcal{F})$ as ξ is affine. This shows

$$H^p(\tilde{X}, \xi^* \mathcal{F}) \cong H^p(X, \xi_* \xi^* \mathcal{F}) \cong H^p\left(X, \bigoplus_{n \geq 0} I^n \mathcal{F}\right) \cong \bigoplus_{n \geq 0} H^p(X, I^n \mathcal{F}) = K^p.$$

Note that direct sums usually *don't* commute with cohomology, but here they do, because X is quasi-compact and $\bigoplus_{n \geq 0} I^n \mathcal{F}$ is quasi-coherent (for which we need quasi-compactness as well), so we may compute $H^p(X, \bigoplus_{n \geq 0} I^n \mathcal{F})$ via finite affine Čech covers. In this case, the products in the Čech complex are all finite, hence commute with the direct sum, which is what we needed.

Now $\tilde{f}: \tilde{X} \rightarrow \tilde{Y} = \operatorname{Spec} R$ is proper (as a base change of the proper morphism f), hence the right-hand side is a finitely generated R -module by our finiteness results for the cohomology of proper morphisms (cf. [AG2, Theorem 5]). We win. \square

Remark 1. Note that in the lecture Franke used $\mathcal{K}_n \cong \mathcal{J}^n \mathcal{F}$ instead of $I^n \mathcal{F}$, where $\mathcal{J} = f^{-1} \mathcal{I}$ is the inverse image (in the sense of Definition 1). But $\mathcal{J}^n \mathcal{F} \cong I^n \mathcal{F}$ – which is not that surprising, since the I^n -action on \mathcal{F} is given via the algebraic component $\mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ of f , so $I^n \mathcal{F} = \mathcal{J}^n \mathcal{F}$ is pretty obvious from the construction of f^{-1} described in the proof of Lemma 1 below. I prefer the notation $I^n \mathcal{F}$ – in particular, this is how Grothendieck denotes it in [EGAIII, (4.1.7)], so I believe it's my right to do so as well. Nevertheless, Lemma 1 is *perhaps worthwhile to know* (if you get what I mean), so we will include it now.

Definition 1. Let $f: X \rightarrow Y$ be any morphism of preschemes and $\mathcal{J} \subseteq \mathcal{O}_Y$ a sheaf of ideals on Y . Then define $f^{-1} \mathcal{J}$ to be the image of $f^* \mathcal{J} \rightarrow \mathcal{O}_X$ (which is obtained as the composition of the pull-back of $\mathcal{J} \rightarrow \mathcal{O}_Y$ with the isomorphism $f^* \mathcal{O}_Y \cong \mathcal{O}_X$).

Lemma 1. *Let $f: X \rightarrow Y$ be any morphism of preschemes and $\mathcal{J} \subseteq \mathcal{O}_Y$ quasi-coherent.*

- (a) $f^{-1} \mathcal{J} \subseteq \mathcal{O}_X$ is quasi-coherent.
- (b) Let Y_0 and X_0 be the closed subpreschemes of Y and X defined by \mathcal{J} , $f^{-1} \mathcal{J}$ respectively. Then $X_0 \cong X \times_Y Y_0$.
- (c) For all $n \geq 0$ we have $f^{-1}(\mathcal{J}^n) \cong (f^{-1} \mathcal{J})^n$.

Sketch of a proof. The question is easily seen to be local on both X and Y . So let's consider the affine situation where $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, and $\mathcal{J} = \tilde{J}$ for some ideal $J \subseteq A$. Let $\varphi: A \rightarrow B$ be the morphism of rings corresponding to f . Then $f^{-1} \mathcal{J} = \tilde{I}$ where I is the image of $B \otimes_A J \rightarrow B$ sending $b \otimes j \mapsto b \cdot \varphi(j)$. All three assertions are then easily checked. \square

Remark 2. Recall that for a morphism $f: X \rightarrow Y$ of preschemes and a point $y \in Y$ the **fibre** $f^{-1}\{y\}$ of f at y is defined as the prescheme $f^{-1}\{y\} = X \times_Y \operatorname{Spec} \hat{\mathcal{R}}(y)$. This makes sense, since $f^{-1}\{y\}$ is indeed – topologically – the preimage of y , as proved in [AG1, Corollary 1.3.3]. Moreover, $\operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n \rightarrow Y$ is immersive for all $n \geq 1$ and has image $\{y\}$ as well. So [AG1, Corollary 1.3.3] is applicable again and shows that $X_n = X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n)$ has $f^{-1}\{y\}$ as underlying topological space too (but, of course, the prescheme structure differs in general). We may thus think of X_n as the $(n-1)^{\text{st}}$ infinitesimal thickening of $f^{-1}\{y\}$.

Using this, Theorem 1 can be restated as follows.

Theorem 1a. *Let $f: X \rightarrow Y$ be a proper morphism between locally noetherian¹ preschemes. Let \mathcal{F} be a coherent \mathcal{O}_X -module. For every $y \in Y$ let $X_n = X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n)$ be the $(n-1)^{\text{st}}$ infinitesimal thickening of $f^{-1}\{y\}$. Then there is an isomorphism*

$$(R^p f_* \mathcal{F})_y^\wedge \xrightarrow{\sim} \varprojlim_{n \geq 1} H^p(X_n, \mathcal{F}|_{X_n}),$$

where $^\wedge$ denotes the $\mathfrak{m}_{Y,y}$ -adic completion.

¹Franke only assumes Y to be locally noetherian, but f being of (locally) finite type implies that X is locally noetherian as well by Hilbert's Basissatz. This happens multiple times throughout the text.

Proof. We may assume that $Y = \operatorname{Spec} A$ is affine, and that A is a noetherian ring. Indeed, replacing Y by an affine neighbourhood $U \cong \operatorname{Spec} A$ and X by $f^{-1}(U)$ doesn't change $(R^p f_* \mathcal{F})_y$ (because the construction of $R^p f_* \mathcal{F}$ is base-local) and also X_n is preserved since $f^{-1}\{y\}$ is already contained in $f^{-1}(U)$ (by [AG2, postnote]).

In this case, $R^p f_* \mathcal{F} = H^p(X, \mathcal{F})^\sim$ by [AG2, Proposition 1.5.1(d)]. Let $\mathfrak{p} \in \operatorname{Spec} A$ be the prime ideal associated to y . Then $R^p f_* \mathcal{F} \cong H^p(X, \mathcal{F})_{\mathfrak{p}}$ and $\mathcal{O}_{Y,y} \cong A_{\mathfrak{p}}$ is flat over A . Let $\mathfrak{m} = \mathfrak{p}A_{\mathfrak{p}} \cong \mathfrak{m}_{Y,y}$ be its maximal ideal. We denote $\pi: \operatorname{Spec} A_{\mathfrak{p}} \rightarrow \operatorname{Spec} A$. Applying [AG2, Fact 4.1.1] to π gives

$$H^p(X \times_Y \operatorname{Spec} A_{\mathfrak{p}}, \pi^* \mathcal{F}) \cong H^p(X, \mathcal{F})_{\mathfrak{p}} \cong (R^p f_* \mathcal{F})_y.$$

Also

$$\begin{aligned} (X \times_Y \operatorname{Spec} A_{\mathfrak{p}}) \times_{\operatorname{Spec} A_{\mathfrak{p}}} \operatorname{Spec}(A_{\mathfrak{p}}/\mathfrak{m}^n) &\cong X \times_Y \left(\operatorname{Spec} A_{\mathfrak{p}} \times_{\operatorname{Spec} A_{\mathfrak{p}}} \operatorname{Spec}(A_{\mathfrak{p}}/\mathfrak{m}^n) \right) \\ &\cong X \times_Y \operatorname{Spec}(A_{\mathfrak{p}}/\mathfrak{m}^n) \\ &\cong X_n \end{aligned}$$

by a bit abstract nonsense. Now Theorem 1 may be applied to $X \times_Y \operatorname{Spec} A_{\mathfrak{p}} \rightarrow \operatorname{Spec} A_{\mathfrak{p}}$ (the base change of f) and the assertion follows. \square

1.3. Application to Zariski's main theorem

1.3.1. A lot of (not necessarily main) theorems by Zariski

Out there in the real world, there are multiple *main theorems* of Zariski around, and usually they're only loosely related. Professor Franke recommends Mumford's *The red book of varieties and schemes* for a discussion of various such version.

Corollary 1. *Let $f: X \rightarrow Y$ be any proper morphism between locally noetherian preschemes and let $d = \sup_{y \in Y} \dim(f^{-1}\{y\})$. If \mathcal{F} is a coherent \mathcal{O}_X -module and $p > d$, then $R^p f_* \mathcal{F} = 0$.*

Proof. Since $R^p f_* \mathcal{F}$ is coherent (this is [AG2, Theorem 5]), $(R^p f_* \mathcal{F})_y$ is a finitely generated $\mathcal{O}_{Y,y}$ -module, hence it vanishes iff its $\mathfrak{m}_{Y,y}$ -adic completion vanishes by Fact A.1.1(b). But $X_n = X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n)$ has underlying space $f^{-1}\{y\}$ (as explained in Remark 1.2.2), hence $H^p(X_n, \mathcal{F}|_{X_n}) = 0$ when $p > d$ by Grothendieck's theorem on cohomological dimension (cf. [AG2, Proposition 1.4.1]). The assertion now follows from Theorem 1a. \square

Definition 1. A morphism $f: X \rightarrow Y$ of finite type is called **quasi-finite at $x \in X$** if x is discrete in its fibre, i.e., if $\{x\}$ is an open and closed subset of $f^{-1}\{y\}$ where $y = f(x)$. We call f **quasi-finite** if it is quasi-finite at every $x \in X$.

The following fact wasn't mentioned in the lecture, but it's *definitely* (in particular, not only *perhaps*) *worthwhile to know!*

Fact 1. *Let $f: X \rightarrow Y$ be a morphism of finite type. Let $x \in X$ be open in its fibre $f^{-1}\{y\}$, where $y = f(x)$. Then f is already quasi-finite at x .*

Proof. Choose an affine open neighbourhood $y \in U \cong \operatorname{Spec} A$. Then $f^{-1}\{y\}$ is contained in $f^{-1}(U)$, so we may w.l.o.g. assume that $Y = \operatorname{Spec} A$ is affine. Put $k = \mathfrak{K}(y)$. Since X may be covered by affine open subsets $\operatorname{Spec} R$, where R is of finite type over A , we may cover the fibre product $f^{-1}\{y\} = X \times_Y \operatorname{Spec} k$ by affine open subsets $\operatorname{Spec}(R \otimes_A k)$, in which $R \otimes_A k$ is a k -algebra of finite type, hence a Jacobson ring. This proves that $f^{-1}\{y\}$ is a Jacobson prescheme as in [AG1, Definition 2.4.2(c)]. But then x is a closed point of the open subset $\{x\} \subseteq f^{-1}\{y\}$, hence also a closed point of $f^{-1}\{y\}$ by [AG1, Fact 2.4.1(c)]. \square

Fact 2. (a) *Any finite morphism is quasi-finite.*

(b) *If k is a field, a morphism $f: X \rightarrow \operatorname{Spec} k$ of finite type is quasi-finite iff it is finite.*

(c) *Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be morphisms of finite type such that g is quasi-finite at $y = f(x)$ for some $x \in X$. Then gf is quasi-finite at x iff f is quasi-finite at x .*

(d) *Base changes of a quasi-finite morphism is quasi-finite again.*

Proof. Maybe that's my bad, but the proof of this is actually annoyingly laborious. We begin with part (a). Let $f: X \rightarrow Y$ be a finite morphism, $x \in X$ and $y = f(x)$. Then the morphism

$$f^{-1}\{y\} = X \times_Y \operatorname{Spec} \mathfrak{K}(y) \longrightarrow \operatorname{Spec} \mathfrak{K}(y)$$

is finite again, as a base changes of finite morphisms are finite again (cf. [AG1, Corollary 1.5.1]). Letting $k = \mathfrak{K}(y)$ this puts us in the situation from (b), so it's sufficient to prove (b).

In the case of (b) we have $f^{-1}\{y\} = X$, so what we need to show is that X carries the discrete topology if f is finite. We know that $X \cong \operatorname{Spec} R$ where R is some finite-dimensional k -algebra (using finiteness of f). For $x \in X$ let \mathfrak{p} be the corresponding prime ideal of R . Then R/\mathfrak{p} is a domain and a finite-dimensional k -vector space, hence a finite field extension of k by Hilbert's Nullstellensatz. This means that \mathfrak{p} is a maximal ideal of R . Consequently, all points of X are closed, so it suffices to show that X has finitely many points. Let $\{x_1, \dots, x_n\}$ be any finite subset of X and $\{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$ the corresponding maximal ideals of R . For every i , we may choose an element $\alpha_i \in \mathfrak{m}_i$ which is not contained in any \mathfrak{m}_j for $j \neq i$ (e.g. by the prime avoidance lemma, cf. [Alg1, Lemma 2.5.1]). Put $\beta_i = \prod_{j \neq i} \alpha_j$ (so that $\beta_i \in \mathfrak{m}_j$ for all $j \neq i$ but $\beta_i \notin \mathfrak{m}_i$). We claim that β_1, \dots, β_n are k -linearly independent. Indeed, if $\lambda_1 \beta_1 + \dots + \lambda_n \beta_n = 0$ for some coefficients $\lambda_1, \dots, \lambda_n \in k$, then reducing modulo \mathfrak{m}_i gives $\lambda_i \beta_i = 0$ in $R/\mathfrak{m}_i = \mathfrak{K}(\mathfrak{m}_i)$. But $\beta_i \neq 0$ in $\mathfrak{K}(\mathfrak{m}_i)$, so $\lambda_i = 0$ for all $i = 1, \dots, n$. This proves $\dim_k R \geq n$. But R is finite-dimensional over k , hence X must have finitely many points, as claimed.

Conversely, assume that $f: X \rightarrow \operatorname{Spec} k$ is quasi-finite. Then X is discrete, so it must have finitely many points. Indeed, f being of finite type implies it is quasi-compact (by definition), so X is quasi-compact because $\operatorname{Spec} k$ is, and any discrete quasi-compact space is finite. Let $X = \{x_1, \dots, x_n\}$. Every point $x_i \in X$ together with the restriction $\mathcal{O}_X|_{\{x_i\}}$ of the structure

sheaf is a prescheme again, hence affine (because $x_i \in \{x_i\}$ must have an affine neighbourhood). Let $\{x_i\} \cong \operatorname{Spec} R_i$. Then

$$X \cong \coprod_{i=1}^n \operatorname{Spec} R_i \cong \operatorname{Spec} \left(\bigoplus_{i=1}^n R_i \right)$$

is affine. This shows that f is affine, but finiteness is yet to prove. Clearly, it suffices that each R_i is a finite-dimensional k -vector space. Note that R_i has precisely one prime ideal \mathfrak{m}_i (corresponding to x_i), which is then automatically maximal. Since f is of finite type, R_i has finite type over k . In particular R_i is noetherian and we may choose generators r_1, \dots, r_m of \mathfrak{m}_s . Since \mathfrak{m}_i is the only prime ideal of R_s , we have $\mathfrak{m}_i = \operatorname{nil} R_i$. Consequently, there is an $N \in \mathbb{N}$ such that $r_\ell^N = 0$ for all ℓ . Moreover, R_i/\mathfrak{m}_i is a field extension of finite type over k , hence a finite field extension by Hilbert's Nullstellensatz. Let $\beta_1, \dots, \beta_d \in R_i$ be elements whose images modulo \mathfrak{m}_i form a k -basis of R_i/\mathfrak{m}_i . Then it is straightforward to check that R is generated as a k -vector space by the elements

$$\beta_j \cdot r_1^{e_1} r_2^{e_2} \cdots r_n^{e_n} \quad \text{where } 0 \leq e_\ell < N \text{ for all } \ell.$$

This shows $\dim_k R < \infty$, hence f is finite.

Part (c). Since g is quasi-finite at y , the subset $\{y\} \subseteq g^{-1}\{g(y)\}$ is open and closed, hence $f^{-1}\{y\} \subseteq (gf)^{-1}\{g(y)\}$ is open and closed. This means that $\{x\}$ is open and closed in the fibre $(gf)^{-1}\{g(y)\}$ iff it is open and closed in $f^{-1}\{y\}$ and we win.

Part (d). Let $f: X \rightarrow S$ be quasi-finite and $g: Y \rightarrow S$ another morphism of preschemes. The base change $\tilde{f}: \tilde{X} = X \times_S Y \rightarrow Y$ is of finite type by [AG1, Fact 2.2.2]. Now let $y \in Y$ and put $s = g(y)$. By (b) we see that $X \times_S \operatorname{Spec} \mathfrak{K}(s)$ is finite over $\operatorname{Spec} \mathfrak{K}(s)$. Then

$$\tilde{X} \times_Y \operatorname{Spec} \mathfrak{K}(y) \cong X \times_S (\operatorname{Spec} \mathfrak{K}(s) \times_S \operatorname{Spec} \mathfrak{K}(y)) \cong (X \times_S \operatorname{Spec} \mathfrak{K}(s)) \times_{\operatorname{Spec} \mathfrak{K}(s)} \operatorname{Spec} \mathfrak{K}(y)$$

is finite over $\operatorname{Spec} \mathfrak{K}(y)$, so \tilde{f} is indeed quasi-finite by (b) again. \square

Theorem 2 (Grothendieck's version of Zariski's main theorem). (a) *Let $f: X \rightarrow Y$ be a quasi-finite proper morphism between locally noetherian preschemes. Then f is finite.*

(b) *Let $f: X \rightarrow Y$ be a quasi-finite and separated morphism between noetherian preschemes. Then there exists a factorization $X \xrightarrow{j} \overline{X} \xrightarrow{g} Y$ of f where j is an open immersion and g is finite.*

(c) *If $f: X \rightarrow Y$ is any morphism of finite type between locally noetherian preschemes, then*

$$U = \{x \in X \mid f \text{ is quasi-finite at } x\}$$

is open in X , and the restriction $f|_U$ is quasi-finite (by definition).

