Algebraic Geometry II

Ferdinand Wagner

Sommersemester 2018

This text consists of notes on the lecture Algebraic Geometry II, taught at the University of Bonn by Professor Jens Franke in the summer term (Sommersemester) 2018.

Please report bugs, typos etc. through the *Issues* feature of github.

Contents

Introduction

This lecture will develop the cohomology of (quasi)coherent sheaves of modules. Professor Franke assumes familiarity with the contents of last term's Algebraic Geometry I. In particular, this includes the category of (pre)schemes, equalizers and fibre products of preschemes as well as in arbitrary categories and quasi-coherent $\mathcal{O}_X$-modules. If you want to brush up your knowledge about these topics, the *lecture notes from Algebraic Geometry I* [1] might be your friend.

Professor Franke started the lecture with an example of sheaf cohomology entering the game. Let X be a topological space, $\mathcal{C}_X$ the sheaf of continuous $\mathbb{C}$-valued functions on X and $\underline{\mathbb{Z}}_X$ the sheaf of locally constant (i.e., continuous) functions on X with values in $\mathbb{Z}$. Then there is a short exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_X \xrightarrow{\cdot 2\pi \mathrm{i}} \mathcal{C}_X \xrightarrow{\exp} \mathcal{C}_X^{\times} \longrightarrow 0$$

of sheaves of abelian groups. In general, taking global section doesn't preserve exactness but gives rise to a long exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_X(X) \longrightarrow \mathcal{C}_X(X) \longrightarrow \mathcal{C}_X^{\times}(X) \xrightarrow{d} H^1(X,\underline{\mathbb{Z}}_X) \longrightarrow H^1(X,\mathcal{C}_X) \longrightarrow \dots$$

$$\downarrow \wr \qquad\qquad \downarrow \wr \qquad\qquad \downarrow \wr \qquad\nearrow$$

$$0 \longrightarrow H^0(X,\underline{\mathbb{Z}}_X) \longrightarrow H^0(X,\mathcal{C}_X) \longrightarrow H^0(X,\mathcal{C}_X^{\times})$$

in which the $H^k(X,\underline{\mathbb{Z}}_X)$, $H^k(X,\mathcal{C}_X)$, and $H^k(X,\mathcal{C}_X^{\times})$ are *sheaf cohomology groups*. There is the more general notion of *derived functors* (Grothendieck, Tôhoku paper), but this won't appear in the lecture.

Background in homological algebra is not required safe for cohomology groups of cochain complexes, the long exact cohomology sequence and the following famous lemma.

Lemma (Five lemma)**.** *Given a diagram*

$$
\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \varepsilon\downarrow \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}
$$

of (abelian) groups/R-modules/etc. with exact rows, in which β and δ are isomorphisms, α is an epimorphism and ε a monomorphism, then γ is an isomorphism as well.

Proof. Easy diagram chase. *q.e.d.*

1. Cohomology of quasi-coherent sheaves of modules

1.1. Recollection of basic definitions and results

Definition 1 ([1, Definition 1.5.2 and Definition 1.5.9(*b*)])**.** (*a*) A **prescheme** (Franke uses "EGA termology") is a locally ringed space $(X, \mathcal{O}_X)$ which locally has the form $\operatorname{Spec} R$ for some rings R.

  (*b*) A prescheme X is called a **scheme**, if, for any prescheme T and any pair of morphisms $T \overset{a}{\underset{b}{\rightrightarrows}} X$, the equalizer $\operatorname{Eq}\left(T \overset{a}{\underset{b}{\rightrightarrows}} X\right)$ is a closed subprescheme of X.

Remark. Equivalently, a prescheme X is a scheme iff the diagonal $\Delta \colon X \xrightarrow{(\mathrm{id}_X, \mathrm{id}_X)} X \times X$ is a closed immersion (cf. [1, Fact 1.5.8]). In other words, schemes are *separated* preschemes

Proposition 1. *If U and V are affine open subsets of a scheme X, then their intersection $U \cap V$ is again affine (and open of course).*

Proof. This was proved in [1, Proposition 1.5.4].   *q.e.d.*

Note that open subsets of the form $\operatorname{Spec}(R_f) \simeq \operatorname{Spec} R \setminus V(f)$ form a topology base on $\operatorname{Spec} R$ and that the *saturation* of $\{1, f, f^2, \ldots\}$ (i.e. the largest multiplicative subset of R which still gives the same localization) depends only on $\operatorname{Spec} R \setminus V(f)$. Hence, for any R-module M, the localization M_f depends (up to canonical isomorphism) only on $\operatorname{Spec} R \setminus V(f)$ and M. One defines a sheaf of modules $\widetilde{M}$ on $\operatorname{Spec} R$ as the sheafification of $\operatorname{Spec}(R_f) \mapsto M_f$. Then

$$\widetilde{M}(U) = \left\{ (m_{\mathfrak{p}}) \in \prod_{\mathfrak{p} \in U} M_{\mathfrak{p}} \;\middle|\; \begin{array}{l} \text{for every } \mathfrak{q} \in U \text{ there are } f \in R \setminus \mathfrak{q} \text{ and } \mu \in M_f \text{ such that} \\ m_{\mathfrak{p}} = (\text{image of } \mu \text{ under } M_f \to M_{\mathfrak{p}}) \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R_f) \end{array} \right\}.$$

In the following, I will deviate from Franke's numbering in the lecture, but I simply refuse to call a definition "remark".

Definition 2 ([1, Definition 1.4.2])**.** A sheaf of modules $\mathcal{M}$ on $\operatorname{Spec} R$ is called **quasi-coherent** if $\mathcal{M} \simeq \widetilde{M}$ for some R-module M.

Definition 3 ([1, Definition 1.5.3])**.** A prescheme is called **quasi-compact** if the underlying topological space is quasi-compact and **quasi-separated** if the intersection of any two quasi-compact open subsets is quasi-compact.

Definition 4 ([1, Definition 1.5.4])**.** Let X be a prescheme. An $\mathcal{O}_X$-module $\mathcal{M}$ is called **quasi-coherent** if it satisfies the following equivalent conditions.

(a) X may be covered by affine open subsets U such that $\mathcal{M}|_U$ is quasi-coherent in the sense of Definition 2.

(b) For any affine open $U \subseteq X$, $\mathcal{M}|_U$ is quasi-coherent.

(c) For any quasi-compact and quasi-separated open $U \subseteq X$ and $f \in \mathcal{O}_X(U)$, the canonical morphism

$$\mathcal{M}(U)_f \longrightarrow \mathcal{M}(U \setminus V(f)) \tag{1}$$

(coming from the universal property of localization) is an isomorphism.

(d) The morphism (1) is an isomorphism when U is quasi-compact and quasi-separated and injective when U is only quasi-compact.

(e) When $U \subseteq X$ is affine, the canonical morphism

$$\mathcal{M}(U)_{\mathfrak{p}_x} \longrightarrow \mathcal{M}_x \tag{2}$$

is an isomorphism for all $x \in U$, where $\mathfrak{p}_x = \{f \in \mathcal{O}_X(U) \mid x \in V(f)\}$ is the prime ideal in $\operatorname{Spec} \mathcal{O}_X(U)$ corresponding to x.

Remark. To be fair: Despite Professor Franke's usual fondness of proving definitions (or rather stating definitions in a way they need a proof), the equivalence of (a) to (d) wasn't proposed as a definition in Algebraic Geometry I.

Proof of Definition 4. The equivalence of (a) to (d) was proved in [1, Proposition 1.5.1], but property (e) is something we haven't seen yet. Recall the *adjunction* (cf. Definition A.1.5)

$$\operatorname{Hom}_{\mathcal{O}_{\operatorname{Spec} R}}(\widetilde{M}, \mathcal{N}) \overset{\sim}{\longrightarrow} \operatorname{Hom}_R(M, \mathcal{N}(\operatorname{Spec} R)) \tag{3}$$

for M an R-module and $\mathcal{N}$ a sheaf of $\mathcal{O}_{\operatorname{Spec} R}$-modules (cf. [1, Proposition 1.4.3]). When (2) is an isomorphism for all $x \in U \simeq \operatorname{Spec} R$, it follows that the canonical morphism $\widetilde{M} \to \mathcal{M}|_U$ (with $M = \mathcal{M}(U)$) coming from (3) is an isomorphism on stalks, hence an isomorphism. This shows (e) $\Rightarrow$ (b).

Conversely, if (b) holds, then $\widetilde{M} \to \mathcal{M}|_U$ (with $M = \mathcal{M}(U)$) is an isomorphism for all affine open $U \subseteq X$, hence induces isomorphisms on stalks, which shows (b) $\Rightarrow$ (e). Hence, (e) is equivalent to the other properties. *q.e.d.*

Let $\mathcal{A}$ be the category R-Mod and $\mathcal{B}$ be $\mathcal{O}_{\operatorname{Spec} R}$-Mod, then the functor L given by $M \mapsto \widetilde{M}$ and the functor $\mathcal{M} \mapsto \mathcal{M}(\operatorname{Spec} R)$ are an adjoint pair of functors by (3). It follows that L commutes with cokernels and coproducts. In particular, the full subcategory $\operatorname{QCoh}(X) \subseteq \mathcal{O}_X$-Mod of quasi-coherent $\mathcal{O}_X$-modules is closed under taking cokernels and direct sums for $X = \operatorname{Spec} R$, and by locality of quasi-coherentness this holds for all preschemes X.

Definition 5 ([1, Definition 2.1.1 and Definition 2.1.2]). A morphism $X \overset{f}{\longrightarrow} Y$ of preschemes is **quasi-compact** if it satisfies the following equivalent conditions.

(a) For quasi-compact open $U \subseteq Y$, $f^{-1}(U)$ is quasi-compact.

(b) For affine open $U \subseteq Y$, $f^{-1}(U)$ is quasi-compact.

(c) One can cover Y by affine open U such that $f^{-1}(U)$ is quasi-compact.

It is called **quasi-separated** if it satisfies the following equivalent conditions.

(d) For an open quasi-separated $U \subseteq Y$, $f^{-1}(U)$ is quasi-separated again.

(e) For affine open subsets $U \subseteq Y$, $f^{-1}(U)$ is quasi-separated.

(f) It is possible to cover Y by affine open U such that $f^{-1}(U)$ is quasi-separated.

Proof. Equivalence was proved in [1, Fact 2.1.1] for quasi-compactness and [1, Lemma 2.1.1] for quasi-separatedness.   *q.e.d.*

Proposition 2. *If $X \xrightarrow{f} Y$ is quasi-compact and quasi-separated morphism of preschemes and $\mathcal{M} \in \mathrm{Ob}(\mathrm{QCoh}(X))$, then $f_*\mathcal{M} \in \mathrm{Ob}\left(\mathrm{QCoh}(Y)\right)$.*

Proof. This is [1, Proposition 1.5.2(b)].   *q.e.d.*

Proposition 3.   (a) *The full subcategory $\mathrm{QCoh}(X) \subseteq \mathcal{O}_X$-Mod of quasi-coherent sheaves of $\mathcal{O}_X$-modules on a prescheme X is closed under taking kernels and cokernels of morphisms and under taking (finite) direct sums.*

(b) *If $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module and $U \subseteq X$ open, then $\mathcal{M}|_U \in \mathrm{Ob}\left(\mathrm{QCoh}(U)\right)$.*

Proof. Part (a). For cokernels and finite direct sums (which are finite coproducts since $\mathcal{O}_X$-Mod is an abelian category by Proposition A.1.1), consider the case $X = \mathrm{Spec}\, R$ first. Then R-Mod $\xrightarrow{L} \mathcal{O}_X$-Mod, $M \mapsto \widetilde{M}$ and $\mathcal{O}_X$-Mod $\to R$-Mod, $\mathcal{M} \mapsto \mathcal{M}(X)$ are adjoint functors by (3). By Remark A.1.4, L preserves cokernels and coproducts. By locality of quasi-coherentness, this follows for all preschemes X.

Closedness under taking kernels was proved in [1, Fact 1.5.3]. It's worth pointing out that in fact, the proof given there shows that $M \mapsto \widetilde{M}$ preserves kernels as well.

Part (b) follows immediately from (e.g.) Definition 4(b).   *q.e.d.*

Corollary 1. *Let X be a prescheme, $\mathcal{M} \xrightarrow{f} \mathcal{N}$ a morphism of quasi-coherent $\mathcal{O}_X$-modules and $U \subseteq X$ open, then*

$$\ker\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right) \simeq \ker(f)(U) .$$

If U is, in addition, affine, then

$$\mathrm{coker}\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right) \simeq \mathrm{coker}(f)(X) .$$

Proof. The first assertion holds by our explicit construction of $\ker(f)$ in Lemma A.1.2. For the second one, we may assume $X = U = \mathrm{Spec}\, R$. Denoting $M = \mathcal{M}(X)$, $N = \mathcal{N}(X)$, then

$$\mathrm{coker}\left(\mathcal{M} \xrightarrow{f} \mathcal{N}\right) = \mathrm{coker}\left(\widetilde{M} \xrightarrow{f} \widetilde{N}\right) \simeq \left(\mathrm{coker}\left(M \xrightarrow{f} N\right)\right)^{\widetilde{}}$$

as $M \mapsto \widetilde{M}$ preserves cokernels.   *q.e.d.*

4

Corollary 2. *Let $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ be a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules on a prescheme X and $U \subseteq X$ be affine open, then*

$$0 \longrightarrow \mathcal{M}'(U) \longrightarrow \mathcal{M}(U) \longrightarrow \mathcal{M}''(U) \longrightarrow 0$$

is exact as well.

Proof. Follows from Corollary 1. *q.e.d.*

Remark. It turns out to be sufficient to assume that two of the above three sheaves $\mathcal{M}', \mathcal{M}, \mathcal{M}''$ are quasi-coherent. Indeed, we proved in Proposition 3 that kernels and cokernels of morphisms between quasi-coherent sheaves are quasi-coherent again, so the only case in question is where $\mathcal{M}$ is not required to be quasi-coherent. This case, however, will be treated by cohomological methods.

Our plan is to associate to any quasi-coherent $\mathcal{O}_X$-module on a scheme X *cohomology groups* $H^i(X, \mathcal{M})$ such that

- $H^0(X, \mathcal{M}) \simeq \mathcal{M}(X)$.

- when $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is an exact sequence of $\mathcal{O}_X$-modules, we have a canonical long exact sequence

$$0 \longrightarrow H^0(X, \mathcal{M}') \longrightarrow H^0(X, \mathcal{M}) \longrightarrow H^0(X, \mathcal{M}'')$$
$$\xrightarrow{d} H^1(X, \mathcal{M}') \longrightarrow H^1(X, \mathcal{M}) \longrightarrow H^1(X, \mathcal{M}'') \xrightarrow{d} H^2(X, \mathcal{M}') \longrightarrow \ldots$$

But before we do this, we to introduce the notion of *coherent $\mathcal{O}_X$-modules*.

Proposition 4. *If X is a prescheme, associating to (the isomorphism class of) a closed embedding $Y \xrightarrow{i} X$ the sheaf of ideals $\mathcal{J} = \ker\left(\mathcal{O}_X \xrightarrow{i^*} i_* \mathcal{O}_X\right)$ gives a bijection between the set of closed subpreschemes of X and the quasi-coherent sheaves of ideals in $\mathcal{O}_X$.*

Proof. This is [1, Proposition 1.5.3]. *q.e.d.*

Lemma 1. *For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ on a prescheme X, the following conditions are equivalent.*

(a) *For any affine open $U \subseteq X$, $\mathcal{M}(U)$ is a finitely generated $\mathcal{O}_X(U)$-module.*

(b) *It is possible to cover X by affine open subsets $U \subseteq X$, for which $\mathcal{M}(U)$ is a finitely generated $\mathcal{O}_X(U)$-module.*

Proof. This will follow from Lemma 2 and Lemma 3 below. *q.e.d.*

Lemma 2. *Let $\mathcal{P}$ be a property of affine open subsets of a prescheme X such that*

(α) *If $U \subseteq X$ is affine and $f \in \mathcal{O}_X(U)$, then $\mathcal{P}(U)$ implies $\mathcal{P}(U \setminus V(f))$.*

(β) *If U is affine and $f_1, \ldots, f_n \in \mathcal{O}_X(U)$ are such that $\bigcap_{i=1}^n V(f_i) = \emptyset$ and such that $\mathcal{P}(U \setminus V(f_i))$ holds for all $i = 1, \ldots, n$, then $\mathcal{P}(U)$ holds.*

Then the following assertions about X are equivalent.

(a) *If $U \subseteq X$ is affine open, $\mathcal{P}(U)$ holds.*

(b) *X may be covered by affine open U for which $\mathcal{P}(U)$ holds.*

Proof. We proved this in [1, Lemma 2.2.2]. $\hspace{2cm}$ *q.e.d.*

Lemma 3. (a) *If M is a finitely generated R-module, then M_f is a finitely generated R_f-module.*

(b) *If M is an R-module and $f_1, \ldots, f_n \in R$ such that $\bigcap_{i=1}^{n} V(f_i) = \emptyset$ in $\operatorname{Spec} R$ and such that M_{f_i} is finitely generated over R_{f_i}, then M is finitely generated over R.*

Proof. Part (a) is trivial, as the images of R-generators of M in M_f generate it as an R_f-module.

Now for part (b). As M_{f_i} is finitely generated over R_{f_i}, there are $k \in \mathbb{N}$ and $m_{i,j} \in M$, $j = 1, \ldots, N_i$ such that $m_{i,j} f^{-k}$ generate M_{f_i} over R_{f_i} (as there are only finitely many generators, we can choose a common exponent k for all of them). Then also the $m_{i,j}$ generate M_{f_i} since f_i is a unit in R_{f_i}. We claim that the $\{m_{i,j} \mid i = 1, \ldots, n \text{ and } j = 1, \ldots, N_i\}$ generate M as an R-module. Indeed, let $m \in M$, then

$$m = \sum_{j=1}^{N_i} \frac{r_{i,j}}{f_i^{\ell}} m_{i,j} \quad \text{in } M_{f_i} \ ,$$

where $r_{i,j} \in R$ and $\ell \in \mathbb{N}$ (again, we can choose a common exponent ℓ). Then there is some $\ell' \in \mathbb{N}$ such that

$$f_i^{\ell+\ell'} m = \sum_{j=1}^{N_i} r_{i,j} f_i^{\ell'} m_{i,j} \quad \text{in } M \ .$$

Replacing ℓ by $\ell + \ell'$ and $r_{i,j}$ by $f_i^{\ell'} r_{i,j}$ we may assume $\ell' = 0$, i.e.

$$f_i^{\ell} m = \sum_{j=1}^{N_i} r_{i,j} m_{i,j} \quad \text{in } M \ .$$

We now have $\bigcap_{i=1}^{n} V(f_i^{\ell}) = \bigcap_{i=1}^{n} V(f_i) = \emptyset$, hence the ideal generated by the f_i^{ℓ} is R and we thus find $g_1, \ldots, g_n \in R$ such that $\sum_{i=1}^{n} f_i^{\ell} g_i = 1$ in R. It follows that

$$m = \sum_{i=1}^{n} f_i^{\ell} g_i m = \sum_{i=1}^{n} \sum_{j=1}^{N_i} r_{i,j} g_i m_{i,j}$$

is an element of the submodule generated by the $m_{i,j}$. $\hspace{2cm}$ *q.e.d.*

Definition 6. (a) We call a quasi-coherent $\mathcal{O}_X$-module **locally finitely generated** it it satisfies the equivalent conditions from Lemma 1.

(b) When X is locally Noetherian (cf. [1, Definition 2.2.2]), an $\mathcal{O}_X$-module is called **coherent** if it is quasi-coherent and locally finitely generated.

Remark. There is a general definition of *coherent* sheaves of modules on arbitrary ringed spaces, which in the case of a locally Noetherian prescheme is equivalent to the above.

1.2. Čech cohomology

Let $\mathcal{U}\colon X = \bigcup_{i\in I} U_i$ be an open cover of a topological space X. In the following, we will use the convention

$$U_{i_0,\ldots,i_n} = \bigcap_{k=0}^{n} U_{i_k} \ . \tag{1}$$

Definition 1. For an open cover $\mathcal{U}$ of a topological space X (e.g., a prescheme) and $\mathcal{M}$ a presheaf of abelian groups (e.g., a quasi-coherent $\mathcal{O}_X$-module) on X the **Čech complex** $\check{C}^\bullet(\mathcal{U},\mathcal{M})$ is the cochain complex defined as follows. Let

$$\check{C}^n(\mathcal{U},\mathcal{M}) \coloneqq \prod_{(i_0,\ldots,i_n)\in I^{n+1}} \mathcal{M}(U_{i_0,\ldots,i_n}) \ .$$

Let the elements of $\check{C}^n(\mathcal{U},\mathcal{M})$ be denoted $\psi = (\psi_{i_0,\ldots,i_n})_{(i_0,\ldots,i_n)\in I^{n+1}}$. The differentials $\check{C}^n(\mathcal{U},\mathcal{M}) \xrightarrow{\check{d}^n} \check{C}^{n+1}(\mathcal{U},\mathcal{M})$ are defined by

$$(\check{d}^n\psi)_{i_0,\ldots,i_{n+1}} = \sum_{j=0}^{n+1}(-1)^j\psi_{i_0,\ldots,\hat{i}_j,\ldots,i_{n+1}}\big|_{U_{i_0,\ldots,i_{n+1}}}$$

where $\hat{i}_j$ denotes the omission of the index i_j. For instance,

$$(\check{d}^0\psi)_{i,j} = \psi_j|_{U_{i,j}} - \psi_i|_{U_{i,j}} \quad \text{and} \quad (\check{d}^1\psi)_{i,j,k} = \psi_{j,k}|_{U_{i,j,k}} - \psi_{i,k}|_{U_{i,j,k}} + \psi_{i,j}|_{U_{i,j,k}} \ .$$

The **Čech cohomology** $\check{H}^\bullet(\mathcal{U},\mathcal{M})$ is defined as the cohomology of the Čech complex, i.e.,

$$\check{H}^i(\mathcal{U},\mathcal{M}) = H^i\left(\check{C}^\bullet(\mathcal{U},\mathcal{M})\right) \ .$$

To see that $\check{C}^\bullet(\mathcal{U},\mathcal{M})$ is indeed a cochain complex, we need to prove $\check{d}^2 = 0$ – and we won't do this in a remark!

Proof of Definition 1. For $\ell = 0,\ldots,n+1$ let $\check{C}^n(\mathcal{U},\mathcal{M}) \xrightarrow{d_\ell} \check{C}^{n+1}(\mathcal{U},\mathcal{M})$ be given by

$$(d_\ell\psi)_{i_0,\ldots,i_{n+1}} = \psi_{i_0,\ldots,\hat{i}_\ell,\ldots,i_{n+1}}\big|_{U_{i_0,\ldots,i_{n+1}}} \ .$$

Again, $\hat{i}_\ell$ denotes the omission of the index i_ℓ.

Step 1. We prove that

$$d_m d_\ell = d_{\ell+1} d_m \quad \text{when } \ell \geq m \ . \tag{2}$$

Indeed, we have $(d_m d_\ell\psi)_{\boldsymbol{i}} = \psi_{\boldsymbol{j}}|_{U_{\boldsymbol{i}}}$, where $\boldsymbol{j}$ is obtained from $\boldsymbol{i}$ by omitting the indices i_ℓ and i_m when $\ell < m$ and the indices $i_{\ell+1}$ and i_m when $\ell \geq m$. The assertion follows.

