Homological Methods in Commutative Algebra

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Introduction

Professor Franke started the lecture giving an idea of what the Tor and Ext functors do. Let R be a commutative ring with 1. For an exact sequence of R-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and T another R-module, the sequence

$$M' \otimes_R T \longrightarrow M \otimes_R T \longrightarrow M'' \otimes_R T \longrightarrow 0 \tag{1}$$

is exact but usually can't be extended by 0 on the left end. The same is true for

$$0 \longrightarrow \mathrm{Hom}_R(T, M') \longrightarrow \mathrm{Hom}_R(T, M) \longrightarrow \mathrm{Hom}_R(T, M'') \tag{2}$$

and

$$0 \longrightarrow \mathrm{Hom}_R(M'', T) \longrightarrow \mathrm{Hom}_R(M, T) \longrightarrow \mathrm{Hom}_R(M', T)\ , \tag{3}$$

but again, they can't be extended by 0 on the right in general.

Example. Take $R = \mathbb{Z}$ and consider the exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$.

(a) Let $T = \mathbb{Z}/2\mathbb{Z}$ in (1). Then $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/2\mathbb{Z}$ is the zero morphism, showing that injectivity on the left end fails in (1).

(b) Let $T = \mathbb{Z}/2\mathbb{Z}$ in (2). We claim that surjectivity fails on the right end. Indeed, if it was surjective, then $\mathrm{id}_{\mathbb{Z}/2\mathbb{Z}} \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ would have to have a lift

$$\begin{tikzcd} & \mathbb{Z} \arrow[d] \\ \mathbb{Z}/2\mathbb{Z} \arrow[ur, dashed] \arrow[r, equal] & \mathbb{Z}/2\mathbb{Z} \end{tikzcd}$$

which it hasn't as $\mathbb{Z}$ is 2-torsion free and thus every morphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ must be 0.

(c) Let $T = \mathbb{Z}$ in (3). We claim that that surjectivity fails on the right end, or more specifically, that $\mathrm{id}_{\mathbb{Z}} \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ has no preimage. Indeed, if $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ is a preimage of $\mathrm{id}_{\mathbb{Z}}$, i.e. the composition $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{f} \mathbb{Z}$ equals $\mathrm{id}_{\mathbb{Z}}$, then f must be given by $f(n) = \frac{n}{2}$ on $2\mathbb{Z}$, but this can't be extended to all of $\mathbb{Z}$, contradiction!

To handle this deficiency, one constructs *derived functors* Tor and Ext, which give rise to long exact sequences

$$\begin{aligned}\dots \longrightarrow \operatorname{Tor}_2^R(M'',T) \longrightarrow \operatorname{Tor}_1^R(M',T) \longrightarrow \operatorname{Tor}_1^R(M,T) \longrightarrow \operatorname{Tor}_1^R(M'',T)\\ \longrightarrow M'\otimes_R T \longrightarrow M\otimes_R T \longrightarrow M''\otimes_R T \longrightarrow 0\ ,\end{aligned}$$

as well as

$$\begin{aligned}0 \longrightarrow \operatorname{Hom}_R(T,M') \longrightarrow \operatorname{Hom}_R(T,M) \longrightarrow \operatorname{Hom}_R(T,M'')\\ \longrightarrow \operatorname{Ext}^1_R(T,M') \longrightarrow \operatorname{Ext}^1_R(T,M) \longrightarrow \operatorname{Ext}^1_R(T,M'') \longrightarrow \operatorname{Ext}^2_R(T,M') \longrightarrow \dots\end{aligned}$$

and

$$\begin{aligned}0 \longrightarrow \operatorname{Hom}_R(M'',T) \longrightarrow \operatorname{Hom}_R(M,T) \longrightarrow \operatorname{Hom}_R(M',T)\\ \longrightarrow \operatorname{Ext}^1_R(M'',T) \longrightarrow \operatorname{Ext}^1_R(M,T) \longrightarrow \operatorname{Ext}^1_R(M',T) \longrightarrow \operatorname{Ext}^2_R(M'',T) \longrightarrow \dots\ .\end{aligned}$$

extending the open ends of (1), (2), and (3) respectively.

A highlight of this lecture will be *Serre's characterization of regularity.*

Theorem. *For a Noetherian local ring R with maximal ideal $\mathfrak{m}$ and residue field k, the following are equivalent.*

(*a*) $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim R$ *(i.e., R is regular).*

(*b*) *There is some vanishing bound for* $\operatorname{Tor}_*^R(-,-)$.

(*c*) *… and* $\dim R$ *is such a vanishing bound.*

(*d*) *There is some vanishing bound for* $\operatorname{Ext}_R^*(-,-)$.

(*e*) *… and* $\dim R$ *is again such a vanishing bound.*

From this, one can deduce the following

Corollary. *If R is a regular Noetherian local ring and $\mathfrak{p} \in \operatorname{Spec} R$, then $R_\mathfrak{p}$ is regular as well.*

We will also introduce the notion of *Cohen–Macaulay rings* and prove that they are *universally catenary* (which is quite a generalization of what we did in Algebra I, cf. [1, Theorem 10]).

Theorem. *If R is a regular Noetherian local ring or, more generally, a Cohen–Macaulay ring, then it is* **universally catenary***: If A is an R-algebra of finite type and $X \subseteq Y \subseteq Z$ are irreducible closed subsets of* $\operatorname{Spec} A$*, then*

$$\operatorname{codim}(X,Y) + \operatorname{codim}(Y,Z) = \operatorname{codim}(X,Z)\ .$$

1. Tor and Ext of R-modules

From now on, unless otherwise stated, our rings are commutative with 1.

1.1. Injective and projective modules and properties of Ext_R^*

Proposition 1 (Baer's criterion)**.** *For an R-module N, the following are equivalent.*

(*a*) *The functor* $\mathrm{Hom}_R(-, N)$ *is exact.*

(*b*) *For any embedding $M' \hookrightarrow M$ of R-modules,* $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N)$ *is surjective.*

(*c*) *Property* (*b*) *holds for $R = M$. In other words, if $I \subseteq R$ is any ideal, then any morphism $I \to N$ of R-modules extends to a morphism $R \to N$.*

Remark 1. (*a*) Since there is a bijection

$$\begin{aligned} \mathrm{Hom}_R(R, M) &\xrightarrow{\sim} M \\ (r \mapsto r \cdot m) &\longleftarrow m \\ \left(R \xrightarrow{\varphi} M\right) &\longmapsto \varphi(1)\ , \end{aligned}$$

Proposition 1(*c*) can be reformulated as that any morphism $I \to N$ for $I \subseteq R$ an ideal has the form $i \mapsto i \cdot m$ for some $m \in M$.

(*b*) Note that Proposition 1(*c*) is trivial when $I = 0$.

(*c*) When $R = \mathbb{Z}$, every ideal $I \subseteq \mathbb{Z}$ has the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$ and a morphism $n\mathbb{Z} \xrightarrow{\varphi} N$ is uniquely determined by $\varphi(n)$. Thus, an extension $\hat{\varphi}$ of φ to $\mathbb{Z}$ exists iff there is an element $\nu \in N$ such that $n \cdot \nu = \varphi(n)$ (in that case, put $\hat{\varphi}(1) = \nu$). Hence, Proposition 1(*c*) amounts to whether the abelian group N is *divisible*, that is, whether $N \xrightarrow{\cdot n} N$ is surjective for all $n \in \mathbb{Z}$ (also cf. Definition 2).

Definition 1. (*a*) An R-module is called **injective** if it satisfies the equivalent conditions from Proposition 1.