Proof. Part (a). We may assume that $Y = \operatorname{Spec} A$ is affine (indeed, all involved properties are base-local). Let $\mathcal{J} \subseteq \mathcal{O}_X$ be a sheaf of ideals, then \mathcal{J} is coherent as X is locally noetherian. Since f is quasi-finite, all fibres carry the discrete topology. In particular, they are zero-dimensional and Corollary 1 shows that $R^1 f_* \mathcal{J} = 0$. Then also $0 = R^1 f_* \mathcal{J}(Y) = H^1(X, \mathcal{J})$ (using [AG2,

Proposition 1.5.1(d)], hence X is affine by Serre's affinity criterion. This shows that f is affine. Moreover, $f_*\mathcal{O}_X$ is a coherent \mathcal{O}_Y -module by [AG2, Theorem 5], hence f is finite.

Part (b) is hard, see the discussion on page 12. We only prove a special case there, which, however, is sufficient to prove (c). But before we can do this, we need to prove some more theorems of Zariski. \square

Theorem 3 (Zariski's connectedness theorem). *Let $f: X \rightarrow Y$ be a proper morphism between locally noetherian schemes, whose algebraic component $f^*: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is an isomorphism.*

(a) *The fibres $f^{-1}\{y\}$ are connected for all $y \in Y$.*

(b) *The set*

$$U = \{x \in X \mid \{x\} = f^{-1}\{f(x)\}\} = \{x \in X \mid f \text{ is quasi-finite at } x\}$$

is open in X , and the restriction $f|_U$ is quasi-finite (by definition).

Proof. Part (a). Assume $f^{-1}\{y\}$ is not connected, say, $f^{-1}\{y\} = U_1 \cup U_2$ for disjoint non-empty open subsets $U_1, U_2 \subseteq f^{-1}\{y\}$. Since all infinitesimal thickenings $X_n = X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n)$ have underlying topological space $f^{-1}\{y\}$, there is a unique $\varepsilon_n \in \mathcal{O}_{X_n}(X_n) = H^0(X_n, \mathcal{O}_{X_n})$ such that $\varepsilon_n|_{U_1} = 0$ and $\varepsilon_n|_{U_2} = 1$. The sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ clearly defines an element ε of

$$\varprojlim_{n \geq 1} H^0(X_n, \mathcal{O}_{X_n}) \cong (f_*\mathcal{O}_X)_y^\wedge \cong \widehat{\mathcal{O}}_{Y,y}.$$

The left isomorphism here is due to Theorem 1a and the fact that $\mathcal{O}_{X_n} = \mathcal{O}_X|_{X_n}$, and the right one holds by assumption. Hence $\widehat{\mathcal{O}}_{Y,y}$ is a local ring (by Corollary A.1.7) with an idempotent $\varepsilon \neq 0, 1$. Then $1 - \varepsilon \neq 0, 1$ is another non-trivial idempotent. Both ε and $1 - \varepsilon$ can't be units in $\widehat{\mathcal{O}}_{Y,y}$, otherwise $\varepsilon^2 = \varepsilon$ implies $\varepsilon = 1$ (and similar for $1 - \varepsilon$). But then they are elements of the maximal ideal \mathfrak{m} , so $\varepsilon + (1 - \varepsilon) = 1$ is an element of \mathfrak{m} as well, contradiction!

Part (b). By (a), any point $x \in X$ is open and closed in its fibre iff $f^{-1}\{f(x)\} = \{x\}$. Therefore the two definitions of U indeed coincide.

We must show that U is open. This is a local question with respect to Y , hence we may assume that $Y = \operatorname{Spec} A$ is affine. Let $x \in U$ and $V \subseteq X$ an affine open neighbourhood of x . Put $Z = X \setminus V$. Then $Z \subseteq X$ is closed and disjoint from $f^{-1}\{f(x)\} = \{x\}$. As f is proper, $Z' = f(Z) \subseteq Y$ is closed, and $y = f(x) \notin Z'$. There's an $\alpha \in A$ such that $y \notin V(\alpha)$ and $V(\alpha) \supseteq Z'$. Let $Y_1 = Y \setminus V(\alpha)$. Note that $Y_1 \cong \operatorname{Spec} A_\alpha$ is affine and $x \in X_1 = f^{-1}(Y_1) \subseteq V$. Then $X_1 = X \setminus V(f^*\alpha) = V \setminus V(f^*\alpha)$ is affine as well, so the restriction $f|_{X_1}: X_1 \rightarrow Y_1$ of f is affine and proper. But every affine proper morphism is finite (because $f|_{X_1,*}\mathcal{O}_{X_1}$ is a coherent \mathcal{O}_{Y_1} -module by [AG2, Theorem 5]), so $f|_{X_1}$ is, in particular, quasi-finite by Fact 2 and $U \cap X_1 = X_1$. This proves that U is open. \square

Remark 1. On first glance, the argument from Theorem 3(b) might look like it proves that every proper morphism is affine, but what it actually shows is the following: If $f: X \rightarrow Y$ is a proper morphism such that for each $x \in X$ the fibre $f^{-1}\{f(x)\}$ is contained in some affine subset $V \subseteq X$, then f is already affine (and hence finite).

Remark 2. Recall that a prescheme X is called **normal** if it is integral and all local rings $\mathcal{O}_{X,x}$ (which are domains if X is integral) are normal (cf. [AG1, Definition 2.4.5]). This is the case iff $\mathcal{O}_X(U)$ is a normal domain for all affine $U \subseteq X$, cf. the discussion in [AG1, Remark 2.5.1].

Corollary 2 (Zariski's birationality theorem). *Let $f: X \rightarrow Y$ be a proper morphism between locally noetherian preschemes, where Y is normal. Suppose that f is **birational** in the sense that there is a dense open subset $U \subseteq Y$ such that the restriction $f|_{f^{-1}(U)}: f^{-1}(U) \xrightarrow{\sim} U$ is an isomorphism and $f^{-1}(U)$ is dense in X . Then all assertions from Theorem 3 apply to f . In particular, f has connected fibres.*

Proof. First note that U is irreducible as an open subset of the irreducible space Y (irreducibility of Y is implied by Y being normal). Hence X is irreducible because it has the dense irreducible subset $f^{-1}(U) \cong U$. Let $\text{Spec } A \cong V \subseteq Y$ be an affine open subset, where A is a domain. Then $f^{-1}(V)$ is open in X , hence dense in X and thus irreducible. Since U is dense in Y , the intersection $U \cap V$ is non-empty, hence $f^{-1}(U \cap V) \subseteq f^{-1}(V)$ is a non-empty open subset and thereby dense again. This shows that we can actually reduce to the case $Y = \text{Spec } A$ (all the other involved properties are clearly base-local). Moreover, we may assume that X is integral. Indeed, the assertions from Theorem 3 are purely topological, so we may replace X by its reduction $X^{\text{red}} = V(\text{nil}(\mathcal{O}_X))$ to obtain an X which is irreducible and reduced (hence integral) and has the same underlying topological space as the original one.

Claim 1. The ring $B = \mathcal{O}_X(X)$ is a domain in the above situation, and A and B have the same field of quotients K . Moreover, we have $A \subseteq B$ as subrings of K .

Believing this for the moment, the proof can be finished as follows. Since B is finitely generated as an A -module (because $f_*\mathcal{O}_X = \tilde{B}$ is coherent by [AG2, Theorem 5]), it is integral over A . But A is integrally closed in K , hence $A \subseteq B$ implies $A = B$. We conclude $f_*\mathcal{O}_X \cong \mathcal{O}_Y$, as needed.

Unfortunately, the proof of Claim 1 wasn't discussed in the lecture, but I think it should have been. Since X and Y are irreducible, they have unique generic points η_X and η_Y . As η_Y is dense in Y , we have $\eta_Y \in U$ and similarly $\eta_X \in f^{-1}(U)$. Hence $f(\eta_X) = \eta_Y$ and the induced morphism $\mathcal{O}_{Y,\eta_Y} \xrightarrow{\sim} \mathcal{O}_{X,\eta_X}$ is an isomorphism by the birationality assumption. Moreover, η_Y corresponds to $0 \in \text{Spec } A$, hence $\mathcal{O}_{Y,\eta_Y} \cong K$ is the quotient field of A . So we should prove that \mathcal{O}_{X,η_X} is the quotient field of $B = \mathcal{O}_X(X)$ as well.

It's clear that B is a domain because X is integral (cf. [AG1, Proposition 2.1.4(b)]). Since $U \subseteq \text{Spec } A$ is open, we find an affine open subset $V = \text{Spec } A \setminus V(\alpha) \subseteq U$. Then

$$f^{-1}(V) = X \setminus V(f^*\alpha) = f^{-1}(U) \setminus V(f^*\alpha) \cong V$$

is affine again by birationality of f . We know that X is quasi-compact and separated since so are f and $\text{Spec } A$. In particular, [AG1, Proposition 1.5.1(c)] is applicable to \mathcal{O}_X and gives

$$\mathcal{O}_X(f^{-1}(V)) = \mathcal{O}_X(X \setminus V(f^*\alpha)) \cong \mathcal{O}_X(X)_{f^*\alpha},$$

so B and $\mathcal{O}_X(f^{-1}(V))$ have the same quotient field. But $\mathcal{O}_X(f^{-1}(V)) \cong \mathcal{O}_Y(V) \cong A_\alpha$ has quotient field K , so we win.

The fact that $A \subseteq B$ as subrings of K follows from the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \mathcal{O}_{Y,\eta_Y} & \xrightarrow{\sim} & \mathcal{O}_{X,\eta_X} \end{array}$$

in which every arrow except the top one is injective, hence $A \rightarrow B$ is injective as well. \square

Fact 3. *Every proper morphism $f: X \rightarrow Y$ between locally noetherian preschemes can be factorized via the **Stein factorization** as*

$$f: X \xrightarrow{\tilde{f}} \mathbf{Spec}(f_*\mathcal{O}_X) \xrightarrow{g} Y .$$

In this composition, g is finite and the assumptions of Theorem 3 hold for \tilde{f} .

Sketch of a proof. It's pretty obvious that this factorization exists (to construct \tilde{f} , use the adjunction from [AG2, Proposition 1.6.2(b)]). To show that \tilde{f} and g have the required properties, we look at things locally and assume that $Y = \mathbf{Spec} A$ is affine (and A is noetherian). Then the factorization looks like

$$f: X \xrightarrow{\tilde{f}} \mathbf{Spec} \mathcal{O}_X(X) \xrightarrow{g} \mathbf{Spec} A ,$$

so g is affine. Moreover, $\mathcal{O}_X(X)$ is a finitely generated A -module because $f_*\mathcal{O}_X$ is a coherent \mathcal{O}_Y -module (by [AG2, Theorem 5], as usual), so g is actually finite. Also proving that $\tilde{f}_*\mathcal{O}_X = \mathcal{O}_{\mathbf{Spec} \mathcal{O}_X(X)}$ is straightforward, so it remains to show that \tilde{f} is proper. But g is finite, hence separated, and $g\tilde{f} = f$ is proper, so \tilde{f} is proper as well by [AG2, Proposition 2.4.1]. \square

1.3.2. Proof of Zariski's main theorem

Proof of Theorem 2(c). Let's assume that $f: X \rightarrow Y$ factors over

$$f: X \xrightarrow{j} \overline{X} \xrightarrow{\overline{f}} Y , \tag{1}$$

where j is an open immersion and \overline{f} is proper. Then

$$\{x \in X \mid f \text{ is quasi-finite at } x\} = X \cap \left\{x \in \overline{X} \mid \overline{f} \text{ is quasi-finite at } x\right\} . \tag{2}$$

Indeed, a point $x \in X$ is open in $\overline{f}^{-1}\{y\}$ (where $y = f(x)$) iff it is open in the open subset $f^{-1}\{y\} = X \cap \overline{f}^{-1}\{y\} \subseteq \overline{f}^{-1}\{y\}$. In view of Fact 1 this shows (2). We thus have reduced (c) (under the assumption that \overline{f} exists) to the case of proper morphisms.

If $f: X \rightarrow Y$ is proper, then consider its Stein factorization. Since g is finite, it's quasi-finite as well by Fact 2(a). So Fact 2(c) shows that

$$\{x \in X \mid f \text{ is quasi-finite at } x\} = \{x \in X \mid \tilde{f} \text{ is quasi-finite at } x\} .$$

But the right-hand side is open in X by Fact 3 and Theorem 3(b) and we're happy!

Note that such an \bar{f} always exists when X and Y are affine. Indeed, if X has finite type over Y and both are affine, we get a closed embedding $X \hookrightarrow \mathbb{A}_Y^n$ for some $n \in \mathbb{N}$. Together with the open embedding $\mathbb{A}_Y^n \hookrightarrow \mathbb{P}_Y^n$ this makes X a closed subprescheme of an open subprescheme of \mathbb{P}_Y^n . But then X is also an open subprescheme of some closed subprescheme $\bar{X} \subseteq \mathbb{P}_Y^n$. This gives a factorization

$$f: X \xrightarrow{j} \bar{X} \xrightarrow{\bar{f}} Y$$

in which $\bar{f}: \bar{X} \hookrightarrow \mathbb{P}_Y^n \rightarrow Y$ is (strongly) projective, hence proper by [AG2, Proposition 2.4.2]. But (c) is completely local on both X and Y (thanks to Fact 1), so by checking the affine case we have actually covered all of (c). \square

We will now sketch a proof of Theorem 2(b), which occupies the rest of the subsection. The proof is split into three parts. In Part I, we will introduce the notion of quasi-affine morphisms and reduce Zariski's main theorem to the question whether quasi-finite separated morphisms are quasi-affine. In Part II, we show that Zariski's main theorem holds when Y is the spectrum of a complete noetherian local ring. Finally, Part III shows how the general case can be reduced to Part II.

So much for the battle plan, now let's get into action!

Part I – proving quasi-affinity is sufficient. We begin with the observation that things become much easier when we are proper.

Lemma 1. *Let $f: X \rightarrow Y$ be a quasi-finite separated morphism between noetherian schemes, and assume that f factors as in (1). Then Zariski's main theorem holds for f .*

Remark 3. It can be shown that such a factorization (1) always exists for morphisms of finite type between noetherian schemes, for which Professor Franke refers to notes of *Brian Conrad* or *Paul Vojta*, although he isn't sure whether using their results to prove Theorem 2(b) doesn't involve any circular reasoning.

Proof of Lemma 1. Let $\bar{f} = g \circ \tilde{f}$ be the Stein factorization of \bar{f} . Put $\bar{Y} = \mathbf{Spec}(\bar{f}_* \mathcal{O}_{\bar{X}})$ for convenience. Since $g: \bar{Y} \rightarrow Y$ is finite by Fact 3 (for which we need properness of \bar{f}), we're done if we show that the composition $\tilde{f}|_X: X \hookrightarrow \bar{X} \rightarrow \bar{Y}$ is an open embedding. Since $f = g\tilde{f}|_X$ and g are quasi-finite (g is even finite), Fact 2(c) shows that $\tilde{f}|_X$ is quasi-finite as well.

In particular $\tilde{f}|_X$ is injective, and for all $x \in X$ we have $\tilde{f}^{-1}\{\tilde{f}(x)\} = \{x\}$, because $\mathcal{O}_{\bar{Y}} \cong \tilde{f}_* \mathcal{O}_{\bar{X}}$ (by Fact 3), so the argument from Theorem 3(b) can be applied. If V is any open neighbourhood of $x \in X$, then $\tilde{f}(\bar{X} \setminus V) \subseteq \bar{Y}$ is closed because \tilde{f} is proper (by Fact 3), hence closed. Moreover, $\tilde{f}(\bar{X} \setminus V)$ doesn't contain $\tilde{f}(x)$ as $\tilde{f}^{-1}\{\tilde{f}(x)\} = \{x\}$. Thus, the complement U of $\tilde{f}(\bar{X} \setminus V)$ in \bar{Y} is open and $\tilde{f}^{-1}(U) \subseteq V$ is an open neighbourhood of x . This shows that $\tilde{f}|_X$ is actually an open map! Together with $\mathcal{O}_{\bar{Y}} \cong \tilde{f}_* \mathcal{O}_{\bar{X}}$ we see that $\tilde{f}|_X$ is an open embedding, as claimed. \square

Definition 2. A morphism $f: X \rightarrow Y$ of preschemes is called **quasi-affine**, if it satisfies the following equivalent conditions:

- (a) For all affine open $U \subseteq Y$, $f^{-1}(U)$ is a quasi-affine scheme (that is, a quasi-compact open subscheme of an affine scheme).
- (b) Y can be covered by affine open U such that $f^{-1}(U)$ is quasi-affine.
- (c) f factors as $X \xrightarrow{j} \overline{X} \xrightarrow{\overline{f}} Y$, where j is an open embedding and \overline{f} is affine.
- (d) The canonical morphism $X \rightarrow \mathbf{Spec}(f_*\mathcal{O}_X)$ in the Stein factorization is an open immersion.

For a proof of equivalence, check out [Stacks, Tag 01SJ]. Note that Franke apparently had a different proof (for the case of quasi-affine morphisms of finite type) in mind than the Stacks Project guys, but I have no idea how this was going to work (whereas the Stacks Project proof is pretty clear to me and works without restrictions on f). Nevertheless, the following lemma he suggested is perhaps worthwhile to know.

Lemma 2. *Let X be a noetherian prescheme, \mathcal{M} a quasi-coherent \mathcal{O}_X -module, and $U \subseteq X$ an open subset. Let $\mathcal{N} \subseteq \mathcal{M}|_U$ be a coherent submodule. Then there is a coherent $\tilde{\mathcal{N}} \subseteq \mathcal{M}$ such that $\tilde{\mathcal{N}}|_U = \mathcal{N}$. In particular, \mathcal{M} is the union of its coherent submodules.*

Proof. We proceed by noetherian induction. Because Y is noetherian, there is a \subseteq -maximal open subset U to which \mathcal{N} extends (and, with slight abuse of notation, we denote some fixed extension to U by \mathcal{N} as well). Assuming $U \neq X$, we will derive a contradiction. Pick $x \in X \setminus U$ and let $V \cong \mathrm{Spec} A$ be an affine open neighbourhood of x .