Step 2. We prove the following. Let $C^\bullet$ be any family of abelian groups (or objects of an abelian category) and $C^n \xrightarrow{d_\ell} C^{n+1}$ morphisms for $\ell = 0,\ldots,n+1$. Suppose that $C^n = 0$ for $n < 0$ and that (2) holds. Then $d^\bullet$ with

$$d^n = \sum_{i=0}^{n+1}(-1)^i d_i$$

satisfies $d^{n+1}d^n = 0$. Indeed,

$$\begin{aligned}
d^{n+1}d^n &= \sum_{m=0}^{n+2}\sum_{\ell=0}^{n+1}(-1)^{\ell+m}d_m d_\ell = \sum_{m=0}^{n+2}\sum_{\ell=0}^{m-1}(-1)^{\ell+m}d_m d_\ell + \sum_{m=0}^{n+2}\sum_{\ell=m}^{n+1}(-1)^{\ell+m}d_m d_\ell \\
&= \sum_{m=0}^{n+2}\sum_{\ell=0}^{m-1}(-1)^{\ell+m}d_m d_\ell + \sum_{m=0}^{n+1}\sum_{\ell=m}^{n+1}(-1)^{\ell+m}d_{\ell+1} d_m \\
&= \sum_{i>j}(-1)^{i+j}d_i d_j + \sum_{i>j}(-1)^{i+j-1}d_i d_j = 0 \ ,
\end{aligned}$$

as required. $\hspace{12cm}$ *q.e.d.*

Remark. Our program is to show that $\check{H}^\bullet(\mathcal{U}, \mathcal{M})$ is independent of $\mathcal{U}$ and has the desired properties, when X is a scheme, $\mathcal{M} \in \mathrm{Ob}(\mathrm{QCoh}(X))$ and $\mathcal{U}$ is an affine open cover.

Remark. For instance, the cohomology of $\mathbb{P}_R^1 = \mathrm{Proj}(R[X_0, X_1])$ can be calculated using the affine open cover

$$U_i = \mathbb{P}_R^1 \setminus V(X_i) \simeq \mathrm{Spec}\left(R[X_0, X_1]_{X_i}\right)_0 \simeq \mathrm{Spec}\, R[t_i] \quad \text{where } t_i = \begin{cases} X_1 \cdot X_0^{-1} & \text{if } i = 0 \\ X_0 \cdot X_1^{-1} & \text{if } i = 1 \end{cases}.$$

Unfortunately, calculations become complicated by the fact that there are infinitely many non-zero terms in $\check{C}^\bullet(\mathcal{U}, \mathcal{M})$.

Let $\check{C}_{\mathrm{alt}}^n(\mathcal{U}, \mathcal{M}) \subseteq \check{C}^n(\mathcal{U}, \mathcal{M})$ be the subgroup containing all $\psi \in \check{C}^n(\mathcal{U}, \mathcal{M})$ such that

$$\psi_{i_{\pi(0)},\ldots,i_{\pi(n)}} = \mathrm{sgn}(\pi)\psi_{i_0,\ldots,i_n} \in \mathcal{M}(U_{i_0,\ldots,i_n}) \quad \text{and} \quad \psi_{i_0,\ldots,i_{n-1},i_{n-1}} = 0 \in \mathcal{M}(U_{i_0,\ldots,i_{n-1}})$$

for all permutations $\pi \in \mathfrak{S}_n$. Note that $U_{i_0,\ldots,i_n} = U_{i_{\pi(0)},\ldots,i_{\pi(n)}}$ as permuting indices doesn't change intersections, so the first property makes sense. Also note that both properties together imply that $\psi_{i_0,\ldots,i_n} = 0$ whenever $(i_0,\ldots,i_n)$ contains a repeated index.

Definition 2. $\check{C}_{\mathrm{alt}}^\bullet(\mathcal{U}, \mathcal{M}) \subseteq \check{C}^\bullet(\mathcal{U}, \mathcal{M})$ is a subcomplex, called the **alternating Čech complex**.

Proof. We need to confirm that the differential $\check{d}^n$ maps $\check{C}_{\mathrm{alt}}^n(\mathcal{U}, \mathcal{M})$ into $\check{C}_{\mathrm{alt}}^{n+1}(\mathcal{U}, \mathcal{M})$. To do this, define *codegeneracy maps*

$$\check{C}^n(\mathcal{U}, \mathcal{M}) \xrightarrow{s_\ell} \check{C}^{n-1}(\mathcal{U}, \mathcal{M}) \ , \quad (s_\ell\psi)_{i_0,\ldots,i_{n-1}} = \psi_{i_0,\ldots,i_\ell,i_\ell,\ldots,i_{n-1}} \quad \text{for } \ell = 0,\ldots,n-1$$

(i.e., s_ℓ repeats the ℓth index) as well as *transposition maps*

$$\check{C}^n(\mathcal{U}, \mathcal{M}) \xrightarrow{t_\ell} \check{C}^n(\mathcal{U}, \mathcal{M}) \ , \quad (t_\ell\psi)_{i_0,\ldots,i_n} = \psi_{i_0,\ldots,i_{\ell-1},i_{\ell+1},i_\ell,i_{\ell+2},\ldots,i_{n-1}} \quad \text{for } \ell = 0,\ldots,n-1$$

(i.e., t_ℓ swaps the ℓth and $(\ell+1)^{\mathrm{st}}$ index). As any permutation may be expressed as a composition of elementary transpositions, $\check{C}_{\mathrm{alt}}^n(\mathcal{U}, \mathcal{M}) \subseteq \check{C}^n(\mathcal{U}, \mathcal{M})$ is given by the relations

$$s_\ell\psi = 0 \quad \text{and} \quad t_\ell\psi = -\psi \quad \text{for } \ell = 0,\ldots,n-1$$

So what we need to check to confirm that $\check{C}_{\mathrm{alt}}^\bullet(\mathcal{U}, \mathcal{M})$ is indeed a subcomplex of $\check{C}^\bullet(\mathcal{U}, \mathcal{M})$ is that the above relations are preserved by the differential $\check{d}$.

8

One may easily check the relations

$$
s_\ell d_i = \begin{cases} d_i s_{\ell-1} & \text{if } i < \ell \\ \text{id} & \text{if } i = \ell \text{ or } i = \ell+1 \\ d_{i-1} s_\ell & \text{if } i > \ell+1 \end{cases} \tag{3}
$$

and

$$
t_\ell d_j = \begin{cases} d_j t_\ell & \text{if } \ell < j-1 \\ d_\ell & \text{if } \ell = j-1 \\ d_{\ell+1} & \text{if } \ell = j \\ d_j t_{\ell-1} & \text{if } \ell > j \end{cases}. \tag{4}
$$

Now let $\psi \in \check{C}^n(\mathcal{U}, \mathcal{M})$ such that $t_j \psi = -\psi$ for all $j = 0, \ldots, n-1$. Using (4), we get

$$
\begin{aligned}
t_\ell \check{d} \psi &= \sum_{j=0}^{\ell-1} (-1)^j t_\ell d_j \psi + (-1)^\ell t_\ell d_\ell \psi + (-1)^{\ell+1} t_\ell d_{\ell+1} \psi + \sum_{j=\ell+2}^{n} (-1)^j t_\ell d_j \psi \\
&= \sum_{j=0}^{\ell-1} (-1)^j d_j t_{\ell-1} \psi + (-1)^\ell d_{\ell+1} \psi + (-1)^{\ell+1} d_\ell \psi + \sum_{j=\ell+2}^{n} (-1)^j d_j t_\ell \psi \\
&= -\sum_{j=0}^{\ell-1} (-1)^j d_j \psi - (-1)^\ell d_\ell \psi - (-1)^{\ell+1} d_{\ell+1} \psi - \sum_{j=\ell+2}^{n} (-1)^j d_j \psi \\
&= -\check{d}\psi \,.
\end{aligned}
$$

Similarly, one can check that $s_\ell \check{d} \psi = 0$ when $s_j \psi = 0$ for all $j = 0, 1, \ldots, n-1$. This shows that $\check{d}$ restricts to a differential on $\check{C}^n_{\text{alt}}(\mathcal{U}, \mathcal{M})$, as required. *q.e.d.*

It will eventually turn out that the cohomology groups $\check{H}^i_{\text{alt}}(\mathcal{U}, \mathcal{M}) = H^i\left(\check{C}^\bullet_{\text{alt}}(\mathcal{U}, \mathcal{M})\right)$ obtained from the alternating Čech complex are the same as the regular Čech cohomology groups $\check{H}^i(\mathcal{U}, \mathcal{M})$.

Remark 1. A *cosimplicial object* of a category $\mathcal{A}$ is a sequence of objects $(X^n)_{n \geq 0}$ with morphisms $d_j \colon X^n \to X^{n+1}$ for $j = 0, \ldots, n+1$ satisfying (2) and $s_j \colon X^n \to X^{n-1}$ for $j = 0, \ldots, n$ satisfying a version of (2) together with (3). In other words, a cosimplicial object is a covariant functor from the *simplex category* Δ to $\mathcal{A}$.

There is a *Dold–Puppe correspondence* between cochain complexes concentrated in nonnegative degrees and cosimplicial objects of an abelian category.

Example 1. (*a*) By the sheaf axiom,

$$
\begin{aligned}
\mathcal{M}(X) &\simeq \left\{ (m_i)_{i \in I} \in \prod_{i \in I} \mathcal{M}(U_i) \;\middle|\; m_i|_{U_{i,j}} = m_j|_{U_{i,j}} \right\} \\
&= \ker\left(\check{C}^0(\mathcal{U}, \mathcal{M}) \xrightarrow{\check{d}^0} \check{C}^1(\mathcal{U}, \mathcal{M}) \right) \simeq \check{H}^0(\mathcal{U}, \mathcal{M}) \\
&= \ker\left(\check{C}^0_{\text{alt}}(\mathcal{U}, \mathcal{M}) \xrightarrow{\check{d}^0} \check{C}^1_{\text{alt}}(\mathcal{U}, \mathcal{M}) \right) \simeq \check{H}^0_{\text{alt}}(\mathcal{U}, \mathcal{M})
\end{aligned}
$$

(b) For the trivial cover $\mathcal{U}_0\colon X = X$, the Čech complex $\check{C}^\bullet(\mathcal{U}_0, \mathcal{M})$ has the form

$$\mathcal{M}(X) \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$$

and the alternating Čech complex $\check{C}^\bullet_{\mathrm{alt}}(\mathcal{U}_0, \mathcal{M})$ looks like

$$\mathcal{M}(X) \xrightarrow{0} \mathcal{M}(X) \xrightarrow{\mathrm{id}} \mathcal{M}(X) \xrightarrow{0} \ldots \, ,$$

so

$$\check{H}^n(\mathcal{U}_0, \mathcal{M}) = \check{H}^n_{\mathrm{alt}}(\mathcal{U}_0, \mathcal{M}) = \begin{cases} \mathcal{M}(X) & \text{if } n = 0 \\ 0 & \text{else} \end{cases} .$$

(c) If $\mathcal{M}$ and $\mathcal{N}$ are presheaves of modules on X and $\mathcal{U}$ is an open cover of X, then

$$\check{C}^\bullet(\mathcal{U}, \mathcal{M} \oplus \mathcal{N}) = \check{C}^\bullet(\mathcal{U}, \mathcal{M}) \oplus \check{C}^\bullet(\mathcal{U}, \mathcal{N})$$

and, more general,

$$\check{C}^\bullet\left(\mathcal{U}, \prod_{i \in I} \mathcal{M}_i\right) = \prod_{i \in I} \check{C}^\bullet(\mathcal{U}, \mathcal{M}_i) \, .$$

The same holds for $\check{C}^\bullet_{\mathrm{alt}}(\mathcal{U}, -)$.

If $\mathcal{U}\colon X = \bigcup_{i \in I} U_i$ is a cover of a scheme X by affine open subsets U_i, then all intersections $U_{i_0,\ldots,i_n}$ are affine again by Proposition 1.1.1. If $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is a short exact sequence in $\mathrm{QCoh}(X)$, Corollary 1.1.2 provides short exact sequences

$$0 \longrightarrow \check{C}^\bullet(\mathcal{U}, \mathcal{M}') \longrightarrow \check{C}^\bullet(\mathcal{U}, \mathcal{M}) \longrightarrow \check{C}^\bullet(\mathcal{U}, \mathcal{M}'') \longrightarrow 0$$

and

$$0 \longrightarrow \check{C}^\bullet_{\mathrm{alt}}(\mathcal{U}, \mathcal{M}') \longrightarrow \check{C}^\bullet_{\mathrm{alt}}(\mathcal{U}, \mathcal{M}) \longrightarrow \check{C}^\bullet_{\mathrm{alt}}(\mathcal{U}, \mathcal{M}'') \longrightarrow 0$$

of chain complexes. For $\check{C}^\bullet(\mathcal{U}, -)$ this is immediate from Definition 1 and from the fact that products of short exact sequences are short exact again. To see this for $\check{C}^\bullet_{\mathrm{alt}}(\mathcal{U}, -)$, choose any linear ordering of I and note that

$$\check{C}^n_{\mathrm{alt}}(\mathcal{U}, \mathcal{M}) \simeq \prod_{i_0 < \ldots < i_n \in I} \mathcal{M}(U_{i_0,\ldots,i_n}) \, ,$$

then the same argument may be applied.

Taking long exact cohomology sequences we just proved

Proposition 1. *If $\mathcal{U}$ is an affine open cover of a scheme X and $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ a short exact sequence of quasi-coherent sheaves of $\mathcal{O}_X$-modules, then there is a long exact cohomology sequence*

$$0 \longrightarrow \check{H}^0(\mathcal{U}, \mathcal{M}') \longrightarrow \check{H}^0(\mathcal{U}, \mathcal{M}) \longrightarrow \check{H}^0(\mathcal{U}, \mathcal{M}'') \xrightarrow{d} \check{H}^1(\mathcal{U}, \mathcal{M}') \longrightarrow \check{H}^1(\mathcal{U}, \mathcal{M}) \longrightarrow \ldots$$

$$\begin{array}{ccccc} & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ 0 \longrightarrow & \mathcal{M}'(X) & \longrightarrow & \mathcal{M}(X) & \longrightarrow & \mathcal{M}''(X) \end{array}$$

and similar for $\check{H}^\bullet_{\mathrm{alt}}(\mathcal{U}, -)$.

Remark 2. For arbitrary preschemes, the situation is more difficult (cf. Thomason, *The Grothendieck Festschrift*).

Definition 3. An open cover $\mathcal{V}\colon X = \bigcup_{j\in J} V_j$ is a **refinement** of $\mathcal{U}\colon X = \bigcup_{i\in I} U_i$ if there is a map $v\colon J \to I$ such that $V_j \subseteq U_{v(j)}$ for all $j \in J$. Such a v is called a **refinement map** for the pair $(\mathcal{V},\mathcal{U})$.

Note that Professor Franke isn't sure whether *refinement map* is the usual term. Assuming the axiom of choice, the existence of v is equivalent to every V_j being contained in some U_i.

A refinement map v induces a morphism

$$\check{C}^\bullet(\mathcal{U},\mathcal{M}) \xrightarrow{v^*} \check{C}^\bullet(\mathcal{V},\mathcal{M}) , \quad (v^n\psi)_{j_0,\dots,j_n} = \psi_{v(j_0),\dots,v(j_n)}|_{V_{j_0,\dots,j_n}} \quad \text{for } \psi \in \check{C}^n(\mathcal{U},\mathcal{M}) .$$

Clearly, v^* commutes with the d_j, s_j, and t_j, hence restricts to a morphism of chain complexes $\check{C}^\bullet_{\mathrm{alt}}(\mathcal{U},\mathcal{M}) \xrightarrow{v^*} \check{C}^\bullet_{\mathrm{alt}}(\mathcal{V},\mathcal{M})$.

Lemma 1. (a) *A refinement $\mathcal{W}$ of a refinement $\mathcal{V}$ of $\mathcal{U}$ is a refinement of $\mathcal{U}$ and if v and w are associated refinement maps for $(\mathcal{V},\mathcal{U})$ and $(\mathcal{W},\mathcal{U})$, then vw is a refinement map for $(\mathcal{W},\mathcal{U})$. Moreover, $(vw)^* = w^*v^*$ and the identity id_I is a refinement map for $(\mathcal{U},\mathcal{U})$ and $\mathrm{id}_I^* = \mathrm{id}_{\check{C}^*(\mathcal{U},-)}$.*

(b) *Two arbitrary open covers have a common refinement. When X is a prescheme, this common refinement can be chosen affine.*

(c) *If $v_1, v_2\colon J \to I$ are two refinement maps for $(\mathcal{V},\mathcal{U})$, then v_1^* and v_2^* induce the same morphism on Čech cohomology.*

Proof. Part (a) is obvious. For (b), let $\mathcal{U}$ and $\mathcal{V}$ be open covers of X. Then $X = \bigcup_{(i,j)\in I\times J} U_i\cap V_j$ is a common refinement. When X is a prescheme, we may cover each $U_i \cap V_j$ by affine open subsets and thus obtain a common affine refinement of $\mathcal{U}$ and $\mathcal{V}$.

Now for part (c). Define maps $h^n\colon \check{C}^n(\mathcal{U},\mathcal{M}) \to \check{C}^{n-1}(\mathcal{V},\mathcal{M})$ as follows: We put

$$h^n = \sum_{\ell=0}^{n-1}(-1)^\ell h_\ell ,$$

where $h_\ell\colon \check{C}^n(\mathcal{U},\mathcal{M}) \to \check{C}^{n-1}(\mathcal{V},\mathcal{M})$ is given by

$$(h_\ell\psi)_{j_0,\dots,j_{n-1}} = \psi_{v_1(j_0),\dots,v_1(j_\ell),v_2(j_\ell),\dots,v_2(j_{n-1})}|_{V_{j_0,\dots,j_{n-1}}} \quad \text{for } \psi \in \check{C}^n(\mathcal{U},\mathcal{M}) .$$

Then it's a straightforward but tedious check that the following relations hold:

$$h_\ell d_k = \begin{cases} d_k h_{\ell-1} & \text{if } 0 \le k < \ell \\ h_{\ell-1} d_k & \text{if } 0 < k = \ell \\ v_2^n & \text{if } 0 = k = \ell \\ h_{\ell+1} d_k & \text{if } k = \ell+1 < n \\ v_1^n & \text{if } k = \ell+1 = n \\ d_{k-1} h_\ell & \text{if } k > \ell+1 \end{cases} . \tag{5}$$

(I tried my best to get the indices right and I claim my hit ratio is way higher than Franke's). Our goal is to show

$$\check{d}^{n-1}h^n + h^{n+1}\check{d}^n = v_2^n - v_1^n \,,$$

for then h is a cochain homotopy between v_1^* and v_2^* and it's a well-known fact from homological algebra that cochain homotopic maps induce same morphisms on cohomology. Indeed, using (5) we get

$$
\begin{aligned}
h^{n+1}\check{d}^n &= \sum_{\ell=0}^{n}\sum_{k=0}^{n+1}(-1)^{\ell+k}h_\ell d_k \\
&= \sum_{\ell=0}^{n}\sum_{k=0}^{\ell-1}(-1)^{\ell+k}h_\ell d_k + (-1)^0 h_0 d_0 + \sum_{\ell=1}^{n}(-1)^{2\ell}h_\ell d_\ell \\
&\quad + \sum_{\ell=0}^{n-1}(-1)^{2\ell+1}h_\ell d_{\ell+1} + (-1)^{2n+1}h_n d_{n+1} + \sum_{\ell=0}^{n}\sum_{k=\ell+2}^{n+1}(-1)^{\ell+k}h_\ell d_k \\
&= \sum_{\ell=0}^{n}\sum_{k=0}^{\ell-1}(-1)^{\ell+k}d_k h_{\ell-1} + v_2^n + \sum_{\ell=1}^{n}h_{\ell-1}d_\ell \\
&\quad - \sum_{\ell=0}^{n-1}h_\ell d_{\ell+1} - v_1^n + \sum_{\ell=0}^{n}\sum_{k=\ell+2}^{n+1}(-1)^{\ell+k}d_{k-1}h_\ell \\
&= \sum_{\ell=0}^{n-1}\sum_{k=0}^{\ell}(-1)^{\ell+1+k}d_k h_\ell + \sum_{\ell=0}^{n}\sum_{k=\ell+1}^{n}(-1)^{\ell+k+1}d_k h_\ell + v_2^n - v_1^n \\
&= -\sum_{k=0}^{n}\sum_{\ell=0}^{n-1}(-1)^{\ell+k}d_k h_\ell + v_2^n - v_1^n \\
&= -\check{d}^{n-1}h^n + v_2^n - v_1^n \,,
\end{aligned}
$$

as required. *q.e.d.*

Corollary 1. *Let $\mathcal{M}$ be a presheaf of modules on a space X and $\mathcal{U}, \mathcal{V}$ be open covers of X.*

(a) *If $\mathcal{V}$ is a refinement of $\mathcal{U}$, we have a canonical morphism*

$$\tau_{\mathcal{U},\mathcal{V}} \colon \check{H}^\bullet(\mathcal{U}, \mathcal{M}) \longrightarrow \check{H}^\bullet(\mathcal{V}, \mathcal{M})$$

satisfying $\tau_{\mathcal{V},\mathcal{W}}\tau_{\mathcal{U},\mathcal{V}} = \tau_{\mathcal{U},\mathcal{W}}$ if $\mathcal{W}$ is another open cover of X which is a refinement of $\mathcal{V}$, as well as $\tau_{\mathcal{U},\mathcal{U}} = \mathrm{id}$.

(b) *If $\mathcal{U}$ is a refinement of $\mathcal{V}$ and $\mathcal{V}$ a refinement of $\mathcal{U}$, then $\tau_{\mathcal{U},\mathcal{V}}$ and $\tau_{\mathcal{V},\mathcal{U}}$ are isomorphisms which are inverse to each other.*

(c) *If there is an $i^* \in I$ such that $U_{i^*} = X$, then $\check{H}^n(\mathcal{U}, \mathcal{M}) = 0$ for $n \geq 1$.*

Proof. Part (a) follows from Lemma 1. Part (b) follows from (a) as $\tau_{\mathcal{V},\mathcal{U}}\tau_{\mathcal{U},\mathcal{V}} = \tau_{\mathcal{U},\mathcal{U}} = \mathrm{id}$. Part (c) follows from (b) and Example 1 as $\mathcal{U}$ and the trivial cover $\mathcal{U}_0 \colon X = X$ are refinements of each other in this case. *q.e.d.*

Remark 3. In general, the cochain homotopy used in the proof of Lemma 1 won't preserve the subcomplex $\check{C}_{\mathrm{alt}}^\bullet(-, \mathcal{M}) \subseteq \check{C}^\bullet(-, \mathcal{M})$. However, Corollary 1(c) for $\check{H}_{\mathrm{alt}}^\bullet(\mathcal{U}, \mathcal{M})$ can still be obtained, using the *cochain contraction* $h \colon \check{C}^\bullet(\mathcal{U}, \mathcal{M}) \to \check{C}^\bullet(\mathcal{U}, \mathcal{M})$, where $h^n \colon \check{C}^n(\mathcal{U}, \mathcal{M}) \to \check{C}^{n-1}(\mathcal{U}, \mathcal{M})$ for $n \geq 1$ is given by

$$(h^n \psi)_{i_0, \ldots, i_n} = \psi_{i^*, i_0, \ldots, i_n} \ .$$

It's a straightforward check that h preserves $\check{C}_{\mathrm{alt}}^\bullet(\mathcal{U}, \mathcal{M}) \subseteq \check{C}^\bullet(\mathcal{U}, \mathcal{M})$. Moreover, one has the relations

$$h^n d_k = \begin{cases} \mathrm{id} & \text{if } k = 0 \\ d_{k-1} h^{n-1} & \text{if } k > 0 \end{cases} ,$$

hence

$$\begin{aligned} \check{d}^{n-1} h^n + h^{n+1} \check{d}^n &= \sum_{k=0}^n (-1)^k d_k h^n + \sum_{k=0}^{n+1} (-1)^k d_k h^{n+1} \\ &= \mathrm{id} + \sum_{k=0}^n \left((-1)^k + (-1)^{k+1} \right) d_k h^n \\ &= \mathrm{id} \ , \end{aligned}$$

proving that h is indeed a cochain contraction, hence $\check{H}_{\mathrm{alt}}^i(\mathcal{U}, \mathcal{M}) = 0$ for $i > 0$. For our purposes, this will turn out to be sufficient.