(*b*) In an arbitrary category $\mathcal{A}$, an object I is called **injective** if for every monomorphism $X \hookrightarrow Y$, the induced map $\mathrm{Hom}_{\mathcal{A}}(Y, I) \to \mathrm{Hom}_{\mathcal{A}}(X, I)$ is surjective, that is, for every morphism $X \xrightarrow{\varphi} I$ there is a (usually non-unique) lift

$$\begin{tikzcd} I & \\ X \arrow[u, "\varphi"] \arrow[r, hook] & Y \arrow[ul, dashed, "\exists\, \hat{\varphi}"'] \end{tikzcd}$$

Proof of Proposition 1. The implication $(b) \Rightarrow (c)$ is trivial. Let's prove $(c) \Rightarrow (b)$. Let $M \xrightarrow{f} N$ be a morphism of R-modules and consider

$$\mathfrak{M} = \left\{(Q, \varphi) \mid M \subseteq Q \subseteq M' \text{ and } \widetilde{\varphi} \in \mathrm{Hom}_R(Q, N) \text{ such that } \varphi|_M = f\right\} .$$

$\mathfrak{M}$ becomes a partially ordered set via $(Q_1, \varphi_1) \preceq (Q_2, \varphi_2) \Leftrightarrow Q_1 \subseteq Q_2$ and $\varphi_2|_{Q_1} = \varphi_1$. Then it's easy to see that Zorn's lemma is applicable, hence $\mathfrak{M}$ has a $\preceq$-maximal element (Q_*, φ_*). If (c) is satisfied and $Q_* \subsetneq M'$, there is an $m \in M' \setminus L_*$. Let $I = \{r \in R \mid rm \in Q_*\}$ and let $I \xrightarrow{g} N$ be given by $g(r) = \varphi_*(rm)$. By (c), there is a morphism $R \xrightarrow{\gamma} N$ extending g, i.e., a $\nu \in N$ such that $\varphi_*(rm) = r\nu$ when $r \in I$ (using Remark 1(a)). Let $\widetilde{Q} = Q_* + Rm$ and $\widetilde{\varphi}(m_* + rm) = \varphi_*(m_*) + r\nu$ for $m_* \in Q_*$ and $r \in R$, then it's easy to see that $\widetilde{\varphi}$ is well-defined and $(Q_*, \varphi_*) \prec (\widetilde{Q}, \widetilde{\varphi})$, a contradiction.

The equivalence $(a) \Leftrightarrow (b)$ is easy to see as for any short exact sequence $0 \to X \to Y \to Z \to 0$, the sequence $0 \to \mathrm{Hom}_R(Z, N) \to \mathrm{Hom}_R(Y, N) \to \mathrm{Hom}_R(X, N)$ is exact anyways and (b) implies exactness at the right end. *q.e.d.*

Definition 2. If R is a domain and M an R-module, then M is called **divisible** if $M \xrightarrow{r\cdot} M$ is surjective for all $r \in R \setminus \{0\}$

Corollary 1. (a) *When R is a domain, the property from Proposition 1(c) for principal ideals I is equivalent do divisibility of N.*

(b) *Any injective module N is divisible in the following sense: If $r \in R$ is not a zero divisor, $N \xrightarrow{r\cdot} N$ is surjective.*

(c) *In particular, if N is injective and $S \subseteq R$ a multiplicative subset not containing zero divisors, then the morphism $N \to N_S$ to the localization of N at S is surjective.*

Proof. Part (a) can be seen using the arguments from Remark 1(c). For (b), note that $R \xrightarrow{r\cdot} R$ is injective when r is no zero divisor, hence, for any $n \in N$, the morphism $\varphi \in \mathrm{Hom}_R(R, N)$ given by $\varphi(1) = n$ extends to $\hat{\varphi} \in \mathrm{Hom}_R(R, N)$ such that $\varphi = r\hat{\varphi}$. Then $\hat{\varphi}(1)$ is a preimage of n under $N \xrightarrow{r\cdot} N$. Part (c) follows from (b) and the universal property of localization. *q.e.d.*

Remark. Note that $R = \mathbb{Z}/p^2\mathbb{Z}$, for $p \in \mathbb{Z}$ a prime, is injective over itself, but $R \xrightarrow{p\cdot} R$ fails to be injective. Indeed, the only ideal of R where Baer's criterion is in question is $(p) \subseteq R$. We need to show that any R-morphism $(p) \to R$ extends to an R-morphism $R \to R$. But any $(p) \xrightarrow{\varphi} R$ maps p to the p-torsion part of R, i.e., to (p) itself, hence is given by $\varphi(p) = rp$ for some $r \in R$ and can be extended via $\hat{\varphi}$ given by $\hat{\varphi}(1) = r$. This shows that Corollary 1(b) is somewhat sharp.

Corollary 2. *A module over a principal ideal domain is injective iff it is divisible.*

Proof. Follows from Corollary 1(a). *q.e.d.*

Remark. The same holds for Dedekind domains, see Corollary 6 (which is not there yet).

Corollary 3. *When R is a principal ideal domain, then any quotient of an injective module is injective again. The category of R-modules has* **sufficiently many injective objects** *in* *the sense that for any object X there is a monomophism $X \hookrightarrow I$ with I injective. Thus, any R-module X has an* **injective resolution**, *i.e., an exact sequence*

$$0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots$$

with injective objects $I^0, I^1, I^2, \dots$. In fact, any R-module, for R a principal ideal domain, has an injective resolution $0 \to X \to I^0 \to I^1 \to 0$ of length 1.

Proof. The first assertion follows as the quotient of divisible modules is divisible again. Note that K/R is divisible, K being the quotient field of R, hence it is injective. If M is any R-module and $m \in M \setminus \{0\}$. We have to distinguish to cases.

Case 1. Suppose $\mathrm{Ann}_R(m)$ is non-zero, i.e., $\mathrm{Ann}_R(m) = (\alpha)$ for some $\alpha \in R \setminus \{0\}$ (remember we have a principal ideal domain). Then we have a morphism from $Rm \subseteq M$ to K/R given by $rm \mapsto \frac{r}{\alpha} \bmod R$ (note that modding out R is necessary for this to be well-defined – we couldn't just have used K). By injectivity of K/R, there is an extension $M \xrightarrow{\varphi_m} K/R$, satisfying $\varphi_m(m) \neq 0$. Let $I_m \subseteq K/R$ be the target of φ_m.

Case 2. If $\mathrm{Ann}_R(m) = 0$, we get a morphism from $Rm \subseteq M$ to K instead, sending $rm \mapsto r$ (this time, using K doesn't cause problems thanks to $\mathrm{Ann}_R(m) = 0$). By injectivity of K, this extends to a morphism $M \xrightarrow{\varphi_m} K$ such that $\varphi_m(m) \neq 0$. Let $I_m = K$ be the target of φ_m.

Now put $I = \prod_{m \in M \setminus \{0\}} I_m$. Then I is divisible (since every I_m is), hence injective, and $M \to I$, $\mu \mapsto (\varphi_m(\mu))_{m \in M \setminus \{0\}}$ is a monomorphism. As a quotient of $I^0 = I$, $I^1 = \mathrm{coker}\,(M \to I^0)$ is injective as well, hence $0 \to M \to I^0 \to I^1 \to 0$ is an injective resolution of length 1. *q.e.d.*

Proposition 2 (a.k.a. "Satz 2")**.** *For any ring R, the category of R-modules has sufficiently many injective objects.*

Proof. This will follow from Lemma 1(b) and (c) below. *q.e.d.*

Remark. This holds in vast more generality, and in particular, Proposition 2 follows immediately from the following theorem, which, however, we are not going to prove in this lecture.

Theorem (Grothendieck)**.** *Any AB5 category with a generator has sufficiently many injective objects.*

Lemma 1. *Let R be any ring.*

(a) *The forgetful functor from R-Mod to the category $\mathbb{Z}$-Mod of abelian groups has a right-adjoint functor, namely $\mathrm{Hom}_{\mathbb{Z}}(R, -)$. That is, there is a bijection*

$$\mathrm{Hom}_{\mathbb{Z}}(M, A) \xrightarrow{\sim} \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, A)) \tag{$*$}$$

for any R-module M and any abelian group A. Here, we equip $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ with an R-module structure via $(r \cdot \varphi)(x) = \varphi(xr)$ for $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(R, A)$ and $r, x \in R$.

(b) *For any injective abelian group I, $\mathrm{Hom}_{\mathbb{Z}}(R, I)$ is an injective R-module.*

(c) *Let M be any R-module and I and abelian group and $M \overset{\varphi}{\hookrightarrow} I$ a monomorphism of abelian groups, then the R-morphism $M \to \mathrm{Hom}_{\mathbb{Z}}(R, I)$ obtained by applying ($*$) is injective.*

Proof. Part (a). The proof given in the lecture was rather computational, so I decided to include a more elegant one. It is easy to see that $\mathrm{Hom}_{\mathbb{Z}}(R, -)$ is indeed a functor $\mathbb{Z}$-Mod $\to R$-Mod. From the well-known tensor-hom adjunction we obtain a canonical bijection

$$\mathrm{Hom}_{\mathbb{Z}}(M \otimes_R R, A) \xrightarrow{\sim} \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, A)) \; .$$

But M is an R-module and so $M \otimes_R R \simeq M$ canonically, proving $(*)$.