Claim. There is a coherent $\mathcal{Q} \subseteq \mathcal{M}|_V$ such that $\mathcal{Q}|_{U \cap V} \supseteq \mathcal{N}|_{U \cap V}$.

Indeed, $V \cong \mathrm{Spec} A$ is a noetherian topological space (because Y is noetherian), so the open subset $U \cap V$ is quasi-compact. This means that we can cover it by finitely many affine open subsets $V_i \cong \mathrm{Spec} A_{f_i}$ for $i = 1, \dots, n$. For every i we know that $\mathcal{N}(V_i) \subseteq \mathcal{M}(V_i) \cong \mathcal{M}(V)_{f_i}$ is a finitely generated A_{f_i} -module (because \mathcal{N} is coherent), so we can choose finitely many $\mu_{i,j} \in \mathcal{M}(V)$, $j = 1, \dots, k_i$, whose images in $\mathcal{N}(V_i)$ form a set of generators. Let $\mathcal{Q} \subseteq \mathcal{M}|_V$ be the subsheaf generated by $\{\mu_{i,j} \mid i \leq n, j \leq k_i\} \in \mathcal{M}(V)$. Then \mathcal{Q} has the required property, proving the claim.

Let $j: U \cap V \hookrightarrow V$ be the obvious inclusion and put $\overline{\mathcal{N}} = \mathcal{Q} \cap j_*\mathcal{N}|_{U \cap V}$. This guy is coherent (because subsheaves of a coherent one are coherent again on locally noetherian preschemes) and satisfies $\overline{\mathcal{N}}|_{U \cap V} = \mathcal{N}|_{U \cap V}$. But then \mathcal{N} can be extended to a coherent sheaf $\tilde{\mathcal{N}}$ on $U \cup V$ via $\tilde{\mathcal{N}}|_U = \mathcal{N}$ and $\tilde{\mathcal{N}}|_V = \overline{\mathcal{N}}$. This contradicts maximality of U . \square

Lemma 3. *If $f: X \rightarrow Y$ is a quasi-finite and quasi-affine morphism between noetherian preschemes, then Theorem 2(b) holds for f .*

Proof. Note that quasi-affine morphisms are automatically separated. We will show that $g: \mathbf{Spec}(f_*\mathcal{O}_X) \rightarrow Y$ is proper, which shows that f factors as in (1). Then Theorem 2(b) holds by Lemma 1. To show properness of g , we define

$$\mathcal{R} = \bigoplus_{n \geq 0} f_*\mathcal{O}_X^n,$$

viewed as a graded quasi-coherent \mathcal{O}_Y -algebra. Note that $\mathbf{Spec}(f_*\mathcal{O}_X) \cong \mathbf{Proj}(\mathcal{R})$ as Y -preschemes (indeed, locally this reduces to [AG1, Example 2.6.2]). Moreover, \mathcal{R} is locally of finite type over \mathcal{O}_Y . In fact, the 0th homogeneous component $\mathcal{R}_0 \cong f_*\mathcal{O}_X$ is locally of finite type over \mathcal{O}_Y (because f is of finite type), and \mathcal{R} is generated by \mathcal{R}_0 and the element $1 \in \mathcal{R}_1(Y)$ (which acts as a shift). Therefore, $\mathbf{Proj}(\mathcal{R}) \rightarrow Y$ is proper by [AG2, Proposition 2.4.2], whence we're done. \square

So we see that to prove Zariski's main theorem it is sufficient to show that any quasi-finite and separated morphism between noetherian preschemes is quasi-affine (note that this is actually *weaker* than Theorem 2(b)).

Part II – the case of complete noetherian local rings. In this part Professor Franke follows [SGA1, Exposé VIII.6]. Assume $Y = \mathbf{Spec} A$, where A is a noetherian complete local ring (with respect to its maximal ideal \mathfrak{m}) and that Zariski's main theorem is true for morphisms $f': X' \rightarrow Y'$, where in addition to the other assumptions we have $\dim Y' < \dim Y$. Let s be the unique closed point of A (given by \mathfrak{m}).

Lemma 4. *If B is any finite A -algebra, then $\mathbf{Spec} B$ has a decomposition $\mathbf{Spec} B = \bigcup_{i=1}^n U_i$, where the U_i are disjoint open subsets such that the only closed point s of $\mathbf{Spec} A$ has precisely one preimage in each U_i .*

Proof. By Corollary A.1.9, B has finitely many maximal ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ and these are precisely the prime ideals over \mathfrak{m} . For every \mathfrak{q}_i we will construct an idempotent $e_i \in B$ such that $e_i \in \mathfrak{q}_i$ but $e_i \notin \mathfrak{q}_j$ for $j \neq i$. Then $\mathfrak{q}_i \in V(e_i)$ but $\mathfrak{q}_j \notin V(e_i)$ for $j \neq i$, and $V(e_i)$ is an open and closed subset of B (because e_i is an idempotent), so we see that the \mathfrak{q}_i are contained in distinct connected components of $\mathbf{Spec} B$. But B is noetherian, so $\mathbf{Spec} B$ has finitely many connected components (as pointed out in [AG1, Lemma 2.4.2]). Then every connected component is open and we can construct the required U_i as unions of suitable connected components of $\mathbf{Spec} B$.

Let $\bar{\mathfrak{q}}_i = \mathfrak{q}_i/\mathfrak{m}$ be the prime ideals of $B/\mathfrak{m}B$ (which are automatically maximal). Then the intersection $\bigcap_{j=1}^n \bar{\mathfrak{q}}_j$ is the nilradical $\mathrm{nil}(B/\mathfrak{m}B)$, so by the Chinese remainder theorem we find an element $\bar{e}_i \in (B/\mathfrak{m}B)/\mathrm{nil}(B/\mathfrak{m}B)$ such that $\bar{e}_i \equiv 0 \pmod{\bar{\mathfrak{q}}_i}$ and $\bar{e}_j \equiv 1 \pmod{\bar{\mathfrak{q}}_j}$ for $j \neq i$. Note that $\bar{e}_i^2 = \bar{e}_i$. Since $B/\mathfrak{m}B$ is noetherian, there is an $N \in \mathbb{N}$ such that $\mathrm{nil}(B/\mathfrak{m}B)^N = 0$. Hence $B/\mathfrak{m}B$ is $\mathrm{nil}(B/\mathfrak{m}B)$ -adically complete, which means we can lift \bar{e}_i to an idempotent $\tilde{e}_i \in B/\mathfrak{m}B$ by Hensel's lemma (cf. Proposition A.1.1). Now B is $\mathfrak{m}B$ -adically complete by Proposition A.1.4, so using Hensel's lemma once again we can lift \tilde{e}_i to an idempotent $e_i \in B$ with the required properties. We're done. \square

Let $f: X \rightarrow Y = \mathbf{Spec} A$ be quasi-finite and separated, and let $x \in f^{-1}\{s\}$. Because the fibre $f^{-1}\{s\}$ is discrete, there is an affine open $U \subseteq X$ such that $U \cap f^{-1}\{s\} = \{x\}$. Zariski's main theorem applies to the affine morphism $f|_U: U \rightarrow \mathbf{Spec} A$, which therefore factors as

$$U \xhookrightarrow{j} \mathbf{Spec} B \longrightarrow Y$$

where B is as above and j is an open embedding. By Lemma 4 (and shrinking U if necessary) we may assume that B has only one closed point $j(x)$, i.e. is local itself. But then the only open

subset of $\operatorname{Spec} B$ containing the closed point $j(x)$ is $\operatorname{Spec} B$ itself, hence j is an isomorphism. We have thus found an affine open neighbourhood $U \cong \operatorname{Spec} B$ of x which is finite over Y . Then $U \rightarrow X$ is proper (as $U \rightarrow Y$ is finite, hence proper, and f is separated, so [AG2, Proposition 2.4.1] applies), so U is also closed. Putting $U = U_1$ and $X \setminus U = X_1$, what we proved is that $X = X_1 \amalg U_1$ is the disjoint union of its open subsets X_1 and U_1 .

Iterating this for the remaining preimages² of s in X provides a decomposition

$$X = X' \amalg \coprod_{i=1}^n U_i$$

into disjoint open subsets, where $\coprod_{i=1}^n U_i \cong \coprod_{i=1}^n \operatorname{Spec} \mathcal{O}_X(U_i) \cong \operatorname{Spec} (\oplus_{i=1}^n \mathcal{O}_X(U_i))$ is finite over Y and $f' = f|_{X'}: X' \rightarrow Y$ has image in $Y' = Y \setminus \{s\}$, hence the induction assumption applies to $f': X' \rightarrow Y'$. This means that f' can be written as $f': X' \hookrightarrow \overline{X'} \rightarrow Y'$, where X' is an open subscheme of $\overline{X'}$, which in turn is finite over Y' . Still we aren't done yet, as we need something finite over Y rather than Y' .

Lemma 5. *Let A be a noetherian ring and $S \subseteq \operatorname{Spec} A$ an open subscheme. If $f: X \rightarrow S$ is a finite morphism, then there is a finite A -algebra B such that X is an open subscheme of $\operatorname{Spec} B$ and the diagram*

$$\begin{array}{ccc} \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A \\ \uparrow & & \uparrow \\ X & \xrightarrow{f} & S \end{array}$$

commutes.

Proof. By Lemma 3 it is enough to show that $X \rightarrow \operatorname{Spec} A$ is quasi-affine (which removes the condition that B is finite over A). The Stein factorization of $X \rightarrow \operatorname{Spec} A$ reads

$$X \longrightarrow \operatorname{Spec} \mathcal{O}_X(X) \longrightarrow \operatorname{Spec} A ,$$

so it is enough to show that the canonical morphism $X \rightarrow \operatorname{Spec} \mathcal{O}_X(X)$ is an open embedding.

Since $\operatorname{Spec} A$ is a noetherian space, its open subset S is quasi-compact. We thus find a finite cover $S = \bigcup_{i=1}^n S_i$, where $S_i \cong \operatorname{Spec} A_{\alpha_i}$ (for some $\alpha_i \in A$) are affine open subschemes. Putting $X_i = f^{-1}(S_i)$ we get an affine open cover $X = \bigcup_{i=1}^n X_i$ of X . Note that X is quasi-compact because S and f are. Moreover, S is separated as an open subscheme of a separated prescheme, hence X is separated as well, as finite morphisms are separated. If we denote the image of α_i under the restriction $A \rightarrow \mathcal{O}_S(S)$ by α_i as well, we see that $S_i = S \setminus V(\alpha_i)$, so $A_{\alpha_i} \cong \mathcal{O}_S(S \setminus V(\alpha_i)) \cong \mathcal{O}_S(S)_{\alpha_i}$. This works because S is quasi-compact and separated, so [AG1, Proposition 1.5.1(c)] applies. By the same argument applied to X , we get $\mathcal{O}_X(X_i) = \mathcal{O}_X(X \setminus V(f^*\alpha_i)) \cong \mathcal{O}_X(X)_{f^*\alpha_i}$, so $X_i \cong \operatorname{Spec} \mathcal{O}_X(X)_{f^*\alpha_i}$ because the X_i are affine.

²There are finitely many of them. Indeed, $f^{-1}\{s\} \subseteq X$ is closed (as the preimage of the closed point s), hence quasi-compact (because X is noetherian, hence quasi-compact). But the fibre $f^{-1}\{s\}$ is supposed to be discrete, so it must be finite.

Now $X = \bigcup_{i=1}^n X_i \cong \bigcup_{i=1}^n \operatorname{Spec} \mathcal{O}_X(X)_{f^*\alpha_i}$ is a union of open subschemes of $\operatorname{Spec} \mathcal{O}_X(X)$, hence an open subscheme itself, and we are done. \square

This finishes the case of complete noetherian local rings.

Part III – reduction to complete noetherian local rings. We will show that Zariski's main theorem is valid for (quasi-finite separated) $f: X \rightarrow Y$ with target Y (which is locally noetherian) if for all $y \in Y$ it is valid with $\operatorname{Spec} \widehat{\mathcal{O}}_{Y,y}$ as target. But before that, some technical preparations need to be done.

Professor Franke intended to use Chevalley's theorem (Proposition A.2.2) to prove Lemma 6, but I don't understand how that's supposed to work. The proof I came up with doesn't need Chevalley's theorem, but it's surely worthwhile to know, so I put it in the appendix (page 39).

Lemma 6. *Let $f: X \rightarrow Y$ be a morphism of locally finite type between locally noetherian preschemes such that f induces an injective map of topological spaces and isomorphisms $f^*: \mathcal{O}_{Y,f(x)} \xrightarrow{\sim} \mathcal{O}_{X,x}$ on stalks. Then f is an open embedding.*

Proof. It is sufficient to show that X carries the induced topology and is open in Y . All we need to do for this is to prove that f is an open morphism. This will be done by several reduction steps until we arrive at a nice enough situation.

The first of these reductions is that f being an open morphism is a local question both on source and target, so we may assume $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ to be affine, where A and B are noetherian rings with B of finite type over A . Let Z_1, \dots, Z_n be the irreducible components of Y and let $\mathfrak{p}_i \subseteq A$ be the prime ideals such that $Z_i = V(\mathfrak{p}_i)$. Note that a subset $V \subseteq Y$ is open iff its intersections $V \cap Z_i$ are all open (indeed, in this case $Y \setminus V$ is closed as a finite union of the closed subsets $Z_i \setminus V$). Therefore, it suffices to show that every $x \in X$ has an open neighbourhood U_i such that for all open neighbourhoods $U \subseteq U_i$ of x the intersection $f(U) \cap Z_i$ is open (because then $f(U) \subseteq X$ is an open neighbourhood of $f(x)$ for $U = U_1 \cap \dots \cap U_n$). For those irreducible components Z_i such that $x \notin Z_i$ this is easy – just take $U_i = f^{-1}(Y \setminus Z_i)$. Now suppose $x \in Z_i$. Since $f(U) \cap Z_i = f(U \cap f^{-1}(Z_i))$ it suffices to find such a neighbourhood U_i of x in $f^{-1}(Z_i)$ rather than in X . That is, we may replace A by A/\mathfrak{p}_i and B by $B/\mathfrak{p}_i B$ (clearly we still get isomorphisms on stalks), so we may henceforth assume that A is a domain.

Note that $X = \operatorname{Spec} B$ has finitely many connected components (by [AG1, Lemma 2.4.2]). Let C be one of them. Then C is open in X , connected, and the local rings $\mathcal{O}_{X,x} \cong \mathcal{O}_{Y,f(x)}$ are domains for all $x \in X$ (the $\mathcal{O}_{Y,y}$ are domains for all $y \in Y$ because A is a domain), so C is an integral prescheme by [AG1, Proposition 2.1.4(d)]. If we restrict to an affine open subset of C (we can do this because everything is local on X) we have reduced the situation to the case where B is a domain as well.

Note that $A \rightarrow B$ is injective. Indeed, suppose that $a \in A$ is contained in the kernel. Choose any prime ideal $\mathfrak{q} \in \operatorname{Spec} B$ and let $\mathfrak{p} \in \operatorname{Spec} A$ be its preimage in A . Then a is contained in the kernel of $A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}$ as well, so $a = 0$ in $A_{\mathfrak{p}}$ because f induces isomorphisms on stalks. But $A \rightarrow A_{\mathfrak{p}}$ is injective when A is a domain, proving that $a = 0$ holds in A as well. Thus, we may regard A as a subring of B . Note that both have the same quotient field K because f induces

an isomorphism between the stalks at the generic points. Now it's time to remember that f is of locally finite type, so B has finite type over A . If $x_1, \dots, x_m \in B$ are generators over A , then we can write $x_i = \frac{a_i}{s_i}$ for $a_i, s_i \in A$, because the x_i are elements of K . Putting $s = s_1 \cdots s_m$ we deduce that $B = A[s^{-1}]$ is the localization of A at s . Then $\text{Spec } B \rightarrow \text{Spec } A$ is an open embedding, which means we're done, finally. \square

Proof of Theorem 2(b) (FINALLY). Now let $f: X \rightarrow Y$ be a quasi-finite separated morphism between noetherian preschemes such that Zariski's main theorem holds with Y replaced by $\text{Spec } \hat{\mathcal{O}}_{Y,y}$. Note that $\pi_Y: \text{Spec } \hat{\mathcal{O}}_{Y,y} \rightarrow Y$ is flat (in the sense of the upcoming Definition 1.4.1) because of Corollary A.1.3(b). Therefore, in the pullback diagram

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\hat{f}} & \text{Spec } \hat{\mathcal{O}}_{Y,y} \\ \downarrow & \lrcorner & \downarrow \pi_Y \\ X & \xrightarrow{f} & Y \end{array} \quad (3)$$

we obtain

$$(f_* \mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} \hat{\mathcal{O}}_{Y,y} \cong \hat{f}_* \mathcal{O}_{\hat{X}}(\text{Spec } \hat{\mathcal{O}}_{Y,y}). \quad (4)$$

Indeed, this is just the base change morphism (1.4.3) from Proposition 1.4.1, applied to $\mathcal{F} = \mathcal{O}_X$ and $p = 0$ – and also we've taken global sections and we used that $(f_* \mathcal{O}_X)(U) \otimes_{\mathcal{O}_Y(U)} \hat{\mathcal{O}}_{Y,y} \cong (f_* \mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} \hat{\mathcal{O}}_{Y,y}$ when U is an affine open neighbourhood of y in Y .