We now arrive at the main result of this section.

Proposition 2. *Let X be a quasi-compact scheme, $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module and $\mathcal{U}$ and affine open cover of X.*

(a) *If $\mathcal{V}$ is another affine open cover of X which is a refinement of $\mathcal{U}$, then*

$$\tau_{\mathcal{U}, \mathcal{V}} \colon \check{H}^\bullet(\mathcal{U}, \mathcal{M}) \overset{\sim}{\longrightarrow} \check{H}^\bullet(\mathcal{V}, \mathcal{M}) \ .$$

(b) *The inclusion $\check{C}_{\mathrm{alt}}^\bullet(\mathcal{U}, \mathcal{M}) \subseteq \check{C}^\bullet(\mathcal{U}, \mathcal{M})$ of cochain complexes induces an isomorphism*

$$\check{H}_{\mathrm{alt}}^\bullet(\mathcal{U}, \mathcal{M}) \overset{\sim}{\longrightarrow} \check{H}^\bullet(\mathcal{U}, \mathcal{M}) \ .$$

(c) *If X is affine and $i > 0$, then $\check{H}_{\mathrm{alt}}^i(\mathcal{U}, \mathcal{M}) = \check{H}^i(\mathcal{U}, \mathcal{M}) = 0$.*

Before the proof, a lemma.

Lemma 2. *For an open cover $\mathcal{U} \colon X = \bigcup_{i \in I} \mathcal{U}_i$ of a topological space X and a continuous map $f \colon Y \to X$, let $f^{-1}(\mathcal{U})$ be the cover $Y = \bigcup_{i \in I} f^{-1}(U_i)$. Let $\mathcal{F}$ be a sheaf of abelian groups on Y.*

(a) *We have an isomorphism of cochain complexes $\check{C}^\bullet(f^{-1}(\mathcal{U}), \mathcal{F}) \simeq \check{C}^\bullet(\mathcal{U}, f_* \mathcal{F})$ and this isomorphism restricts to an isomorphism $\check{C}_{\mathrm{alt}}^\bullet(f^{-1}(\mathcal{U}), \mathcal{F}) \simeq \check{C}_{\mathrm{alt}}^\bullet(\mathcal{U}, f_* \mathcal{F})$.*

(b) *If the image of f is contained in one of the open subsets U_i, then $\check{H}^i(\mathcal{U}, f_* \mathcal{F}) = \check{H}_{\mathrm{alt}}^i(\mathcal{U}, f_* \mathcal{F}) = 0$ for $i > 0$.*

Proof. Part (a) is pretty much tautological. Let $\mathcal{V} = f^{-1}(\mathcal{U})$, $V_i = f^{-1}(U_i)$, then we have $V_{i_0,\ldots,i_n} = f^{-1}(U_{i_0,\ldots,i_n})$, hence $f_*\mathcal{F}(V_{i_0,\ldots,i_n}) = \mathcal{F}(U_{i_0,\ldots,i_n})$ by definition of the *direct image $f_*\mathcal{F}$*. The differentials $\check{d}$ as well as the degeneracy and transposition maps s_ℓ, t_ℓ of $\check{C}^\bullet(f^{-1}(\mathcal{U}), \mathcal{F})$ and $\check{C}^\bullet(\mathcal{U}, f_*\mathcal{F})$ clearly coincide, proving the asserted isomorphisms of cochain complexes.

Part (b) follows from (a), Corollary 1(c) and Remark 3. $\hfill q.e.d.$

Proof of Proposition 2. Part (a). We fix the covers $\mathcal{U}$ and $\mathcal{V}$ and consider the following conditions on $\mathcal{M}$:

$\mathcal{M}$ satisfies $A_i(\mathcal{M})$ iff $\check{H}^j(\mathcal{U}, \mathcal{M}) \xrightarrow{\tau_{\mathcal{U},\mathcal{V}}} \check{H}^j(\mathcal{V}, \mathcal{M})$ is an isomorphism for $j < i$ and injective for $i = j$.

Claim 1. For a short exact sequence $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ of quasi-coherent $\mathcal{O}_X$-modules one has the implication

$$\left[A_{i+1}(\mathcal{M}) \text{ and } A_i(\mathcal{M}'')\right] \Longrightarrow A_{i+1}(\mathcal{M}') .$$

To prove Claim 1, consider the commutative diagram

$$
\begin{array}{ccccccccc}
\check{H}^{j-1}(\mathcal{U}, \mathcal{M}) & \longrightarrow & \check{H}^{j-1}(\mathcal{U}, \mathcal{M}'') & \longrightarrow & \check{H}^j(\mathcal{U}, \mathcal{M}') & \longrightarrow & \check{H}^j(\mathcal{U}, \mathcal{M}) & \longrightarrow & \check{H}^j(\mathcal{U}, \mathcal{M}'') \\
\alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow & & \varepsilon \downarrow \\
\check{H}^{j-1}(\mathcal{V}, \mathcal{M}) & \longrightarrow & \check{H}^{j-1}(\mathcal{V}, \mathcal{M}'') & \longrightarrow & \check{H}^j(\mathcal{V}, \mathcal{M}') & \longrightarrow & \check{H}^j(\mathcal{V}, \mathcal{M}) & \longrightarrow & \check{H}^j(\mathcal{V}, \mathcal{M}'')
\end{array}
$$

When $j \leq i$, then α, β, and δ are isomorphisms, while ε is injective, proving γ to be an isomorphism by the five lemma. When $j = i + 1$, α is an isomorphism by $A_{i+1}(\mathcal{M})$ while β and δ are injective, proving γ to be injective by the four lemma. Claim 1 follows.

Claim 2. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ there are a quasi-coherent $\mathcal{O}_X$-module $\mathcal{N}$ such that $A_i(\mathcal{N})$ holds for arbitary i, and a monomorphism $\mathcal{M} \hookrightarrow \mathcal{N}$.

To show this, let $j_W \colon W \to X$ be the embedding of any affine open subset W contained in one of the open subsets forming the cover $\mathcal{V}$. As X is a scheme, the morphism j_W is affine, i.e., preimages of affine open subsets are affine (cf. [1, Definition 2.5.1]). Indeed, since preimages in this case are intersections with W, this follows from Proposition 1.1.1. Being an open embedding, j_W is also quasi-separated. By Proposition 1.1.2 and Proposition 1.1.3(b), $(j_W)_*\mathcal{M}|_W$ is a quasi-coherent $\mathcal{O}_X$-module. Since X is quasi-compact, we find finitely many affine open subsets $W_1, \ldots, W_n$ covering X, each W_i contained in an element of $\mathcal{V}$ (that is, $X = \bigcup_{i=1}^n W_i$ is a finite affine refinement of $\mathcal{V}$). Then

$$\mathcal{N} := \bigoplus_{i=1}^n (j_{W_i})_*\mathcal{M}|_{W_i}$$

is quasi-coherent by Proposition 1.1.3(a).

We have morphisms

$$\mathcal{M}(U) \longrightarrow \mathcal{N}(U) = \bigoplus_{i=1}^{n} (j_{W_i})_* (\mathcal{M}|_{W_i})(U) = \bigoplus_{i=1}^{n} \mathcal{M}(U \cap W_i)$$
$$m \longmapsto (m|_{U \cap W_i})_{i=1}^{n}$$

which are injective as $\mathcal{M}$ satisfies the sheaf axiom and form a morphism of $\mathcal{O}_X$-modules $\mathcal{M} \to \mathcal{N}$, which is thus a monomorphism.

By Example $1(c)$,

$$\check{H}^k(\mathcal{U}, \mathcal{N}) = \bigoplus_{i=1}^{n} \check{H}^k(\mathcal{U}, (j_{W_i})_* \mathcal{M}|_{W_i})$$

and same for $\mathcal{V}$. But Example $1(a)$ and Lemma $2(b)$ (which applies here as every W_i is contained in an element of $\mathcal{V}$, thus also of $\mathcal{U}$) show that

$$\check{H}^k(\mathcal{U}, (j_{W_i})_* \mathcal{M}|_{W_i}) = \check{H}^k(\mathcal{V}, (j_{W_i})_* \mathcal{M}|_{W_i}) = \begin{cases} \mathcal{M}(W_i) & \text{if } k = 0 \\ 0 & \text{else} \end{cases},$$

hence $(j_{W_i})_* \mathcal{M}|_{W_i}$ has property A_i for arbitrary i. Now $\mathcal{N}$ inherits the property A_i from its summands, proving Claim 2.

We are now able to finish the proof of Proposition $2(a)$. We prove by induction on i that $A_i(\mathcal{M})$ holds for arbitrary quasi-coherent $\mathcal{O}_X$-modules $\mathcal{M}$. For $i = 0$, note that $\check{H}^0(\mathcal{U}, \mathcal{M}) \simeq \check{H}^0(\mathcal{V}, \mathcal{M}) \simeq \mathcal{M}(X)$ for all $\mathcal{O}_X$-modules $\mathcal{M}$ (quasi-coherent or not), i.e., $A_0(\mathcal{M})$ holds.

Now suppose $i \geq 1$ and A_{i-1} holds for every quasi-coherent $\mathcal{O}_X$-module. Using Claim 2, we choose an embedding $\mathcal{M} \xhookrightarrow{\iota} \mathcal{N}$ where $\mathcal{N}$ is quasi-coherent and satisfies A_i, then apply Claim 1 to the exact sequence

$$0 \longrightarrow \mathcal{M} \xrightarrow{\iota} \mathcal{N} \longrightarrow \operatorname{coker}(\iota) \longrightarrow 0$$

to derive $A_i(\mathcal{M})$ from $A_i(\mathcal{N})$ and $A_{i-1}(\operatorname{coker}(\iota))$. Here, Proposition $1.1.3(a)$ ensures that $\operatorname{coker}(\iota)$ is quasi-coherent as well.

Part (b). The proof of part (a) can be carried over with the following modifications.

- $A_i(\mathcal{M})$ is now the condition that $\check{H}^j_{\mathrm{alt}}(\mathcal{U}, \mathcal{M}) \to \check{H}^j(\mathcal{U}, \mathcal{M})$ is an isomorphism when $j < i$ and injective for $j = i$.

- Claim 1 also holds for the new A_i.

- If $W \subseteq X$ is affine and contained in one of the V_i and $\mathcal{F} \in \mathrm{Ob}(\mathrm{Qcoh}(W))$, then $(j_W)_* \mathcal{F}$ satisfies all A_i (which follows like above from Lemma $2(b)$ and Example 1).

Part (c). Apply (a) with $\mathcal{U}$ the trivial cover $X = X$ and $\mathcal{V}$ any affine open cover of X to conclude $\check{H}^i(\mathcal{V}, \mathcal{M}) = 0$ when $i > 0$ and $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module. Then one uses (b) to show that $\check{H}^\bullet_{\mathrm{alt}}(\mathcal{V}, \mathcal{M})$ also vanishes in positive degrees. $q.e.d.$

Definition 4. Let X be a quasi-compact scheme and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module. We put

$$H^i(X, \mathcal{M}) \coloneqq \check{H}^i(\mathcal{V}_0, \mathcal{M}) \,,$$

where $\mathcal{V}_0$ is the affine open cover of X consisting of all affine open subsets of X.

Summing up our previous work, we obtain the following Theorem 1 in full glory.

Theorem 1. *Let X be a quasi-compact scheme and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module.*

(*a*) *If $\mathcal{U}$ is any affine open over of X, then*

$$H^\bullet(X, \mathcal{M}) \xrightarrow[\tau_{\mathcal{U}, \mathcal{V}_0}]{\sim} \check{H}^\bullet(\mathcal{U}, \mathcal{M}) \xleftarrow{\sim} \check{H}^\bullet_{\mathrm{alt}}(\mathcal{U}, \mathcal{M})$$

are isomorphisms compatible with the (iso)morphisms $\check{H}^\bullet(\mathcal{U}, \mathcal{M}) \xrightarrow{\tau_{\mathcal{U}, \mathcal{V}}} \check{H}^\bullet(\mathcal{V}, \mathcal{M})$ when $\mathcal{V}$ is an affine refinement of $\mathcal{U}$.

(*b*) *When X is affine, $H^i(X, \mathcal{M}) = 0$ when $i > 0$.*

(*c*) *There is a canonical isomorphism $H^0(X, \mathcal{M}) \simeq \mathcal{M}(X)$.*

(*d*) *When $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules, one has a long exact cohomology sequence*

$$
\begin{array}{ccccc}
0 \longrightarrow \mathcal{M}'(X) \longrightarrow & \mathcal{M}(X) \longrightarrow & \mathcal{M}''(X) \\
\Big\updownarrow \wr & \Big\updownarrow \wr & \Big\updownarrow \wr \\
0 \longrightarrow H^0(X, \mathcal{M}') \longrightarrow H^0(X, \mathcal{M}) \longrightarrow & H^0(X, \mathcal{M}'') \xrightarrow{d} H^1(X, \mathcal{M}') \longrightarrow \dots
\end{array}
$$

$$\dots \longrightarrow H^{i-1}(X, \mathcal{M}'') \xrightarrow{d} H^i(X, \mathcal{M}') \longrightarrow H^i(X, \mathcal{M}) \longrightarrow H^i(X, \mathcal{M}'') \xrightarrow{d} \dots$$

Proof. Part (*a*) follows from Proposition 2 and the fact that every affine open cover $\mathcal{U}$ is a refinement of $\mathcal{V}_0$. Part (*b*) is Proposition 2(*c*). Part (*c*) was seen in Example 1(*a*) and (*d*) follows from Proposition 1. *q.e.d.*

Corollary 2. *Let $0 \to \mathcal{M}' \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\beta} \mathcal{M}'' \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules, where X is any prescheme. Then if two of $\mathcal{M}', \mathcal{M}, \mathcal{M}''$ are quasi-coherent, so is the third, and for any affine open subset $U \subseteq X$ the sequence*

$$0 \longrightarrow \mathcal{M}'(U) \xrightarrow{\alpha} \mathcal{M}(U) \xrightarrow{\beta} \mathcal{M}''(U) \longrightarrow 0 \qquad\qquad (*)$$

is exact.

Proof. We have seen in Proposition 1.1.3(*a*) that the category $\mathrm{QCoh}(X)$ of quasi-coherent $\mathcal{O}_X$-modules is stable under kernels and cokernels. Therefore, $\mathcal{M}'$ is quasi-coherent when $\mathcal{M}$ and $\mathcal{M}''$ are and $\mathcal{M}''$ is quasi-coherent when $\mathcal{M}'$ and $\mathcal{M}$ are. We may thus assume that $\mathcal{M}'$ and $\mathcal{M}''$ are the quasi-coherent ones.

To show exactness of $(*)$, we may replace X by U and the involved sheaves by their restriction to U and thus assume that $X = U = \operatorname{Spec} R$ is affine. This may also be assumed in the proof of

quasi-coherentness of $\mathcal{M}$, which is a local question. We then only need to show surjectivity of $\mathcal{M}(X) \xrightarrow{\beta} \mathcal{M}''(X)$. If we had the long exact cohomology sequence available, this would be an immediate consequence of the vanishing of $H^1(X, \mathcal{M}')$ – but we haven't, so what the proof does is mimicking the cohomological argument on foot. For this, let $m'' \in \mathcal{M}''(X)$. As $\mathcal{M} \xrightarrow{\beta} \mathcal{M}''$ is an epimorphism and the affine open subsets form a topology base in X, there is an affine open cover $\mathcal{V} \colon X = \bigcup_{i \in I} V_i$ such that $m''|_{V_i}$ is in the image of $\beta(m_i)$ for some $m_i \in \mathcal{M}(V_i)$.

The m_i satisfy $\beta(m_i|_{V_{i,j}} - m_j|_{V_{i,j}}) = 0 \in \mathcal{M}''(V_{i,j})$, hence there are $m'_{i,j} \in \mathcal{M}(V_{i,j})$ such that $\alpha(m'_{i,j}) = m_i|_{V_{i,j}} - m_j|_{V_{i,j}}$, the sequence $0 \to \mathcal{M}'(V_{i,j}) \xrightarrow{\alpha} \mathcal{M}(V_{i,j}) \xrightarrow{\beta} \mathcal{M}''(V_{i,j})$ being exact. Consider the elements $\mu = (m_i)_{i \in I} \in \check{C}^0(\mathcal{V}, \mathcal{M})$ and $\mu' = (m'_{i,j})_{(i,j) \in I^2} \in \check{C}^1(\mathcal{V}, \mathcal{M}')$. Then $\alpha(\mu') = \check{d}\mu$. Thus

$$\alpha(\check{d}\mu') = \check{d}\alpha(\mu') = \check{d}^2\mu = 0 \ ,$$

hence $\check{d}\mu' = 0$ by injectivity of α. This means that μ' is a cocycle, and by the vanishing of $\check{H}^1(\mathcal{V}, \mathcal{M}')$ (due to Proposition 2(c)), also a coboundary. That is, $\mu' = \check{d}\lambda'$ for some $\lambda \in \check{C}^0(\mathcal{V}, \mathcal{M}')$ (note that at this point Professor Franke run out of primes, though there are apparently infinitely many of them). Let $\lambda = \alpha(\lambda')$. As $\check{d}(\mu - \lambda) = \alpha(\mu') - \alpha(\mu') = 0$, the $\mu_i - \lambda_i$ lift by the sheaf axiom to a single element $m^* \in \mathcal{M}(X)$ such that $m^*|_{V_i} = \mu_i - \lambda_i$. Since $\beta(\mu - \lambda) = \beta(\mu) - \beta(\alpha(\lambda')) = \beta(\mu)$, we have $\beta(m^*) = m''$. This proves the surjectivity of $\mathcal{M}(X) \to \mathcal{M}''(X)$ and thus $(*)$.

To show that $\mathcal{M}$ is quasi-coherent, consider $0 \to M' \to M \to M'' \to 0$ where $M' = \mathcal{M}'(X)$, $M = \mathcal{M}(X)$, and $M'' = \mathcal{M}''(X)$ which is an exact sequence of R-modules (as we just proved). As pointed out in the proof of Proposition 1.1.3(a), the functor $R\text{-Mod} \to \mathrm{QCoh}(\mathrm{Spec}\,R)$ mapping $N \mapsto \tilde{N}$ preserves kernels and cokernels, hence the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{M}' & \longrightarrow & \tilde{M} & \longrightarrow & \tilde{M}'' & \longrightarrow & 0 \\
& & \downarrow{\wr} & & \downarrow & & \downarrow{\wr} & & \\
0 & \longrightarrow & \mathcal{M}' & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}'' & \longrightarrow & 0
\end{array}$$

has exact rows. Since $\tilde{M}' \to \mathcal{M}'$ and $\tilde{M}'' \to \mathcal{M}''$ are isomorphisms, so is $\tilde{M} \to \mathcal{M}$ by the five lemma. *q.e.d.*

In the following, if $\mathcal{U} \colon X = \bigcup_{i \in I} U_i$ is an open cover and $Y \subseteq X$, we will write $\mathcal{U} \cap Y$ for the open cover $Y = \bigcup_{i \in I}(Y \cap U_i)$.

Corollary 3. *Let X be a quasi-compact scheme.*

(a) *If $W \xhookrightarrow{j} X$ is the embedding of the affine open subset W and $\mathcal{M} \in \mathrm{Ob}(\mathrm{QCoh}(W))$, then $H^p(X, j_*\mathcal{M}) = 0$ for $p > 0$.*

(b) *Let $W_i \xhookrightarrow{j_i} X$ be the embeddings of affine open subsets W_i for $i = 1, \ldots, n$. Let $W = \bigcup_{i=1}^n W_i$. If $\mathcal{M} \in \mathrm{Ob}(\mathrm{QCoh}(X))$, then*

$$H^p\left(X, \bigoplus_{i=1}^n (j_i)_*\mathcal{M}|_{W_i}\right) = 0 \quad \text{for } p > 0.$$

and if $\mathcal{K}$ denotes the kernel of $\mathcal{M} \to \bigoplus_{i=1}^n (j_i)_* \mathcal{M}|_{W_i}$ sending $m \in \mathcal{M}(U)$ to

$$(m|_{U \cap W_i})_{i=1}^n \in \bigoplus_{i=1}^n \mathcal{M}(U \cap W_i) = \bigoplus_{i=1}^n (j_i)_* \mathcal{M}|_{W_i}(U) \,,$$

then $\mathcal{K}$ satisfies $\mathcal{K}|_W = 0$.

Proof. Part (a). Let $\mathcal{U}$ be any affine open cover of X. By Proposition 1.1.1, the $U_i \cap W$ are affine and we have $\check{H}^p(\mathcal{U} \cap W, \mathcal{M}) \simeq \check{H}^p(\mathcal{U}, j_* \mathcal{M}) = 0$ by Lemma 2(a) and Proposition 2(c).

Part (b) follows from the fact that each j_i is a monomorphism and the sheaf axiom.   *q.e.d.*

Lemma 3. *Let $\mathcal{U}$ be a finite open cover of the quasi-separated prescheme X by quasi-compact open subsets and $f \in \mathcal{O}_X(X)$. By the universal property of localization, we have morphisms $\mathcal{M}(U)_f \to \mathcal{M}(U \setminus V(f))$ for arbitrary $\mathcal{O}_X$-modules $\mathcal{M}$ and open $U \subseteq X$. When applied to $U = U_{i_0,\dots,i_n}$, this gives a morphism*

$$\check{C}^\bullet(\mathcal{U}, \mathcal{M})_f \longrightarrow \check{C}^\bullet\big(\mathcal{U} \cap (X \setminus V(f)), \mathcal{M}\big)$$

of Čech complexes. When $\mathcal{M}$ is quasi-coherent, this is an isomorphism.

Remark. The structure of an $\mathcal{O}_X(X)$-module on $\check{C}^\bullet(\mathcal{U}, \mathcal{M})$ is obtained using the $\mathcal{O}_X(U_{i_0,\dots,i_n})$-module structures on $\mathcal{M}(U_{i_0,\dots,i_n})$ followed by $\mathcal{O}_X(X) \to \mathcal{O}_X(U_{i_0,\dots,i_n})$. Taking the product over all $(i_0,\dots,i_n) \in I^{n+1}$ gives the structure of a cochain complex of $\mathcal{O}_X(X)$-modules on $\check{C}^\bullet(\mathcal{U}, \mathcal{M})$, which is used in the formulation of Lemma 3. The morphism under investigation is a morphism of cochain complexes of $\mathcal{O}_X(X)_f$-modules.