Part (b). Since the forgetful functor R-Mod $\to \mathbb{Z}$-Mod clearly preserves injectivity of morphisms (i.e., monomorphisms), this comes down to the following more general fact about adjoint pairs of functors.

Fact 1. Let $\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{B}$ be an adjoint pairs of functors. Suppose that L preserves monomorphisms. Then R preserves injective objects.

Proof of Fact 1. Let $I \in \mathrm{Ob}(\mathcal{B})$ be injective and $X \hookrightarrow Y$ be a monomorphism in $\mathcal{A}$. By assumption, $LX \hookrightarrow LY$ is a monomorphism in $\mathcal{B}$. In the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(Y, RI) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_{\mathcal{A}}(X, RI) \arrow[d, "\wr"] \\
\mathrm{Hom}_{\mathcal{B}}(LY, I) \arrow[r] & \mathrm{Hom}_{\mathcal{B}}(LX, I)
\end{tikzcd}$$

the lower horizontal arrow is surjective by injectivity of I, hence so is the upper horizontal arrow. *q.e.d.*

Back to the proof of Lemma 1 and let's prove (c). Let $M \overset{\varphi}{\hookrightarrow} I$ be a monomorphism of abelian groups. The corresponding morphism $M \xrightarrow{\psi} \mathrm{Hom}_{\mathbb{Z}}(R, I)$ sends $m \in M$ to $\psi(m)\colon R \to I$ given by $\psi(m)(r) = \varphi(rm)$. If $\psi(m)$ is the zero morphism for some $m \in M$, then $0 = \psi(m)(1) = \varphi(m)$, proving $m = 0$ by injectivity of φ. Then ψ is also injective. *q.e.d.*

A. Appendix – category theory corner

A.1. Derived functors and Ext_R^*

Definition 1. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories (cf. [2, Definition A.1.4]). A **homological ∂-functor** $F_*\colon \mathcal{A} \to \mathcal{B}$ is a sequence $(F_n)_{n\geq 0}$ of additive functors $\mathcal{A} \xrightarrow{F_i} \mathcal{B}$ together with a natural transformation $\partial = \partial_F\colon F_{i+1}(A'') \to F_i(A')$ on the category of short exact sequences on $\mathcal{A}$, such that the sequence

$$\begin{aligned} \ldots \longrightarrow F_{i+1}(A'') \xrightarrow{\partial} F_i(A') \longrightarrow F_i(A) \longrightarrow F_i(A'') \xrightarrow{\partial} \ldots \\ \ldots \longrightarrow F_1(A'') \xrightarrow{\partial} F_0(A') \longrightarrow F_0(A) \longrightarrow F_0(A'') \longrightarrow 0\,. \end{aligned}$$

is exact whenever $0 \to A' \to A \to A'' \to 0$ is a short exact sequence in $\mathcal{A}$.

A **morphism** $F_* \xrightarrow{\varphi} G_*$ of homological ∂-functors is a sequence $(\varphi_n)_{n\geq 0}$ of natural transformations $F_i \xrightarrow{\varphi_i} G_i$ such that for any short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$ the diagram

$$\begin{tikzcd} F_{i+1}(A'') \arrow[r, "\partial_F"] \arrow[d, "\varphi_{i+1}"'] & F_i(A') \arrow[d, "\varphi_i"] \\ G_{i+1}(A'') \arrow[r, "\partial_G"] & G_i(A') \end{tikzcd}$$

commutes.

Similarly, a **cohomological ∂-functor** $F^*\colon \mathcal{A} \to \mathcal{B}$ is a sequence $(F^n)_{n\geq 0}$ of additive functors $\mathcal{A} \xrightarrow{F^i} \mathcal{B}$ together with connecting morphism $\partial = \partial_F\colon F^i(A'') \to F^{i+1}(A')$ such that for a short exact sequence $0 \to A' \to A \to A'' \to 0$, the sequence

$$\begin{aligned} 0 \longrightarrow F^0(A') \longrightarrow F^0(A) \longrightarrow F^0(A'') \xrightarrow{\partial} F^1(A') \longrightarrow \ldots \\ \ldots \xrightarrow{\partial} F^i(A') \longrightarrow F^i(A) \longrightarrow F^i(A'') \xrightarrow{\partial} F^{i+1}(A') \longrightarrow \ldots \end{aligned}$$

is required to be exact. And the notion of a **morphism** $F^* \xrightarrow{\varphi} G^*$ of cohomological ∂-functors is defined in the obvious way.

Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ be a *right-exact* functor, i.e., for any short exact sequence $0 \to A' \to A \to A'' \to 0$, the sequence $FA' \to FA \to FA'' \to 0$ is exact (but that's not quite it, cf. Definition A.2.3). A **left-derived functor** of F is a homological functor L_*F from $\mathcal{A}$ to $\mathcal{B}$ with a natural isomorphism $L_0F \simeq F$ such that for any homological functor $\Phi_*\colon \mathcal{A} \to \mathcal{B}$, any natural transformation $\Phi_0 \to L_0F$ extends in a unique way to a morphism $\Phi_* \to L_*F$ of homological functors.

Similar, a functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is *right-exact* functor if $0 \to FA' \to FA \to FA''$ is exact for any short exact sequence $0 \to A' \to A \to A'' \to 0$ (and F is *additive*, cf. Definition A.2.3). A **right-derived functor** of F is a homological functor R^*F from $\mathcal{A}$ to $\mathcal{B}$ with a natural isomorphism $R^0F \simeq F$ such that for any homological functor $\Psi^* \colon \mathcal{A} \to \mathcal{B}$, any natural transformation $R^0F \to \Psi^0$ extends in a unique way to a morphism $R^*F \to \Psi^*$ of homological functors.

Remark 1. (*a*) It follows (by the usual Yoneda argument) that derived functors are unique up to unique isomorphism of (co)homological functors if they exist.

(*b*) If F is left-exact in the above sense, it preserves monomorphisms and it can be shown that $0 \to FX' \to FX \to FX''$ is exact even when only $0 \to X' \to X \to X''$ is exact (a nasty technical proof which won't appear here). Similar for right-exact functors.

(*c*) A generalized definition drops the exactness assumptions and requires $F \to L_0F$ with the universal property that any diagram

$$\begin{tikzcd} F \arrow[rr] \arrow[dr] & & L_0F \\ & \Phi_0 \arrow[ur] & \end{tikzcd}$$

can be uniquely extended to a morphism $\Phi_* \to L_*F$ of homological functors. Similar for right-derived functors.

Example 1. If F is an exact functor, then left- and right-derived functors of F are given by $L_0F = R^0F = F$ and $L_iF = R^iF = 0$ for $i \geq 1$.

Definition 2. An object I in an abelian category $\mathcal{A}$ is **injective** iff the following equivalent conditions hold.

(*a*) When $X \overset{\xi}{\hookrightarrow} Y$ is a monomorphism, then any morphism $X \xrightarrow{\iota} I$ extends to a morphism $Y \to I$ (i.e., I is injective in the sense of Definition 1.1.1(*b*)).

(*b*) Any short exact sequence $0 \to I \to X \to X'' \to 0$ splits.

Proof. To see (*a*) $\Rightarrow$ (*b*), extend id_I to $X \xrightarrow{\pi} I$, then π gives a split of the exact sequence (the argument used in the case of R-modules still works in arbitrary abelian categories).