Note that the local rings $\hat{\mathcal{O}}_{Y,y}$ and $\mathcal{O}_{Y,y}$ have the same residue field $\mathfrak{K}(y)$. Hence $\text{Spec } \mathfrak{K}(y) \rightarrow Y$ can be factored through $\pi_Y: \text{Spec } \hat{\mathcal{O}}_{Y,y} \rightarrow Y$ and together with the Stein factorizations of f and \hat{f} this gives a diagram

$$\begin{array}{ccccccc} X \times_Y \text{Spec } \mathfrak{K}(y) & \longrightarrow & \text{Spec } ((f_* \mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} \mathfrak{K}(y)) & \longrightarrow & \text{Spec } \mathfrak{K}(y) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \hat{X} & \xrightarrow{\hat{f}} & \text{Spec } (\hat{f}_* \mathcal{O}_{\hat{X}}) & \xrightarrow{\hat{g}} & \text{Spec } \hat{\mathcal{O}}_{Y,y} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \pi_Y \\ X & \xrightarrow{f} & \text{Spec } (f_* \mathcal{O}_X) & \longrightarrow & Y \end{array} \quad (5)$$

Claim 2. In the diagram (5) every subrectangle (not only those indicated) is a pullback.

Indeed, it is clear that the whole diagram as well as its bottom half (both can be viewed as subrectangles) are pullbacks, since that's how $X \times_Y \text{Spec } \mathfrak{K}(y)$ and \hat{X} are defined. Moreover, we have the relation $\text{Spec } (\pi_Y^*(f_* \mathcal{O}_X)) \cong \text{Spec } (f_* \mathcal{O}_X) \times_Y \text{Spec } \hat{\mathcal{O}}_{Y,y}$ by [AG2, Corollary 1.6.2], which together with (4) implies $\text{Spec } (\hat{f}_* \mathcal{O}_{\hat{X}}) \cong \text{Spec } (f_* \mathcal{O}_X) \times_Y \text{Spec } \hat{\mathcal{O}}_{Y,y}$. This shows that the bottom right rectangle in (5) is a pullback. By a similar argument the right half rectangle is a pullback as well. Now it's a matter of abstract nonsense to show that the remaining subrectangles are pullbacks too.

Because quasi-finiteness and separatedness are preserved under base change, \hat{f} satisfies all assumptions from Theorem 2(b) and has target $\mathrm{Spec} \hat{\mathcal{O}}_{Y,y}$. Hence Zariski's main theorem may be applied to \hat{f} . In particular, this means that \hat{f} is quasi-affine, so $\hat{j}: \hat{X} \rightarrow \mathbf{Spec}(\hat{f}_* \mathcal{O}_{\hat{X}})$ is an open embedding by Definition 2(d).

Note that Claim 2 implies that the fibres $f^{-1}\{y\}$ and $\hat{f}^{-1}\{y\}$ coincide, as do $g^{-1}\{y\}$ and $\hat{g}^{-1}\{y\}$. Moreover we know $\hat{j}: \hat{X} \rightarrow \mathbf{Spec}(\hat{f}_* \mathcal{O}_{\hat{X}})$ is injective on the fibre $\hat{f}^{-1}\{y\}$ (in fact, it is injective everywhere because we just showed it's an open embedding). By diagram (5) we conclude that

$$f^{-1}\{y\} = \hat{f}^{-1}\{y\} = X \times_Y \mathrm{Spec} \mathfrak{K}(y) \longrightarrow \mathrm{Spec} \left((f_* \mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} \mathfrak{K}(y) \right) = \hat{g}^{-1}\{y\} = g^{-1}\{y\}$$

is injective on points. But this means that $j: X \rightarrow \mathbf{Spec}(f_* \mathcal{O}_X)$ is injective on $f^{-1}\{y\}$ – and since y was chosen arbitrarily, we see that j is actually injective on points! So the only thing left to do is to show that j induces isomorphisms on stalks, then Lemma 6 does the rest.

Claim 3. Let $x \in X$ be such that $y = f(x)$. Then x has a unique preimage $\hat{x} \in \hat{X}$ and the local ring $\mathcal{O}_{\hat{X},\hat{x}}$ is given by $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \hat{\mathcal{O}}_{Y,y}$.

The easiest way to see that \hat{x} is unique is probably that the fibres $f^{-1}\{y\}$ and $\hat{f}^{-1}\{y\}$ coincide, but it also follows from the fact that $\mathfrak{m}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \hat{\mathcal{O}}_{Y,y}$ is a maximal ideal of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \hat{\mathcal{O}}_{Y,y}$ (by Proposition A.1.5(a)) and our explicit description of fibre products in [AG1, p. 24]. By Proposition A.1.5(b), the ring $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \hat{\mathcal{O}}_{Y,y}$ is already local, hence it is isomorphic to $\mathcal{O}_{\hat{X},\hat{x}}$ (again by the explicit construction of fibre products).

In the same way we see that $\mathcal{O}_{\mathbf{Spec}(\hat{f}_* \mathcal{O}_{\hat{X}}), \hat{j}(\hat{x})} \cong \mathcal{O}_{\mathbf{Spec}(f_* \mathcal{O}_X), j(x)} \otimes_{\mathcal{O}_{Y,y}} \hat{\mathcal{O}}_{Y,y}$. This means that the morphism on stalks $j^*: \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{\mathbf{Spec}(f_* \mathcal{O}_X), j(x)}$ becomes an isomorphism after tensoring with $\hat{\mathcal{O}}_{Y,y}$, because \hat{j} is an open embedding and induces isomorphisms on stalks. But the completion $\hat{\mathcal{O}}_{Y,y}$ is faithfully flat over $\mathcal{O}_{Y,y}$ by Corollary A.1.3(c), hence j already induces isomorphisms on stalks.

By Lemma 6, $j: X \rightarrow \mathbf{Spec}(f_* \mathcal{O}_X)$ is an open embedding, hence f is quasi-affine by Definition 2(d). Then we're done by Lemma 1. \square

1.4. Flat morphisms

Definition 1. Let $f: X \rightarrow Y$ be a morphism of preschemes. A quasi-coherent \mathcal{O}_X -module \mathcal{F} is called **flat** over \mathcal{O}_Y iff it has the following equivalent properties.

- (a) If $U \subseteq X$ and $V \subseteq Y$ are affine opens such that $f(U) \subseteq V$, then $\mathcal{F}(U)$ is flat as an $\mathcal{O}_Y(V)$ -module.
- (b) It is possible to cover X by affine open subsets for which such V may be found.
- (c) For any $x \in X$ the stalk \mathcal{F}_x is a flat $\mathcal{O}_{Y,y}$ -module, where $y = f(x)$.

We call f a **flat morphism** if \mathcal{O}_X is flat over \mathcal{O}_Y , and **faithfully flat** if f is flat and surjective.

Sketch of a proof of equivalence. The implication $(a) \Rightarrow (b)$ is trivial. If you think about it, showing $(b) \Rightarrow (c) \Rightarrow (a)$ comes down to the following fact from commutative algebra: Let B be an A -algebra and F a B -module. Then F is flat over A and only if $F_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$ algebra for all primes $\mathfrak{p} \in \operatorname{Spec} A$ and $\mathfrak{q} \in \operatorname{Spec} B$ above \mathfrak{p} .

Indeed, if F is flat over A , then so is $F_{\mathfrak{q}}$ (by exactness of localization), hence $F_{\mathfrak{q}}$ is also flat over $A_{\mathfrak{p}}$. Conversely, if $F_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$, then also over A because $A_{\mathfrak{p}}$ is flat over A (exactness of localization again). Now let $M' \rightarrow M \rightarrow M''$ be an exact sequence of A -modules. Then the sequence of B -modules $M' \otimes_A F \rightarrow M \otimes_A F \rightarrow M'' \otimes_A F$ is exact iff its localization at any prime $\mathfrak{q} \in \operatorname{Spec} B$ is exact. But $(M \otimes_A F)_{\mathfrak{q}} \cong M \otimes_A F_{\mathfrak{q}}$ (and same for M', M''), so the localizations are indeed exact by flatness of the $F_{\mathfrak{q}}$, proving that F is flat itself. \square

1.4.1. Flat base change and cohomology

Let A be a ring, B an A -algebra and $f: X \rightarrow \operatorname{Spec} A$ a quasi-compact and separated morphism. Let

$$\tilde{f}: \tilde{X} = X \times_{\operatorname{Spec} A} \operatorname{Spec} B \longrightarrow \operatorname{Spec} B$$

its base change along $\operatorname{Spec} B \rightarrow \operatorname{Spec} A$. Also let $\pi_X: \tilde{X} \rightarrow X$ be projection to the other fibre product factor. We want to investigate the relation between the cohomology of X and \tilde{X} . For a quasi-coherent \mathcal{O}_X -module \mathcal{F} and an affine Čech cover $\mathcal{U}: X = \bigcup_{i \in I} U_i$ of X , the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$ calculates $H^\bullet(X, \mathcal{F})$. Pulling back \mathcal{U} gives a Čech cover

$$\pi_X^{-1}\mathcal{U}: \tilde{\mathcal{U}} = \bigcup_{i \in I} U_i \times_{\operatorname{Spec} A} \operatorname{Spec} B,$$

of \tilde{X} , whose components $U_i \times_{\operatorname{Spec} A} \operatorname{Spec} B$ are affine again (as fibre products of affine schemes), hence $\pi_X^{-1}\mathcal{U}$ may be used to compute $H^\bullet(\tilde{X}, \pi_X^* \mathcal{F})$. Explicitly, we obtain

$$\check{C}^\bullet(\pi_X^{-1}\mathcal{U}, \pi_X^* \mathcal{F}) \cong \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A B. \quad (1)$$

This gives a canonical morphism

$$H^\bullet(X, \mathcal{F}) \otimes_A B \longrightarrow H^\bullet(\tilde{X}, \pi_X^* \mathcal{F}), \quad (2)$$

which clearly is an isomorphism if B is flat over A .

Proposition 1. *Consider a cartesian diagram*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\pi_X} & X \\ \tilde{f} \downarrow & \lrcorner & \downarrow f \\ \tilde{Y} & \xrightarrow{\pi_Y} & Y \end{array}$$

in which f is quasi-compact and separated, and let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Then we get a morphism of $\mathcal{O}_{\tilde{Y}}$ -modules, called the **base change morphism**,

$$\pi_Y^* R^p f_* \mathcal{F} \longrightarrow R^p \tilde{f}_* (\pi_X^* \mathcal{F}) \quad (3)$$

which is an isomorphism when π_Y is flat in the sense of Definition 1.

Proof. We may check (3) locally, where it is just (2). \square

1.4.2. The base change spectral sequence

Proposition 1a. *Suppose that $Y = \operatorname{Spec} A$ and $\tilde{Y} = \operatorname{Spec} B$ are affine in the situation from Proposition 1. If \mathcal{F} is flat over \mathcal{O}_Y in the sense of Definition 1, then the base change morphism (3) is part of a spectral sequence*

$$E_2^{p,q} = \operatorname{Tor}_{-p}^A(B, H^q(X, \mathcal{F})) \implies H^{p+q}(\tilde{X}, \pi_X^* \mathcal{F}) .$$

Remark 1. The minus sign in $\operatorname{Tor}_{-p}^A$ is not a typo. This spectral sequence is supported in the 2nd quadrant and we interpret cohomology or Tor in negative degrees to be zero.

Proof of Proposition 1a. Let $J^{\bullet, \bullet}$ be a Cartan–Eilenberg resolution of the alternating Čech complex $\check{C}_{\text{alt}}^{\bullet}(\mathcal{U}, \mathcal{F})$, where \mathcal{U} is an arbitrary finite affine open cover of X (which is quasi-compact, so this is fine). That is, $J^{\bullet, \bullet}$ is a 2nd quadrant double complex of A -modules such that

- $J^{\bullet, q} \rightarrow \check{C}_{\text{alt}}^q(\mathcal{U}, \mathcal{F})$ is a projective resolution for all $q \geq 0$ (note that \bullet ranges through all *non-positive* integers here, because we want $J^{\bullet, q}$ to be a cochain complex).
- The vertical morphisms $d_{\text{vert}}^{\bullet, q} : J^{\bullet, q} \rightarrow J^{\bullet, q+1}$ are *split* in the sense that there is a decomposition $J^{\bullet, q} \cong B^{\bullet, q} \oplus H^{\bullet, q} \oplus B^{\bullet, q+1}$ and $d_{\text{vert}}^{\bullet, q}$ corresponds to the projection onto the summand $B^{\bullet, q+1}$.
- The induced cochain complex $H_{\text{vert}}^q(J^{\bullet, \bullet}) \cong H^{\bullet, q} \rightarrow \check{H}_{\text{alt}}^q(\mathcal{U}, \mathcal{F}) \cong H^q(X, \mathcal{F})$ is a projective resolution as well.

It's well-known that Cartan–Eilenberg resolutions exist in the category $\mathbf{Mod}(A)$ (in fact, in any abelian category with sufficiently many projectives). Moreover, since $\check{C}_{\text{alt}}^q(\mathcal{U}, \mathcal{F}) = 0$ when q is sufficiently large, say $q \geq q_0$, we may choose $J^{\bullet, \bullet}$ such that $J^{\bullet, q} = 0$ when $q \geq q_0$. Then $J^{\bullet, \bullet}$ is bounded from above in both directions (that is, a 3rd quadrant double complex, except it's supported completely in the 2nd quadrant – you know what I mean).

The same is true for the double complex $J^{\bullet, \bullet} \otimes_A B$, hence both its horizontal and its vertical spectral sequence converge – and even to the same limit. Note that $H_{\text{vert}}^q(J^{\bullet, \bullet} \otimes_A B) \cong H^{\bullet, q} \otimes_A B$ because of the nice splitness condition for $d_{\text{vert}}^{\bullet, q}$. We thus have

$${}^{\text{vert}} E_2^{p,q} = H_{\text{hor}}^p(H_{\text{vert}}^q(J^{\bullet, \bullet} \otimes_A B)) \cong H^p(H^{\bullet, q} \otimes_A B) \cong \operatorname{Tor}_{-p}^A(B, H^q(X, \mathcal{F})) ,$$

since $H^{\bullet, q}$ is a (negatively indexed) projective resolution of $H^q(X, \mathcal{F})$. Similarly

$${}^{\text{hor}} E_2^{p,q} = H_{\text{vert}}^q(H_{\text{hor}}^p(J^{\bullet, \bullet} \otimes_A B)) = H^q(\operatorname{Tor}_{-p}^A(B, \check{C}_{\text{alt}}^{\bullet}(\mathcal{U}, \mathcal{F})))$$

since $J^{\bullet, q}$ is a projective resolution of $\check{C}^q(\mathcal{U}, \mathcal{F})$. But $\check{C}_{\text{alt}}^q(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_q} \mathcal{F}(U_{i_0, \dots, i_q})$ is a product of flat A -modules because \mathcal{F} is flat over \mathcal{O}_Y , hence $\check{C}_{\text{alt}}^q(\mathcal{U}, \mathcal{F})$ is a flat A -module itself (by [Hom, Example 1.2.6] – actually, the products occurring here are finite, so it's a lot easier to see that flatness is preserved). Hence

$$\operatorname{Tor}_{-p}^A(B, \check{C}_{\text{alt}}^q(\mathcal{U}, \mathcal{F})) \cong \begin{cases} 0 & \text{if } p < 0 \\ \check{C}_{\text{alt}}^q(\mathcal{U}, \mathcal{F}) \otimes_A B & \text{if } p = 0 \end{cases} .$$

By (1) we get ${}^{\text{hor}} E^{0,q} \cong H^q(\tilde{X}, \pi_X^* \mathcal{F})$ and ${}^{\text{hor}} E^{p,q} = 0$ if $p < 0$. Therefore the horizontal spectral sequence immediately degenerates, and its limit is $H^{p+q}(\tilde{X}, \pi_X^* \mathcal{F})$. That's precisely what we need. \square

1.4.3. The Grauert–Grothendieck theorem

Convention 1. Let $f: X \rightarrow Y$ be a morphism of preschemes. Let's make the following conventions.

- (a) If $y \in Y$, then X_y denotes the fibre $f^{-1}\{y\} = X \times_Y \operatorname{Spec} \mathfrak{K}(y)$.
- (b) We keep the abuse of notation that $\mathcal{F}|_{X_y} = i_y^* \mathcal{F}$ when \mathcal{F} is a quasi-coherent sheaf of \mathcal{O}_X -modules. Here $i_y: X_y = X \times_Y \operatorname{Spec} \mathfrak{K}(y) \rightarrow X$ is the canonical projection to the first fibre product factor.

Theorem 4. Let $f: X \rightarrow Y$ be a proper morphism between locally noetherian preschemes and let \mathcal{F} be a coherent \mathcal{O}_X -module which is flat over \mathcal{O}_Y .

- (a) If $Y = \operatorname{Spec} A$ is affine, there exists a complex

$$P^\bullet: 0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \dots \longrightarrow P^d \longrightarrow 0$$

of finitely generated projective A -modules with the following property. If B is an A -algebra and $\pi_X: \tilde{X} = X \times_Y \operatorname{Spec} B \rightarrow X$ is the projection of the fibre product to its first factor, then

$$H^p(\tilde{X}, \pi_X^* \mathcal{F}) \cong H^p(P^\bullet \otimes_A B) .$$

It is possible to choose d to be the largest number with $H^d(X, \mathcal{F}) \neq 0$, or $P^\bullet = 0$ if $H^\bullet(X, \mathcal{F}) = 0$. Moreover, if $d' \geq 0$ is such that $H^p(X_y, \mathcal{F}|_{X_y}) = 0$ for all $y \in Y$ when $p < d'$, then it is possible to choose P^\bullet such that $P^i = 0$ unless $d' \leq i \leq d$.

- (b) The function $h^p(-, \mathcal{F}): Y \rightarrow \mathbb{N}$ given by

$$h^p(y, \mathcal{F}) = \dim_{\mathfrak{K}(y)} H^p(X_y, \mathcal{F}|_{X_y})$$

is upper-semicontinuous in the sense that $\{y \in Y \mid h^p(y, \mathcal{F}) \leq c\}$ is an open subset of Y for all $c \in \mathbb{R}$. Moreover, the **Euler–Poincaré characteristic** $\chi(-, \mathcal{F}): Y \rightarrow \mathbb{N}$ of the fibres, which is given by

$$\chi(y, \mathcal{F}) = \sum_{p \geq 0} (-1)^p h^p(y, \mathcal{F}) ,$$

is locally constant on Y .