Proof of Lemma 3. As $\check{C}^n(\mathcal{U}, \mathcal{M}) = \prod_{(i_0,\dots,i_n) \in I^{n+1}} \mathcal{M}(U_{i_0,\dots,i_n})$ is a finite product and localization commutes with finite products, it is sufficient to show that

$$\mathcal{M}(U_{i_0,\dots,i_n})_f \xrightarrow{\sim} \mathcal{M}(U_{i_0,\dots,i_n} \setminus V(f)) \,.$$

By Definition 1.1.4(c), this will hold if $U_{i_0,\dots,i_n}$ is quasi-compact and quasi-separated. It is quasi-separated as an open subset of the quasi-separated prescheme X. It is quasi-compact as an intersection of the quasi-compact subsets $U_{i_0},\dots,U_{i_n}$ using quasi-separatedness of X.   *q.e.d.*

Proposition 3. *Let X be a quasi-compact scheme, $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module, and $f \in \mathcal{O}_X(X)$. Then we have a canonical isomorphism*

$$H^\bullet(X, \mathcal{M})_f \xrightarrow{\sim} H^\bullet\left(X \setminus V(f), \mathcal{M}|_{X \setminus V(f)}\right) \,.$$

Proof. Follows from Lemma 3. Note that the open subset $X \setminus V(f) \subseteq X$ is a scheme again by [1, Fact 1.5.9(c)] and quasi-compact by [1, Fact 2.1.2].   *q.e.d.*

Remark. In general, there is the notion of *derived functors* (from Grothendieck's famous *Tôhoku paper*) applicable to the left-exact functors on an abelian category with sufficiently many injective objects, like $\mathcal{R}$-Mod ($\mathcal{R}$ being any sheaf of rings on any topological space) and $\mathrm{QCoh}(X)$ (X being any prescheme). When $\mathcal{M}$ is an $\mathcal{R}$-module on any topological space X, the derived functors taken on the category of $\mathcal{R}$-modules or sheaves of abelian groups are canonically isomorphic. When X is quasi-compact (or paracompact, this is actually sufficient) scheme, these are isomorphic to the cohomology introduced here and also to the derived functor of the global section functor taken for $\mathrm{QCoh}(X)$.

1.3. The affinity criterion of Serre

Proposition 1 (Serre's affinity criterion)**.** *For a quasi-compact scheme X, the following conditions are equivalent.*

   (*a*) *X is affine.*

   (*b*) *$H^p(X, \mathcal{M}) = 0$ for $p > 0$ when $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module.*

   (*c*) *$H^1(X, \mathcal{J}) = 0$ when $\mathcal{J}$ is a quasi-coherent sheaf of ideals on X.*

Remark. In EGA III, this is shown under the weaker assumptions that X is quasi-compact and quasi-separated (instead of separated), using $H^\bullet$ defined as a derived functor on $\mathcal{O}_X$-Mod.

To prove Proposition 1 we need the following.

Proposition 2.   (*a*) *If Z is a non-empty quasi-compact closed subset of a prescheme X, it contains a closed point.*

   (*b*) *If $Z \subseteq X$ is a closed subset of a prescheme X, then $\mathcal{J}(U) := \{ f \in \mathcal{O}_X(U) \mid Z \cap U \subseteq V(f) \}$ defines a quasi-coherent sheaf of ideals.*

   (*c*) *If $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{M}$ are quasi-coherent subsheaves of the quasi-coherent sheaf of modules $\mathcal{M}$, $(\mathcal{N}_1 \cap \mathcal{N}_2)(U) := \mathcal{N}_1(U) \cap \mathcal{N}_2(U)$ defines a quasi-coherent sheaf of modules.*

Proof. Part (*a*) is proved in [1, Proposition 2.1.1]

Part (*b*). Let $U \subseteq X$ be quasi-compact and quasi-separated, $f \in \mathcal{O}_X(U)$ and $\varphi \in \mathcal{J}(U)$ such that $\varphi|_{U \cap V(f)} = 0$. Then, as $\mathcal{O}_X$ is quasi-coherent, there exists $n \in \mathbb{N}$ such that $f^n \varphi = 0$ by Definition 1.1.4(*c*). It follows that the canonical map $\mathcal{J}(U)_f \to \mathcal{J}(U \setminus V(f))$ is injective.

If $\psi \in \mathcal{J}(U \setminus V(f))$, then as $\mathcal{O}_X$ is quasi-coherent, there is $n \in \mathbb{N}$ such that $f^n \psi$ extends to some $g \in \mathcal{O}_X(U)$ (using Definition 1.1.4(*c*) again). Now fg extends $f^{n+1}\psi$ and is in $\mathcal{J}(U)$. Indeed, we need to show $Z \cap U \subseteq V(fg)$ and this follows from

$$(Z \cap U) \setminus V(f) \subseteq V(\psi) \subseteq V\left(f^{n+1}\psi\right) = V\left(fg|_{U \setminus V(f)}\right) \subseteq V(fg)$$

(the first inclusion coming from $\psi \in \mathcal{J}(U \setminus V(f))$) and $V(f) \subseteq V(fg)$. Thus we get an isomorphism $\mathcal{J}(U)_f \xrightarrow{\sim} \mathcal{J}(U \setminus V(f))$, which is the criterion from Definition 1.1.4(*c*) once again.

Part (*c*). It is directly possible to verify $(\mathcal{N}_1 \cap \mathcal{N}_2)(U)_f \xrightarrow{\sim} (\mathcal{N}_1 \cap \mathcal{N}_2)(U \setminus V(f))$ similar as above. Alternatively, note that $\mathcal{N}_1 \cap \mathcal{N}_2 = \ker(\mathcal{N}_1 \to \mathcal{M} \to \mathcal{M}/\mathcal{N}_2)$ and use that $\mathrm{QCoh}(X)$ has kernels and cokernels by Proposition 1.1.3(*a*). *q.e.d.*

Proof of Proposition 1. The implication (*a*) $\Rightarrow$ (*b*) follows from Theorem 1(*c*) and (*b*) $\Rightarrow$ (*c*) is Trivial.

Let's do (*c*) $\Rightarrow$ (*a*). We first derive (*c*) $\Rightarrow$ (*c*$^{++}$), where (*c*$^{++}$) is the condition

(*c*$^{++}$) If $\mathcal{M} \subseteq \mathcal{O}_X^n$ is a quasi-coherent submodule, then $H^1(X, \mathcal{M}) = 0$.

We prove (c^{++}) by induction on n. The case $n = 0$ is trivial. Now let $n \geq 1$ and assume that (c^{++}) holds for $n-1$. If $\mathcal{O}_X^{n-1}(U)$ is identified with $\{(f_1, \ldots, f_{n-1}, 0) \in \mathcal{O}_X(V)^n\} \subseteq \mathcal{O}_X(U)^n$, then $\mathcal{M}' = \mathcal{M} \cap \mathcal{O}_X^{n-1}$ is quasi-coherent by Proposition 2(c). Moreover,

$$\mathcal{M}'' := \mathcal{M}/\mathcal{M}' \longrightarrow \mathcal{O}_X$$
$$(f_1, \ldots, f_n) \bmod \mathcal{M} \longmapsto f_n$$

is a monomorphism. Hence $H^1(X, \mathcal{M}') = 0$ by induction and $H^1(X, \mathcal{M}'') = 0$ by (c). Now looking at the piece

$$\ldots \longrightarrow H^1(X, \mathcal{M}') \longrightarrow H^1(X, \mathcal{M}) \longrightarrow H^1(X, \mathcal{M}'') \longrightarrow \ldots$$

of the long exact cohomology sequence, we get $H^1(X, \mathcal{M}) = 0$, completing the induction.

Now we prove $(c^{++}) \Rightarrow (a)$. Let $R = \mathcal{O}_X(X)$, so we have a morphism $X \xrightarrow{p} \operatorname{Spec} R$ corresponding to id_R under the adjunction

$$\operatorname{Hom}_{\mathrm{LRS}}(X, \operatorname{Spec} R) \simeq \operatorname{Hom}_{\mathrm{Ring}}(R, \mathcal{O}_X(X))$$

from [1, Proposition 1.4.2]. We have to show that p is an isomorphism. For this it is sufficient to show the following.

(α) If $f \in R$ is chosen such that $X \setminus V(f)$ is affine, then $X \setminus V(f) = p^{-1}(\operatorname{Spec} R \setminus V(f))$ is isomorphically mapped to $\operatorname{Spec} R \setminus V(f) \simeq \operatorname{Spec}(R_f)$ by p.

(β) The open subsets $X \setminus V(f)$ with f as in (α) cover X.

(γ) If $f_1, \ldots, f_n \in R$ are as in (α) and $X = \bigcup_{i=1}^n (X \setminus V(f_i))$ then

$$\operatorname{Spec} R = \bigcup_{i=1}^n (\operatorname{Spec} R \setminus V(f_i)) \,.$$

By (β) and the quasi-compactness of X, it is possible to find $f_1, \ldots, f_n \in R$ to which (γ) may be applied. Then the open subsets $U_i = (\operatorname{Spec} R \setminus V(f_i))$ cover $\operatorname{Spec} R$ and $p^{-1}(U_i) \xrightarrow[p]{\sim} U_i$. Then we get $\operatorname{Spec} R \xrightarrow[p]{\sim} X$.

Proof of (α). The fact $X \setminus V(f) = p^{-1}(\operatorname{Spec} R \setminus V(f))$ follows from the explicit construction of p. Namely, p maps a point $x \in X$ to the prime ideal $\mathfrak{p} \subseteq R$ which is the preimage of the maximal ideal $\mathfrak{m}_{X,x} \subseteq \mathcal{O}_{X,x}$ under $R = \mathcal{O}_X(X) \to \mathcal{O}_{X,x}$. Hence $x \in X \setminus V(f) \Leftrightarrow f \notin p(x) \Leftrightarrow p(x) \in \operatorname{Spec} R \setminus V(f)$, showing that p is a bijection on the underlying sets.

If $X \setminus V(f)$ is affine, we get

$$
\begin{array}{ccccc}
X \setminus V(f) & \xrightarrow{\ \sim\ } & \operatorname{Spec}\left(\mathcal{O}_X(X \setminus V(f))\right) & \xrightarrow{\ \sim\ } & \operatorname{Spec}(R_f) \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{\quad p \quad} & \operatorname{Spec} \mathcal{O}_X(X) & =\!=\!=\!=\!= & \operatorname{Spec} R
\end{array}
$$

The first isomorphism comes from the fact that $X \setminus V(f)$ is affine. For the second, we use $\mathcal{O}_X(X \setminus V(f)) \xrightarrow{\ \sim\ } \mathcal{O}_X(X)_f = R_f$, which follows from Definition 1.1.4(c) using that $\mathcal{O}_X$ is quasi-coherent, and X is both quasi-compact and quasi-separated. This shows (α).

Proof of (β). Let $U \subseteq X$ be the union of the affine open subsets of the form $X \setminus V(f)$ and $Z = X \setminus U$. If $Z \neq \emptyset$, there is a closed point $z \in Z$ by Proposition 2(a). Choose an affine open neighbourhood V of z. Let $Y_1 \subseteq X$ be the closed subset $X \setminus V$, let $Y_2 = \{z\}$ and $Y = Y_1 \cup Y_2$. The sheaf of ideals $\mathcal{J}$ of functions $f \in \mathcal{O}_X(W)$ such that $Y \cap W \subseteq V(f)$ is quasi-coherent by Proposition 2(b). By (c), we have $H^1(X, \mathcal{J}) = 0$. Note that Y is the closed subprescheme of X defined by $\mathcal{J}$ and denote by $Y \overset{i}{\hookrightarrow} X$ its embedding, i.e., $\mathcal{J} = \ker\left(\mathcal{O}_X \overset{i^*}{\to} i_*\mathcal{O}_Y\right)$. We get some $\varphi \in \mathcal{O}_Y(Y) = i_*\mathcal{O}_Y(X)$ such that $\varphi|_{Y_1} = 0$ and $\varphi|_{Y_2} = 1$ as Y is the disjoint union of its open subsets Y_1, Y_2.

By the cohomology sequence

$$0 \longrightarrow \mathcal{J}(X) \longrightarrow \mathcal{O}_X(X) \overset{i^*}{\longrightarrow} i_*\mathcal{O}_Y(X) \longrightarrow H^1(X, \mathcal{J}) = 0 \,,$$

there is some $f \in \mathcal{O}_X(X)$ such that $i^*f = \varphi$. Hence $Y_1 \subseteq V(f)$ and $z \notin V(f)$, so $z \in X \setminus V(f) = (X \setminus Y_1) \setminus V(f) = V \setminus V(f)$, which is an affine open subset of X. We get $z \in U$, a contradiction.

Proof of (γ). Let $f_1, \ldots, f_n$ be as requested. Consider the morphisms $\mathcal{O}_X \overset{\cdot f_i}{\longrightarrow} \mathcal{O}_X$ given by multiplication with f_i and let $f\colon \mathcal{O}_X^n \to \mathcal{O}_X$ be the morphism induced by the $\cdot f_i$. Then $\cdot f_i$ is an isomorphism on $U_i = X \setminus V(f_i)$, hence induces isomorphisms on stalks $\mathcal{O}_{X,x}$ for $x \in U_i$, and the U_i cover X. Thus f is an epimorphism on stalks, hence an epimorphism of sheaves. Let $\mathcal{M}$ be the kernel of this epimorphism. We have

$$
\begin{array}{ccc}
(r_1, \ldots, r_n) \longmapsto \sum r_i f_i \\
R^n \longrightarrow R \\
\Big\downarrow \wr \qquad\qquad \Big\downarrow \wr \\
\mathcal{O}_X(X)^n \overset{f}{\longrightarrow} \mathcal{O}_X(X) \longrightarrow H^1(X, \mathcal{M})
\end{array}
$$

The lower row is an extract of a long exact cohomology sequence. Note that $H^1(X, \mathcal{M}) = 0$ by (c^{++}), hence the f_i generate R as an ideal of R and $\bigcap_{i=1}^n V(f_i) = \emptyset$ in $\operatorname{Spec} R$ as stated.   *q.e.d.*

Remark. $H^1(X, \mathcal{O}_X) = 0$ is not enough (even when X is a quasi-compact scheme) for affinity of X as it holds, e.g., for $X = \mathbb{P}_R^n$.

1.4. Cohomological dimension

Proposition 1 (Grothendieck)**.** *Let X be a quasi-compact scheme and $Z \subseteq X$ a closed subset which is Noetherian as topological space. Let $\mathcal{M}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules satisfying $\mathcal{M}|_{X \setminus Z} = 0$ and let $p > \dim(Z)$ (the Krull dimension of Z, which may be infinite). Then $H^p(X, \mathcal{M}) = 0$.*

Remark. Grothendieck shows $H^p(Z, \mathcal{G}) = 0$ when Z is a Noetherian topological space, $\mathcal{G}$ a sheaf of abelian groups on it and $p > \dim(Z)$. This, however, requires the construction of H^p as a derived functor.

In our situation, the bound may fail to be sharp.

Proof of Proposition 1. There is nothing to show if $\dim(Z) = \pm\infty$, so we may show this by induction on $\dim(Z)$.

Let $\dim(Z) = 0$ and $Z = \bigcup_{i=1}^{n} Z_i$ the decomposition of Z into its irreducible components. Each Z_i is Noetherian (as Z is), hence quasi-compact, and closed, thus contains a closed point z_i by Proposition 1.3.2(a). But then $Z_i = \{z_i\}$ as $\{z_i\}$ is irreducible and $\dim(Z_i) = 0$.

As the affine open subsets form a topology base on X, there are affine open neighbourhoods $W_i \subseteq X$ of z_i such that $Z \cap W_i = \{z_i\}$. Let $\mathcal{M}_i := (\iota_i)_* (\mathcal{M}|_{W_i})$ where $W_i \overset{\iota_i}{\hookrightarrow} X$ is the corresponding open immersion. Let $\mathcal{F} = \bigoplus_{i=1}^{n} \mathcal{M}_i$. We have seen in Corollary 1.2.3(b) that $H^p(X, \mathcal{F}) = 0$ when $p > 0$. Moreover, $(\mathcal{M}_j)_{z_i} = 0$ when $i \neq j$ while $\mathcal{F} \to \mathcal{M}_i$ is an isomorphism on W_i, hence induces an isomorphism $\mathcal{F}_{z_i} \overset{\sim}{\longrightarrow} (\mathcal{M}_i)_{z_i} = \mathcal{M}_{z_i}$ on stalks at z_i. Moreover, since $\mathcal{M}$ and $\mathcal{M}_i$ vanish on $X \setminus Z$, we see that $\mathcal{F} \overset{\sim}{\longrightarrow} \mathcal{M}$ and the assertion follows.

For the inductive step, let $d = \dim(Z)$ be finite and the assertion be proved for quasi-coherent $\mathcal{M}$ supported on closed subsets of smaller Krull dimension. Let $Z = \bigcup_{i=1}^{n} Z_i$ be the decomposition into irreducible components Z_i. Then $Z_i \setminus \bigcup_{j \neq i} Z_j$ is non-empty, as $Z_i \subseteq Z_j$ otherwise by irreducibility of Z_i. Hence we find affine open subsets $W_i \subseteq X$ such that W_i intersects Z_i but no Z_j fpr $j \neq i$.

Let $\mathcal{N}_i = (j_i)_* \mathcal{M}|_{W_i}$ where $W_i \overset{j_i}{\hookrightarrow} X$ is the inclusion of the open subset $W_i \subseteq X$ and let $\mathcal{N} = \bigoplus_{i=1}^{n} \mathcal{N}_i$. We consider the morphism $\mathcal{M} \to \mathcal{N}$ given by

$$\mathcal{M}(U) \longrightarrow \mathcal{N}(U) = \bigoplus_{i=1}^{n} (j_i)_*(\mathcal{M}|_{W_i})(U) = \bigoplus_{i=1}^{n} \mathcal{M}(U \cap W_i)$$
$$m \longmapsto (m|_{U \cap W_i})_{i=1}^{n} .$$

As was seen in Corollary 1.2.3(b), $H^p(X, \mathcal{N}) = 0$ when $p > 0$ and also $\mathcal{M} \to \mathcal{N}$ is a monomorphism outside $\widetilde{X} = X \setminus \bigcup_{i=1}^{n} W_i$ (well, this is the same construction as in Claim 1.2.2). Let $\widetilde{Z} = Z \cap \widetilde{X}$. We have $\mathcal{M}|_{X \setminus Z} = 0 = \mathcal{N}|_{X \setminus Z}$, hence $\mathcal{M} \to \mathcal{N}$ is also a monomorphism outside Z, hence a monomorphism outside $\widetilde{Z}$. We claim:

Claim 1. The morphism $\mathcal{M} \to \mathcal{N}$ is also an epimorphism outside $\widetilde{Z}$.

Assuming this for the moment, we have an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \mathcal{C} \longrightarrow 0 \tag{§}$$

(with $\mathcal{K} = \ker(\mathcal{M} \to \mathcal{N})$ and $\mathcal{C} = \mathrm{coker}(\mathcal{M} \to \mathcal{N})$), where $\mathcal{K}$ and $\mathcal{C}$ vanish outside $\widetilde{Z}$. Since W_i intersects Z_i, we have $Z_i \not\subseteq \widetilde{X}$, hence no irreducible component of Z is contained in $\widetilde{Z}$ and we obtain $\dim(\widetilde{Z}) < \dim(Z) = d$. If $d = 0$, this implies $\widetilde{Z} = 0$, hence $\mathcal{M} \simeq \mathcal{N}$ has vanishing cohomology in positive degrees. Otherwise, the induction assumption implies $H^p(X, \mathcal{K}) = 0 = H^p(X, \mathcal{C})$ when $p \geq d$. We split (§) into $0 \to \mathcal{B} \to \mathcal{N} \to \mathcal{C} \to 0$ and $0 \to \mathcal{K} \to \mathcal{M} \to \mathcal{B} \to 0$, where $\mathcal{B} = \ker(\mathcal{N} \to \mathcal{C})$ which is also the image of $\mathcal{M} \to \mathcal{N}$. These give

$$\ldots \longrightarrow H^{p-1}(X, \mathcal{C}) \longrightarrow H^p(X, \mathcal{B}) \longrightarrow H^p(X, \mathcal{N}) \longrightarrow \ldots ,$$

hence $H^p(X, \mathcal{B}) = 0$ when $p > d$, and then

$$\ldots \longrightarrow H^p(X, \mathcal{K}) \longrightarrow H^p(X, \mathcal{M}) \longrightarrow H^p(X, \mathcal{B}) \longrightarrow \ldots ,$$

implying $H^p(X, \mathcal{M}) = 0$ when $p > d$. The vanishing assertion follows.

It remains to prove Claim 1 that $\mathcal{C}|_{X \setminus \widetilde{Z}}$ vanishes. For this, we use that

$$(\mathcal{N}_i)_x \simeq \begin{cases} \mathcal{M}_x & \text{if } x \in Z_i \cap W_i \\ 0 & \text{if } x \notin Z_i \end{cases} . \tag{\%}$$

When $x \in Z$ this implies $\mathcal{N}_x = 0$, hence $\mathcal{M}_x \to \mathcal{N}_x$ is surjective. When $x \in Z \setminus \widetilde{Z}$, then $x \in Z \cap W_i$ for some i, hence $x \in Z_i$ since W_i doesn't intersect Z_j for $j \neq i$. Thus

$$(\mathcal{N}_j)_x \simeq \begin{cases} \mathcal{M}_x & \text{if } j = i \\ 0 & \text{else} \end{cases}$$

by $(\%)$ and $\mathcal{M}_x \xrightarrow{\sim} \mathcal{N}_x$. To see $(\%)$, note that $\mathcal{N}_i(U) = \mathcal{M}(U)$ when $U \subseteq W_i$ and $x \in Z_i \cap W_i$, hence $(\mathcal{N}_i)_x \simeq \mathcal{M}_x$ in this case. Suppose $x \notin Z_i$. When $x \notin W_i$, then clearly $(\mathcal{N}_i)_x = 0$. When $x \in W_i \setminus Z_i$, then also $x \notin Z$ as W_i doesn't intersect the other irreducible components, hence $(\mathcal{N}_i)_x = 0$ follows from the fact that $\mathcal{M}|_{X \setminus Z} = 0$. $\hspace{1cm} q.e.d.$

1.5. Cohomology of morphisms and functorial properties of cohomology

Let $X \xrightarrow{f} Y$ be a quasi-compact and separated morphism of preschemes. The set $\mathcal{B}$ of quasi-compact open subsets $U \subseteq Y$ which are schemes form a topology base on Y (well, it contains all affine open subsets). By well-known properties of quasi-compact and of separated morphisms, $f^{-1}(U)$ is a quasi-compact scheme when $U \in \mathcal{B}$ (quasi-compactness is just Definition 1.1.5 and $f^{-1}(U) \to \operatorname{Spec} \mathbb{Z}$ is separated as the composition of the separated morphisms f and $U \to \operatorname{Spec} \mathbb{Z}$, cf. [1, Fact 1.5.7(c)]). We thus have for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ a presheaf

$$\Phi^p \mathcal{M} : U \mapsto H^p \left(f^{-1}(U), \mathcal{M} \right) \tag{$*$}$$

on $\mathcal{B}$. The presheaf structure on $(*)$ is defined as follows. Let $V \subseteq U$ be elements of $\mathcal{B}$ and $\mathcal{U}, \mathcal{V}$ the covers of $f^{-1}(U)$ and $f^{-1}(V)$ by their affine open subsets. Then the morphism of restricting to the intersection with $f^{-1}(V)$ gives $\check{C}^\bullet(\mathcal{U}, \mathcal{M}) \to \check{C}^\bullet(\mathcal{U} \cap f^{-1}(V), \mathcal{M})$ which may be followed by any morphism $\check{C}^\bullet(\mathcal{U} \cap f^{-1}(V), \mathcal{M}) \to \check{C}^\bullet(\mathcal{V}, \mathcal{M})$ coming from a refinement map for the pair $(\mathcal{V}, \mathcal{U} \cap f^{-1}(V))$ (we proved in Lemma 1.2.1(c) that any two refinement maps induce the same morphism on cohomology, so this choice doesn't matter), which gives

$$H^p \left(f^{-1}(U), \mathcal{M} \right) = \check{H}^p(\mathcal{U}, \mathcal{M}) \longrightarrow \check{H}^p(\mathcal{V}, \mathcal{M}) = H^p \left(f^{-1}(V), \mathcal{M} \right) ,$$

and this is the desired presheaf structure.

Definition 1. Sheafifying $\Phi^p \mathcal{M}$ gives an $\mathcal{O}_Y$-module called the $p^{\mathbf{th}}$ **direct image** of $\mathcal{M}$ under f and denoted $R^p f_* \mathcal{M}$.