For (*b*) $\Rightarrow$ (*a*) consider $C = \operatorname{coker}\left(X \xrightarrow{\iota \times \xi} I \oplus Y\right)$ and let $I \oplus Y \xrightarrow{p} C$ be the associated morphism. Let $i = \mathrm{id}_I \times 0$ and $j = 0 \times \mathrm{id}_Y$ be the canonical inclusions $I \to I \oplus Y$ and $Y \to I \oplus Y$. We claim that the composition

$$I \xrightarrow{i} I \oplus Y \xrightarrow{p} C$$

is a monomorphism. First note that $X \xrightarrow{\iota \times \xi} I \oplus Y$ is a monomorphism (since its composition with the projection to Y equals ξ, which was supposed to be monic), hence it's the kernel of its own cokernel as we are working in an abelian category and thus every monomorphism is an *effective monomorphism*, cf. [2, Definition A.1.3(*d*) and Definition A.1.4]. That is, $X = \ker(p)$.

Suppose now that $T \xrightarrow{\tau} I$ is a morphism satisfying $pi\tau = 0$, then $i\tau$ factors over $X = \ker(p)$. We thus have a diagram

$$\begin{tikzcd} T \arrow[r, "\tau"] \arrow[d, dashed, "\exists!\, \vartheta"'] & I \arrow[d, "i"] \\ X \arrow[r, "\iota \times \xi"] & I \oplus Y \end{tikzcd} \tag{$*$}$$

Postcomposing with the canonical projection $I \oplus Y \xrightarrow{\pi} Y$ we see that $\pi i \tau = 0 \circ \tau = 0$, hence also $\xi\vartheta = 0$ as $(*)$ commutes and $\pi \circ (\iota \times \xi) = \xi$. But ξ is a monomorphism, hence $\vartheta = 0$. By $(*)$, this implies $\tau = 0$ as i is a monomorphism. This shows that α is indeed a monomorphism.

We thus obtain a short exact sequence

$$0 \longrightarrow I \longrightarrow C \longrightarrow \operatorname{coker}(\alpha) \longrightarrow 0$$

which splits due to (b), i.e., $C \simeq I \oplus \operatorname{coker}(\alpha)$. Let $C \xrightarrow{q} I$ be the associated projection. Consider the composition

$$Y \xrightarrow{j} I \oplus Y \xrightarrow{p} C \xrightarrow{q} I\ .$$

We claim that $\upsilon = -qpj$ is a morphism $Y \xrightarrow{\upsilon} I$ extending $X \xrightarrow{\iota} I$. We have $qp \circ (\iota \times \xi) = q \circ 0 = 0$ since C is precisely the cokernel of $\iota \times \xi$. Also $qp \circ (\iota \times 0) = \iota$ as $qpi = \mathrm{id}_I$ by construction of q. Then

$$\upsilon\xi = -qpj\xi = -qp \circ (0 \times \xi) = qp \circ ((\iota \times 0) - (\iota \times \xi)) = \iota - 0 = \iota\ ,$$

hence υ has indeed the required property. *q.e.d.*

Theorem A. *Let $\mathcal{A}$ be an abelian category with sufficiently many injective objects.*

(a) *Any left-exact functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ from $\mathcal{A}$ to any abelian category $\mathcal{B}$ has a right-derived functor.*

(b) *Let $\Phi^*\colon \mathcal{A} \to \mathcal{B}$ be a cohomological functor, then Φ^* is a right-derived functor of Φ^0 iff $\Phi^p I = 0$ for any injective object $I \in \mathrm{Ob}(\mathcal{A})$ and all $p > 0$.*

(c) *Let $\mathcal{F}\colon 0 \to F' \to F \to F'' \to 0$ be a sequence of left-exact functors $\mathcal{A} \to \mathcal{B}$ and functor morphisms between them such that $0 \to F'I \to FI \to F''I \to 0$ is exact when I is an injective object of $\mathcal{A}$. Then there is a unique sequence of natural transformations $R^iF'' \xrightarrow{d_{\mathcal{F}}} R^{i+1}F'$ such that*

$$0 \longrightarrow F'X \longrightarrow FX \longrightarrow F''X \xrightarrow{d_{\mathcal{F}}} R^1F'X \longrightarrow \dots$$
$$\dots \longrightarrow R^{i-1}F''X \xrightarrow{d_{\mathcal{F}}} R^iF'X \longrightarrow R^iFX \longrightarrow R^iF''X \xrightarrow{d_{\mathcal{F}}} R^{i+1}F'X \longrightarrow \dots$$

is exact for arbitrary $X \in \mathrm{Ob}(\mathcal{A})$ and such that the diagram

$$\begin{tikzcd} R^iF''(Y) \arrow[r, "d_{\mathcal{F}}"] \arrow[d, "\partial_{R^*F''}"'] & R^{i+1}F'(Y) \arrow[d, "-\partial_{R^*F'}"] \\ R^{i+1}F''(X) \arrow[r, "d_{\mathcal{F}}"] & R^{i+2}F'(X) \end{tikzcd}$$

commutes when $0 \to X \to Z \to Y \to 0$ is a short exact sequence in $\mathcal{A}$.

Proof. We start by proving the *if* part of (b). Assume that $\Phi^p I = 0$ for any injective I and all $p > 0$. Let $\Phi^0 \xrightarrow{\alpha^0} \Psi^0$ be given. By induction on n, we construct $\Phi^k \xrightarrow{\alpha^k} \Psi^k$ for $k \leq n$ such that $\alpha^k \partial_\Phi = \partial_\Psi \alpha^{k-1}$ for $k = 1, \dots n$. For $n = 0$, this is trivial. Let $n > 0$ and α^k be constructed for $k < n$. To construct α^n, we consider any object X of $\mathcal{A}$ and choose a monomorphism $X \overset{\iota}{\hookrightarrow} I$ where I is injective. When $n > 1$, we have a part of the long exact cohomology sequence for $0 \to X \xrightarrow{\iota} I \to X' = \operatorname{coker}(\iota) \to 0$,

$$0 = \Phi^{n-1} I \longrightarrow \Phi^{n-1} X' \xrightarrow{\partial_\Phi} \Phi^n X \longrightarrow \Phi^n I = 0\,,$$

giving an isomorphism $\partial_\Phi = \partial_{X,\iota} \colon \Phi^{n-1} X' \xrightarrow{\sim} \Phi^n X$. When $n = 1$, we still have $\Phi^1 I = 0$ and thus an isomorphism $\partial_{X,\iota} \colon \operatorname{coker}\left(\Phi^0 I \to \Phi^0 X'\right) \xrightarrow{\sim} \Phi^1 X$.

We have $\Psi^{n-1} X' \xrightarrow{\partial_\Psi} \Psi^n X$ and put $\alpha^n_{X,\iota} = \partial_\Psi \alpha^{n-1}_{X'} \partial^{-1}_{X,\iota}$ when $n > 1$. When $n = 1$, α^0 induces a morphism

$$\operatorname{coker}\left(\Phi^0 I \longrightarrow \Phi^0 X'\right) \xrightarrow{\overline{\alpha}^0} \operatorname{coker}\left(\Psi^0 I \longrightarrow \Psi^0 X'\right)$$

and we put $\alpha^1_{X,\iota} = \overline{\partial}_\Psi \overline{\alpha}^0 \partial^{-1}_{X,\iota}$, where $\overline{\partial}_\Psi \colon \operatorname{coker}\left(\Psi^0 I \to \Psi^0 X'\right) \to \Psi^1 X$ is obtained from $\Psi^0 X' \xrightarrow{\partial_\Psi} \Psi^1 X$ using the universal property of cokernels.

We want to show that $\alpha^n_{X,\iota}$ does not depend on ι and that $\alpha_X \coloneqq \alpha^n_{X,\iota}$ induces a natural transformation $\Phi^n \xrightarrow{\alpha^n} \Psi^n$. We can show both assertions at once by considering monomorphisms $X \overset{\iota}{\hookrightarrow} I$ and $Y \overset{\kappa}{\hookrightarrow} K$ into injective objects I, K and any morphism $X \xrightarrow{\xi} Y$ and showing $\Psi^n(\xi)\alpha^n_{X,\iota} = \alpha^n_{Y,\kappa}\Psi^n(\xi)$. When $X = Y$ and $\xi = \mathrm{id}_X$, this shows that $\alpha^n_{X,\iota}$ is independent of ι and the general case implies that α^n is a natural transformation.