- (c) If Y is reduced and $h^p(-, \mathcal{F})$ is locally constant on Y , then the base change morphism

$$\pi_Y^* R^p f_* \mathcal{F} \xrightarrow{(3)} R^p \tilde{f}_* (\pi_X^* \mathcal{F})$$

(with notation from Proposition 1) is an isomorphism. In particular, for $\tilde{Y} = \operatorname{Spec} \mathfrak{K}(y)$ we get

$$(R^p f_* \mathcal{F})_y \xrightarrow{\sim} H^p(X_y, \mathcal{F}|_{X_y}) . \tag{4}$$

- (d) If reducedness wasn't required in (c), the canonical morphism (4) is bijective (for given p and y) if and only if it is surjective. Moreover, for any $p \geq 0$ the set U_p of all $y \in Y$ for which this is the case is open in Y .

(e) Let $y \in U_p$, then the following assertions are equivalent.

(α) We also have $y \in U_{p-1}$.

(β) $R^p f_* \mathcal{F}$ is locally free in some neighbourhood of y .

Remark 2. (a) Note that in Theorem 4(a), $H^p(X, \mathcal{F}) = 0$ for p sufficiently large, because X is quasicompact, hence we can calculate its cohomology from the antisymmetric Čech complex of a finite affine open cover (even though X could have infinite dimension).

(b) For instance, the assumptions of Theorem 4 are fulfilled when f is a proper and flat morphism and \mathcal{F} a vector bundle on X .

(c) From Theorem 4(d) it follows that $R^p f_* \mathcal{F} = 0$ when $H^p(X_y, \mathcal{F}|_{X_y}) = 0$ for all $y \in Y$.

(d) Note that B in Theorem 4(a) is completely arbitrary. In particular, it needn't be noetherian!

The strategy for Theorem 4(a) is to modify the anti-symmetric Čech complex $\check{C}^\bullet = \check{C}_{\text{alt}}^\bullet(\mathcal{U}, \mathcal{F})$, where \mathcal{U} is some fixed affine open cover of X . As \mathcal{F} is flat over \mathcal{O}_Y , the $\mathcal{F}(U_{i_0, \dots, i_n})$ are flat A -modules, hence \check{C}^\bullet is a complex of flat A -modules, but the finite generation assertions will typically fail, as will the vanishing assertions. However, from (1) we know that

$$H^p(\tilde{X}, \pi_X^* \mathcal{F}) \cong H^p(\check{C}^\bullet \otimes_A B) \quad (5)$$

when B and \tilde{X} are as in (a). We will construct cochain complexes P^\bullet and Q^\bullet with morphisms $c: Q^\bullet \rightarrow \check{C}^\bullet$ and $p: Q^\bullet \rightarrow P^\bullet$ such that P^\bullet has all the required properties and c, p induce isomorphisms on $H^\bullet(- \otimes_A M)$ for arbitrary A -modules M .

The rest of the proof of (a) doesn't concern any \check{C}^\bullet -specific properties except flatness and the base change property. However, we will rather freely use facts from homological algebra. If you want to brush up your knowledge, have a look at [Hom].

We start with a sequence of lemmas now.

Lemma 1. Let A be a (not necessarily noetherian) ring and $\varphi: C^\bullet \rightarrow D^\bullet$ a quasi-isomorphism³ between bounded-from-above cochain complexes of flat A -modules. Then, for any A -module M ,

$$\varphi \otimes \text{id}_M: C^\bullet \otimes_A M \longrightarrow D^\bullet \otimes_A M$$

is a quasi-isomorphism again.

Proof. Using the short exact cone sequence $0 \rightarrow D^\bullet \rightarrow \text{Cone}(\varphi) \rightarrow C[1]^\bullet \rightarrow 0$ (cf. [AG2, Definition 2.1.1]), we see that φ is a quasi-isomorphism iff $\text{Cone}(\varphi)$ has vanishing cohomology. But $\text{Cone}(\varphi \otimes \text{id}_M) \cong \text{Cone}(\varphi) \otimes_A M$, so it's sufficient to prove that $\text{Cone}(\varphi) \otimes_A M$ is acyclic. It's clear from the construction that $\text{Cone}(\varphi)$ consists of flat A -modules again and is bounded from above as well. But then $\text{Cone}(\varphi)$ gives a flat resolution of the zero A -module, so the cohomology of $\text{Cone}(\varphi) \otimes_A M$ computes $\text{Tor}_p^A(0, M) = 0$ for all $p \geq 0$. We're done. \square

³That is, φ induces isomorphisms on cohomology

Remark 3. (a) If you're uncomfortable with the Tor shortcut argument at the end of the proof above, Professor Franke suggests an alternative proof as follows. If Z^p is the kernel of $\text{Cone}(\varphi)^p \rightarrow \text{Cone}(\varphi)^{p+1}$, then one can show that all Z^p are flat (using boundedness from above and downward induction on p). Then the short exact sequences

$$0 \longrightarrow Z^p \longrightarrow \text{Cone}(\varphi)^p \longrightarrow Z^{p+1} \longrightarrow 0$$

stay exact after tensoring with M , and splicing together the tensored sequences gives $\text{Cone}(\varphi) \otimes_A M$.

(b) When A is regular, the boundedness assumption may be dropped. Indeed, since $\text{Cone}(\varphi) \otimes_A M$ is exact iff its localizations at any $\mathfrak{p} \in \text{Spec } A$ are exact, we may assume A to be regular and local. But then $\text{Tor}_p^A(-, -)$ vanishes when $p > \dim A$ (cf. [Hom, Theorem 1]), so the Tor argument from the proof of Lemma 1 works even if we're not resolving the zero module.

However, boundedness can't be dropped in general. For example, consider $A = \mathbb{Z}/p^2\mathbb{Z}$ for some prime p , and let

$$C^\bullet: \dots \xrightarrow{p^\cdot} A \xrightarrow{p^\cdot} A \xrightarrow{p^\cdot} A \xrightarrow{p^\cdot} \dots,$$

$D^\bullet = 0$, and $M = \mathbb{Z}/p\mathbb{Z}$. Then $D^\bullet \otimes_A M = 0$, but $C^\bullet \otimes_A M$ has cohomology $\mathbb{Z}/p\mathbb{Z}$ in every degree.

Recall that if C^\bullet is a cochain complex, one can form its **soft truncation from above**

$$\tau_{\leq d} C^\bullet: \dots \longrightarrow C^n \longrightarrow C^{n+1} \longrightarrow \dots \longrightarrow C^{d-1} \longrightarrow Z^d \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots,$$

where $Z^d = \ker(C^d \rightarrow C^{d+1})$. We have $H^p(\tau_{\leq d} C^\bullet) = H^p(C^\bullet)$ if $p \leq d$ and $H^p(\tau_{\leq d} C^\bullet) = 0$ otherwise.

Lemma 2. *Let F^\bullet be a bounded-from-above cochain complex of flat modules over an arbitrary ring A . If F is acyclic in degrees $> d$, then $\tau_{\leq d} F^\bullet \hookrightarrow F^\bullet$ is a quasi-isomorphism and $\tau_{\leq d} F^\bullet$ is still flat over A .*

Proof. Using induction, we may assume F^\bullet to be concentrated in degrees $\leq d+1$. Then $0 \rightarrow Z^d \rightarrow C^d \rightarrow C^{d+1} \rightarrow 0$ is exact. Let M be any A -module. Since C^d, C^{d+1} are flat, we get

$$0 = \text{Tor}_2^A(C^{d+1}, M) \longrightarrow \text{Tor}_1^A(Z^d, M) \longrightarrow \text{Tor}_1^A(C^d, M) = 0$$

as part of the long exact Tor sequence. Hence $\text{Tor}_1^A(Z^d, M) = 0$ for all M , so Z^d is flat as well, proving that $\tau_{\leq d} F^\bullet$ is flat over A . The other assertion about $\tau_{\leq d} F^\bullet$ is trivial. \square

Remark 4. (a) In particular, Lemma 1 applies to $\tau_{\leq d} F^\bullet \hookrightarrow F^\bullet$.

(b) Without boundedness, Lemma 2 counterexample from Remark 3 still applies.

Lemma 3. *Let A be a noetherian ring and F^\bullet a flat cochain complex of A -modules which is concentrated in degrees $< d$. Then there is a cochain complex Q^\bullet , consisting of finitely generated free A -modules and concentrated in degrees $< d$, together with a quasi-isomorphism $\rho^\bullet: Q^\bullet \rightarrow F^\bullet$.*

Proof. Denote the differentials of F^\bullet and Q^\bullet by d_F^\bullet , d_Q^\bullet (the latter of which is yet to be constructed). We will construct the Q^i by downward induction. We start by putting $Q^i = 0$ when $i \geq d$. Now let $k \in \mathbb{Z}$ and suppose we have already constructed Q^i and d_Q^i for $i \geq k$ together with $\rho^i: Q^i \rightarrow F^i$ (compatible with the differentials) such that

- $H^p(Q^\bullet) \xrightarrow{\sim} H^p(F^\bullet)$ is an isomorphism when $p > k$.
- $\ker(d_Q^k) \rightarrow H^k(F^\bullet)$ is surjective.

Now it's about constructing suitable Q^{k-1} , d_Q^{k-1} such that the above conditions remain valid for $k-1$. Let $\beta_1, \dots, \beta_\ell$ be generators of $\{x \in Q^k \mid \rho^k(x) \in \text{Im}(d_F^{k-1})\}$ (that's where we really need that A is noetherian) and let $\varphi_i \in F^{k-1}$ such that $d_F^{k-1}(\varphi_i) = \rho^k(x)$ for all $i = 1, \dots, \ell$. Also choose $\psi_1, \dots, \psi_m \in \ker(d_F^{k-1})$ whose images in $H^{k-1}(F^\bullet)$ generate that A -module. Put $Q^{k-1} = A^{\ell+m}$ and let e_i be the i^{th} standard basis vector. Now define $d_Q^{k-1}: Q^{k-1} \rightarrow Q^k$ and $\rho^{k-1}: Q^{k-1} \rightarrow F^{k-1}$ by

$$d_Q^{k-1}(e_i) = \begin{cases} \beta_i & \text{if } i \leq \ell \\ 0 & \text{if } \ell < i \leq \ell + m \end{cases} \quad \text{and} \quad \rho^{k-1}(e_i) = \begin{cases} \varphi_i & \text{if } i \leq \ell \\ \psi_{i-\ell} & \text{if } \ell < i \leq \ell + m \end{cases}.$$

This does it, as is easily checked. \square

Let C^\bullet be any cochain complex. Similar to the soft truncation from above (from above), there's a **soft truncation from below** given by

$$\tau_{\geq d} C^\bullet: \dots \longrightarrow 0 \longrightarrow 0 \longrightarrow C^d/B^d \longrightarrow C^{d+1} \longrightarrow C^{d+2} \longrightarrow \dots,$$

in which $B^d = \text{Im}(C^{d-1} \rightarrow C^d)$. It has the property that $H^p(\tau_{\geq d} C^\bullet) = H^p(C^\bullet)$ if $p \geq d$ and $H^p(\tau_{\geq d} C^\bullet) = 0$ otherwise.

Lemma 4. *Let A be a Noetherian ring and Q^\bullet be a cochain complex of finitely generated projective A -modules. Suppose there's $d' \in \mathbb{Z}$ such that $H^p(Q^\bullet \otimes_A \mathfrak{K}(\mathfrak{m})) = 0$ for all $p < d'$ and all maximal ideals $\mathfrak{m} \subseteq A$. Then $\tau_{\geq d'} Q^\bullet$ is a cochain complex of projective A -modules. Also $Q^\bullet \rightarrow \tau_{\geq d'} Q^\bullet$ is a quasi-isomorphism if $H^p(Q^\bullet) = 0$ for all $p \leq d'$.*

Proof. W.l.o.g. let $d' = 0$ and $B^0 = \text{Im}(Q^{-1} \rightarrow Q^0)$. Then

$$\dots \longrightarrow Q^{-2} \longrightarrow Q^{-1} \longrightarrow Q^0 \longrightarrow Q^0/B^0 \longrightarrow 0$$

is a projective resolution of Q^0/B^0 . Hence $\text{Tor}_p^A(Q^0/B^0, \mathfrak{K}(\mathfrak{m})) = H^{-p}(Q^\bullet \otimes_A \mathfrak{K}(\mathfrak{m})) = 0$ for all $\mathfrak{m} \in \mathfrak{m}\text{-Spec } A$ and all $p > 0$. Since Q^0/B^0 is finitely generated over the noetherian ring A , this implies that Q^0/B^0 is projective (this is essentially [Hom, Propositions 1.3.1 and 1.3.2], but you should combine them with [Hom, Fact 1.2.4] first). The other assertion about $Q^\bullet \rightarrow \tau_{\geq d'} Q^\bullet$ is trivial again. \square

Exercise 1. Under the assumptions of Theorem 4, derive Theorem 1 for $f: X \rightarrow Y$ and \mathcal{F} from what has been shown until Lemma 3.

Lemma 5. *If d' is as in Theorem 4(a), i.e., has the property that $H^p(X_y, \mathcal{F}|_{X_y}) = 0$ for all $y \in Y$ and $p < d'$, then also $H^p(X, \mathcal{F}) = 0$ when $p < d'$.*

Proof. Professor Franke outsourced a part of this proof to an external fact – however, this fact looked rather fishy to me (if not to say *wrong* unless f is flat). I think the following captures what he actually had in mind.

By Corollary A.1.5 it's sufficient to show that the \mathfrak{m} -adic completions $H^p(X, \mathcal{F})^\wedge$ vanish for all maximal ideals $\mathfrak{m} \subseteq A$ and all $p < d'$. Let $y \in Y$ be the point corresponding to \mathfrak{m} . For all $n \leq 0$ let

$$X_y^{(n)} = X \times_Y \operatorname{Spec} A/\mathfrak{m}^{n+1}$$

be the n^{th} infinitesimal thickening of the fibre X_y , and let $\kappa_n: X_y^{(n)} \rightarrow X$ be the projection to the first fibre product factor. We will show $H^p(X_y^{(n)}, \mathcal{F}|_{X_y^{(n)}}) = 0$ for all $n \geq 1$. Note that

$$H^p(X_y^{(n)}, \mathcal{F}|_{X_y^{(n)}}) \cong H^p(X, \kappa_{n,*} \kappa_n^* \mathcal{F})$$

and $\kappa_{n,*} \kappa_n^* \mathcal{F} \cong \mathcal{F}/\mathfrak{m}^{n+1} \mathcal{F}$ (we have already seen this in the discussion before Theorem 1). Now $H^p(X, \mathcal{F}/\mathfrak{m}^{n+1} \mathcal{F}) = 0$ follows by induction on n as follows. The case $n = 0$ is just the assumption that $H^p(X_y, \mathcal{F}|_{X_y}) = 0$. Now let $n \geq 1$ and suppose that $H^p(X, \mathcal{F}/\mathfrak{m}^n \mathcal{F}) = 0$ has already been shown. Consider the short exact sequence

$$0 \longrightarrow \mathfrak{m}^n \mathcal{F}/\mathfrak{m}^{n+1} \mathcal{F} \longrightarrow \mathcal{F}/\mathfrak{m}^{n+1} \mathcal{F} \longrightarrow \mathcal{F}/\mathfrak{m}^n \mathcal{F} \longrightarrow 0.$$

If we show that $H^p(X, \mathfrak{m}^n \mathcal{F}/\mathfrak{m}^{n+1} \mathcal{F}) = 0$, then $H^p(X, \mathcal{F}/\mathfrak{m}^{n+1} \mathcal{F}) = 0$ follows from the long exact cohomology sequence and the induction hypothesis. Note that $\mathfrak{m}^n \otimes_A \mathcal{F} \cong \mathfrak{m}^n \mathcal{F}$ (and same for $n + 1$). Indeed, this follows from [Hom, Proposition 1.2.3(d)] because \mathcal{F} is flat over \mathcal{O}_Y . Hence

$$\mathfrak{m}^n \mathcal{F}/\mathfrak{m}^{n+1} \cong \mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes_A \mathcal{F} \cong \mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes_{\mathfrak{K}(\mathfrak{m})} \mathcal{F}/\mathfrak{m} \mathcal{F}$$

is isomorphic to a direct sum of $\dim_{\mathfrak{K}(\mathfrak{m})}(\mathfrak{m}^n/\mathfrak{m}^{n+1})$ copies of $\mathcal{F}/\mathfrak{m} \mathcal{F}$, because $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a finite-dimensional $\mathfrak{K}(\mathfrak{m})$ -vector space. By the $n = 0$ case, this shows $H^p(X, \mathfrak{m}^n \mathcal{F}/\mathfrak{m}^{n+1} \mathcal{F}) = 0$ and we're done. \square

Remark 5. As the above proof shows, it's actually sufficient to have $H^p(X_y, \mathcal{F}|_{X_y}) = 0$ for all *closed* $y \in Y$.

We now have all the technicalities together to prove Theorem 4 (ok, actually we there are still three commutative algebra lemmas to follow, but we can start now anyway).

Proof of Theorem 4. The first step will be to prove (a), from which the other assertions will be deduced. In (a), Y is affine and the other assertions are all base-local, hence without losing generality we may assume $Y = \operatorname{Spec} A$ for the remainder of this proof.