We have a morphism

$$H^p\left(f^{-1}(U),\mathcal{M}\right) \longrightarrow R^p f_* \mathcal{M}(U) = H^0(U, R^p f_* \mathcal{M}) \tag{1}$$

(for $U \in \mathcal{B}$) as a special case of the morphism from a presheaf to its sheafification.

Proposition 1. *Let $X \xrightarrow{f} Y$ is a quasi-compact and separated morphism of preschemes and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module.*

(a) $R^0 f_* \mathcal{M} \simeq f_* \mathcal{M}$ *canonically.*

(b) *The $\mathcal{O}_X$-modules $R^p f_* \mathcal{M}$ are quasi-coherent.*

(c) *For a short exact sequence $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ of quasi-coherent $\mathcal{O}_X$-modules one has a long exact cohomology sequence*

$$
\begin{array}{ccccc}
0 \longrightarrow f_* \mathcal{M}' \longrightarrow & f_* \mathcal{M} \longrightarrow & f_* \mathcal{M}'' \\
\wr \uparrow \qquad\qquad & \wr \uparrow \qquad & \wr \uparrow \\
0 \longrightarrow R^0 f_* \mathcal{M}' \longrightarrow & R^0 f_* \mathcal{M} \longrightarrow & R^0 f_* \mathcal{M}'' \xrightarrow{\delta} R^1 f_* \mathcal{M}' \longrightarrow \dots
\end{array}
$$

$$\dots \longrightarrow R^{p-1} f_* \mathcal{M}'' \xrightarrow{\delta} R^p f_* \mathcal{M}' \longrightarrow R^p f_* \mathcal{M} \longrightarrow R^p f_* \mathcal{M}'' \xrightarrow{\delta} \dots$$

(d) *When $U \subseteq Y$ is affine, (1) becomes an isomorphism $H^p\left(f^{-1}(U),\mathcal{M}\right) \xrightarrow{\sim} R^p f_* \mathcal{M}(U)$.*

Proof. Part (a). We have $\Phi^0(U) = H^0(f^{-1}(U),\mathcal{M}) \simeq \mathcal{M}(f^{-1}(U)) = f_* \mathcal{M}(U)$, hence $\Phi^0 \simeq f_* \mathcal{M}|_{\mathcal{B}}$. Then sheafifying gives $f_* \mathcal{M}$ again by [1, Proposition 1.2.1(c),(d)].

Part (c). By Theorem 1 we have a similar long exact cohomology sequence of presheaves on $\mathcal{B}$. Sheafifying it gives the desired sequence, as sheafification is an exact functor (exactness may be tested on stalks and sheafification preserves stalks by [1, Proposition1.2.1(a)]).

It is easy to see that our construction of $R^p f_*$ is base-local, i.e.,

$$R^p f_* \mathcal{M}|_U \simeq R^p \left(f|_{f^{-1}(U)}\right)_* \mathcal{M}|_{f^{-1}(U)}.$$

Thus we may assume $Y = \operatorname{Spec} R$ to be affine (quasi-coherentness is a local property) for (b) and $Y = U = \operatorname{Spec} R$ in (d).

Let $\mathcal{B} = \{U \subseteq Y \mid U$ open and a quasi-compact scheme$\}$ the topology base on which Φ^p is defined, then $\widetilde{\mathcal{B}} = \{Y \setminus V(\lambda) \mid \lambda \in R\}$ is another topology base of Y, and $(\mathcal{G}|_{\widetilde{\mathcal{B}}})^{\mathrm{sh}} = \mathcal{G}^{\mathrm{sh}}$ for any presheaf $\mathcal{G}$ on $\mathcal{B}$ by [1, Proposition 1.2.1(c)]. But for $U = Y \setminus V(\lambda) \in \widetilde{\mathcal{B}}$ we have

$$H^p\left(f^{-1}(U),\mathcal{M}\right) = H^p\left(f^{-1}(Y \setminus V(\lambda)),\mathcal{M}\right) = H^p(X \setminus V(f^*\lambda),\mathcal{M}) \simeq H^p(X,\mathcal{M})_{f^*\lambda} = M_\lambda$$

by Proposition 1.2.3, where we consider $M = H^p(X,\mathcal{M})$ as an $R = \mathcal{O}_Y(Y)$-module via $f^*: \mathcal{O}_Y(Y) \to f_* \mathcal{O}_X(Y) = \mathcal{O}_X(X)$. We also used $f^{-1}(Y \setminus V(\lambda)) = X \setminus V(f^*\lambda)$ as f is a morphism of locally ringed spaces (cf. [1, Definition 1.3.4(b)]). It follows that $R^p f_* \mathcal{M}$ is canonically isomorphic to the sheafifification of $U \setminus V(\lambda) \mapsto M_\lambda$, which is just $\widetilde{M}$ by definition of the twiddlelization ([1, Definition 1.5.1], actually). As $\widetilde{M}$ is a quasi-coherent $\mathcal{O}_Y$-module, (b) follows. Then $M \simeq \widetilde{M}(Y)$, which shows (d). *q.e.d.*

In general, the morphism (1) fails to be an isomorphism: The *restriction to fibres* (for Y a quasi-compact scheme)

$$H^p(X, \mathcal{M}) \longrightarrow R^p f_* \mathcal{M} = E_2^{0,p} \tag{2}$$

as a special case of (1) is normally not an isomorphism when Y is not affine. There is another morphism, the *pull-back* (for Y again a quasi-compact scheme)

$$E_2^{p,0} = H^p(Y, f_* \mathcal{M}) \longrightarrow H^p(X, \mathcal{M}) \tag{3}$$

(also not an isomorphism unless f is affine) which can be constructed as follows. Let $\mathcal{U}_Y$ be the open cover of Y by its affine subsets and $\mathcal{V}$ any affine refinement of $f^{-1}(\mathcal{U}_Y)$. Then, using Lemma 1.2.2(b) and Proposition 1.2.2(a) we obtain

$$H^p(Y, f_* \mathcal{M}) = \check{H}^p(\mathcal{U}_Y, f_* \mathcal{M}) = \check{H}^p\left(f^{-1}(\mathcal{U}_Y), \mathcal{M}\right) \longrightarrow \check{H}^p(\mathcal{V}, \mathcal{M}) \overset{\sim}{\longleftarrow} H^p(X, \mathcal{M})$$

defining the above pull-back morphism (3).

There is an also inverse image functor $f^* \colon \mathcal{O}_Y\text{-Mod} \to \mathcal{O}_X\text{-Mod}$ which preserves quasi-coherence and is left-adjoint to f_*[1]. Thus, for $\mathcal{N}$ a quasi-coherent $\mathcal{O}_Y$-module, there is a canonical morphism $\mathcal{N} \to f_* f^* \mathcal{M}$ which gives another type of pull-back morphism

$$H^p(Y, \mathcal{N}) \longrightarrow H^p(Y, f_* f^* \mathcal{N}) \overset{(3)}{\longrightarrow} H^p(X, f^* \mathcal{N}) \ .$$

In general, (2) and (3) will fail to be isomorphisms but are part of the *Leray spectral sequence*

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{M}) \Longrightarrow H^{p+q}(X, \mathcal{M})$$

converging "to $H^{p+q}(X, \mathcal{M})$" (actually to some limit filtration of this).

This means the following: One has a sequence $(E_k^{\bullet,\bullet})_{k=2}^\infty$ of doubly graded abelian groups (or R-modules or objects of any other abelian category) equipped with morphisms (the *differentials*) $d_k = d_k^{p,q} \colon E_k^{p,q} \to E_k^{p+k,q+1-k}$ such that

$$E_k^{p,q} \overset{d_k}{\longrightarrow} E_k^{p+k,q+1-k} \overset{d_k}{\longrightarrow} E_k^{p+2k,q+1-2k}$$

vanishes and such that $E_{k+1}^{\bullet,\bullet} = H^\bullet(E_k^{\bullet,\bullet}, d_k^{\bullet,\bullet})$ in the sense that

$$E_{k+1}^{p,q} = \ker\left(E_k^{p,q} \overset{d_k}{\longrightarrow} E_k^{p+k,q+1-k}\right) \big/ \operatorname{Im}\left(E_k^{p-k,q+k-1} \overset{d_k}{\longrightarrow} E_k^{p,q}\right) . \tag{η}$$

Thus $E_k^{p,q} = 0$ for all $k \geq 2$ when $E_2^{p,q} = 0$. When $E_2^{\bullet,\bullet}$ is supported in the 1st quadrant (i.e., $E_2^{p,q} = 0$ if $p < 0$ or $q < 0$), the same is true for all $E_k^{\bullet,\bullet}$ and (η) becomes

$$E_{k+1}^{p,q} = \operatorname{coker}\left(E_k^{p-k,q+k-1} \overset{d_k}{\longrightarrow} E_k^{p,q}\right) \quad \text{for } k > q+1 \tag{η'}$$

and

$$E_{k+1}^{p,q} = \ker\left(E_k^{p,q} \overset{d_k}{\longrightarrow} E_k^{p+k,q-k+1}\right) \quad \text{for } k > p \tag{η''}$$

---

[1] We constructed f^* on exercise sheet #10 of Algebraic Geometry I and also proved the relevant properties there. It is given by $f^* \mathcal{M} = f^\sharp \mathcal{M} \otimes_{f^\sharp \mathcal{O}_Y} \mathcal{O}_X$, where $f^\sharp$ is constructed as in [1, Remark 1.2.4].

and we deduce $E^{p,q}_{k+1} = E^{p,q}_k$ for $k > \max\{p, q+1\}$.

For 1st quadrant sequences (like Leray's) we put $E^{p,q}_\infty = E^{p,q}_k$ where k is large enough. When the spectral sequence *converges to some filtration on its limit* $L^\bullet$ (we have $L^\bullet = H^\bullet(X, \mathcal{M})$ for Leray), it is meant that there is a filtration $F^\bullet L^\bullet : L^\bullet = F^0 L^\bullet \supseteq F^1 L^\bullet \supseteq \ldots$ with $E^{p,q}_\infty \simeq F^p L^{p+q}/F^{p+1} L^{p+q}$. Thus, for 1st quadrant sequences,

$$F^1 L^p = \ker\left(L^p \longrightarrow E^{0,p}_\infty\right) = \ker\left(L^p \longrightarrow E^{p,q}_2\right)$$

(where we use $E^{0,p}_\infty \subseteq E^{0,p}_2$ by (η'')) has (2) as a special case. Also, using $F^{p+1} L^p = 0$ for 1st quadrant sequences we obtain an epimorphism

$$F^p L^p \simeq E^{p,q}_\infty \longleftarrow E^{p,0}_2$$

by (η'). The resulting morphism $E^{p,0}_2 \to L^p$ has (3) as a special case.

There is also a Leray spectral sequence $R^p g_* R^q f_* \mathcal{M} \Rightarrow R^{p+q}(gf)_* \mathcal{M}$.

1.6. Affine morphisms and the relative Spec functor

Proposition 1. *Let* $X \xrightarrow{f} Y$ *be a morphism of preschemes, then the following conditions are equivalent.*

  (a) *If $U \subseteq Y$ is an affine open subset, then so is $f^{-1}(U) \subseteq X$.*

  (b) *It is possible to cover Y by affine open U such that $f^{-1}(U)$ is affine open.*

When X is, in addition, quasi-separated, this is moreover equivalent to

  (c) *f is separated and quasi-compact and we have $R^p f_* \mathcal{M} = 0$ when $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module and $p > 0$.*

Proof. An unconditional proof of $(a) \Leftrightarrow (b)$ was given in [1, Lemma 2.5.1].

The implication $(a) \Rightarrow (b)$ is trivial. For $(b) \Rightarrow (c)$, it is obvious that f is quasi-compact and separatedness was proved in [1, Fact 2.6.1]. To show $R^p f_* \mathcal{M} = 0$, note that if $U \subseteq Y$ is affine open such that $f^{-1}(U)$ is affine and $\lambda \in \mathcal{O}_X(U)$, then $f^{-1}(U \setminus V(\lambda)) = f^{-1}(U) \setminus V(f^* \lambda)$ is affine as well. In particular, the $U \subseteq Y$ such that $f^{-1}(U)$ is affine form a topology base on Y. By Theorem 1(b) and Proposition 1.5.1(b), $\Phi^p \mathcal{M}$ thus vanishes on a topology base of Y. Hence $R^p f_* \mathcal{M} = 0$ as the sheafification of $\Phi^p \mathcal{M}$. In particular, we didn't need the additional assumption.

Now for $(c) \Rightarrow (a)$ assuming that X is quasi-separated. If $U \subseteq Y$ is affine and $\mathcal{M} \in \mathrm{Ob}(\mathrm{QCoh}(f^{-1}(U)))$, then $\mathcal{M}$ extends to X by Lemma 1 and by Proposition 1.5.1(d)

$$H^p\left(f^{-1}(U), \mathcal{M}\right) \simeq R^p f_* \mathcal{M}(U) = 0.$$

By Serre's affinity criterion (Proposition 1.3.1), $f^{-1}(U)$ is affine. $\quad$ *q.e.d.*

Lemma 1. *Let X be a quasi-separated prescheme and $\mathcal{N}$ a quasi-coherent $\mathcal{O}_U$-module for some quasi-compact open subset $U \subseteq X$. Then there is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ such that $\mathcal{M}|_U \simeq \mathcal{N}$.*

Proof. The open immersion $U \xhookrightarrow{j} X$ is quasi-separated (in fact, separated as is any monomorphism between preschemes, cf. [1, Fact 1.5.7(a)]) and quasi-compact since X is quasi-separated and U is quasi-compact. Hence $\mathcal{M} = j_*\mathcal{N}$ is quasi-coherent by Proposition 1.1.2 and $\mathcal{M}|_U \simeq \mathcal{N}$. *q.e.d.*

Definition 1. (*a*) A morphism $X \xrightarrow{f} Y$ between preschemes is called **affine** if it satisfies the equivalent conditions from Proposition 1(*a*) and (*b*).

(*b*) It is called **finite** if in addition for any affine open $U \subseteq Y$, $\mathcal{O}_X(f^{-1}(U))$ is a finitely generated $\mathcal{O}_Y(U)$-module, or, equivalently, if this holds for sufficiently many U to cover Y (by Lemma 1.1.1).

Remark. In other words, f is finite iff f is affine and $f_*\mathcal{O}_X$ is a locally finitely generated $\mathcal{O}_Y$-module in the sense of Definition 1.1.6(*a*). If Y is Noetherian, this is equivalent to f being affine and $f_*\mathcal{O}_X$ a *coherent* $\mathcal{O}_Y$-module in the sense of Definition 1.1.6(*b*).

Corollary 1. *If $X \xrightarrow{f} Y$ is an affine morphism and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module, then $R^p f_* \mathcal{M} = 0$ when $p > 0$. When Y is a quasi-compact scheme, the morphism*

$$H^p(Y, f_*\mathcal{M}) \xrightarrow{(1.5.3)} H^p(X, \mathcal{M})$$

is an isomorphism.

Proof. The first assertion follows from the fact that $R^p f_* \mathcal{M}$ is the sheafification of $U \mapsto H^p(f^{-1}(U), \mathcal{M})$, which vanishes on the topology base of affine open subsets $U \subseteq Y$ when $p > 0$ by Theorem 1(*b*).

When Y is a quasi-compact scheme, then X is a scheme as well since $X \to \operatorname{Spec}\mathbb{Z}$ is separated as the composition of the separated morphisms f and $Y \to \operatorname{Spec}\mathbb{Z}$. Also, X is quasi-compact as f and Y are quasi-compact. Let $\mathcal{U}_Y$ be the open cover of Y by its affine open subsets, then

$$H^p(Y, f_*\mathcal{M}) \simeq \check{H}^p(\mathcal{U}_Y, f_*\mathcal{M}) \simeq \check{H}^p\left(f^{-1}(\mathcal{U}_Y), \mathcal{M}\right) \simeq H^p(X, \mathcal{M})$$

(which was our description of (1.5.3)) where the last isomorphism holds as f is affine, hence $f^{-1}(\mathcal{U})$ is an affine open cover of X.

This proves the second assertion under the assumptions which we constructed the relevant cohomology groups. *q.e.d.*

Remark. Using the general derived functor construction it follows from the Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q f_* \mathcal{M}) \implies H^{p+q}(Y, \mathcal{M})$$

and the vanishing of $R^q f_* \mathcal{M}$ when $q > 0$, $\mathcal{M}$ is quasi-coherent and f affine.

Let X be any prescheme. Recall the construction of the $\mathbf{Spec}_X(-)$ functor from [1, Subsection 2.5.1], which is given as follows. When $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_X$-algebra, one has schemes $\operatorname{Spec}\mathcal{A}(U) \to \operatorname{Spec}\mathcal{O}_X(U) \simeq U$ when $U \subseteq X$ is an affine open subset. When $V \subseteq U$ is another affine open subset, then $\operatorname{Spec} A(V)$ is isomorphic to the preimage of V under $\operatorname{Spec}\mathcal{A}(U) \to U$. This allows one to glue the different $\operatorname{Spec}\mathcal{A}(U)$ together to one prescheme $\mathbf{Spec}_X(\mathcal{A}) \to X$ which also admits the following description:

- *Points.* As a set,

$$\mathbf{Spec}_X(\mathcal{A}) = \left\{ (x, \mathfrak{p}) \;\middle|\; \begin{array}{l} x \in X,\ \mathfrak{p} \in \operatorname{Spec}(\mathcal{A}_x) \text{ such that the preimage of } \mathfrak{p} \\ \text{under } \mathcal{O}_{X,x} \to \mathcal{A}_x \text{ is } \mathfrak{m}_{X,x} \text{ (the maximal ideal of } \mathcal{A}_x) \end{array} \right\}$$

- *Topology base.* Let

$$\Omega(U, \lambda) = \{(x, \mathfrak{p}) \mid x \in U \text{ and } \mathfrak{p} \not\ni (\text{image of } \lambda \text{ under } \mathcal{A}(U) \to \mathcal{A}_x)\}$$

where $U \subseteq X$ is open and $\lambda \in \mathcal{A}(U)$. As $\Omega(U, \lambda) \cap \Omega(V, \vartheta) = \Omega(U \cap V, \lambda|_{U \cap V} \cdot \vartheta|_{U \cap V})$ this is indeed a base for some topology.

- *Local rings.* There is only one reasonable choice and this is $\mathcal{O}_{Y,[x,\mathfrak{p}]} := (\mathcal{A}_x)_{\mathfrak{p}}$. Here, $Y = \mathbf{Spec}_X(\mathcal{A})$ for brevity.

- *Structure sheaf.* For $\Omega \subseteq Y$ open, let

$$\mathcal{O}_Y(\Omega) = \left\{ \alpha = (\alpha_{x,\mathfrak{p}}) \in \prod_{(x,\mathfrak{p}) \in \Omega} \mathcal{O}_{Y,[x,\mathfrak{p}]} \;\middle|\; \begin{array}{l} \alpha \text{ fulfills the } \textit{cohe-} \\ \textit{rence condition}^{\text{TM}} \end{array} \right\} .$$

Here, the *coherence condition*$^{\text{TM}}$ says that every $\omega \in \Omega$ has a neighbourhood $\Omega(U, \lambda)$ together with an $a \in \mathcal{A}(U)_\lambda$ such that whenever $(x, \mathfrak{p}) \in \Omega \cap \Omega(U, \lambda)$, we have

$$\alpha_{x,\mathfrak{p}} = \left(\begin{array}{c} \text{image of } a \text{ under} \\ \mathcal{A}(U)_\lambda \to (\mathcal{A}_x)_\lambda \to (\mathcal{A}_x)_{\mathfrak{p}} = \mathcal{O}_{Y,[x,\mathfrak{p}]} \end{array} \right) .$$

Proposition 2. *Let Y be a prescheme, $\mathcal{A}$ an $\mathcal{O}_Y$-algebra and $X \xrightarrow{\xi} Y$ a morphism of preschemes.*

(a) *The morphism $\operatorname{Spec}_Y(\mathcal{A}) \to Y$ is affine.*

(b) *One has a bijection*

$$\operatorname{Hom}_{\mathcal{O}_Y\text{-Alg}}(\mathcal{A}, \xi_* \mathcal{O}_X) \xleftarrow{\sim} \operatorname{Hom}_{Y\text{-PreSch}}(X, \mathbf{Spec}_Y(\mathcal{A})) \tag{1}$$

by gluing the adjunctions

$$\operatorname{Hom}_{\mathcal{O}_Y(U)\text{-Mod}}\left(\mathcal{A}(U), \mathcal{O}_X(\xi^{-1}U)\right) \simeq \operatorname{Hom}_{\text{PreSch}}\left(\xi^{-1}U, \operatorname{Spec}\mathcal{A}(U)\right)$$

together.

(c) *The morphism ξ is affine iff $\xi_* \mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_Y$-algebra and the morphism $X \to \mathbf{Spec}_Y(\xi_* \mathcal{O}_X)$ corresponding to $\operatorname{id}_{\xi_* \mathcal{O}_X}$ under (1) is an isomorphism.*

Sketch of a proof. For (a), the preimage of an affine open $U \subseteq Y$ is $\operatorname{Spec} \mathcal{A}(U)$, hence affine. Part (b) was sketched in [1, Proposition 2.5.1] and part (c) is [1, Lemma 2.5.1]. *q.e.d.*

Corollary 2. *Let $\widetilde{Y} \xrightarrow{f} Y$ be a morphism of preschemes. Let $f^*\colon \mathcal{O}_Y\text{-Alg} \to \mathcal{O}_{\widetilde{Y}}\text{-Alg}$ be the inverse image functor from* [this](#) *footnote. If $\mathcal{A}$ is a $\mathcal{O}_Y$-algebra, we have $\mathbf{Spec}_{\widetilde{Y}}(f^*\mathcal{A}) \simeq \mathbf{Spec}_Y(\mathcal{A}) \times_Y \widetilde{Y}$.*

Proof. Recall that f^* is left-adjoint to f_* Let $X \xrightarrow{\xi} \widetilde{Y}$ be a $\widetilde{Y}$-prescheme. Repeatedly applying Proposition 2(b) we get

$$
\begin{aligned}
\operatorname{Hom}_{\widetilde{Y}\text{-PreSch}}\left(X, \mathbf{Spec}_{\widetilde{Y}}(f^*\mathcal{A})\right) &\simeq \operatorname{Hom}_{\mathcal{O}_{\widetilde{Y}}\text{-Alg}}\left(f^*\mathcal{A}, \xi_* \mathcal{O}_X\right) \\
&\simeq \operatorname{Hom}_{\mathcal{O}_Y\text{-Alg}}\left(\mathcal{A}, f_* \xi_* \mathcal{O}_X\right) \\
&\simeq \operatorname{Hom}_{Y\text{-PreSch}}\left(X, \mathbf{Spec}_Y(\mathcal{A})\right) .
\end{aligned}
$$

In the last line, the Y-prescheme structure on X is given by $X \xrightarrow{\xi} \widetilde{Y} \xrightarrow{f} Y$. If you think about it for a moment, a morphism of Y-preschemes from X to $\operatorname{Spec}_Y(\mathcal{A})$, where the Y-prescheme structure on X factors over $\widetilde{Y}$ is exactly the commuting square from the universal property of the fibre product $\mathbf{Spec}_Y(\mathcal{A}) \times_Y \widetilde{Y}$ and we're done. *q.e.d.*

1.7. The relation between H^1 and torsors

The usual term for these things seems to be *torsor* rather than *torseur*, so we will deviate from the lecture's notation here.