Now let's do this! By injectivity of K there exists some morphism $I \xrightarrow{\hat{\xi}} K$ such that $\hat{\xi}\iota = \kappa\xi$. We have an induced morphism $\xi' \colon X' \to Y'$ on the cokernels $X' = \operatorname{coker}(\iota)$ and $Y' = \operatorname{coker}(\kappa)$. This gives a commutative diagram

$$\begin{tikzcd}
\mathcal{X}\colon & 0 \arrow[r] & X \arrow[r, "\iota"] \arrow[d, "\xi"'] & I \arrow[r] \arrow[d, dashed, "\hat{\xi}"'] & X' \arrow[r] \arrow[d, dotted, "\xi'"'] & 0 \\
\mathcal{Y}\colon & 0 \arrow[r] & Y \arrow[r, "\kappa"] & K \arrow[r] & Y' \arrow[r] & 0
\end{tikzcd}$$

with exact rows $\mathcal{X}, \mathcal{Y}$. Let $n > 1$. Our goal $\Psi^n(\xi)\alpha^n_{X,\iota} = \alpha^n_{Y,\kappa}\Psi^n(\xi)$ is precisely that the right face in the following cute little diagram commutes.

$$\begin{tikzcd}
& \Phi^{n-1}Y' \arrow[rr, "\partial_\Phi = \partial_{Y,\kappa}"] \arrow[dd, "\alpha^{n-1}_{Y'}"' near end] & & \Phi^n Y \arrow[dd, "\alpha^n_{Y,\kappa}"] \\
\Phi^{n-1}X' \arrow[ur, "\Phi^{n-1}(\xi')"] \arrow[rr, "\partial_\Phi = \partial_{X,\iota}" near start] \arrow[dd, "\alpha^{n-1}_{X'}"'] & & \Phi^n X \arrow[ur, "\Phi^n(\xi)"] \arrow[dd, "\alpha^n_{X,\iota}" near start] & \\
& \Psi^{n-1}Y' \arrow[rr, "\partial_\Psi" near start] & & \Psi^n Y' \\
\Psi^{n-1}X' \arrow[ur, "\Psi^{n-1}(\xi')"'] \arrow[rr, "\partial_\Psi"'] & & \Psi^n X \arrow[ur, "\Psi^n(\xi)"'] &
\end{tikzcd}$$

Indeed, the front and back faces commute by definition of $\alpha^n_{X,\iota}$ and $\alpha^n_{Y,\kappa}$. The top and bottom face commute since the connecting homomorphisms ∂_Φ and ∂_Ψ are natural, i.e., compatible with morphisms of short exact sequences by Definition 1. The left face commutes by the induction assumption. As $\partial_{X,\iota}$ and $\partial_{Y,\kappa}$ are isomorphisms, this implies commutativity of the right face, as required. For $n = 1$, a slight modification of the argument works again.

We have seen that α^n is a well-defined natural transformation. It remains to show $\alpha^n\partial_\Phi = \partial_\Psi\alpha^{n-1}$. Let $0 \to X' \to X \to X'' \to 0$ be a short exact sequence. We choose an embedding $X' \xhookrightarrow{\iota} I$ into some injective object ι. This gives a commutative diagram with exact rows

$$\begin{tikzcd}
\mathcal{X}\colon & 0 \arrow[r] & X' \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, dashed, "i"'] & X'' \arrow[r] \arrow[d, dotted, "i''"'] & 0 \\
\mathcal{Y}\colon & 0 \arrow[r] & X' \arrow[r, "\iota"] & I \arrow[r] & \mathrm{coker}(\iota) \arrow[r] & 0
\end{tikzcd}$$

where i exists by injectivity of I and i'' is the induced morphism on cokernels. Now the claim $\alpha^n\partial_\Phi = \partial_\Psi\alpha^{n-1}$ follows by another cubic diagram chase. We have

$$\begin{tikzcd}
& \Phi^{n-1}\big(\mathrm{coker}(\iota)\big) \arrow[rr, "\partial_\Phi"] \arrow[dd, "\alpha^{n-1}_{\mathrm{coker}(\iota)}"' near start] & & \Phi^n X' \arrow[dd, "\alpha^n_{X'a}"] \\
\Phi^{n-1}X'' \arrow[ur, "\Phi^{n-1}(i'')"] \arrow[rr, "\partial_\Phi" near end] \arrow[dd, "\alpha^{n-1}_{X''}"'] & & \Phi^n X \arrow[ur, equal] \arrow[dd, "\alpha^n_{X'}" near start] & \\
& \Psi^{n-1}\big(\mathrm{coker}(\iota)\big) \arrow[rr, "\partial_\Psi" near end] & & \Psi^n X' \\
\Psi^{n-1}X'' \arrow[ur, "\Psi^{n-1}(i'')"'] \arrow[rr, "\partial_\Psi"'] & & \Psi^n X \arrow[ur, equal] &
\end{tikzcd}$$

The top and bottom face commute by naturality of ∂_Φ and ∂_Ψ. The left face commutes because α^{n-1} is a natural transformation. The back face commutes by Definition of $\alpha^n_{X'}$ and on the right face nothing really happens. Hence the front face commutes as well. This shows the desired equality, thus completing the inductive construction of the α^n and showing the existence part of the universal property of a right-derived functor for Φ^0.

For uniqueness, let $\alpha^*\colon \Phi^* \to \Psi^*$ be any morphism of cohomological functors. Let X be any object and $0 \to X \xrightarrow{\iota} I \to X'' \to 0$ any short exact sequence with I injective. As α^* is a morphism of cohomological functors, it is compatible with ∂_Φ and ∂_Ψ for this short exact sequence. That is,

$$\alpha^n_X\partial_\Phi = \partial_\Psi\alpha^{n-1}_{X'} \; .$$

Again, for $n > 1$ the connecting morphism $\partial_\Phi = \partial_{X,\iota}$ is an isomorphism by vanishing of $\Phi^{n-1}I$ and $\Phi^n I$, hence $\alpha^n_X = \partial_\Psi\alpha^{n-1}_{X'}\partial^{-1}_{X,\iota}$. For $n = 1$, we need to make the same modification as above. In either case, this shows that the above construction of α^n from α^0 is the only possible one. This shows the uniqueness part and thus the *if* part of (b).

Before we can pursue the proof, we need some homological algebra, namely, the horseshoe lemma.

Definition 3. An **injective resolution** of an object X of an abelian category $\mathcal{A}$ is a long exact sequence

$$0 \longrightarrow X \xrightarrow{\xi} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \dots$$

with $I^0, I^1, \dots$ injective.

Fact 1. In an abelian category $\mathcal{A}$ with sufficiently many injective objects, injective resolutions exist for any object.

Proof. Indeed, for $X \in \mathrm{Ob}(\mathcal{A})$ choose a monomorphism $X \overset{\xi}{\hookrightarrow} I^0$, then a monomorphism $\operatorname{coker}(\xi) \hookrightarrow I^1$, then a monomorphism $\operatorname{coker}\left(I^0 \to I^1\right) \hookrightarrow I^2$ and so on. *q.e.d.*

Proposition 1 (Horseshoe lemma)**.** *Let $X \xrightarrow{\xi} I^*$ and $Y \xrightarrow{\upsilon} J^*$ be injective resolutions of X and Y.*

(*a*) *If $X \xrightarrow{f} Y$ is any morphism, then there is a morphism $I^* \xrightarrow{\varphi^*} J^*$ compatible with f in the sense that $\upsilon f = \varphi^0 \xi$.*

(*b*) *The extension from* (*a*) *is unique up to cochain homotopy. If ψ^* is a different morphism of cochain complexes with the same property then there is a cochain homotopy s^*, where $s^n \colon I^n \to J^{n-1}$, between φ^* and ψ^*. That is,*

$$d_{J^*}^{n-1} s^n + s^{n+1} d_{I^*}^n = \psi^n - \varphi^n$$

and $s^0 = 0$.

(*c*) *If $0 \to X \xrightarrow{\alpha} Z \xrightarrow{\beta} Y \to 0$ is a short exact sequence, there exists an injective resolution $Z \xrightarrow{\zeta} K^*$ of Z such that*

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\alpha"] \arrow[d, "\xi"'] & Z \arrow[r, "\beta"] \arrow[d, "\zeta"'] & Y \arrow[r] \arrow[d, "\upsilon"'] & 0 \\
0 \arrow[r] & I^0 \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^1 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{§}$$

is a commutative diagram whose rows are short exact sequences which are split safe for the first one.