Fix a finite affine open cover \mathcal{U} of X and let $\check{C}^\bullet = \check{C}_{\text{alt}}^\bullet(\mathcal{U}, \mathcal{F})$. Using Lemma 2 we see that $\tau_{\leq d} \check{C}^\bullet$ consists of flat A -modules again. Then we choose a finitely generated free approximation $Q^\bullet \rightarrow \tau_{\leq d} \check{C}^\bullet$ as in Lemma 3. By Lemma 1 we get

$$H^p(Q^\bullet \otimes_A \mathfrak{K}(\mathfrak{m})) \cong H^p(\tau_{\geq d} \check{C} \otimes_A \mathfrak{K}(\mathfrak{m})) \cong H^p(\check{C} \otimes_A \mathfrak{K}(\mathfrak{m})) \stackrel{(5)}{\cong} H^p(X_y, \mathcal{F}|_{X_y}),$$

where $y \in Y$ on the right-hand side is the point corresponding to the maximal ideal \mathfrak{m} , so that we have $X_y \cong X \times_Y \text{Spec } \mathfrak{K}(\mathfrak{m})$. In particular, we see $H^p(Q^\bullet \otimes_A \mathfrak{K}(\mathfrak{m})) = 0$ if $p < d'$ and also $H^p(Q^\bullet) \cong H^p(X, \mathcal{F}) = 0$ if $p < d'$ by Lemma 5, hence Lemma 4 may be applied. Therefore, $P^\bullet = \tau_{\geq d'} Q^\bullet$ is complex of finitely generated projective A -modules with is supported only between d' and d , and P^\bullet has the required base change property because of (5) and Lemma 1. In other words, P^\bullet is what we want. This proves (a).

Note that for the rest of the proof we may even assume that P^\bullet is a complex of *free* modules. Indeed, our construction of P^\bullet provided free modules except in degree d' , where $P^{d'}$ might only be projective. However, assertions (b) to (e) are base-local, so we may replace $\text{Spec } A$ by $\text{Spec } A_f$ such that $P_f^{d'}$ is free over A_f .

Let $Q^p = \text{coker}(P^{p-1} \rightarrow P^p)$. Then $\text{coker}(P^{p-1} \otimes_A B \rightarrow P^p \otimes_A B) \cong Q^p \otimes_A B$. If we put $\delta^p: Q^p \rightarrow P^{p+1}$, then $H^p(P^\bullet \otimes_A B) \cong \ker(\delta^p \otimes \text{id}_B)$ for all A -algebras B (actually, we don't even need B to be an algebra).

Now for part (b). We prove that $\chi(-, \mathcal{F})$ is locally constant on Y first. Indeed, since the P^i are free over A (or rather we may assume them to be, as explained above) we have

$$\begin{aligned} \chi(y, \mathcal{F}) &= \sum_{p \geq 0} (-1)^p \dim_{\mathfrak{K}(y)} H^p(X_y, \mathcal{F}|_{X_y}) = \sum_{p \geq 0} (-1)^p \dim_{\mathfrak{K}(y)} H^p(P^\bullet \otimes_A \mathfrak{K}(y)) \\ &= \sum_{p \geq 0} (-1)^p \dim_{\mathfrak{K}(y)} (P^p \otimes_A \mathfrak{K}(y)) \\ &= \sum_{p \geq 0} (-1)^p \text{rank } P^p \end{aligned}$$

(the transition from the first to the second line is a well-known fact from homological algebra). The last line is clearly constant on Y and we're done.

To prove the rest of (b) as well as (c), we invoke another lemma.

Lemma 6. *Let A be a noetherian ring and $\delta: M \rightarrow F$ a morphism of finitely generated A -modules, of which F is free of rank n .*

(a) *The function $g: \text{Spec } A \rightarrow \mathbb{N}$ given by*

$$g(\mathfrak{p}) = \dim_{\mathfrak{K}(\mathfrak{p})} \ker(\delta_{\mathfrak{p}} = \delta \otimes \text{id}_{\mathfrak{K}(\mathfrak{p})}: M \otimes_A \mathfrak{K}(\mathfrak{p}) \longrightarrow F \otimes_A \mathfrak{K}(\mathfrak{p}))$$

is upper-semicontinuous on $\text{Spec } A$.

(b) *If in addition g is locally constant and A is reduced, then M , $\ker \delta$, and $\text{coker } \delta$ are all projective A -modules, and the canonical morphism*

$$\ker \delta \otimes_A N \longrightarrow \ker(\delta \otimes \text{id}_N)$$

is an isomorphism for all A -modules N .

Proof. Part (a). Let $\mu_1, \dots, \mu_s \in M$ be elements whose images $\bar{\mu}_1, \dots, \bar{\mu}_s$ in $M \otimes_A \mathfrak{K}(\mathfrak{p})$ form a basis of that $\mathfrak{K}(\mathfrak{p})$ -vector space.⁴ Moreover, we may choose them in such a way that their images $\delta_{\mathfrak{p}}(\bar{\mu}_1), \dots, \delta_{\mathfrak{p}}(\bar{\mu}_r)$ (for some $r \leq s$) in $F \otimes_A \mathfrak{K}(\mathfrak{p})$ vanish while the images $\delta_{\mathfrak{p}}(\bar{\mu}_{r+1}), \dots, \delta_{\mathfrak{p}}(\bar{\mu}_s)$ form a basis of $\text{Im } \delta_{\mathfrak{p}}$.

Then we have $r = g(\mathfrak{p})$. Since the assertions are all local near any given $\mathfrak{p} \in \text{Spec } A$, we are free to replace A by A_f for $f \notin \mathfrak{p}$.

Claim 1. Choosing f appropriately we can assure that μ_1, \dots, μ_s generate M as an A -module, and also that their images in $M \otimes_A \mathfrak{K}(\mathfrak{q})$ stay $\mathfrak{K}(\mathfrak{q})$ -linearly independent for all $\mathfrak{q} \in \text{Spec } A$.

The first of these two assertions is a Nakayama-style argument that you can find in [Alg2, Lemma 1.5.1]. The latter, however, needs a bit more care, and we'll explain how this works now. Choose a basis of F (which is free of rank n by assumption) and represent $\delta(\mu_{r+1}), \dots, \delta(\mu_s)$ as linear combinations of that basis. The resulting coefficient matrix $C \in A^{(s-r) \times n}$ has a $(s-r) \times (s-r)$ -minor $D \in A^{(s-r) \times (s-r)}$ whose image in $\mathfrak{K}(\mathfrak{p})^{(s-r) \times (s-r)}$ is invertible, because $\delta_{\mathfrak{p}}(\bar{\mu}_{r+1}), \dots, \delta_{\mathfrak{p}}(\bar{\mu}_s)$ are $\mathfrak{K}(\mathfrak{p})$ -linearly independent. In particular, we must have $\det D \not\equiv 0 \pmod{\mathfrak{p}}$, so we may localize A at $\det D$ to make D invertible over A . Then the images of μ_{r+1}, \dots, μ_s in $M \otimes_A \mathfrak{K}(\mathfrak{q})$ are linearly independent for all $\mathfrak{q} \in \text{Spec } A$, because they are still linearly independent after applying $\delta_{\mathfrak{q}}$. This justifies the above Claim 1.

Now we have $g(\mathfrak{q}) = \dim_{\mathfrak{K}(\mathfrak{q})} \ker \delta_{\mathfrak{q}} = \dim_{\mathfrak{K}(\mathfrak{q})} (M \otimes_A \mathfrak{K}(\mathfrak{q})) - \dim_{\mathfrak{K}(\mathfrak{q})} \text{Im } \delta_{\mathfrak{q}}$, in which the inequalities

$$\dim_{\mathfrak{K}(\mathfrak{q})} (M \otimes_A \mathfrak{K}(\mathfrak{q})) \leq s \quad \text{and} \quad \dim_{\mathfrak{K}(\mathfrak{q})} \text{Im } \delta_{\mathfrak{q}} \geq s - r \quad (6)$$

hold true because of Claim 1. Hence $g(\mathfrak{q}) \leq r$ for all $\mathfrak{q} \in \text{Spec } A$, and this proves upper-semicontinuity. We're done with (a).

Part (b). Replacing A by some localization again, we may assume that $g(\mathfrak{q}) = r$ for all $\mathfrak{q} \in \text{Spec } A$. Then equality must hold in both inequalities from (6). In particular, $\dim_{\mathfrak{K}(\mathfrak{q})} (M \otimes_A \mathfrak{K}(\mathfrak{q})) = s$ for all $\mathfrak{q} \in \text{Spec } A$, which implies that M is locally free (of rank s) by the upcoming Lemma 7. Similarly, we have

$$\begin{aligned} \dim_{\mathfrak{K}(\mathfrak{q})} (\text{coker } \delta \otimes_A \mathfrak{K}(\mathfrak{q})) &= \dim_{\mathfrak{K}(\mathfrak{q})} \text{coker } \delta_{\mathfrak{q}} = \dim_{\mathfrak{K}(\mathfrak{q})} (F \otimes_A \mathfrak{K}(\mathfrak{q})) - \dim_{\mathfrak{K}(\mathfrak{q})} \text{Im } \delta_{\mathfrak{q}} \\ &= n - s + r \end{aligned}$$

(where $n = \text{rank } F$, remember?), so $\text{coker } \delta$ is locally free by Lemma 7 again. In particular, $\text{coker } \delta$ is projective, so $F \twoheadrightarrow \text{coker } \delta$ splits and we get $F \cong \text{Im } \delta \oplus \text{coker } \delta$ (*non-canonically*). Then $\text{Im } \delta$ is projective as well, so $M \twoheadrightarrow \text{Im } \delta$ splits as well, hence $M \cong \ker \delta \oplus \text{Im } \delta$ and $\ker \delta$ lines up with the other projective modules. Therefore, δ has the form $\delta: \ker \delta \oplus \text{Im } \delta \rightarrow \text{Im } \delta \oplus \text{coker } \delta$, and this nice splitting behaviour is preserved after tensoring with some A -module N . This shows (b). \square

Die-hard Franke fans might remember the next lemma from Franke's Algebra II lecture, where it appeared as a remark though; see [Alg2, p. 25]. We'll prove it here again.

⁴Why can we do this? Start with any basis of $M \otimes_A \mathfrak{K}(\mathfrak{p}) \cong M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ and lift it to $M_{\mathfrak{p}}$. Then the numerators (which are elements of M) will do. We will use this argument implicitly in the proofs to come.

Lemma 7. *Let A be a reduced noetherian ring and M a finitely generated A -module, such that the function*

$$\mathfrak{p} \longmapsto \dim_{\mathfrak{K}(\mathfrak{p})} M \otimes_A \mathfrak{K}(\mathfrak{p})$$

is locally constant on $\operatorname{Spec} A$. Then M is locally free.

Proof. Again, the question is local on $\operatorname{Spec} A$, so after localizing A appropriately we may assume that $\dim_{\mathfrak{K}(\mathfrak{p})}(M \otimes_A \mathfrak{K}(\mathfrak{p})) = \ell$ for all $\mathfrak{p} \in \operatorname{Spec} A$ and some $\ell \in \mathbb{N}$. Choose some $\mathfrak{p} \in \operatorname{Spec} A$ and let $\mu_1, \dots, \mu_\ell \in M$ be elements whose images form a basis of $M \otimes_{\mathfrak{K}} \mathfrak{K}(\mathfrak{p})$. Without losing generality (or rather after another localization, using [Alg2, Lemma 1.5.1]) the μ_i generate M as an A -module. We claim they do so freely. Indeed, if $\alpha_1 \mu_1 + \dots + \alpha_\ell \mu_\ell = 0$ for some $\alpha_i \in A$ is any relation in M , then $\alpha_i \in \mathfrak{q}$ for any $\mathfrak{q} \in \operatorname{Spec} A$ as $\dim_{\mathfrak{K}(\mathfrak{q})}(M \otimes_A \mathfrak{K}(\mathfrak{q})) < \ell$ otherwise. Hence $\alpha_i \in \bigcap_{\mathfrak{q} \in \operatorname{Spec} A} \mathfrak{q} = \operatorname{nil} A = 0$ and we're done. \square

We come back to the proof of Theorem 4(b) and (c). The upper-semicontinuity part of (b) follows immediately from Lemma 6(a) applied to $M = Q^p$ and $F = P^{p+1}$. For part (c), we may assume the base change was done along $\pi_Y: \tilde{Y} = \operatorname{Spec} B \rightarrow Y = \operatorname{Spec} A$ because the question is local on Y and \tilde{Y} . Then $R^p \tilde{f}_*(\pi_X^* \mathcal{F})$ is a quasi-coherent $\mathcal{O}_{\tilde{Y}}$ -module given by

$$H^p(\tilde{X}, \pi_X^* \mathcal{F}) \cong H^p(P^\bullet \otimes_A B) \cong \ker \left(\delta^p \otimes \operatorname{id}_B: Q^p \otimes_A B \longrightarrow P^{p+1} \otimes_A B \right),$$

and $\pi_Y^* R^p f_* \mathcal{F}$ is given by

$$H^p(X, \mathcal{F}) \otimes_A B \cong H(P^\bullet) \otimes_A B \cong \ker \left(\delta^p: Q^p \longrightarrow P^{p+1} \right) \otimes_A B.$$

That these two agree is precisely what Lemma 6(c) says.

For (d) and (e) we need yet another lemma, but I promise this will be the last technical commutative algebra lemma required to prove Theorem 4.

Lemma 8. *In the above situation, the following conditions on a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ are equivalent.*

(a) *The canonical morphism*

$$\ker \delta^p \otimes_A \mathfrak{K}(\mathfrak{p}) \longrightarrow \ker \left(\delta_{\mathfrak{p}}^p = \delta^p \otimes \operatorname{id}_{\mathfrak{K}(\mathfrak{p})}: Q^p \otimes_A \mathfrak{K}(\mathfrak{p}) \longrightarrow P^{p+1} \otimes_A \mathfrak{K}(\mathfrak{p}) \right)$$

is surjective.

(b) *Same as (a), but the morphism is even bijective.*

(c) *The localization $Q_{\mathfrak{p}}^{p+1}$ is a free $A_{\mathfrak{p}}$ -module.*

Proof. We use similar arguments as in Lemma 6. Let's do the implication (a) \Rightarrow (c) first. If assertion (a) holds, then we may choose $\mu_1, \dots, \mu_r \in \ker \delta^p$ whose images $\bar{\mu}_i$ in $\ker (\delta^p \otimes \operatorname{id}_{\mathfrak{K}(\mathfrak{p})})$ form a basis. We may extend μ_1, \dots, μ_r by some $\mu_{r+1}, \dots, \mu_s \in Q^p$ such that $\bar{\mu}_1, \dots, \bar{\mu}_s$ form a basis of $Q^p \otimes_A \mathfrak{K}(\mathfrak{p})$. Then the images $\delta_{\mathfrak{p}}^p(\bar{\mu}_{r+1}), \dots, \delta_{\mathfrak{p}}^p(\bar{\mu}_s) \in P^{p+1} \otimes_A \mathfrak{K}(\mathfrak{p})$ are $\mathfrak{K}(\mathfrak{p})$ -linearly

independent, so we may choose $\pi_1, \dots, \pi_n \in P^{p+1}$ whose images in $P^{p+1} \otimes_A \mathfrak{K}(\mathfrak{p})$ extend $\delta_{\mathfrak{p}}^p(\bar{\mu}_{r+1}), \dots, \delta_{\mathfrak{p}}^p(\bar{\mu}_s)$ to a basis.

By the same argument as in Claim 1 from the proof of Lemma 6(a), we can replace A by some localization so that $\delta^p(\mu_{r+1}), \dots, \delta^p(\mu_s)$ and π_1, \dots, π_n generate P^{p+1} – and we may even assume they do so *freely*! Indeed, write $\delta^p(\mu_{r+1}), \dots, \delta^p(\mu_s)$ and π_1, \dots, π_n as linear combinations of some basis of P^{p+1} . Then the coefficient matrix C becomes invertible when reduced to $\mathfrak{K}(\mathfrak{p})$, hence after localizing at $\det C$ we may assume that C is already invertible over A . Therefore, after suitable localization, we see that $\text{Im } \delta^p$ is a direct summand of P^{p+1} . Then $Q^{p+1} = \text{coker } \delta^p$ is a direct summand of $P^{p+1} \cong \text{Im } \delta^p \oplus \text{coker } \delta^p$ as well. This proves that Q^{p+1} is projective, hence $Q_{\mathfrak{p}}^{p+1}$ is free over $A_{\mathfrak{p}}$. This proves the implication (a) \Rightarrow (c).

Now for (c) \Rightarrow (b). By [Alg2, Corollary 1.5.1], we may replace A (and Q^{p+1}) by some localization such that Q^{p+1} is already free over A . Then $P^{p+1} \twoheadrightarrow Q^{p+1} = \text{coker } \delta^p$ is split, hence $P^{p+1} \cong \text{Im } \delta^p \oplus \text{coker } \delta^p$ (non-canonically). Then $\text{Im } \delta^p$ is projective as well, so $Q^p \cong \ker \delta^p \oplus \text{Im } \delta^p$ (non-canonically). That is, δ^p splits as $\delta^p: \ker \delta^p \oplus \text{Im } \delta^p \rightarrow \text{Im } \delta^p \oplus \text{coker } \delta^p$, and this nice splitting behaviour is preserved after tensoring with $\mathfrak{K}(\mathfrak{p})$. This shows $\ker \delta^p \otimes_A \mathfrak{K}(\mathfrak{p}) \cong \ker (\delta^p \otimes \text{id}_{\mathfrak{K}(\mathfrak{p})})$, i.e., (b) holds.

Finally (b) \Rightarrow (a) is trivial and we're done. \square

Now there isn't really much left to prove of (d) and (e), which we'll do in the next lecture. \square

Corollary 1. *In the situation of Theorem 4, if $h^p(y, \mathcal{F}) = 0$, then $R^p f_* \mathcal{F}$ vanishes in some neighbourhood of x .*

Proof. By Theorem 4(d) we have $(R^p f_* \mathcal{F})_y = 0$, hence $R^p f_* \mathcal{F} = 0$ near y by Nakayama-style arguments (to be precise, this follows from [Alg2, Corollary 1.5.1]). \square

A. Appendix

A.1. Some prerequisites about completions

We briefly recall the most important facts about completions. An excellent introduction to this subject can be found in [AM94, Section 10].

A.1.1. Definitions and Hensel's lemma

Definition 1. Let A be a ring (commutative with 1), $I \subseteq A$ and ideal, and M an A -module.