Definition 1. Let X be any topological space, $\mathcal{G}$ a sheaf of groups on X. A $\mathcal{G}$**-torsor** or $\mathcal{G}$**-principal homogeneous space** is a sheaf of sets $\mathcal{X}$ on X with a morphism

$$
\mathcal{G} \times \mathcal{X} \longrightarrow \mathcal{X}
$$

of sheaves of sets indicated by $(g, \xi) \mapsto g \cdot \xi$, with the property that $1_{\mathcal{G}(U)} \cdot \xi = \xi$ and $g \cdot (h \cdot \xi) = (gh) \cdot \xi$ for $g, h \in \mathcal{G}(U)$ and $\xi \in \mathcal{X}(U)$ and such that $\mathcal{X}_x \neq \emptyset$ for any $x \in X$ and the action of $\mathcal{G}_x$ on $\mathcal{X}_x$ is simply transitive for each $x \in X$.

The torsor is called **trivial** if $\mathcal{X}(X) \neq \emptyset$ (and *yes*, the $\neq$ is correct here).

Example 1. $\mathcal{G}$ with its group multiplication is a $\mathcal{G}$-torsor and trivial as $1_{\mathcal{G}(X)} \in \mathcal{G}(X)$.

Remark 1. (a) If $\xi \in \mathcal{X}(X)$ then

$$
\begin{aligned}
\mathcal{G} &\longrightarrow \mathcal{X} \\
g \in \mathcal{G}(U) &\longmapsto g \cdot \xi|_U
\end{aligned}
$$

is an isomorphism (by the simple transitivity stuff it is an isomorphism on stalks).

(b) If $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{G}$-torsors then any morphism $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ of sheaves of sets compatible with the $\mathcal{G}$-actions induces isomorphisms on stalks, i.e., is automatically an isomorphism of sheaves. Thus, the category of $\mathcal{G}$-torsors is a *groupoid* – a category, in which every morphism is an isomorphism.

(c) I think the following should be pointed out separately (though it wasn't in the lecture). Let $\mathcal{X}$ be a $\mathcal{G}$-torsor, $U \subseteq X$ open. Then $\mathcal{G}(U)$ acts simply transitively on $\mathcal{X}(U)$. Indeed, If $\mathcal{X}(U) = \emptyset$ this is trivial. Otherwise, $\mathcal{X}|_U$ clearly is a trivial $\mathcal{G}|_U$-torsor by (a) and the assertion is just as trivial.

For $\xi, \upsilon \in \mathcal{G}(U)$ the unique element g such that $g \cdot \upsilon = \xi$ will be denoted $g = \frac{\xi}{\upsilon}$ (or $\xi - \upsilon$, if the $\mathcal{G}$-action is written additively).

Example 2. (a) Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathcal{L}$ is a *line bundle* on X (i.e., a locally free $\mathcal{O}_X$-module of rank 1) then

$$U \mapsto \mathcal{L}^\times(U) = \left\{ \ell \in \mathcal{L}(U) \mid \mathcal{O}_U \xrightarrow{\cdot \ell} \mathcal{L}|_U \text{ is an isomorphism} \right\}$$

$$= \left\{ \ell \in \mathcal{L}(U) \mid \begin{array}{c} \text{the image of } \ell \text{ in } \mathcal{L}_x \text{ is a generator of} \\ \text{that free } \mathcal{O}_{X,x}\text{-module for all } x \in U \end{array} \right\}$$

is an $\mathcal{O}_X^\times$-torsor when equipped with the $\mathcal{O}_X^\times$-action given by $(f, \ell) \mapsto f \cdot \ell$.

If $(X, \mathcal{O}_X)$ is locally ringed with maximal ideals $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ and $\ell \in \mathcal{L}(U)$ for some open subset $U \subseteq X$, we put

$$V(\ell) = \{x \in U \mid (\text{image of } \ell \text{ in } \mathcal{L}_x) \in \mathfrak{m}_x \mathcal{L}_x\} \ .$$

If $\lambda \in \mathcal{L}(U)$ happens to be a free generator of $\mathcal{L}|_U$, then $V(\ell)$ coincides with $V\left(\frac{\ell}{\lambda}\right)$, the latter vanishing set being taken in $\mathcal{O}_X$ as usual (that is, as in [2, Definition 1.3.3]). Then

$$\mathcal{L}^\times(U) = \{\ell \in \mathcal{L}(U) \mid V(\ell) = \emptyset\} \ .$$

The line bundle $\mathcal{L}$ is trivial iff $\mathcal{L}^\times$ has a global section, i.e., iff it's a *trivial* torsor in the sense of Definition 1. It is obvious that for an isomorphism $\varphi \colon \mathcal{L} \xrightarrow{\sim} \mathcal{M}$ of line bundles, we have an isomorphism $\Phi \colon \mathcal{L}^\times \xrightarrow{\sim} \mathcal{M}^\times$ sending $\ell \in \mathcal{L}^\times(U)$ to $\varphi(\ell) \in \mathcal{M}^\times(U)$. Vice versa, if $\Phi \colon \mathcal{L}^\times \xrightarrow{\sim} \mathcal{M}^\times$ is an isomorphism of $\mathcal{O}_X^\times$-torsors (or just a morphism – it's automatically an isomorphism by Remark 1(b)), there is a unique isomorphism $\varphi \colon \mathcal{L} \xrightarrow{\sim} \mathcal{M}$ such that $\Phi(\ell) = \varphi(\ell)$ for all open $U \subseteq X$ and $\ell \in \mathcal{L}^\times(U)$.

We thus get a bijection between the isomorphism classes of line bundles and the isomorphism classes of $\mathcal{O}_X^\times$-torsors. As was pointed out, we need to check that every $\mathcal{O}_X^\times$-torsor is given as $\mathcal{L}^\times$ for some line bundle $\mathcal{L}$. If $\mathcal{X}$ is an $\mathcal{O}_X^\times$-torsor, define $\mathcal{L}$ as the sheafification of

$$U \mapsto (f, \xi)/_\sim \quad \text{where } f \in \mathcal{O}_X(U),\ \xi \in \mathcal{X}(U)$$

and the equivalence relation $\sim$ is defined by $(f, \xi) \sim (g, \upsilon)$ iff $f = g \cdot \frac{\xi}{\upsilon}$. $\mathcal{L}$ becomes an $\mathcal{O}_X$-module via $h \cdot (f, \xi)/_\sim = (fh, \xi)/_\sim$ and $(f, \xi)/_\sim + (g, \upsilon)/_\sim = \left(g + f \cdot \frac{\xi}{\upsilon}, \upsilon\right)/_\sim$.

(b) Let $(X, \mathcal{O}_X)$ be a locally ringed space again. Define a sheaf $\mathrm{GL}_n(\mathcal{O}_X)$ by

$$\mathrm{GL}_n(\mathcal{O}_X)(U) = \{g \in \mathcal{O}_X(U)^{n \times n} \mid V(\det g) = \emptyset\}$$
$$= \{g \in \mathcal{O}_X(U)^{n \times n} \mid g \text{ has an inverse matrix}\}$$

Then there is a similar bijection

$$
\left\{ \begin{array}{c} \text{isomorphism classes of } n\text{-dimen-} \\ \text{sional vector bundles on } X \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \mathrm{GL}_n(\mathcal{O}_X)\text{-torsors} \end{array} \right\}
$$

$$
[\mathcal{E}] \longmapsto \left[\left\{ e \in \mathcal{E}(U)^n \ \middle| \ \begin{array}{c} e \text{ is a vector of free} \\ \text{generators of } \mathcal{E}|_U \end{array} \right\} \right] ,
$$

where $\mathrm{GL}_n(\mathcal{O}_X)$ acts by right multiplication with the column vector e.

Let $\mathcal{G}$ be a sheaf of abelian groups, $\mathcal{X}$ a $\mathcal{G}$-torsor, $\mathcal{U}\colon X = \bigcup_{i\in I} U_i$ an open cover such that $\mathcal{X}|_{U_i}$ is trivial (we will then say that $\mathcal{X}$ is trivial on $\mathcal{U}$). To be compatible with our notation for Čech cohomology we will write the $\mathcal{G}$-action on $\mathcal{X}$ additively. If $\xi_i \in \mathcal{X}(U_i)$ for all $i \in I$ are given, there are unique $\psi_{i,j} \coloneqq \xi_i|_{U_{i,j}} - \xi_j|_{U_{i,j}} \in \mathcal{G}(U_{i,j})$ as in Remark 1(c). Let $\psi = (\psi_{i,j})_{(i,j)\in I^2} \in \check{C}^1(\mathcal{U},\mathcal{G})$. One easily verifies the correctness of the formal calculation

$$
\begin{aligned}
(\check{d}^1\psi)_{i,j,k} &= \psi_{j,k}|_{U_{i,j,k}} - \psi_{i,k}|_{U_{i,j,k}} + \psi_{i,j}|_{U_{i,j,k}} \\
&= \left(\xi_j|_{U_{i,j,k}} - \xi_k|_{U_{i,j,k}} \right) - \left(\xi_i|_{U_{i,j,k}} - \xi_k|_{U_{i,j,k}} \right) + \left(\xi_i|_{U_{i,j,k}} - \xi_j|_{U_{i,j,k}} \right) \\
&= 0 ,
\end{aligned}
$$

that is, ψ is a cocycle.

Proposition 1. (a) *The cohomology class $[\psi] \in \check{H}^1(\mathcal{U},\mathcal{G})$ depends only on the isomorphism class of the $\mathcal{G}$-torsor $\mathcal{X}$, not on the choice of trivializations.*

(b) *One gets a bijection between the isomorphism classes of $\mathcal{G}$-torsors trivial on $\mathcal{U}$ and $\check{H}^1(\mathcal{U},\mathcal{G})$.*

Proof. Part (a). If $\widetilde{\xi}$ is another trivialization yielding $\widetilde{\psi}_{i,j} = \widetilde{\xi}_i|_{U_{i,j}} - \widetilde{\xi}_j|_{U_{i,j}}$, then $\widetilde{\xi}_i = \gamma_i + \xi_i$ where $\gamma_i = \widetilde{\xi}_i - \xi_i \in \mathcal{G}(U_i)$ (Remark 1(c)) and putting $\gamma = (\gamma_i)_{i\in I} \in \check{C}^0(\mathcal{U},\mathcal{G})$ we get $\widetilde{\psi} = \check{d}^0\gamma + \psi$.

Part (b). By (a) it is clear that we have a map from isomorphism classes of $\mathcal{G}$-torsors trivial on $\mathcal{U}$ to $\check{H}^1(\mathcal{U},\mathcal{G})$. To construct a converse map, let $\eta \in \check{H}^1(\mathcal{U},\mathcal{G})$ and let $\psi \in \check{C}^1(\mathcal{U},\mathcal{G})$ be a representative of η (so that $\check{d}^1\psi = 0$) and let $\mathcal{X}_\psi$ be the sheaf of sets on X given by

$$
\mathcal{X}_\psi(V) = \left\{ \gamma \in \check{C}^0(\mathcal{U}\cap V, \mathcal{G}) \ \middle| \ \check{d}^0\gamma = \psi|_V \right\} .
$$

We get a $\mathcal{G}$-action on $\mathcal{X}_\psi$ via $g + \gamma = (g|_{U_i\cap V} + \gamma_i)_{i\in I}$ for $g \in \mathcal{G}(V)$, $\gamma \in \check{C}^0(\mathcal{U}\cap V, \mathcal{G})$. We verify that $\mathcal{G}$ acts simply transitively on stalks and that $\mathcal{G}$ trivializes on $\mathcal{U}$. Note that if $\gamma, \gamma' \in \mathcal{X}(V)$, then $\check{d}^0(\gamma - \gamma') = 0$, hence $\gamma - \gamma' \in \check{C}^0(\mathcal{U}\cap V, \mathcal{G})$ corresponds to (unique) element of $\mathcal{G}(V)$. So the action on $\mathcal{X}_\psi(V)$ is simply transitive for every $V \subseteq X$ open and the only thing left to check is triviality on $\mathcal{U}$. For this, note that $\check{H}^1(\mathcal{U}\cap V, \mathcal{G}|_V) = 0$ by Corollary 1.2.1(c), so $\psi|_V$ has a preimage in $\check{C}^0(\mathcal{U}\cap V, \mathcal{G})$ and consequently $\mathcal{X}_\psi(V) \neq \emptyset$. In particular, the stalks $(\mathcal{X}_\psi)_x$ are non-empty and $\mathcal{X}_\psi$ is a $\mathcal{G}$-torsor trivial on $\mathcal{U}$.

If $\widetilde{\psi} = \psi + \check{d}^0\varphi$ with $\varphi \in \check{C}^0(\mathcal{U},\mathcal{G})$ then we have an isomorphism of $\mathcal{G}$-torsors

$$
\mathcal{X}_\psi \xrightarrow{\sim} \mathcal{X}_{\widetilde{\psi}}
$$
$$
\gamma \in \mathcal{X}_\psi(V) \mapsto \gamma + \varphi|_V .
$$

Therefore, we get a map in the opposite direction

$$\check{H}^1(\mathcal{U}, \mathcal{G}) \longrightarrow \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \mathcal{G}\text{-torsors trivial on } \mathcal{U} \end{array} \right\}$$

$$\eta = [\psi] \longmapsto [\mathcal{X}_\psi] \, .$$

We claim that the maps we constructed from $\check{H}^1(\mathcal{U}, \mathcal{G})$ to the set of isomorphism classes of $\mathcal{U}$-trivial $\mathcal{G}$-torsors and vice versa are inverse to each other.

Indeed, we may choose $\xi_i \in \mathcal{X}_\psi(U_i)$ such that $(\xi_i)_j = \psi_{ij}$ – using $\check{d}^1\psi = 0$ they indeed satisfy $\check{d}^0\xi_i = \psi|_{U_{i,j}}$ – and using $\check{d}^1\psi = 0$ again we find $\xi_i|_{U_{i,j}} - \xi_j|_{U_{i,j}} = \psi_{i,j}$. So we send $[\mathcal{X}_\psi]$ to the cohomology class $[\psi]$ and then back to $[\mathcal{X}_\psi]$.

And for the other direction, if an arbitrary $\mathcal{U}$-trivial $\mathcal{G}$-torsor $\mathcal{X}$ is given, choose trivializations $\xi_i \in \mathcal{X}(U_i)$ and put $\psi = (\xi_i|_{U_{i,j}} - \xi_j|_{U_{i,j}})_{(i,j) \in I^2} \in \check{C}^1(\mathcal{U}, \mathcal{G})$. Then we get an isomorphism $\mathcal{X} \xrightarrow{\sim} \mathcal{X}_\psi$ via

$$\mathcal{X}(V) \longrightarrow \mathcal{X}_\psi(V)$$

$$\upsilon \longmapsto (\upsilon|_{U_i \cap V} - \xi_i|_{U_i \cap V})_{i \in I} \in \check{C}^1(\mathcal{U} \cap V, \mathcal{G})$$

(this is a morphism of $\mathcal{G}$-torsors, hence an isomorphism by Remark 1(*b*)), so $[\mathcal{X}]$ is mapped to $[\psi]$ and then back to $[\mathcal{X}_\psi] = [\mathcal{X}]$ again. *q.e.d.*

Remark. When $\mathcal{G}$ is any sheaf of groups on X, one can define $H^1(X, \mathcal{G})$ as the set of isomorphism classes of $\mathcal{G}$-torsors on X. This is a pointed set, the base point being the isomorphism class of the trivial torsor. Thus, the isomorphism classes of line bundles and of n-dimensional vector bundles on X are in canonical bijection with $H^1(X, \mathcal{O}_X^\times)$ and $H^1(X, \mathrm{GL}_n(\mathcal{O}_X))$ respectively by Example 2.

Corollary 1. *If X is an affine scheme, $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module and $\mathcal{X}$ an $\mathcal{M}$-torsor on X, then $\mathcal{X}$ is trivial.*

Proof. Let $\mathcal{U}$ be an open cover of X on which $\mathcal{X}$ trivializes. We may replace $\mathcal{U}$ by an affine refinement. Then $\check{H}^1(\mathcal{U}, \mathcal{M}) = H^1(X, \mathcal{M}) = 0$ by Theorem 1(*a*) and (*b*), proving that $\mathcal{X}$ is trivial by Proposition 1(*b*). *q.e.d.*

If X is an S-prescheme, we have a homomorphism of abelian groups

$$d \log \colon \mathcal{O}_X^\times \longrightarrow \Omega_{X/S}$$

$$f \longmapsto d \log f = \frac{d_{X/S} f}{f}$$

defining a morphism of Čech cohomology groups $\check{H}^1(X, \mathcal{O}_X^\times) \to \check{H}^1(X, \Omega_{X/S})$. When no open cover is specified, we put $\check{H}^1(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{F})$ for a presheaf $\mathcal{F}$ on X, where the colimit is taken over all open covers $\mathcal{U}$ of X, partially ordered by refinement. We have to use Čech cohomology here since we don't have any other available, unless X is a quasi-compact scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. In this case, the affine open covers are cofinal in the poset of open covers and $\check{H}^1(X, \mathcal{F}) = H^1(X, \mathcal{F})$ is what we expect.

Definition 2. Let $\mathcal{L}$ be line bundle on X. The image under the above map of the element of $\check{H}^1(X, \mathcal{O}_X^\times)$ associated to the torsor $\mathcal{L}^\times$ by Proposition 1(b) is called the **first Chern class** of $\mathcal{L}$ and is denoted

$$c_1(\mathcal{L}) \in \check{H}^1(X, \Omega_{X/S}) \ .$$

Definition 3. Let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles on the locally ringed space $(X, \mathcal{O}_X)$. An **extension** of $\mathcal{W}$ by $\mathcal{V}$ is a short exact sequence

$$0 \longrightarrow \mathcal{V} \xrightarrow{a} \mathcal{E} \xrightarrow{b} \mathcal{W} \longrightarrow 0 \ .$$

A **morphism of extensions** is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{V} & \xrightarrow{a} & \mathcal{E} & \xrightarrow{b} & \mathcal{W} & \longrightarrow & 0 \\
& & \| & & \downarrow{\varepsilon} & & \| & & \\
0 & \longrightarrow & \mathcal{V} & \xrightarrow{a'} & \mathcal{E}' & \xrightarrow{b'} & \mathcal{W} & \longrightarrow & 0
\end{array}
$$

Note that ε in the above diagram is automatically an isomorphism. Indeed, we can apply the snake lemma in the abelian category $\mathcal{O}_X$-Mod to obtain the vanishing of both $\ker(\varepsilon)$ and $\mathrm{coker}(\varepsilon)$.

For sheaves $\mathcal{F}$ and $\mathcal{G}$ on a topological space X, we get another sheaf on X via

$$U \longmapsto \mathrm{Hom}_{\mathrm{Shv}(U)}(\mathcal{F}|_U, \mathcal{G}|_U) \quad \text{for open subsets } U \subseteq X \ .$$

This sheaf is denoted $\mathcal{H}om(\mathcal{F}, \mathcal{G})$. If $\mathcal{R}$ is a sheaf of rings on X and $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{R}$-modules, we can restrict to homomorphisms respecting the $\mathcal{R}$-module structures and obtain a sheaf $\mathcal{H}om_{\mathcal{R}}(\mathcal{F}, \mathcal{G})$ by $\mathcal{H}om_{\mathcal{R}}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathcal{R}|_U\text{-Mod}}(\mathcal{F}|_U, \mathcal{G}|_U)$ for $U \subseteq X$ open.

Proposition 2. *Let $(X, \mathcal{O}_X)$ be a locally ringed space and $\mathcal{V}, \mathcal{W}$ vector bundles on X. Associating to any extension $\mathcal{E}$ of $\mathcal{W}$ by $\mathcal{V}$ the $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{W}, \mathcal{V})$-torsor $\mathcal{X}$ given by*

$$\mathcal{X}(U) = \left\{ \text{morphisms } \mathcal{E}|_U \xrightarrow{\pi} \mathcal{V}|_U \text{ of } \mathcal{O}_U\text{-modules} \ \middle| \ \pi a = \mathrm{id}_{\mathcal{V}|_U} \right\}$$

for $U \subseteq X$ open, whose $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{W}, \mathcal{V})$-action is given by $h \in \mathrm{Hom}_{\mathcal{O}_U\text{-Mod}}(\mathcal{W}|_U, \mathcal{V}|_U)$ sending π to $\pi + hb$) is an equivalence of groupoids between extensions of $\mathcal{W}$ by $\mathcal{V}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{W}, \mathcal{V})$-torsors on X.

Franke didn't prove this in the lecture. For the sake of completeness, however, I will do my best to sketch the important steps.

Sketch of a proof. Denote $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{W}, \mathcal{V})$ for short and let $\mathcal{X}_{\mathcal{E}}$ be the $\mathcal{H}$-torsor associated to an extension $\mathcal{E}$. Let's first give an argument why $\mathcal{X}_{\mathcal{E}}$ is even an $\mathcal{H}$-torsor. If $\pi, \pi' \in \mathcal{X}_{\mathcal{E}}(U)$ are two sections of a on U, then $(\pi - \pi')a = 0$, hence $\pi - \pi'$ factors uniquely over the cokernel of $\mathcal{V}_U \xrightarrow{a} \mathcal{E}|_U$, which is $\mathcal{W}|_U$. That is, there is a unique morphism $\mathcal{W}|_U \xrightarrow{h} \mathcal{V}|_U$ of $\mathcal{O}_U$-modules such that $\pi - \pi' = hb$. This shows that $\mathcal{H}(U)$ acts simply transitively on $\mathcal{X}_{\mathcal{E}}(U)$. It remains to show that the stalks $\mathcal{X}_{\mathcal{E}, x}$ are non-empty. But $\mathcal{V}$ and $\mathcal{W}$ are vector bundles, so every $x \in X$

has a neighbourhood U in which $\mathcal{W}|_U$ and $\mathcal{V}|_U$ are free $\mathcal{O}_U$-modules. In particular, $\mathcal{W}|_U$ is projective and the extension $\mathcal{E}|_U$ is split, i.e., $\mathcal{E}|_U \simeq \mathcal{W}|_U \oplus \mathcal{V}|_U$. Then $\mathcal{X}_\mathcal{E}(V)$ is non-empty for all open subsets $V \subseteq U$, hence $\mathcal{X}_{\mathcal{E},x} \neq \emptyset$.

Now about the functoriality properties. To a morphism $\mathcal{E} \xrightarrow{\varepsilon} \mathcal{E}'$ of extensions we associate the morphism $\mathcal{X}_\mathcal{E} \to \mathcal{X}_{\mathcal{E}'}$ which takes $\pi \in \mathcal{X}_\mathcal{E}(U)$ to $\varepsilon^{-1}\pi \in \mathcal{X}_{\mathcal{E}'}(U)$ (recall that ε is automatically an isomorphism; we could have taken $\varepsilon\pi$ instead to get a contravariant equivalence of groupoids). This gives a functor Φ from the groupoid of extensions of $\mathcal{W}$ by $\mathcal{V}$ to the groupoid of $\mathcal{H}$-torsors.

We show fully faithfulness first. Faithfulness of Φ is straightforward: If $\varepsilon, \varepsilon' \colon \mathcal{E} \to \mathcal{E}'$ are morphisms of extensions such that $\Phi(\varepsilon) = \Phi(\varepsilon')$, then choose an open cover $\mathcal{U} \colon X = \bigcup_{i \in I} U_i$ on which $\mathcal{V}$ and $\mathcal{W}$ trivialize (i.e., $\mathcal{V}|_{U_i}$ is a free $\mathcal{O}_{U_i}$-module for all $i \in U$ and same for $\mathcal{W}$). Then $\mathcal{E}|_{U_i} \simeq \mathcal{W}|_{U_i} \oplus \mathcal{V}|_{U_i}$ is a split extension (and same for $\mathcal{E}'|_{U_i}$), hence the $\mathcal{X}_\mathcal{E}(U_i)$ are non-empty and we may choose $\pi_i \in \mathcal{X}_\mathcal{E}(U_i)$ for all $i \in I$. From $\varepsilon^{-1}\pi_i = (\varepsilon')^{-1}\pi_i$ and splitness of $\mathcal{E}|_{U_i}, \mathcal{E}'|_{U_i}$ it's easy to deduce $\varepsilon = \varepsilon'$ on U_i. Since the U_i cover X, this holds on all of X and we are done.