Remark 2. (*a*) Let $Z^n(I^*) = \ker(d_{I^*}^n)$ and $Z^n(J^*) = \ker(d_{J^*}^n)$. Then φ and ψ induce morphisms $\varphi, \psi \colon Z^n(I^*) \to Z^n(J^*)$ and Proposition 1(*b*) shows that $\psi - \varphi = d_{J^*}^{n-1} s^n$ on $Z^n(I^*)$, that is, they differ by a coboundary and thus induce the same morphisms in cohomology.

(*b*) The assumption that $\mathcal{A}$ has sufficiently many injective objects is not required. It also suffices to have $0 \to Y \xrightarrow{\upsilon} J^*$ a cochain complex with injective J^n (thus dropping exactness) and

$$0 \longrightarrow X \xrightarrow{\xi} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \dots$$

a long exact sequence in which the I^n may fail to be injective.

Before we prove Proposition 1, let's see how the rest of Theorem A follows from it. We first prove (*a*). Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ be left exact. For every object X of $\mathcal{A}$, chose an injective resolution $X \to I_X^*$ of X (and we need the axiom of choice – for *classes!* – to do this). For every morphism $X \xrightarrow{f} Y$ choose an extension $I_X^* \xrightarrow{f^*} I_Y^*$ of f. Let

$$R^pF(X) = H^p(FI_X^*) \quad \text{and} \quad R^pF(f) = H^p\left(FI_X^* \xrightarrow{f^*} FI_Y^*\right).$$

We need to show that the R^pF are functors. That $R^pF(\mathrm{id}_X) = \mathrm{id}_{R^pF(X)}$ (even when $\mathrm{id}_X^* \neq \mathrm{id}_{I_X^*}$) follows from Proposition 1(*b*) and Remark 2(*a*). When $X \xrightarrow{f} Y \xrightarrow{g} Z$, the morphisms

$$I_X^* \underset{(gf)^*}{\overset{g^*f^*}{\rightrightarrows}} I_Z^*$$

are cochain homotopic by Proposition 1(*b*). Applying the additive functor F (F is left-exact, hence additive by Definition A.2.3 and Remark A.2.1) gives cochain homotopic morphisms $F(g^*f^*)$ and $F(g^*)F(f^*)$, showing that the induced morphisms in cohomology coincide, i.e., $R^pF(gf) = R^pF(g) \circ R^pF(f)$. Thus, R^pF is indeed a functor.

To construct the long exact cohomology sequence for R^pF, consider a short exact sequence $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ and construct a short exact sequence $0 \to I_X^* \xrightarrow{a^*} K^* \xrightarrow{b^*} I_Y^* \to 0$ as in Proposition 1(*c*). We obtain a short exact sequence $0 \to FI_X^* \to FK^* \to FI_Y^* \to 0$ (using Proposition A.2.3) and a long exact sequence

$$\begin{tikzcd} \dots \arrow[r] & R^{p-1}F(Y) \arrow[r, "\partial"] & R^pF(X) \arrow[r] \arrow[dr, dashed] & H^p(FK^*) \arrow[r] \arrow[d, "\wr"] & R^pF(X) \arrow[r, "\partial"] & R^{p+1}F(X) \arrow[r] & \dots \\ & & & R^pF(Z) \arrow[ur, dashed] & & & \end{tikzcd} \tag{\#}$$

The vertical arrow is obtained by choosing morphisms $K^* \xrightarrow{\kappa_1^*} I_Z^*$ and $I_Z^* \xrightarrow{\kappa_2^*} K^*$ of cochain complexes (by Proposition 1(*b*)). Applying F gives $FK^* \xrightarrow{F\kappa_1^*} FI_Z^*$ and $FI_Z^* \xrightarrow{F\kappa_2^*} FK^*$ such that $F(\kappa_1^*)F(\kappa_2^*)$ and $F(\kappa_2^*)F(\kappa_1^*)$ are cochain homotopic to $\mathrm{id}_{FI_Z^*}$ and id_{FK^*} respectively. It follows that $F(\kappa_1^*)$ and $F(\kappa_2^*)$ induce isomorphisms on cohomology which are inverse to each other, resulting in the vertical arrow of (#).

Also, to verify commutativity of (#), one notes that α^* and $\kappa_1 a^*$, and hence $F(\alpha^*)$ and $F(\kappa_1 a^*)$, are cochain homotopic by Proposition 1(*a*). Same for β^* and $b^*\kappa_2$.

This shows that R^pF has the required long exact cohomology sequence. In a similar fashion one can show its functoriality on the category of short exact sequences in $\mathcal{A}$.

If X is injective, one can choose $I^*\colon X \to 0 \to 0 \to \dots$ and $X \xrightarrow{\mathrm{id}_X} I^*$ as injective resolution. Applying Proposition 1(b), we get

$$R^pF(X) = H^p(FI_X^*) = H^p(FI^*) = 0 \quad \text{when } p > 0$$

and, for arbitrary $X \in \mathrm{Ob}(\mathcal{A})$ (injective or not), a canonical isomorphism

$$R^0F(X) = \ker(FI_X^0 \longrightarrow FI_X^1) \simeq F\big(\ker(I_X^0 \longrightarrow I_X^1)\big) \simeq F(X)\ .$$

(using that F is left-exact, hence commutes with kernels). By the *if* part of (b) that was already proved, R^*F is a right-derived functor of F.

The *only if* part of (b) is an immediate consequence. Let Φ^* be the right-derived functor of $F = \Phi^0$, then $\Phi^* \simeq R^*F$ by the universal property of derived functors. By the above, $\Phi^pI = R^pF(I) = 0$ when I is injective.

Part (c). Let $0 \to F' \to F \to F'' \to 0$ be a sequence of left-exact functors which is exact on injective objects. Choosing I_X^* as in the above construction, we get a short exact sequence

$$0 \longrightarrow F'I_X^* \longrightarrow FI_X^* \longrightarrow F''I_X^* \longrightarrow 0 \tag{$*$}$$

and this gives a long exact cohomology sequence which is functorial in X as ($*$) is. The anti-commutativity of connecting morphisms (with $0 \to X \to Z \to Y \to 0$ a short exact sequence) comes from the analogous (and well-known) assertion for the connecting morphisms in the diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F'I_X^* \arrow[r] \arrow[d] & F'K^* \arrow[r] \arrow[d] & F'I_Y^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & FI_X^* \arrow[r] \arrow[d] & FK^* \arrow[r] \arrow[d] & FI_Y^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F''I_X^* \arrow[r] \arrow[d] & F''K^* \arrow[r] \arrow[d] & F''I_Y^* \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

(where K^* is the same as in the proof of (a)). *q.e.d.*

Proof of Proposition 1. Part (a). For $n \geq 1$ denote the image of $d_{I^*}^{n-1}\colon I^{n-1} \to I^n$ by $B^n \subseteq I^n$ and let $B^0 \subseteq I^0$ be the image of ξ. As $B^0 \simeq X$ we have a morphism $f^0\colon B^0 \to J^0$ such that $\upsilon f = f^0\xi$. We now construct the required morphisms φ^* inductively. Let $n \geq 0$ and suppose that $\varphi^k\colon I^k \to J^k$ have already been constructed for $k = -1, \dots, n-1$ (where $\varphi^{-1} = f$) as well as $f^n\colon B^n \to J^n$ such that $d_{J^*}^{k-1}\varphi^{k-1} = \varphi^k d_{I^*}^{k-1}$ for $k < n$ and $d_{J^*}^{n-1}\varphi^{n-1} = f^n d_{I^*}^n$ (where we set $d_{I^*}^{-1} = \xi$ and $d_{J^*}^{-1} = \upsilon$). Let $\varphi^n\colon I^n \to J^n$ be any extension of f^n using that J^n is injective and $B^n \hookrightarrow I^n$ a monomorphism. We have $d_{J^*}^n\varphi^n d_{I^*}^n = d_{J^*}^{n-1}f^n d_{I^*}^{n-1} = d_{J^*}^n d_{J^*}^{n-1}\varphi^{n-1} = 0$, hence

$d^n_{J^*}\varphi^n \colon I^n \to J^{n+1}$ factors over $\operatorname{coker}(d^n_{I^*}) \simeq I^n/B^n = I^n/Z^n(I^*) \simeq B^{n+1}$ by exactness of the resolution, which gives $f^{n+1} \colon B^{n+1} \to J^{n+1}$. This completes the induction.