- (a) The **I -adic topology** on M is the unique topology such that $\{I^n\}_{n \in \mathbb{N}}$ is a fundamental system of neighbourhoods of 0 and M (with its additive structure) becomes a topological group in this topology.
- (b) The **completion** of M with respect to the I -adic topology is

$$\widehat{M} = \varprojlim_{n \in \mathbb{N}} M/I^n M.$$

Note that \widehat{A} is a ring again. We call M **complete** in the I -adic topology if the canonical morphism $M \rightarrow \widehat{M}$ is an isomorphism.

Remark 1. M with its I -adic topology is *pseudo-metrizable* via $d(x, y) = e^{-\sup\{n \mid x-y \in I^n\}}$. It is easy to check that \widehat{M} is also the completion of M in the analytical sense, i.e. the set of Cauchy sequences modulo the zero sequences.

Example 1. If $I^n = 0$ for some $n \in \mathbb{N}$, then any A -module is complete in the I -adic topology.

Example 2. If $A = \mathbb{Z}$ and $I = p\mathbb{Z}$ for some prime p , then $\widehat{A} = \mathbb{Z}_p$ is the ring of p -adic integers.

Proposition 1 (Hensel's lemma). *Suppose the ring A is complete in the I -adic topology. Let $P \in A[T]$ be a polynomial and $a_0 \in A$ such that $P(a_0) \equiv 0 \pmod{I}$ and $P'(a_0)$ is a unit in A/I . Then there is a unique $a \in A$ such that $a \equiv a_0 \pmod{I}$ and $P(a) = 0$.*

Proof. Step 1. Consider the special case $I^2 = 0$. For $\delta \in I$ we have $P(a_0 + \delta) = P(a_0) + \delta P'(a_0)$ since all terms of order δ^2 or higher vanish in the binomial expansion. Now $P'(a_0)$ being a unit in A/I gives a unique $\delta \in I$ such that $a = a_0 + \delta$ satisfies $P(a) = 0$.

Step 2. Suppose that $I^{2^n} = 0$ for some $n \in \mathbb{N}$. Using induction on n (with the base case being precisely Step 1) we may assume that Hensel's lemma holds for $A/I^{2^{n-1}}$. In particular, there is

a unique a_{n-1} such that $P(a_{n-1}) \equiv 0 \pmod{I^{2^{n-1}}}$ and $a_{n-1} \equiv a_0 \pmod{I}$. Moreover, $P'(a_{n-1})$ is invertible in $A/I^{2^{n-1}}$. Indeed, this follows from Hensel's lemma applied to $A/I^{2^{n-1}}$ (for which it holds by induction hypothesis) and the polynomial $Q = P'(a_{n-1})T - 1$. The derivative $Q'(a_{n-1})$ equals $P'(a_{n-1})$ which is invertible in A/I since $P'(a_{n-1}) \equiv P'(a_0) \pmod{I}$, so Hensel's lemma is indeed applicable. Now replacing I by $I^{2^{n-1}}$ and a_0 by a_{n-1} reduces the situation to Step 1, proving the inductive step.

Step 3. Now let I be arbitrary. By Step 2 there is for every $n \in \mathbb{N}$ a unique $a_n \in A/I^{2^n}$ such that $P(a_n) \equiv 0 \pmod{I^{2^n}}$ and $a_n \equiv a_0 \pmod{I}$. Then $a_n \equiv a_{n-1} \pmod{I^{2^{n-1}}}$ is forced by uniqueness. Hence $a = (a_n)_{n \in \mathbb{N}}$ defines an element of

$$\varprojlim_{n \in \mathbb{N}} A/I^{2^n} = \varprojlim_{n \geq 1} A/I^n = \widehat{A},$$

providing the desired element $a \in \widehat{A}$. □

Corollary 1. *Let A be complete in the I -adic topology.*

- (a) *If $a \in A$ becomes a unit in A/I , then already $a \in A^\times$.*
- (b) *For every idempotent $\pi \in A/I$ there is a unique idempotent in A whose image modulo I is π . Therefore, $\operatorname{Spec} A$ and $\operatorname{Spec} A/I$ have the same connected components.*
- (c) *I is contained in the Jacobson radical $\operatorname{rad} A$.*

Proof. Part (a) follows from Proposition 1 applied to $P = aT - 1$ (whose derivative a is a unit in A/I by assumption, so this is fine). For (b) we use the polynomial $P = T^2 - T$. Again, $P'(\pi) = 2\pi - 1$ is a unit in A/I since $(2\pi - 1)^2 = 4\pi^2 - 4\pi + 1 = 1$ in A/I . To prove (c) recall the characterization

$$\operatorname{rad} A = \{x \in A \mid 1 - ax \in A^\times \text{ for all } a \in A\}.$$

If $x \in I$, then $1 - ax$ is a unit in A/I , hence also in A by (a). □

Proposition 2. *Let A be noetherian and $N \subseteq M$ finitely generated A -modules. Then the I -adic topology on N coincides with the induced topology by the I -adic topology on M .*

Sketch of a proof. By the Artin–Rees lemma (cf. [Alg2, Proposition 3.4.1]) there exists a number $c \in \mathbb{N}$ such that $N \cap I^{n+c}M \subseteq I^nN$. From this, the assertion is easily deduced. □

A.1.2. Flatness properties

Fact 1. (a) *The canonical morphism $\widehat{M} = \varprojlim M/I^nM \rightarrow M/IM$ is surjective.*

- (b) *If M is finitely generated and I is contained in the Jacobson radical of A , then $\widehat{M} = 0$ implies $M = 0$.*

Proof. For (a), note that the composition $M \rightarrow \widehat{M} \rightarrow M/IM$ equals the projection $M \rightarrow M/IM$ by definition of the limit. Since the latter is surjective, so is $\widehat{M} \rightarrow M/IM$.

In particular, part (a) shows that $\widehat{M} = 0$ implies $M = IM$. In the situation of (b) this is equivalent to $M = 0$ by Nakayama's lemma (which – as we all know – Professor Franke also likes to attribute to Azumaya and Krull, even though he regards Krull as a noob compared to Grothendieck). \square

Corollary 2. *If A is noetherian, then the functor $M \mapsto \widehat{M}$ is exact on the category of finitely generated A -modules.*

Proof. Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be a short exact sequence of finitely generated A -modules. Then $M' + I^n M$ is the kernel of $M \rightarrow M''/I^n M''$. Using $(M' + I^n M)/I^n M \cong M'/(M' \cap I^n M)$ we get short exact sequences

$$0 \longrightarrow M'/(M' \cap I^n M) \longrightarrow M/I^n M \longrightarrow M''/I^n M'' \longrightarrow 0 \quad (*)$$

for every $n \in \mathbb{N}$. Since $M'/(M' \cap I^n M)$ is sandwiched between $M'/I^n M'$ and $M'/I^{n+c} M'$ for some $c \in \mathbb{N}$ by the Artin–Rees lemma, it's easy to see that

$$\varprojlim_{n \in \mathbb{N}} M'/(M' \cap I^n M) = \varprojlim_{n \geq 1} M'/I^{n+c} M' = \varprojlim_{n \geq 1} M'/I^n M' = \widehat{M}'.$$

Moreover, each $M'/(M' + I^{n+1} M) \rightarrow M'/(M' + I^n M)$ is clearly surjective, so Fact 1.1.3 gives

$$\varprojlim_{n \geq 1}^1 M'/(M' \cap I^n M) = 0.$$

Thus, taking the limit over $(*)$ gives a short exact sequence $0 \rightarrow \widehat{M}' \rightarrow \widehat{M} \rightarrow \widehat{M}'' \rightarrow 0$ by Fact 1.1.1. We are done. \square

Corollary 3. *Let A be a Noetherian ring and $\widehat{}$ the I -completion for some ideal $I \subseteq A$.*

- (a) *When M is a finitely generated A -module, then $\widehat{M} \cong M \otimes_A \widehat{A}$.*
- (b) *\widehat{A} is flat as an A -module.*
- (c) *If I is contained in the Jacobson radical of A , then \widehat{A} is **faithfully flat** over A . That is, if one of the sequences of A -modules*

$$\begin{aligned} 0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0 \\ 0 \longrightarrow M' \otimes_A \widehat{A} \xrightarrow{\alpha \otimes \text{id}} M \otimes_A \widehat{A} \xrightarrow{\beta \otimes \text{id}} M'' \otimes_A \widehat{A} \longrightarrow 0 \end{aligned}$$

is exact, then so is the other. In particular, this holds when A is local and $I = \mathfrak{m}$ the maximal ideal of A .

Proof. Part (a). Every finitely generated A -module is finitely presented as well since A is noetherian. So take a representation $M \cong \text{coker}(A^m \rightarrow A^n)$ for some $m, n \in \mathbb{N}$. It's obvious that $(A^n)^\wedge \cong \hat{A}^n \cong A^n \otimes_A \hat{A}$. Since both completion and tensor products commute with cokernels, this shows $\hat{M} \cong M \otimes_A \hat{A}$ as well.

This immediately shows (b). Indeed, by Corollary 2 and (a), $-\otimes_A \hat{A}$ is exact on finitely generated A -modules – which is sufficient for flatness by [Hom, Proposition 1.2.2].

For (c), we first prove an auxiliary statement.

Claim 1. If M is any A -module, then $M \otimes_A \hat{A} = 0$ implies $M = 0$.

Indeed, to see this, let $N \hookrightarrow M$ be any finitely generated submodule. Because \hat{A} is flat over A by (b), $N \otimes_A \hat{A} \rightarrow M \otimes_A \hat{A}$ is still injective, so $0 = N \otimes_A \hat{A} = \hat{N}$ by assumption and part (a). Hence $N = 0$ by Fact 1. But M is the union of its finitely generated submodules, so $M = 0$ as well.

Suppose now that $0 \rightarrow M' \otimes_A \hat{A} \rightarrow M \otimes_A \hat{A} \rightarrow M'' \otimes_A \hat{A} \rightarrow 0$ is exact (the other direction is clear because we proved in (b) that \hat{A} is flat over A). First note that $\beta\alpha = 0$. Indeed, if B is the image of M' in M'' , then $B \otimes_A \hat{A}$ is the image of $M' \otimes_A \hat{A}$ in $M'' \otimes_A \hat{A}$ (using that \hat{A} is flat). Hence $B \otimes_A \hat{A} = 0$, so $B = 0$ by Claim 1.

This induces a morphism $M' \rightarrow \ker \beta$. Let K and Q be its kernel and cokernel. Then $\ker(\beta \otimes \text{id}_{\hat{A}}) \cong \ker \beta \otimes_A \hat{A}$ because \hat{A} is flat, hence the sequence

$$0 \longrightarrow K \otimes_A \hat{A} \longrightarrow M' \otimes_A \hat{A} \longrightarrow \ker(\beta \otimes \text{id}_{\hat{A}}) \longrightarrow Q \otimes_A \hat{A} \longrightarrow 0.$$

is exact (using flatness of \hat{A} once again). But $M' \otimes_A \hat{A} \rightarrow \ker(\beta \otimes \text{id}_{\hat{A}})$ must be an isomorphism, so $K \otimes_A \hat{A} = 0$ and $Q \otimes_A \hat{A} = 0$. By Claim 1 this shows $K = 0$ and $Q = 0$ and we conclude $M' \cong \ker \beta$. In the same way one can show $M'' \cong \text{coker } \alpha$. We are done. \square

Corollary 4. Suppose that I is contained in the Jacobson radical of the noetherian ring A . If $\mu: M \rightarrow N$ is a morphism of finitely generated A -modules whose I -adic completion $\hat{\mu}: \hat{M} \rightarrow \hat{N}$ is an isomorphism, then μ is an isomorphism.

Proof. Follows from Corollary 3(a) and (c). \square

Corollary 5. Let A be noetherian. A sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ of finitely generated A -modules is exact iff for all maximal ideals $\mathfrak{m} \subseteq A$ the sequence $0 \rightarrow \hat{M}' \rightarrow \hat{M} \rightarrow \hat{M}'' \rightarrow 0$ of \mathfrak{m} -adic completions is exact.

Proof. This sequence is exact iff its localizations at the maximal ideals $\mathfrak{m} \subseteq A$ are all exact. Having localized at \mathfrak{m} , we may check exactness after going over to $\mathfrak{m}A_{\mathfrak{m}}$ -adic completions by Corollary 3(c). So all that's left to do is to show that $\hat{M} \cong \hat{M}_{\mathfrak{m}}$ holds for the \mathfrak{m} -adic resp. $\mathfrak{m}A_{\mathfrak{m}}$ -adic completions. This follows from $M/\mathfrak{m}^n M \cong M_{\mathfrak{m}}/\mathfrak{m}^n M_{\mathfrak{m}}$ for all $n \geq 1$. Indeed, A/\mathfrak{m}^n is a local ring with maximal ideal $\mathfrak{m}/\mathfrak{m}^n$, hence multiplication by $a \in A \setminus \mathfrak{m}$ is already bijective on $M/\mathfrak{m}^n M$. \square

Corollary 6. *If $J \subseteq A$ is any ideal and M a finitely generated A -module, then $(JM)^\wedge \rightarrow \widehat{M}$ defines an isomorphism $(JM)^\wedge \xrightarrow{\sim} J\widehat{M}$.*

Proof. We may view $(JM)^\wedge$ as a submodule of \widehat{M} since completion preserves injectivity of the inclusion $JM \subseteq M$ by Corollary 2. It's easy to see that $J\widehat{M}$ is contained in $(JM)^\wedge$. To prove the converse, take generators j_1, \dots, j_n of J . Then completion preserves surjectivity of $(j_1, \dots, j_n): M^n \twoheadrightarrow JM$ and we are done. \square

Corollary 7. *If A is a noetherian local ring with maximal ideal \mathfrak{m} , then \widehat{A} is local with maximal ideal $\mathfrak{m}\widehat{A}$.*

Proof. We proved this in [Hom, Corollary 2.2.2]. \square

A.1.3. Miscellaneous facts about completion

Proposition 3. *Let A be noetherian and $I \subseteq A$ any ideal, then the I -adic completion \widehat{A} is noetherian again.*

To prove this, we need to prove the evil twin of Hilbert's Basissatz first.

Lemma 1. *If A is noetherian, then so is the power series ring $R = A[[T]]$.*

Proof. We can (and will) basically copy the proof of Hilbert's Basissatz. Let $J \subseteq R$ be any ideal and put $J_n = \{a_n \mid \sum_{k=n}^\infty a_k T^k \in J\}$ for $n \geq 0$. Then $(J_n)_{n \in \mathbb{N}}$ form an ascending sequence of ideals in A . Noetherianness of A tells us that this sequence becomes eventually stationary, say, at $n = s$. So we may choose $a^{(i)} = \sum_{k \geq s} a_k T^k \in R$ for $i = 1, \dots, N$ such that $a_s^{(1)}, \dots, a_s^{(N)}$ generate J_s . Then $a^{(1)}, \dots, a^{(N)}$ generate $J \cap T^s R$. Indeed, given any $b = \sum_{k \geq s} b_k T^k \in J$ we can inductively choose coefficients $r_k^{(1)}, \dots, r_k^{(N)} \in A$ such that $r^{(i)} = \sum_{k \geq 0} r_k^{(i)} T^k$ satisfy $r^{(1)}a^{(1)} + \dots + r^{(N)}a^{(N)} = b$ up to degree T^{s+k} . This works because $J_{k+s} = J_s$ for all $k \geq 0$ is generated by $a_s^{(1)}, \dots, a_s^{(N)}$ again.

Now $R/T^s R$ is a finitely generated A -module, hence the image of J in it is finitely generated as well, A being noetherian. We thus may choose $a^{(N+1)}, \dots, a^{(N+M)} \in J$ whose images modulo $T^s R$ generate the image of J in $R/T^s R$. Then $a^{(1)}, \dots, a^{(N+M)}$ generate J and our job's done here. \square

Proof of Proposition 3. Let r_1, \dots, r_n be generators of I . Then sending $X_i \mapsto r_i$ defines a surjective morphism $A[[X_1, \dots, X_n]] \twoheadrightarrow \widehat{A}$. Since $A[[X_1, \dots, X_n]]$ is noetherian by Lemma 1 and induction on n , so is its quotient \widehat{A} . \square

Corollary 8. *Suppose that A is a noetherian local ring and $I \subseteq A$ any (proper) ideal. Then $\dim A = \dim \widehat{A}$. In particular, A is regular iff \widehat{A} is regular.*

Proof. Let \mathfrak{m} be the maximal ideal of A . Then $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{A}$ (this equality holds because of Corollary 6) is the maximal ideal of the local ring \widehat{A} as was shown in the proof of [Hom, Corollary 2.2.2]. Since $I \subseteq \mathfrak{m}$, the quotients $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ already have I -torsion, hence

$$\mathfrak{m}^i/\mathfrak{m}^{i+1} \cong (\mathfrak{m}^i/\mathfrak{m}^{i+1})^\wedge \cong \widehat{\mathfrak{m}}^i/\widehat{\mathfrak{m}}^{i+1}$$

(the last isomorphism follows from exactness of completion). This shows that the associated graded rings $\mathrm{gr}(A, \mathfrak{m})$ and $\mathrm{gr}(\widehat{A}, \widehat{\mathfrak{m}})$ agree, hence (A, \mathfrak{m}) and $(\widehat{A}, \widehat{\mathfrak{m}})$ have the same Hilbert–Samuel polynomials, which shows $\dim A = \dim \widehat{A}$ by [Alg2, Theorem 20].

Now A and \widehat{A} have the same residue field k and $\mathfrak{m}/\mathfrak{m}^2 \cong \widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2$ by the I -torsion arguments we have seen several times now, so $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim_k \widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2$. Clearly this implies that A is regular iff \widehat{A} is. \square

Remark 2. In a similar fashion one can show that a noetherian local ring is Cohen–Macaulay, or Gorenstein, or a complete intersection, iff its I -adic completion is one as well. For example, for Cohen–Macaulayness one would need to show $\mathrm{depth}_A(A) = \mathrm{depth}_{\widehat{A}}(\widehat{A})$, which follows from the isomorphism $\mathrm{Ext}_A^p(k, A) \cong \mathrm{Ext}_{\widehat{A}}^p(k, \widehat{A})$ that was described in the proof of [Hom, Proposition 2.4.2].