Now suppose $\mathcal{X}_\mathcal{E} \xrightarrow{\varphi} \mathcal{X}_{\mathcal{E}'}$ is a morphism of $\mathcal{H}$-torsors. We need to construct a morphism $\mathcal{E} \xrightarrow{\varepsilon} \mathcal{E}'$ of extensions such that $\Phi(\varepsilon) = \varphi$. If $\pi \in \mathcal{X}_\mathcal{E}(U)$ is a section of a on U, then π and $\varphi(\pi)$ induce splittings $\mathcal{E}|_U \simeq \mathcal{W}|_U \oplus \mathcal{V}|_U \simeq \mathcal{E}'|_U$, which immediately gives a morphism $\mathcal{E}|_U \xrightarrow{\varepsilon_U} \mathcal{E}'|_U$ of extensions of $\mathcal{W}|_U$ by $\mathcal{V}|_U$. It can be easily checked that ε_U is actually independent of π. So choosing a trivializing open cover $\mathcal{U}$ as above, we get ε_{U_i} for all $i \in I$ which can be glued together (by naturality of their construction) to a morphism $\mathcal{E} \xrightarrow{\varepsilon} \mathcal{E}'$ which indeed satisfies $\Phi(\varepsilon) = \varphi$.

And essential surjectivity. Let $\mathcal{X}$ be an $\mathcal{H}$-torsor. To construct an extension $\mathcal{E}$ such that $\mathcal{X} \simeq \mathcal{X}_\mathcal{E}$, the idea is – of course – to choose $\mathcal{E}|_U \simeq \mathcal{W}|_U \oplus \mathcal{V}|_U$ on small neighbourhoods where $\mathcal{X}$ trivializes, and then to glue them together somehow. Actually we will take the smallest possible "neighbourhoods" – the stalks $\mathcal{E}_x \simeq \mathcal{W}_x \oplus \mathcal{V}_x$ and impose a suitable coherence condition. To do this, choose an element $\pi_x \in \mathcal{X}_x$ for every $x \in X$ and put

$$\mathcal{E}(U) = \left\{ e = (w_x, v_x)_{x \in U} \in \prod_{x \in U} \mathcal{W}_x \oplus \mathcal{V}_x \;\middle|\; \begin{array}{l} e \text{ fulfills the } \textit{cohe-} \\ \textit{rence condition}^{\text{TM}} \end{array} \right\},$$

where the *coherence condition*$^{\text{TM}}$ says that every $x \in U$ has a neighbourhood $V \subseteq U$ on which π_x, w_x and v_x have preimages $\pi_V \in \mathcal{X}(V)$, $w_V \in \mathcal{W}(V)$ and $v_V \in \mathcal{V}(V)$ such that the following holds: $\pi_{V,y}$, $w_{V,y}$ and $v_{V,y}$ denote the respective images in $\mathcal{X}_y$, $\mathcal{W}_y$ and $\mathcal{V}_y$, then $w_y = w_{V,y}$ and $v_y = v_{V,y} + h(w_y)$, where $h = \pi_y - \pi_{V,y} \in \mathcal{H}_y$. It is possible to show that this indeed defines an extension of $\mathcal{W}$ by $\mathcal{V}$ and that $\mathcal{X} \simeq \mathcal{X}_\mathcal{E}$. *q.e.d.*

2. Cohomology of projective spaces

2.1. Regular sequences and the Koszul complex

We will use the following conventions:

Definition 1. (a) For a cochain complex $(C^\bullet, d_C^\bullet)$ let the **shift** $C[p]^\bullet$ be the cochain complex defined by $C[p]^q = C^{p+q}$ and whose differential $d_{C[p]}^\bullet$ is given by $d_{C[p]}^q = (-1)^p d_C^{p+q}$.

(b) Let $C^\bullet \xrightarrow{\varphi} \widetilde{C}^\bullet$ be a morphism of cochain complexes. Then the **cone** $\mathrm{Cone}(\varphi)$ is the cochain complex defined by

$$\mathrm{Cone}(\varphi)^p = \widetilde{C}^p \oplus C^{p+1} \quad \text{and} \quad d_{\mathrm{Cone}(\varphi)}^p(\widetilde{c}, c) = \left(d_{\widetilde{C}}^p(\widetilde{c}) + \varphi(c), -d_C^{p+1}(c) \right) . \qquad (1)$$

Remark. (a) Obviously, $H^q(C[p]^\bullet) = H^{p+q}(C^\bullet)$.

(b) Note that $d^\bullet = d_{\mathrm{Cone}(\varphi)}^\bullet$ is indeed a differential of cochain complexes, as

$$\begin{aligned}
d^{p+1} d^p(\widetilde{c}, c) &= d^{p+1}\left(d_{\widetilde{C}}^p(\widetilde{c}) + \varphi(c), -d_C^{p+1}(c) \right) \\
&= \left(d_{\widetilde{C}}^{p+1} d_{\widetilde{C}}^p(\widetilde{c}) + d_{\widetilde{C}}^{p+1}\varphi(c) - \varphi d_C^{p+1}(c), d_C^{p+2} d_C^{p+1}(c) \right) \\
&= 0 .
\end{aligned}$$

We get a short exact sequence

$$0 \longrightarrow \widetilde{C}^\bullet \longrightarrow \mathrm{Cone}(\varphi) \longrightarrow C[1]^\bullet \longrightarrow 0$$

of cochain complexes. The connecting homomorphism $H^{p+1}(C^\bullet) = H^p(C[1]^\bullet) \xrightarrow{\partial} H^{p+1}(\widetilde{C}^\bullet)$ coincides with the morphism induced by φ. We thus get a long exact cohomology sequence

$$\ldots \longrightarrow H^p(C^\bullet) \xrightarrow{\varphi} H^p(\widetilde{C}^\bullet) \longrightarrow H^p(\mathrm{Cone}(\varphi)) \longrightarrow H^{p+1}(C^\bullet) \longrightarrow \ldots . \qquad (2)$$

In particular, φ induces isomorphisms on cohomology iff $H^\bullet(\mathrm{Cone}(\varphi))$ vanishes.

Naturally arising in the computation of cohomology of projective spaces, *regular sequences* are interesting enough to be studied on their own.

Definition 2. Let R be a ring and M an R-module. A sequence $(x_0, \ldots, x_n)$ of elements of R is called M**-regular** if for $i = 0, \ldots, n$

$$M/(x_0 M + \ldots + x_{i-1} M) \xrightarrow{x_i \cdot} M/(x_0 M + \ldots + x_{i-1} M)$$

is an injective map. A **regular sequence** in R is an M-regular sequence where $M = R$.

Example 1. (*a*) If S is any ring and $R = S[X_0, \ldots, X_n]$, then the sequence $(X_0, \ldots, X_n)$ is R-regular. Indeed, $R/(X_0 R + \ldots + X_{i-1} R) \simeq S[X_i, \ldots, X_n]$ and multiplication by X_i in the polynomial ring $S[X_i, \ldots, X_n]$ is clearly injective.

(*b*) The sequence $(1, 0)$ is always regular, but $(0, 1)$ is only regular when $R = 0$.

The following definition was lacking the $m = 0$ exception in the lecture (let alone the indices), but I think the following is what Professor Franke intended.

Definition 3. Let R be a ring, M be an R-module, $(x_0, \ldots, x_n)$ a sequence of elements of R. The **Koszul complex** $K^\bullet((x_0, \ldots, x_n), M)$ is a cochain complex defined as follows. Let $[n] = \{0, \ldots, n\}$. For $m > 1$, $K^m((x_0, \ldots, x_n), M)$ is the collection of maps $f \colon [n]^m \to M$ with the following properties:

(*a*) $f(i_1, \ldots, i_m) = 0$ when $i_k = i_\ell$ for some integers $0 \le k < \ell \le m$.

(*b*) $f(i_{\pi(1)}, \ldots, i_{\pi(m)}) = \mathrm{sgn}(\pi) f(i_1, \ldots, i_m)$ for every permutation $\pi \in \mathfrak{S}_m$.

The differential $d^m \colon K^m((x_0, \ldots, x_n), M) \to K^{m+1}((x_0, \ldots, x_n), M)$ is given by

$$d^m f = \sum_{j=0}^{m} (-1)^j d_j f \quad \text{where} \quad d_j f(i_1, \ldots, i_{m+1}) = x_{i_{j+1}} f(i_1, \ldots, \hat{i}_{j+1}, \ldots, i_{m+1}) \, .$$

As in Definition 1.2.1, the hat $\hat{i}_{j+1}$ denotes the omission of i_{j+1}. Moreover, for $m = 0$ we put $K^0((x_0, \ldots, x_n), M) = M$ and $d^0 \mu$ for $\mu \in M$ is the coaugmentation given by $d^0 \mu(i) = x_i \mu$ for $i = 0, \ldots, n$.

Remark. Before we check that $K^\bullet((x_0, \ldots, x_n), M)$ is indeed a cochain complex, let's answer the question *how on earth* Definition 3 is related to the definition you may find in the literature (e.g., [3, p. 423]) ? Well, they are equivalent (duh!) and here is how.

Let $0 \le \ell_1 < \ldots < \ell_m \le n$ and $\delta_{\ell_1, \ldots, \ell_m} \colon [n]^m \to R$ be the map satisfying

$$\delta_{\ell_1, \ldots, \ell_m}(i_1, \ldots, i_m) = \begin{cases} \mathrm{sgn}(\pi) & \text{when } (i_1, \ldots, i_m) = (\ell_{\pi(1)}, \ldots, \ell_{\pi(m)}) \text{ for } \pi \in \mathfrak{S}_m \\ 0 & \text{when } (i_1, \ldots, i_m) \text{ is no permutation of } (\ell_1, \ldots, \ell_m) \end{cases} .$$

Clearly, $K^m((x_0, \ldots, x_n), M)$ is generated by the $\delta_{\ell_1, \ldots, \ell_m} \cdot \mu$ for $\mu \in M$. For $K^m(x_0, \ldots, x_n) \coloneqq K^m((x_0, \ldots, x_n), R)$ the $\delta_{\ell_1, \ldots, \ell_m}$ even form a basis and we easily get an isomorphism

$$K^\bullet((x_0, \ldots, x_n), M) \simeq K^\bullet(x_0, \ldots, x_n) \otimes_R M$$

of cochain complexes. Moreover, if $e_0, \ldots, e_n$ denote the standard basis vectors of R^{n+1}, mapping $\delta_{\ell_1, \ldots, \ell_m} \mapsto e_{\ell_1} \wedge \cdots \wedge e_{\ell_m}$ sends a basis of $K^m(x_0, \ldots, x_n)$ to a basis of $\bigwedge^m R^{n+1}$ and we find that they are isomorphic.

Let us also find out which map d^m corresponds to. Note that $d_j \delta_{\ell_1, \ldots, \ell_m}(i_1, \ldots, i_{m+1})$ is only non-zero when $(i_1, \ldots, i_{m+1})$ is a permutation of $(k, \ell_1, \ldots, \ell_m)$ for some $k \in [n]$ with k at the $(j+1)^{\text{st}}$ position. In this case, it takes the value $\pm x_k$, depending on the sign of the permutation

of the ℓ_i. If you think about this for a while, $d^m \delta_{\ell_1,\ldots,\ell_m}$ corresponds to $\sum_{k=0}^{n} x_k e_k \wedge e_{\ell_1} \wedge \ldots \wedge e_{\ell_m}$. So d^m in $K^\bullet(x_0, \ldots, x_n)$ simply corresponds to the map

$$(x_0, \ldots, x_n) \wedge - : \bigwedge^m R^{n+1} \longrightarrow \bigwedge^{m+1} R^{n+1} .$$

In $K^\bullet((x_0, \ldots, x_n), M) \simeq K^\bullet(x_0, \ldots, x_n) \otimes_R M$, this just gets tensored with M.

Also note that $\bigwedge^0 N = R$ for every R-module N by definition, so the above special case $m = 0$ now fits naturally into the rest.

Remark 1. (a) It is easy to check that the d_j satisfy relations similar to (1.2.2), (1.2.3), and (1.2.4). As in the proofs of Definition 1.2.1 and Definition 1.2.2, we deduce that $d^\bullet$ preserves the antisymmetry conditions and is a differential in a cochain complex.

Alternatively, from the above considerations it's immediately clear that $d^\bullet$ is well defined and satisfies $d^{m+1} d^m = 0$, since wedging with $(x_0, \ldots, x_n)$ two times in a row obviously gives 0.

(b) A short exact sequence of R-modules induces a short exact sequence of Koszul complexes. Indeed, in the representation $K^m((x_0, \ldots, x_n), M) \simeq \bigwedge^m R^{n+1} \otimes_R M \simeq M^{\binom{n+1}{m}}$ this is pretty obvious.

(c) Let φ be the map $\varphi \colon K^\bullet((x_0, \ldots, x_{n-1}), M) \xrightarrow{x_n \cdot} K^\bullet((x_0, \ldots, x_{n-1}), M)$. We have a canonical isomorphism

$$\begin{aligned} K^\bullet((x_0, \ldots, x_n), M) &\xrightarrow{\sim} \operatorname{Cone}(\varphi)[-1] \\ f \in K^m((x_0, \ldots, x_n), M) &\longmapsto \left(f(n, -)|_{[n-1]^{m-1}}, -f|_{[n-1]^m} \right) \end{aligned} \tag{3}$$

(I'm pretty sure there should be a minus in the second component for the differentials to be compatible, i.e., consistent with the *two* sign conventions from Definition 1(a) and (b) which are in play here, though we didn't put a minus there in the lecture). This isomorphism corresponds to the decomposition $\bigwedge^m R^{n+1} = e_n \wedge \bigwedge^{m-1} R^n \oplus \bigwedge^m R^n \simeq \bigwedge^{m-1} R^n \oplus \bigwedge^m R^n$, sending $e_n \wedge y + z$ to $(y, -z)$ (and you have to tensor it with M, of course). If you think about this long enough, this is compatible.

In the following we put $H^j((x_0, \ldots, n), M) := H^j(K^\bullet((x_0, \ldots, x_n), M))$ for "short".

Example 2. (a) For small values of n, we have

$$K^\bullet(\emptyset, M) = \left(0 \longrightarrow M \longrightarrow 0 \longrightarrow \ldots \right)$$

$$K^\bullet((x_0), M) = \left(0 \longrightarrow M \xrightarrow{x_0 \cdot} M \longrightarrow 0 \longrightarrow \ldots \right)$$

$$K^\bullet((x_0, x_1), M) = \left(0 \longrightarrow M \xrightarrow{(x_0, x_1)} M \oplus M \xrightarrow{\binom{-x_1}{x_0}} M \longrightarrow 0 \longrightarrow \ldots \right) .$$

(b) In general, $K^\bullet((x_0,\ldots,x_n),M)$ vanishes in degrees < 0 and $> n+1$ (as $\bigwedge^m R^{n+1}$ is only non-zero for $0 \le m \le n+1$). The differentials d^0, d^n are given by $d^0\mu = \sum_{j=0}^n x_j e_j \otimes \mu$ and $d^n\left(\sum_{j=0}^n (e_0 \wedge \cdots \wedge \hat{e}_j \wedge \cdots \wedge e_n) \otimes \mu_i\right) = \sum_{j=0}^n (-1)^j x_j (e_0 \wedge \cdots \wedge e_n) \otimes \mu_j$. This shows

$$H^0\big((x_0,\ldots,x_n),M\big) \simeq \bigcap_{i=0}^n \ker\left(M \xrightarrow{x_i\cdot} M\right)$$

and

$$H^{n+1}\big((x_0,\ldots,x_n),M\big) \simeq M/(x_0 M + \ldots + x_n M).$$

Fact 1. Let R be a ring, $(x_0,\ldots,x_n)$ a sequence of elements of R and M an R-module.

(a) Let $0 \le i \le n$, then $(x_0,\ldots,x_n)$ is M-regular iff $(x_0,\ldots,x_{i-1})$ is M-regular and $(x_i,\ldots,x_n)$ is $M/(x_0 M + \ldots + x_{i-1}M)$-regular.

(b) A sequence $(x_0,\ldots,x_n)$ is M-regular iff $H^j\big((x_0,\ldots,x_i),M\big) = 0$ for all i,j such that $0 \le i \le n$ and $j \ne i+1$.

(c) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. If $(x_0,\ldots,x_n)$ is M'-regular and M''-regular, then it is M-regular.

(d) If $(x_0,\ldots,x_n)$ is M-regular and $k_0,\ldots,k_n \in \mathbb{N}$ are natural exponents, then $(x_0^{k_0},\ldots,x_n^{k_n})$ is also M-regular.

Proof. Part (a) is obvious. For (b), we use induction on n. For $n = -1$, there is nothing to show. Now suppose that $n \ge 0$ and the assertion holds for $n-1$. From (2) and (3) we get a long exact sequence

$$\ldots \longrightarrow H^{j-1}\big((x_0,\ldots,x_{n-1}),M\big) \xrightarrow{x_n\cdot} H^{j-1}\big((x_0,\ldots,x_{n-1}),M\big) \longrightarrow H^j\big((x_0,\ldots,x_n),M\big)$$
$$\longrightarrow H^j\big((x_0,\ldots,x_{n-1}),M\big) \xrightarrow{x_n\cdot} H^j\big((x_0,\ldots,x_{n-1}),M\big) \longrightarrow \ldots \quad (*)$$

Suppose that $H^j\big((x_0,\ldots,x_i),M\big)$ vanishes for $0 \le i \le n$ and $j \ne i+1$. Then $(x_0,\ldots,x_{n-1})$ is M-regular by induction. Using $H^{n-1}\big((x_0,\ldots,x_{n-1}),M\big) = 0$ and Example 2(b), for $j = n$ the sequence $(*)$ becomes

$$0 \longrightarrow H^n\big((x_0,\ldots,x_n),M\big) \longrightarrow M/(x_0 M + \ldots + x_{n-1}M) \xrightarrow{x_n\cdot} M/(x_0 M + \ldots + x_{n-1}M) \quad (\#)$$

The vanishing of $H^n\big((x_0,\ldots,x_n),M\big)$ in $(\#)$ and exactness show that $x_n\cdot$ is injective on $M/(x_0 M + \ldots + x_{n-1}M)$, hence $(x_0,\ldots,x_n)$ is M-regular.

Conversely, assume that $(x_0,\ldots,x_n)$ is M-regular. Vanishing of $H^j\big((x_0,\ldots,x_i),M\big)$ for $0 \le i \le n-1$ and $j \ne i+1$ follows from induction (in particular, we still get $(\#)$) and for $i = n$ and $j > n+1$ this is also clear. From $(*)$ we get $H^j\big((x_0,\ldots,x_n),M\big) = 0$ for $j < n$ and for $j = n$ this follows from $(\#)$ and injectivity of $x_n\cdot$ on $M/(x_0 M + \ldots + x_{n-1}M)$.

Part (c) follows from (b) and the long exact cohomology sequence associated to the short exact sequence $0 \to K^\bullet\big((x_0,\ldots,x_n),M'\big) \to K^\bullet\big((x_0,\ldots,x_n),M\big) \to K^\bullet\big((x_0,\ldots,x_n),M''\big) \to 0$ of Koszul complexes due to Remark 1(b).

Part (d). We use induction on n. For $n = -1$, there is nothing to show. Let $n \ge 0$ and assume that (d) holds for $n-1$ and any R-module M. Then $M/x_0^{k_0}M$ has a filtration by

submodules $x_0^i M / x_0^{k_0} M$ for $i = 0, \ldots, k_0$ and with filtration quotients $x_0^i M / x_0^{i+1} M$. Since $x_0 \cdot$ is injective on M, $x_0^i \cdot$ induces an isomorphism $x_0^i M / x_0^{i+1} M \xleftarrow{\sim} M / x_0 M$. Since $(x_1, \ldots, x_n)$ is $M / x_0 M$-regular by (a), repeatedly applying (c) implies that $(x_1, \ldots, x_n)$ is $M / x_0^{k_0} M$-regular. By the induction assumption, $(x_1^{k_1}, \ldots, x_n^{k_n})$ is $(M / x_0^{k_0})$-regular as well. An application of (a) completes the proof. *q.e.d.*

Definition 4. Let R be a ring, M and N be finitely generated free (or projective) R-modules. A **non-degenerate pairing** $M \times N \to R$ is a R-bilinear map $M \times N \to R$ such that one of the induced maps $M \to \operatorname{Hom}_R(N, R)$ or $N \to \operatorname{Hom}_R(M, R)$ is an isomorphism.

Corollary 1. *Let R be any ring, $S = R[X_0, \ldots, X_n]$. The Koszul complex $K^\bullet((X_0^\ell, \ldots, X_n^\ell), S)$ is acyclic in cohomological degree $\neq n+1$. Its cohomology in dimension $n+1$ is a free R-module.*

The Koszul complex becomes a complex of graded S-modules via

$$K^m((X_0^\ell, \ldots, X_n^\ell), S)_k := \left\{ f \in K^m((X_0^\ell, \ldots, X_n^\ell), S) \;\middle|\; f([n]^m) \subseteq S_{k+\ell m} \right\},$$

where $S_{k+\ell m}$ denotes the degree $(k + \ell m)$-part of $S = R[X_0, \ldots, X_n]$, and

(a) *$H^{n+1}((X_0^\ell, \ldots, X_n^\ell), S)_k = 0$ when $k > -n - 1$.*

(b) *We have an isomorphism*

$$\tau \colon H^{n+1}((X_0^\ell, \ldots, X_n^\ell)) \xrightarrow{\sim} R$$
$$[f] \longmapsto \text{the coefficient of } (X_0 X_1 \cdots X_n)^{\ell-1} \text{ in } f(0, 1, \ldots, n) \tag{4}$$

(where $[f]$ denotes the cohomology class of $f \in K^{n+1}((X_0^\ell, \ldots, X_n^\ell), S)$ which is a cocycle).

(c) *For $0 < k < \ell$ we have a non-degenerate pairing*

$$S_k \times H^{n+1}((X_0^\ell, \ldots, X_n^\ell), S)_{-n-1-k} \xrightarrow{\text{multiplication}} H^{n+1}((X_0^\ell, \ldots, X_n^\ell), S)_{-n-1} \overset{(4)}{\cong} R. \tag{5}$$

Proof. By Example 1(a), $(X_0, \ldots, X_n)$ is an S-regular sequence, hence so is $(X_0^\ell, \ldots, X_n^\ell)$ by Fact 1(d). Hence the vanishing of $H^i((X_0^\ell, \ldots, X_n^\ell), S)$ is a consequence of Fact 1(b). Moreover, by Example 2(b) we have

$$H^{n+1}((X_0^\ell, \ldots, X_n^\ell), S) \simeq S/(X_0^\ell, \ldots, X_n^\ell)$$

which is a free R-module with basis $[X_0^{\alpha_0} \cdots X_n^{\alpha_n}] = X_0^{\alpha_0} \cdots X_n^{\alpha_n} + (X_0^\ell, \ldots, X_n^\ell)$ for $\alpha_0, \ldots, \alpha_n \in \mathbb{N}_0$ such that $\alpha_i < \ell$. The elements of this basis are homogeneous for the above grading and to get an element of homogeneous degree $-n-1$ we need $\alpha_0 + \ldots + \alpha_n = -n - 1 + \ell(n+1) = (\ell-1)(n+1)$ and there is no other way to have this than $\alpha_i = \ell - 1$ for $i = 0, \ldots, n$. This proves (b).

Along the same lines we see that (a) holds. Indeed, for an element of degree $k > -n - 1$ we would need $\alpha_0 + \ldots + \alpha_n > (\ell-1)(n+1)$ and then there is some $\alpha_i \geq \ell$, contradiction.