Part (b). Professor Franke suggests the horseshoe lemma is best understood if you work out the proof for yourself, so you might want to do just that instead. However, for the sake of completeness of these notes I will now include the proof I obediently worked out for myself.

Let ψ^* be another lift of f. Then $(\psi^0 - \varphi^0)\xi = \upsilon f - \upsilon f = 0$, hence $\psi^0 - \varphi^0$ factors over $\operatorname{coker}(\xi) = I^0/B^0 \simeq B^1$, hence we get a morphism $\sigma^1 \colon B^1 \to J^0$ such that $\sigma^1 d^0_{I^*} = \psi^0 - \varphi^0$. Let $s^1 \colon I^1 \to J^0$ be any extension of σ^1 using injectivity of J^0. We now construct the required cochain homotopy s^* inductively, letting $s^0 = 0$. Let $n \geq 1$ and suppose that $s^k \colon I^k \to J^{k-1}$ has already been constructed for $k \leq n$ such that $d^{k-1}_{J^*}s^k + s^{k+1}d^k_{I^*} = \psi^k - \varphi^k$ for $k < n$. Then

$$\begin{aligned}
\left(\psi^n - \varphi^n - d^{n-1}_{J^*}s^n\right) d^{n-1}_{I^*} &= d^{n-1}_{J^*}\left(\psi^{n-1} - \varphi^{n-1}\right) - d^{n-1}_{J^*}s^n d^{n-1}_{I^*} \\
&= d^{n-1}_{J^*}\left(d^{n-2}_{J^*}s^{n-1} + s^n d^{n-1}_{I^*}\right) - d^{n-1}_{J^*}s^n d^{n-1}_{I^*} \\
&= d^{n-1}_{J^*}d^{n-2}_{J^*}s^{n-1} \\
&= 0\,,
\end{aligned}$$

hence $(\psi^n - \varphi^n - d^{n-1}_{J^*}s^n)d^{n-1}_{I^*}$ factors over $\operatorname{coker}(d^{n-1}_{I^*}) = I^n/B^n \simeq B^{n+1}$, giving a morphism $\sigma^{n+1} \colon B^{n+1} \to J^n$, which we extend to some $s^{n+1} \colon I^{n+1} \to J^n$ with the required property, using that J^n is injective. This finishes the induction step.

Part (c). By Definition 2(b), the rows of (§) except the first are automatically split, so we have no choice but $K^n = I^n \oplus J^n$ for all $n \geq 0$. By Definition 2(a) it is clear that the K^n are injective again. Let $i^n \colon I^n \to K^n$ denote the canonical inclusion and $j^n \colon K^n \to J^n$ the canonical projection. We need to construct the $d^n_{K^*}$ and ζ, which will be yet another inductive lifting argument.

As α is a monomorphism, we can extend ξ to some $\hat{\xi} \colon Z \to I^0$. Let $\zeta = \hat{\xi} \times \upsilon\beta$. Then $d^0_{I^*}\hat{\xi}\alpha = d^0_{I^*}\xi = 0$, hence $d^0_{I^*}\hat{\xi}$ factors over $\operatorname{coker}(\alpha) = Y$. This gives $\delta^0 \colon Y \to I^1$ such that $d^0_{I^*}\hat{\xi} = \delta^0\upsilon\beta$. Since υ is a monomorphism and I^1 injective, we can extend $-\delta^0$ to some $d^0 \colon J^0 \to I^1$. We define $d^0_{K^*} = (d^0_{I^*} + d^0) \times d^0_{J^*}$. By construction, this satisfies $d^0_{K^*}\zeta = 0$.

Let $n \geq 0$ and suppose we have already constructed $d^k_{K^*} = (d^k_{I^*} - d^k) \times d^k_{J^*}$ for $k \leq n$, where $d^k \colon J^k \to I^{k+1}$ satisfies $d^k_{I^*}d^{k-1} + d^k d^{k-1}_{J^*} = 0$ for $k = 1, \dots, n$ and $d^0_{I^*}\hat{\xi} = d^0\upsilon\beta$ (hence $d^k_{K^*}d^{k-1}_{K^*} = 0$ and $d^0_{K^*}\zeta = 0$). Then

$$d^{n+1}_{I^*}d^n d^{n-1}_{J^*} = d^{n+1}_{I^*}\left(-d^n_{I^*}d^{n-1}\right) = 0\,,$$

hence $d^{n+1}_{I^*}d^n$ factors over $\operatorname{coker}(d^{n-1}_{J^*}) = \operatorname{Im}(d^n_{J^*})$. This gives $\delta^{n+1} \colon \operatorname{Im}(d^n_{J^*}) \to I^{n+2}$, and $-\delta^{n+1}$ can be extended to some $d^{n+1} \colon J^{n+1} \to I^{n+2}$ (by injectivity of I^{n+2}) which satisfies $d^{n+1}_{I^*}d^n + d^{n+1}d^n_{J^*} = 0$. Then $d^{n+1}_{K^*} = (d^{n+1}_{I^*} + d^{n+1}) \times d^{n+1}_{J^*}$ fulfills $d^{n+1}_{K^*}d^n_{K^*} = 0$ and the induction is complete.

It remains to verify that $Z \xrightarrow{\zeta} K^*$ is indeed a resolution, i.e., acyclic. But this is an immediate consequence of the long exact cohomology sequence for short exact sequences of chain complexes, since both $H^*(I^*)$ and $H^*(J^*)$ vanish. *q.e.d.*

A.2. Some notes on additive categories and additive functors

The following lists and proves some first properties of additive categories and additive functors that Professor Franke assumed known in the lecture without explicitly mentioning them.

Definition 1. (*a*) A **preadditive** category $\mathcal{A}$ in which each $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ for $X, Y \in \mathrm{Ob}(\mathcal{A})$ is given a group structure which behaves bilinearly under compositions.

(*b*) An **additive category** is a preadditive category which has finite products and coproducts such that the canonical morphism $\coprod_{k=1}^{n} X_k \xrightarrow{c} \prod_{k=1}^{n} X_k$ is an isomorphism for all objects $X_1, \dots, X_n \in \mathrm{Ob}(\mathcal{A})$.

Remark. (*a*) When $\mathcal{A}$ is additive, letting $n = 0$ in Definition 1(*b*) gives an object $* \in \mathrm{Ob}(\mathcal{A})$ which is both an initial and a final object. For $X, Y \in \mathrm{Ob}(\mathcal{A})$, let the *zero morphism* (which we denote 0) $X \xrightarrow{0} Y$ be defined by $X \to * \to Y$.

(*b*) We will construct the canonical morphism $\coprod_{k=1}^{n} X_k \xrightarrow{c} \prod_{k=1}^{n} X_k$ from Definition 1(*b*). Let $X_k \xrightarrow{i_k} \coprod_{k=1} X_k$ and $\prod_{k=1}^{n} X_k \xrightarrow{p_k} X_k$ be the associated inclusion and projection morphisms.

Using the universal property of $\prod_{k=1}^{n} X_k$, we get unique morphisms $X_j \xrightarrow{\alpha_j} \prod_{k=1}^{n} X_k$ such that $p_k \alpha_j = \mathrm{id}_X$ if $k = j$ and 0 else. Then

$$c\colon \coprod_{k=1}^{n} X_k \xrightarrow{\coprod \alpha_k} \prod_{k=1}^{n} X_k$$

is the morphism we are looking for. It is unique with the property that $p_k c i_j = \mathrm{id}_X$, if $k = j$ and 0 else.

(*c*) The isomorphism c is usually suppressed in the notation and one denotes both products and coproducts $\bigoplus_{k=1}^{n} X_k$.

Proposition 1. *Suppose that $\mathcal{A}$ is additive. The group structure on $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ for $X, Y \in \mathrm{Ob}(\mathcal{A})$ is automatically abelian and given as follows. For a pair of morphisms $X \underset{b}{\overset{a}{\rightrightarrows}} Y$, their sum $a + b$ is the composition*

$$X \xrightarrow{\mathrm{id}_X \times \mathrm{id}_X} X \oplus X \xrightarrow{a \amalg b} Y$$

and 0 is the neutral element.