Proposition 4. *Let A be a noetherian ring which is complete in the I -adic topology and let M be a finitely generated A -module. Then M is I -adically complete.*

Proof. Note that this is clearly fulfilled if $M = A^n$ is a finitely generated free A -module. Now let M be arbitrary. Since A is noetherian, M can be represented as $\mathrm{coker}(A^m \rightarrow A^n)$. Because A^m, A^n equal their own completions (as we have just seen) and completion is exact (by Corollary 2), $M = \widehat{M}$ holds as well. \square

Corollary 9. *Let A be a noetherian local ring maximal ideal \mathfrak{m} . Let B be a finite A -algebra.*

- (a) *Then B has only finitely many prime ideals over \mathfrak{m} , and all of them are maximal.*
- (b) *If A is, in addition, \mathfrak{m} -adically complete, then all maximal ideals of B lie over \mathfrak{m} . In particular, B is **semi-local** (i.e. has finitely many maximal ideals). The same is true if $A \subseteq B$.*

Proof. Part (a). If $\mathfrak{q} \in \mathrm{Spec} B$ is a prime ideal over \mathfrak{m} , then B/\mathfrak{q} is a finitely generated domain over the residue field $k = A/\mathfrak{m}$, hence a finite field extension of k , so \mathfrak{q} is maximal. Moreover, $B/\mathfrak{m}B$ is a finite-dimensional k -algebra, hence it has only finitely many maximal ideals by the argument from Fact 1.3.2(b).

Part (b). If A is \mathfrak{m} -adically complete, then B is $\mathfrak{m}B$ -adically complete by Proposition 4, so $\mathfrak{m}B$ is contained in the Jacobson radical $\mathrm{rad} B$ by Corollary 1(c). Then all maximal ideals of B lie over \mathfrak{m} .

Now assume $A \subseteq B$. Let $\mathfrak{q} \in \mathrm{Spec} B$ be a maximal ideal and $\mathfrak{p} = \mathfrak{q} \cap A$. Then $A/\mathfrak{p} \subseteq B/\mathfrak{q}$ is an integral ring extension in which B/\mathfrak{q} is a field, hence so is A/\mathfrak{p} , (by [Alg1, Proposition 1.5.1(d)]) which proves $\mathfrak{p} = \mathfrak{m}$. \square

Proposition 5. *Let A be a noetherian ring and B an A -algebra. Let $\mathfrak{q} \in \operatorname{Spec} B$ be a prime ideal and $\mathfrak{p} \in \operatorname{Spec} A$ its preimage in A . Let $\hat{A}_{\mathfrak{p}}$ denote the $\mathfrak{p}A_{\mathfrak{p}}$ -adic completion of $A_{\mathfrak{p}}$.*

- (a) *The ideal $\mathfrak{q}B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} \hat{A}_{\mathfrak{p}}$ is maximal in $B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} \hat{A}_{\mathfrak{p}}$.*
- (b) *Moreover, assume that B is of finite type over A (so that B is noetherian as well) and that $B \otimes_A \hat{\mathfrak{K}}(\mathfrak{p})$ is a finite-dimensional $\hat{\mathfrak{K}}(\mathfrak{p})$ -vector space. Then $B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} \hat{A}_{\mathfrak{p}}$ equals the $\mathfrak{q}B_{\mathfrak{q}}$ -adic completion $\hat{B}_{\mathfrak{q}}$ of $B_{\mathfrak{q}}$. In particular, $B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} \hat{A}_{\mathfrak{p}}$ is a local ring again.*

Remark 3. The conditions from Proposition 5 say precisely that the induced morphism $\operatorname{Spec} B \rightarrow \operatorname{Spec} A$ of schemes is *quasi-finite* at all primes above \mathfrak{p} , cf. Definition 1.3.1 and Fact 1.3.2(b).

Proof of Proposition 5. Part (a). The reason why $\mathfrak{q}B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} \hat{A}_{\mathfrak{p}}$ is a maximal ideal of $B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} \hat{A}_{\mathfrak{p}}$ is basically that $\hat{A}_{\mathfrak{p}}$ is local again with maximal ideal $\hat{\mathfrak{p}} = \mathfrak{p}\hat{A}_{\mathfrak{p}}$ and has the same residue field $\hat{\mathfrak{K}}(\mathfrak{p})$ as $A_{\mathfrak{p}}$. The details go as follows: We have

$$(B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} \hat{A}_{\mathfrak{p}}) / (\mathfrak{q}B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} \hat{A}_{\mathfrak{p}}) \cong \hat{\mathfrak{K}}(\mathfrak{q}) \otimes_{A_{\mathfrak{p}}} \hat{A}_{\mathfrak{p}}.$$

Denote $\varphi: A \rightarrow B$ the ring morphism that makes B an A -algebra. If $x \in \hat{\mathfrak{K}}(\mathfrak{q}) \otimes_{A_{\mathfrak{p}}} \hat{\mathfrak{p}}$, then x can be written as $x = \sum_{i=1}^n b_i \otimes p_i a_i$ where $b_i \in B_{\mathfrak{q}}$, $a_i \in A_{\mathfrak{p}}$, and $p_i \in \mathfrak{p}$. But then $x = \sum_{i=1}^n \varphi(p_i) b_i \otimes a_i = 0$, because the $\varphi(p_i)$ are elements of \mathfrak{q} as \mathfrak{p} is the preimage of \mathfrak{q} under φ . Hence

$$\hat{\mathfrak{K}}(\mathfrak{q}) \otimes_{A_{\mathfrak{p}}} \hat{A}_{\mathfrak{p}} \cong (\hat{\mathfrak{K}}(\mathfrak{q}) \otimes_{A_{\mathfrak{p}}} \hat{A}_{\mathfrak{p}}) / (\hat{\mathfrak{K}}(\mathfrak{q}) \otimes_{A_{\mathfrak{p}}} \hat{\mathfrak{p}}) \cong \hat{\mathfrak{K}}(\mathfrak{q}) \otimes_{A_{\mathfrak{p}}} \hat{\mathfrak{K}}(\mathfrak{p}) \cong \hat{\mathfrak{K}}(\mathfrak{q})$$

is a field, which shows that $\mathfrak{q}B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} \hat{A}_{\mathfrak{p}}$ is a maximal ideal.

Part (b). Brace yourself, because this proof is *horrible*. I'm trying my best to cite The Stacks Project as rarely as possible, but I make no promises. We will separate the agonizing process into tiny lemmas.

Lemma 2. *In the situation of Proposition 5(b), $B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} \hat{\mathfrak{K}}(\mathfrak{p})$ is a finite-dimensional $\hat{\mathfrak{K}}(\mathfrak{p})$ -vector space as well.*

Proof. We know that $\overline{B} = B \otimes_A \hat{\mathfrak{K}}(\mathfrak{p})$ is a finite-dimensional $\hat{\mathfrak{K}}(\mathfrak{p})$ -algebra. Let $B_{\mathfrak{p}} = B \otimes_A A_{\mathfrak{p}}$ be the localization of B at the image of the multiplicative set $A \setminus \mathfrak{p}$. Put $\overline{\mathfrak{q}} = \mathfrak{q}B_{\mathfrak{p}} / \mathfrak{p}B_{\mathfrak{p}}$, so that $\overline{B}_{\overline{\mathfrak{q}}} \cong B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} \hat{\mathfrak{K}}(\mathfrak{p})$. We know from Fact 1.3.2(b) that $\operatorname{Spec} \overline{B} \rightarrow \operatorname{Spec} \hat{\mathfrak{K}}(\mathfrak{p})$ is quasi-finite, hence $\{\overline{\mathfrak{q}}\}$ is open and closed in $\operatorname{Spec} \overline{B}$. By [Stacks, Tag 00EE] we get an idempotent $e \in \overline{B}$ such that $\{\overline{\mathfrak{q}}\} \cong \operatorname{Spec} \overline{B}_e$. Then $\overline{\mathfrak{q}}$ is the only prime ideal over \overline{B}_e , hence $\overline{B}_e \cong \overline{B}_{\overline{\mathfrak{q}}}$. Also $\overline{B}_e \cong \overline{B}[e^{-1}]$ is a $\hat{\mathfrak{K}}(\mathfrak{p})$ -algebra of finite type again and $\operatorname{Spec} \overline{B}_e \rightarrow \operatorname{Spec} \hat{\mathfrak{K}}(\mathfrak{p})$ is clearly quasi-finite in the sense of Definition 1.3.1. Using Fact 1.3.2(b) again, we see that \overline{B}_e is finite over $\hat{\mathfrak{K}}(\mathfrak{p})$, which is precisely what we wanted to show. \square

Lemma 3. *The $\mathfrak{p}B_{\mathfrak{q}}$ -adic completion of $B_{\mathfrak{q}}$ equals its $\mathfrak{q}B_{\mathfrak{q}}$ -adic completion $\hat{B}_{\mathfrak{q}}$. Moreover, $\hat{B}_{\mathfrak{q}}$ is finitely generated as an $\hat{A}_{\mathfrak{p}}$ -module, and such generators can be chosen from $B_{\mathfrak{q}}$.*

Proof. As we have just seen in Lemma 2, $B_q \otimes_{A_p} \mathfrak{K}(\mathfrak{p}) \cong B_q/\mathfrak{p}B_q$ is a noetherian local ring with only one prime ideal \bar{q} . Then \bar{q} must be the nilradical of $B_q/\mathfrak{p}B_q$, hence $\bar{q}^N = 0$ for some $N \in \mathbb{N}$ (because we are noetherian). This shows $(qB_q)^N \subseteq \mathfrak{p}B_q$, hence the $\mathfrak{p}B_q$ -adic and the qB_q -adic completions of B_q coincide.

Since $B_q/\mathfrak{p}B_q \cong \hat{B}_q/\mathfrak{p}\hat{B}_q$ is finite-dimensional as a vector space over $\mathfrak{K}(\mathfrak{p}) \cong \hat{A}_p/\mathfrak{p}\hat{A}_p$, the second assertion follows from the more general Lemma 4. \square

Lemma 4. *Let A be a noetherian ring which is I -adically complete and let M be an I -adically complete A -module. If M/IM is finitely generated over A/I , then M is finitely generated over A . In fact, lifting a finite set of generators of M/IM over A/I to M gives a (finite) generating set of M .*

Remark 4. The Stacks Project has a more general version of this, cf. [Stacks, Tag 031D].

Proof of Lemma 4. Let x_1, \dots, x_s be lifts of a finite generating set of M/IM and let $N \subseteq M$ be the submodule generated by the x_i . Then N is I -adically complete by Proposition 4.

We claim that also $N/I^n M \rightarrow M/I^n M$ is surjective for all $n \geq 1$. Indeed, for $n = 1$ this is trivial. Using this, it's easy to see that

$$\operatorname{coker} \left(N/I^n N \rightarrow M/I^n M \right) = (I/I^n) \cdot \operatorname{coker} \left(N/I^n N \rightarrow M/I^n M \right).$$

But (I/I^n) is a nilpotent ideal in A/I^n , hence $\operatorname{coker} (N/I^n N \rightarrow M/I^n M) = 0$ (this feels like a dummy version of Nakayama's lemma). We thus get short exact sequences

$$0 \rightarrow (I^n M \cap N)/I^n N \rightarrow N/I^n N \rightarrow M/I^n M \rightarrow 0.$$

Let $x \in I^n M \cap N$. Then x can be written as $x = \sum_i \alpha_i m_i$, where $\alpha_i \in I^n$ and $m_i \in M$. Because $N/IN \rightarrow M/IM$ is surjective, we may write $m_i = n_i + \mu_i$ where $n_i \in N$ and $\mu_i \in IM$. Then $y = \sum_i \alpha_i \mu_i$ is an element in $I^{n+1}M \cap N$ and has the same image in $(I^n M \cap N)/N$ as x . This shows that $((I^n M \cap N)/I^n N)_{n \geq 1}$ has the Mittag-Leffler property from Fact 1.1.3. This proves $\varprojlim^1 (I^n M \cap N)/I^n N = 0$, hence

$$N \cong \varprojlim_{n \geq 1} N/I^n N \rightarrow \varprojlim_{n \geq 1} M/I^n M \cong M$$

is surjective by the six-term exact sequence from Fact 1.1.1. \square

Finally, the proof of Proposition 5(b) can be finished. By [Hom, p. 18], $B_q \otimes_{A_p} \hat{A}_p$ is the union (or rather the colimit) over all $M \otimes_{A_p} \hat{A}_p$ where M ranges through the finitely generated A_p -submodules of B_q . But for such M we have $M \otimes_{A_p} \hat{A}_p \cong \hat{M} \subseteq \hat{B}_q$ by Corollary 3(a). Moreover, by Lemma 3 we can find such an M that satisfies $\hat{M} = \hat{B}_q$. Hence $B_q \otimes_{A_p} \hat{A}_p \cong \hat{B}_q$, and we're finally done. \square

A.2. Generic freeness and Chevalley's theorem

A good reference for this section is [Eis95, Section 14.2 and 14.3].

Professor Franke points out that there are multiple approaches for Generic freeness: a clever *dévissage*-style proof due to Grothendieck (cf. [Eis95, Theorem 14.4]), which Franke says he could not have come up with, or a more natural proof similar to that of Hilbert's Basissatz (cf. [MR89, §24]). However, both references assume R to be noetherian, which we will circumvent by Gröbner basis theory.¹

Proposition 1 (Grothendieck's generic freeness theorem). *Let R be a domain, A an R -algebra of finite type, and M a finitely generated A -module. Then there is $f \in R \setminus \{0\}$ such that M_f is a free R_f -module*

Proof. Every finitely generated A -module M has a filtration $0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = M$ such that M_i/M_{i-1} has the form A/I_i for some ideal $I_i \subseteq A$. If all M_i/M_{i-1} are free R -modules, then each sequence $0 \rightarrow M_{i-1} \rightarrow M_i \rightarrow M_i/M_{i-1} \rightarrow 0$ is split, so M is a free R -module as well. This argument shows that we only need to deal with the case $M \cong A/I$ for some ideal $I \subseteq A$. Replacing A by A/I this can be further reduced to $M = A$.

We can represent A as $A \cong R[X_1, \dots, X_n]/I$ for some ideal $I \subseteq R[X_1, \dots, X_n]$. Let K be the field of quotients of R and $J \subseteq K[X_1, \dots, X_n]$ the ideal generated by the image of I . Let $(\beta_1, \dots, \beta_r)$ be a Gröbner basis of J (for some fixed monomial order), then $\beta_i = \frac{b_i}{d_i}$ for some $b_i \in I$ and $d_i \neq 0$ and the β_i have leading term 1. Replacing R by the localization $R_{d_1 \dots d_r}$ we may assume $\beta_i \in I$. Using generalized division with remainder (i.e. Gröbner basis stuff) one easily derives that the β_i generate I . Then the monomials $X_1^{\alpha_1} \dots X_n^{\alpha_n}$ where $\alpha = (\alpha_1, \dots, \alpha_n)$ has the property that there is no i such that α dominates the leading term of β_i , form a basis of A as an R -module. \square

Proposition 2 (Chevalley's theorem). *If $f: \operatorname{Spec} B \rightarrow \operatorname{Spec} A$ is a morphism of finite type between affine noetherian schemes, then the image of f is contained in a proper closed subset of $\operatorname{Spec} A$ or contains a dense open subset.*

Proof. What Chevalley's theorem actually says is that the image of f is **constructible**, that is, a finite union of open subsets of closed subsets of $\operatorname{Spec} A$ (cf. [Eis95, Corollary 14.6]). This is what we're going to prove now (and we leave it to the reader to show that this implies the assertion – which I found quite a pain, actually).

It suffices to prove the assertion for every irreducible component of $\operatorname{Spec} A$ (as these guys are closed and there are finitely many of them). Hence, if $\mathfrak{p} \in \operatorname{Spec} A$ is a minimal prime corresponding to the irreducible component $V(\mathfrak{p})$ of $\operatorname{Spec} A$, then we may replace A and B by A/\mathfrak{p} and $B/\mathfrak{p}B$. Therefore, we can assume that A is a domain. Moreover, suppose that $A \subseteq B$. Otherwise, A could be replaced by $A/\ker(A \rightarrow B)$, and since that cuts out a closed subset of $\operatorname{Spec} A$, we're fine with that.

¹True Franke fans will remember ...

By Noetherian induction, we may assume that the assertion is true for all closed subsets of $\operatorname{Spec} A$. Let $\alpha \in A$ such that B_α is free as an A_α -module (we can do this by Proposition 1). We claim that any $\mathfrak{p} \in \operatorname{Spec} A \setminus V(\alpha)$ is in the image of f . Indeed, such \mathfrak{p} may be regarded as prime ideals $\mathfrak{p} \in \operatorname{Spec} A_\alpha$. Then $B_\mathfrak{p} \cong B \otimes_A A_\mathfrak{p}$ is still free as an $A_\mathfrak{p}$ -module, say, $B_\mathfrak{p} \cong \bigoplus_{i \in I} A_\mathfrak{p}$ for some indexing set I (which is non-empty as $A_\mathfrak{p} \subseteq B_\mathfrak{p}$). Hence $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \cong B_\mathfrak{p} \otimes_{A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \cong \bigoplus_{i \in I} \mathfrak{K}(\mathfrak{p})$ is not the zero ring because $\mathfrak{K}(\mathfrak{p}) \neq 0$. If $\mathfrak{q} \subseteq B$ is the preimage of any prime ideal of $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$, then it's easily seen that $f(\mathfrak{q}) = \mathfrak{p}$, proving that \mathfrak{p} is in the image of f , as claimed.

By the Noetherian induction hypothesis, the image of f in $V(\alpha)$ – which equals the image of $\bar{f}: \operatorname{Spec}(B/\alpha B) \rightarrow \operatorname{Spec}(A/\alpha A)$ – is constructible. We are done. \square

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