Proof. We first show the characterization of the addition in $\mathrm{Hom}_{\mathcal{A}}(X, Y)$. First thing to check is that 0 is indeed the neutral element. As $*$ is a final object, $\mathrm{Hom}_{\mathcal{A}}(X, *)$ has only one element π and is thus the zero group. In particular, $\pi + \pi = \pi$ and if ι denotes $* \to Y$, then $0 = \iota\pi = \iota(\pi + \pi) = 0 + 0$, hence 0 is indeed the neutral element.

Now let $\Delta = \mathrm{id}_X \times \mathrm{id}_X$ be the diagonal and denote i_1, i_2 the inclusions of X in $X \oplus X$ and p_1, p_2 the projections of $X \oplus X$ to X. We show $\Delta = i_1 + i_2$. Indeed, we have $p_1(i_1 + i_2) = p_1 i_1 + p_1 i_2 = \mathrm{id}_X + 0 = \mathrm{id}_X$ and similarly $p_2(i_1 + i_2) j_2 = \mathrm{id}_X$, which is exactly how Δ is characterized. Hence

$$(a \amalg b) \circ \Delta = (a \amalg b) \circ (i_1 + i_2) = (a \amalg b) \circ i_1 + (a \amalg b) \circ i_2 = a + b\,.$$

For commutativity, we need to show $(a \amalg b) \circ \Delta = (b \amalg a) \circ \Delta$. The universal property of coproducts gives a unique $X \oplus X \xrightarrow{\sigma} X \oplus X$ such that

$$\begin{tikzcd} & X \arrow[dl, "i_2"'] \arrow[dr, "i_1"] \arrow[drrr, bend left, "a"] & & & \\ X \oplus X \arrow[rr, dashed, "\exists!\, \sigma"] & & X \oplus X \arrow[rr, "a \amalg b"] & & Y \\ & X \arrow[ul, "i_1"] \arrow[ur, "i_2"'] \arrow[urrr, bend right, "b"'] & & & \end{tikzcd}$$

commutes. Then σ is easily seen to be an isomorphism and $b \amalg a = (a \amalg b) \circ \sigma$ by the uniqueness of $b \amalg a$. It thus suffices to show $\sigma\Delta = \Delta$. By the uniqueness of Δ, this is equivalent to $p_1\sigma\Delta = \mathrm{id}_X$ and $p_2\sigma\Delta = \mathrm{id}_X$. We claim that $p_1\sigma = p_2$ and vice versa, which would finish the proof. To see this, note that $p_1\sigma = p_2$ is equivalent to $p_1\sigma i_1 = p_2 i_1 = 0$ and $p_1\sigma i_2 = p_2 i_2 = \mathrm{id}_X$ by the universal property of the coproduct $X \oplus X$. This follows from $\sigma i_1 = i_2$ and $\sigma i_2 = i_1$ by definition of σ. *q.e.d.*

Definition 2. A functor between preadditive categories (cf. [2, Definition A.1.1(*b*)]) is **additive** if it preserves finite biproducts together with the canonical inclusion and projection morphism to and from them.

Remark. Definition 2 seems kind of counterintuitive, as one would rather expect an additive functor to induce group homomorphisms on Hom sets. For additive categories, this alternative definition turns out to be equivalent.

Proposition 2. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between additive categories. Then F is additive iff F induces a group homomorphism $\mathrm{Hom}_{\mathcal{A}}(X, Y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{B}}(FX, FY)$ for every $X, Y \in \mathrm{Ob}(\mathcal{A})$.*

Proof. If F is an additive functor, i.e., preserves finite biproducts and the associated structure morphisms, then it is clear that F is compatible with the addition as in constructed Proposition 1.

Conversely, suppose that F is compatible with the addition. Let $X, Y \in \mathrm{Ob}(\mathcal{A})$. Let i_X, i_Y be the inclusions of X, Y in $X \oplus Y$ and p_X, p_Y the projections of $X \oplus Y$ onto X, Y. Define j_X, j_Y and q_X, q_Y analogously for $FX \oplus FY$. By the two universal properties of the biproduct $FX \oplus FY$ we get morphisms $F(X \oplus Y) \xrightarrow{q} FX \oplus FY$ and $FX \oplus FY \xrightarrow{j} F(X \oplus Y)$ such that

$$\begin{tikzcd} & & FX \\ F(X \oplus Y) \arrow[r, dashed, "\exists!\, q"] \arrow[urr, bend left, "F(p_X)"] \arrow[drr, bend right, "F(p_Y)"'] & FX \oplus FY \arrow[ur, "q_X"] \arrow[dr, "q_Y"'] & \\ & & FY \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} FX \arrow[dr, "j_X"] \arrow[drr, bend left, "F(i_X)"] & & \\ & FX \oplus FY \arrow[r, dashed, "\exists!\, j"] & F(X \oplus Y) \\ FY \arrow[ur, "j_Y"'] \arrow[urr, bend right, "F(i_Y)"'] & & \end{tikzcd}$$

are commutative diagrams.

Then $qj = \mathrm{id}_{FX\oplus FY}$ holds since $q_X(qj)i_X = F(p_X)F(i_X) = F(\mathrm{id}_X) = \mathrm{id}_{FX}$ and $q_Y(qj)i_X = F(p_Y)F(i_X) = F(0) = 0$ and similar conditions hold when X and Y switch roles, which precisely characterize $\mathrm{id}_{FX\oplus FY}$. To show $jq = \mathrm{id}_{F(X\oplus Y)}$, write $\mathrm{id}_{FX\oplus FX} = j_X q_X + j_Y q_Y$ to get

$$jq = j(j_X q_X + j_Y q_Y)q = F(i_X)F(p_X) + F(i_Y)F(p_Y) = F(i_X p_X + i_Y p_Y)$$

using that F is a group homomorphism. Now $i_X p_X + i_Y p_Y = \mathrm{id}_{X\oplus Y}$, so we can further deduce $jq = F(\mathrm{id}_{X\oplus Y}) = \mathrm{id}_{F(X\oplus Y)}$. This shows that j and q are inverse to each other, hence $F(X \oplus Y) \simeq FX \oplus FY$ and this is compatible with the structure morphisms. *q.e.d.*

Definition 3. A functor is called **left-exact** if it preserves finite limits, and **right-exact** if it preserves finite colimits.

Remark 1. Again, this is not quite what one would expect, and again, it turns out to be essentially equivalent to the expected definition under the right circumstances. A left-exact functor in the sense of Definition 3 between abelian categories preserves, in particular, kernels, and thus short left-exact sequences. Also, it preserves finite products and their projection morphisms (as certain limits) and hence direct sums, i.e., is additive (one easily checks that the inclusion morphisms are then preserved as well). Conversely, an additive functor between abelian categories that preserves kernels is already left-exact, since every finite limit can be built from finite products and equalizers (which we have, since there are kernels).

Proposition 3. *When $F\colon \mathcal{A} \to \mathcal{B}$ is a left-exact functor between abelian categories $\mathcal{A}, \mathcal{B}$ and $0 \to I \to X \to X'' \to 0$ a short exact sequence in $\mathcal{A}$ with I an injective object of $\mathcal{A}$, then $0 \to FI \to FX \to FX'' \to 0$ is a short exact sequence in $\mathcal{B}$.*

Proof. Since I is injective, the sequence $0 \to I \to X \to X'' \to 0$ splits due to Definition A.1.2(b), i.e., $X \simeq I \oplus X''$. Since F is additive, $FX \simeq FI \oplus FX''$, and $FI \to FX$ and $FX \to FX''$ correspond to the inclusion of FI respectively projection onto FX''. *q.e.d.*

Bibliography

[1] Nicholas Schwab; Ferdinand Wagner. *Algebra I by Jens Franke (lecture notes)*. GitHub: https://github.com/Nicholas42/AlgebraFranke/tree/master/AlgebraI.

[2] Ferdinand Wagner. *Algebraic Geometry II by Jens Franke (lecture notes)*. GitHub: https://github.com/Nicholas42/AlgebraFranke/tree/master/AlgGeoII.