The pairing from (c) pairs $X_0^{\beta_0} \cdot X_n^{\beta_n}$ with $[X_0^{\alpha_0} \cdots X_n^{\alpha_n}] \in H^{n+1}((X_0^\ell, \ldots, X_n^\ell), S)_{-n-1-k}$ to a non-zero element (which is then automatically $(X_0 \cdots X_n)^{\ell-1}$) iff $\alpha + \beta = (\ell-1, \ldots, \ell-1)$. For every α contributing to our basis of $H^{n+1}((X_0^\ell, \ldots, X_n^\ell), S)_{-n-1-k}$ the corresponding

$\beta = (\ell - 1, \ldots, \ell - 1) - \alpha$ is an element of $\mathbb{N}_0^{n+1}$ and $\beta_0 + \ldots + \beta_n = k$. If $k < \ell$ and $\beta \in \mathbb{N}_0^{n+1}$ satisfies $\beta_0 + \ldots + \beta_n = k$, then $\beta_i \leq k \leq \ell - 1$ for all components of β, hence $\alpha = (\ell - 1, \ldots, \ell - 1) - \beta \in \mathbb{N}_0^{n+1}$ contributes to our basis of $H^{n+1}((X_0^\ell, \ldots, X_n^\ell), S)_{-n-1-k}$, showing the non-degeneracy of the pairing. $\hfill q.e.d.$

And finally for a lemma which is perhaps worthwhile to know.

Lemma 1. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$, M a finitely generated R-module and $x_0, \ldots, x_n \in \mathfrak{m}$. Then $(x_0, \ldots, x_n)$ is M-regular iff $H^i((x_0, \ldots, x_n), M) = 0$ for $i \neq n + 1$. In particular, reordering $(x_0, \ldots, x_n)$ will again produce an M-regular sequence.*

Proof. Omitted. $\hfill q.e.d.$

2.2. The scheme $\mathbb{P}_R^n$ and its cohomology

Let R be an $\mathbb{N}$-graded ring. Let $R_+ \subseteq R$ be the ideal of elements of R of positive homogeneous degree. Recall that there is a bijection

$$\{\text{homogeneous } \mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq R_+\} \xrightarrow{\sim} \operatorname{Spec} R_0$$
$$\mathfrak{p} \longmapsto \mathfrak{q} = \mathfrak{p} \cap R_0$$
$$\mathfrak{p} = \mathfrak{q} + R_+ \longleftarrow\!\shortmid \mathfrak{q} \ .$$

As a set, put

$$\operatorname{Proj}(R) = \{\text{homogeneous } \mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \not\subseteq R_+\}$$

and for $I \subseteq R$ a homogeneous ideal, let

$$V_{\mathrm{proj}}(I) = V(I) = \{\mathfrak{p} \in \operatorname{Proj}(R) \mid \mathfrak{p} \supseteq I\} \ .$$

As $\bigcap_{\lambda \in \Lambda} V(I_\lambda) = V\left(\sum_{\lambda \in \Lambda} I_\lambda\right)$ and $V(I) \cup V(J) = V(I \cap J) = V(IJ)$ and $V(R) = V(R_+) = \emptyset$ and $V(0) = \operatorname{Proj}(R)$ the $V(I)$ are the closed subsets of some topology on $\operatorname{Proj}(R)$.

We define a sheaf of graded rings $\mathcal{O}(-)$ on $P = \operatorname{Proj}(R)$ as follows. Let $\mathcal{O}(k)_{[\mathfrak{p}]} = (R_\mathfrak{p})_k$ for homogeneous prime ideals $\mathfrak{p} \subseteq R$ be obtained from R by inverting the *homogeneous* elements of $R_\mathfrak{p}$ only. Note that the subsets of the form $P \setminus V(f)$ with $f \in R$ homogenous form a topology base of P. We put

$$\mathcal{O}(k)(U) = \left\{ (g_\mathfrak{p}) \in \prod_{\mathfrak{p} \in U} (R_\mathfrak{p})_k \ \middle| \ \begin{array}{l} \text{every } x \in U \text{ has a neighbourhood } V = P \setminus V(f) \subseteq U, \\ \text{where } f \text{ is homogeneous, such that there is a } \gamma \in (R_f)_k \\ \text{such that } g_\mathfrak{p} = (\text{image of } \gamma \text{ under } R_f \to R_\mathfrak{p}) \text{ for all} \mathfrak{p} \in V \end{array} \right\} \ .$$

Then there are operations

$$\mathcal{O}(k) \times \mathcal{O}(k) \longrightarrow \mathcal{O}(k) \ , \quad (f_\mathfrak{p})_{\mathfrak{p} \in U} + (g_\mathfrak{p})_{\mathfrak{p} \in U} = (f_\mathfrak{p} + g_\mathfrak{p})_{\mathfrak{p} \in U}$$
$$\mathcal{O}(k) \times \mathcal{O}(\ell) \longrightarrow \mathcal{O}(k + \ell) \ , \quad (f_\mathfrak{p})_{\mathfrak{p} \in U} \cdot (g_\mathfrak{p})_{\mathfrak{p} \in U} = (f_\mathfrak{p} g_\mathfrak{p})_{\mathfrak{p} \in U}$$

and a map

$$\mathcal{O}(k)_\mathfrak{p} \longrightarrow \mathcal{O}(k)_{[\mathfrak{p}]}$$
$$\begin{pmatrix} \text{image of} \\ (q_\mathfrak{q}) \in \mathcal{O}(k)(U) \text{ in } \mathcal{O}(k)_\mathfrak{p} \end{pmatrix} \longmapsto g_\mathfrak{p} \hspace{2cm} (*)$$

Fact 1. The map $(*)$ is an isomorphism.

Proposition 1. (a) $\operatorname{Proj}(R)$ *is a scheme. Open subsets of the form* $\operatorname{Proj}(R) \setminus V(f)$ *where* $f \in R_k$ *with* $k > 0$ *are affine and form a topology base.*

(b) *Every finite subset of* $\operatorname{Proj}(R)$ *is contained in an affine open subset.*

(c) *The sheaf of modules* $\mathcal{O}(k)$ *on* $\operatorname{Proj}(R)$ *is quasi-coherent, and there is a homeomorphism*

$$
\begin{aligned}
\operatorname{Proj}(R) \setminus V(f) &\overset{\sim}{\longrightarrow} \operatorname{Spec}(R_f)_0 \\
\mathfrak{p} &\longmapsto \mathfrak{q} = \mathfrak{p}R_f \cap (R_f)_0 \\
\mathfrak{p} = \sqrt{\mathfrak{q}R_f} \cap R &\longleftarrow\mathfrak{q} \in \operatorname{Spec}(R_f)_0
\end{aligned}
\tag{1}
$$

under which $(R_{\mathfrak{p}})_0 \simeq ((R_f)_0)_{\mathfrak{q}}$. *We thus get an identification* $\mathcal{O}(0)|_{\operatorname{Proj}(R) \setminus V(f)} \simeq \mathcal{O}_{\operatorname{Spec}(R_f)_0}$.

Moreover, under (1) *there is a similar isomorphism* $(R_{\mathfrak{p}})_k \simeq ((R_f)_k)_{\mathfrak{q}}$ *(the latter denotes the localization of the* $(R_f)_0$-*module* $(R_f)_k$ *with respect to* $\mathfrak{q}$, *that is, with respect to the homogeneous elements of* $R_f \setminus \mathfrak{q}$*). This gives an identification*

$$
\mathcal{O}(k)|_{\operatorname{Proj}(R) \setminus V(f)} \simeq ((R_f)_k)^{\sim} .
\tag{2}
$$

(d) *When* R *is generated by* R_1 *as an* R_0-*algebra (or equivalently,* R_+ *by* R_1 *as an ideal in* R*) the sheaves of modules* $\mathcal{O}(k)$ *are line bundles and the morphism*

$$
\begin{aligned}
\mathcal{O}(k) \otimes_{\mathcal{O}_{\operatorname{Proj}(R)}} \mathcal{O}(\ell) &\longrightarrow \mathcal{O}(k+\ell) \\
f \otimes g &\longmapsto fg
\end{aligned}
$$

is an isomorphism.

(e) *We have a morphism of preschemes*

$$
\begin{aligned}
\pi\colon \operatorname{Proj}(R) &\longrightarrow \operatorname{Spec} R_0 \\
\mathfrak{p} &\longmapsto \mathfrak{p} \cap R_0
\end{aligned}
\tag{3}
$$

and $\pi^*\colon \mathcal{O}_{\operatorname{Spec} R_0} \to \pi_* \mathcal{O}_{\operatorname{Proj}(R)}$ *is defined by sending an element* $rf^{-k} \in (R_0)_f \simeq \mathcal{O}_{\operatorname{Spec} R_0}(\operatorname{Spec} R_0 \setminus V(f))$ *to precisely that element in* $\mathcal{O}(0)(\operatorname{Proj}(R) \setminus V(f))$. *The morphism* π *is of finite type iff* R *is an* R_0-*algebra of finite type.*

(f) *If* R *is Noetherian (which is the case iff* R_0 *is Noetherian and* R *of finite type over* R_0*) then* $\operatorname{Proj}(R)$ *is Noetherian and the* $\mathcal{O}(k)$ *are coherent.*
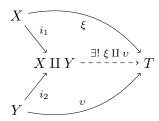
A. Appendix – category theory corner

A.1. Towards abelian categories

Definition 1. (*a*) A **pointed** category is a category with initial and final objects, such that the canonical (unique) morphism from the initial to the final object is an isomorphism.

(*b*) An **additive** category $\mathcal{A}$ is a pointed category which has a product $X \times Y$ (i.e., a fibre product over the final object $*$) and coproduct $X \amalg Y$ (i.e., a dual fibre product with respect to the initial object $*$) such that the canonical morphism $X \amalg Y \to X \times Y$ is an isomorphism for all objects $X, Y \in \mathrm{Ob}(\mathcal{A})$ and such that the resulting addition law on $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ defines a group structure for all $X, Y \in \mathrm{Ob}(\mathcal{A})$.

Remark. (*a*) When $\mathcal{A}$ is a pointed category and $X, Y \in \mathrm{Ob}(\mathcal{A})$, let the *zero morphism* (which we denote 0) $X \xrightarrow{0} Y$ be defined by $X \to * \to Y$, where $*$ is the both initial and final object.

(*b*) We will construct the canonical morphism $X \amalg Y \xrightarrow{c} X \times Y$ from Definition 1(*b*). The product $X \times Y$ comes with canonical projections $X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$ such that given morphisms $T \xrightarrow{\xi} X$ and $T \xrightarrow{\upsilon} Y$ there is a unique $T \xrightarrow{\xi \times \upsilon} X \times Y$ such that

commutes.

Similarly, the coproduct $X \amalg Y$ has morphisms $X \xrightarrow{i_1} X \amalg Y \xleftarrow{i_2} Y$ such that given morphisms $X \xrightarrow{\xi} T$ and $Y \xrightarrow{\upsilon} T$ there is a unique morphism $X \amalg Y \xrightarrow{\xi \amalg \upsilon} T$ such that

commutes.

Using the universal property of $X \times Y$, we get a unique morphism $X \xrightarrow{\alpha} X \times Y$ such that $p_1\alpha = \mathrm{id}_X$, $p_2\alpha = 0$ and a unique morphism $Y \xrightarrow{\beta} X \times Y$ such that $p_1\beta = 0$ and $p_2\beta = \mathrm{id}_Y$. Then

$$c \colon X \amalg Y \xrightarrow{\alpha \amalg \beta} X \times Y$$

is the morphism we are looking for. It is unique with the property that $p_1ci_1 = \mathrm{id}_X$, $p_1ci_2 = 0$, $p_2ci_1 = 0$, and $p_2ci_2 = \mathrm{id}_Y$.
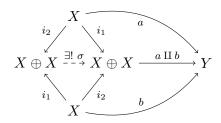
(c) For abelian groups and modules over a ring, both $X \amalg Y$ and $X \times Y$ are given by $\{(x,y) \mid x \in X,\ y \in Y\}$ with component-wise operations and $p_1(x,y) = x$, $p_2(x,y) = y$, $i_1(x) = (x,0)$, and $i_2(y) = (0,y)$.

(d) For an additive category $\mathcal{A}$, it follows that finite products $\prod_{i=1}^n X_i$ and coproducts $\coprod_{i=1}^n X_i$ (of some objects $X_1, \ldots, X_n \in \mathrm{Ob}(\mathcal{A})$) exist and are canonically isomorphic. We typically denote both by $\bigoplus_{i=1}^n X_i$ in that case.

(e) We would like to describe the addition on $\mathrm{Hom}_{\mathcal{A}}(X,Y)$. For a pair of morphisms $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ we denote the composition

$$X \xrightarrow{\mathrm{id}_X \times \mathrm{id}_X} X \oplus X \xrightarrow{a \amalg b} Y$$

by $a + b$. Then 0 is a neutral element and associativity holds, but the existence of inverse elements needs to be imposed to obtain indeed a group structure.

One can show that this is the only possible group structure behaving bilinearly under composition.

(f) It is, however, automatically abelian. What we need to show is $(a \amalg b) \circ \Delta = (b \amalg a) \circ \Delta$ with $\Delta = \mathrm{id}_X \times \mathrm{id}_X$. The universal property of coproducts gives a unique $X \oplus X \xrightarrow{\sigma} X \oplus X$ such that

commutes. Then σ is easily seen to be an isomorphism and $b \amalg a = (a \amalg b) \circ \sigma$ by the uniqueness of $b \amalg a$. It thus suffices to show $\sigma\Delta = \Delta$. By the uniqueness of Δ, this is equivalent to $p_1\sigma\Delta = \mathrm{id}_X$ and $p_2\sigma\Delta = \mathrm{id}_X$. We claim that $p_1\sigma = p_2$ and vice versa, which would finish the proof. To see this, note that $p_1\sigma = p_2$ is equivalent to $p_1\sigma i_1 = p_2 i_1 = 0$ and $p_1\sigma i_2 = p_2 i_2 = \mathrm{id}_X$ by the universal property of the coproduct $X \oplus X$. This follows from $\sigma i_1 = i_2$ and $\sigma i_2 = i_1$ by definition of σ.

Example. The following are additive categories.

(a) Modules over a given ring R (in particular, abelian groups).

(b) Sheaves of modules.

(c) Banach spaces with bounded linear maps as morphisms. The common initial and final object is the zero space and $A \oplus B = \{(a,b) \mid a \in A, \ b \in B\}$ with $\max\{\|a\|, \|b\|\}$ or $\|a\| + \|b\|$ as norm (this category will turn out not to be abelian).

(d) Free or projective modules over a ring R.

Definition 2. Let $A \xrightarrow{\alpha} B$ be a morphism in an additive category $\mathcal{A}$. The **kernel** $\ker(A \xrightarrow{\alpha} B)$ of α (if it exists) comes with a morphism $\ker(\alpha) \xrightarrow{\iota} A$ satisfying the universal property

$$\mathrm{Hom}_{\mathcal{A}}\left(T, \ker\left(A \xrightarrow{\alpha} B\right)\right) \xrightarrow{\sim} \{f \in \mathrm{Hom}_{\mathcal{A}} \mid \alpha f = 0\}$$
$$\left(T \xrightarrow{\tau} A\right) \longmapsto f = \iota\tau$$

for any test object $T \in \mathrm{Ob}(\mathcal{A})$.

Definition 2a. Similarly, the **cokernel** of α (if existent) comes with a morphism $B \xrightarrow{\pi} \mathrm{coker}(\alpha)$ and satisfies

$$\mathrm{Hom}_{\mathcal{A}}\left(\mathrm{coker}\left(A \xrightarrow{\alpha} B\right), T\right) \xrightarrow{\sim} \{g \in \mathrm{Hom}_{\mathcal{A}}(B, T) \mid g\alpha = 0\}$$
$$\left(\mathrm{coker}(\alpha) \xrightarrow{\tau} T\right) \longmapsto g = \tau\pi$$

for any test object $T \in \mathrm{Ob}(\mathcal{A})$.

Remark 1. Kernels and cokernels in an additive category $\mathcal{A}$ are special cases of *equalizers* and *coequalizers* (cf. [1, Definition A.3.2 and Definition A.3.4]), respectively. Indeed, we have

$$\ker\left(A \xrightarrow{\alpha} B\right) = \mathrm{Eq}\left(A \underset{0}{\overset{\alpha}{\rightrightarrows}} B\right) \quad \text{and} \quad \mathrm{coker}\left(A \xrightarrow{\alpha} B\right) = \mathrm{Coeq}\left(A \underset{0}{\overset{\alpha}{\rightrightarrows}} B\right).$$

But we can reconstruct equalizers and coequalizers from kernels and cokernels via

$$\mathrm{Eq}\left(A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B\right) = \ker\left(A \xrightarrow{\alpha-\beta} B\right) \quad \text{and} \quad \mathrm{Coeq}\left(A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B\right) = \mathrm{coker}\left(A \xrightarrow{\alpha-\beta} B\right)$$

(the minus here is the one obtained from additivity of $\mathcal{A}$).

Definition 3. A morphism $A \xrightarrow{i} B$ is an **effective monomorphism**, if the following equivalent conditions hold.

(a) (In any category) We have a bijection

$$\mathrm{Hom}_{\mathcal{A}}(T, A) \xrightarrow{\sim} \left\{ f \in \mathrm{Hom}_{\mathcal{A}}(T, B) \ \middle| \ \begin{array}{c} \alpha f = \beta f \text{ if } B \underset{\beta}{\overset{\alpha}{\rightrightarrows}} S \text{ is any pair of} \\ \text{morphisms such that } \alpha i = \beta i \end{array} \right\}$$
$$t \in \mathrm{Hom}_{\mathcal{A}}(T, A) \longmapsto f = it .$$

(b) (If the category has finite colimits) i is an equalizer of something.

(c) (In additive categories with kernels and cokernels) i is the kernel of an appropriate morphism.

(d) (In additive categories with kernels and cokernels) i is the kernel of its cokernel.

Definition 3a. Dually, $A \xrightarrow{p} B$ is an **effective epimorphism** if the following equivalent conditions hold.

(a) (In any category) We have a bijection

$$\operatorname{Hom}_{\mathcal{A}}(B,T) \xrightarrow{\sim} \left\{ f \in \operatorname{Hom}_{\mathcal{A}}(A,T) \ \middle| \ \begin{array}{l} f\alpha = f\beta \text{ if } S \underset{\beta}{\overset{\alpha}{\rightrightarrows}} A \text{ is any pair of} \\ \text{morphisms such that } p\alpha = p\beta \end{array} \right\}$$
$$t \in \operatorname{Hom}_{\mathcal{A}}(B,T) \longmapsto f = tp \ .$$

(b) (If the category has finite limits) p is a coequalizer of something.

(c) (In additive categories with kernels and cokernels) p is the cokernel of an appropriate morphism.

(d) (In additive categories with kernels and cokernels) p is the cokernel of its kernel.

(e) $B^{\mathrm{op}} \xrightarrow{p^{\mathrm{op}}} A^{\mathrm{op}}$ is an effective monomorphism in the dual category $\mathcal{A}^{\mathrm{op}}$.

In any category, a morphism which is mono and effectively epi (or epi and effectively mono) is an isomorphism, but there are examples of morphisms which are simultaneously mono and epi but not an isomorphism (e.g. $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in the category of rings). This needs to be ruled out by a definition, and that's what is happening now!

Definition 4. A category $\mathcal{A}$ is **abelian**, if it is additive, has kernels and cokernels and such that every monomorphism is effectively mono, every epimorphism is effectively epi, and (thus) any morphism which is both a mono- and an epimorphism is an isomorphism.

Remark. (a) The three conditions on mono- and epimorphisms are not independent. The last condition, i.e. that every morphism which is both a mono- and an epimorphism is an isomorphism, follows from either of the former two.

(b) Since equalizers and coequalizers in an abelian category $\mathcal{A}$ can be constructed from kernels and cokernels (cf. Remark 1) and we already have finite products and coproducts from additivity of $\mathcal{A}$, we deduce that $\mathcal{A}$ has arbitrary finite limits and colimits. Indeed, we proved on exercise sheet #7 from Algebraic Geometry I that finite limits can be constructed from equalizers and finite products, and, given coequalizers and finite coproducts instead, it's just the same for finite colimits.

Conversely, the existence of finite limits and colimits guarantees that $\mathcal{A}$ has equalizers, coequalizers, finite products, and finite products, all of them being special cases of finite limits and colimits.

The category of modules (over a ring R) or sheaves of modules are abelian categories (as we are going to prove in a moment), but not Banach spaces or projective modules over most rings.

Proposition 1. *The category $\mathcal{R}$-Mod of sheaves of modules (over a sheaf of rings $\mathcal{R}$ on some topological space X) is abelian.*

For clarity (and to better distinguish between the proof and Professor Franke's remarks about it), we will chop the proof into some lemmas.

Lemma 1. *The category $\mathcal{R}$-Mod is additive.*

Proof. First note that the zero sheaf 0 is a common initial and final object. A direct sum of $\mathcal{M}, \mathcal{N} \in \mathrm{Ob}(\mathcal{R}\text{-Mod})$ is given by

$$(\mathcal{M} \oplus \mathcal{N})(U) = \{(m,n) \mid m \in \mathcal{M}(U), n \in \mathcal{N}(U)\} \quad \text{for all } U \subseteq X \text{ open}$$

(it's clear that this is a presheaf and it inherits the sheaf axiom from $\mathcal{A}$ and $\mathcal{N}$) with component-wise module operations and with $\mathcal{M} \xleftarrow{p} \mathcal{M} \oplus \mathcal{N} \xrightarrow{q} \mathcal{N}$ and $\mathcal{M} \xrightarrow{i} \mathcal{M} \oplus \mathcal{N} \xleftarrow{j} \mathcal{N}$ given by $p(m,n) = m$, $q(m,n) = n$, $i(m) = (m,0)$, and $j(n) = (0,n)$ on open subsets $U \subseteq X$ and $m \in \mathcal{M}(U)$, $n \in \mathcal{N}(U)$.

If $\mathcal{M} \xrightarrow{\mu} \mathcal{T} \xleftarrow{\nu} \mathcal{N}$ are given, $\mathcal{M} \oplus \mathcal{N} \xrightarrow{\mu \amalg \nu} \mathcal{T}$ sending $(m,n) \in (\mathcal{M} \oplus \mathcal{N})(U)$ to $\mu(m) + \nu(n)$ verifies the universal property of the coproduct for $\mathcal{M} \oplus \mathcal{N}$. Similarly, $\mathcal{T} \xrightarrow{\mu \times \nu} \mathcal{M} \oplus \mathcal{N}$ given by $(\mu \times \nu)(t) = (\mu(t), \nu(t))$ for $t \in \mathcal{T}(U)$ confirms the universal property of the product for $\mathcal{M} \oplus \mathcal{N}$. Also, $c = \mathrm{id}_{\mathcal{M} \oplus \mathcal{N}}$ is the unique endomorphism c of that object such that $pci = \mathrm{id}_{\mathcal{M}}$, $qcj = \mathrm{id}_{\mathcal{N}}$, $pcj = 0$, and $qci = 0$. Thus, $\mathcal{R}$-Mod is additive (the group structure on Hom sets being easily verified). *q.e.d.*
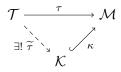
Lemma 2. *The category $\mathcal{R}$-Mod has kernels.*

Proof. Let $\mathcal{M} \xrightarrow{f} \mathcal{N}$ be a morphism of sheaves of $\mathcal{R}$-modules and $\mathcal{K}$ be the sheaf given by

$$\mathcal{K}(U) = \ker\left(\mathcal{M} \xrightarrow{f} \mathcal{N}\right)(U) := \ker\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right)$$

(you should convince yourself that this indeed satisfies the sheaf axiom). Then the inclusion $\mathcal{K} \xrightarrow{\kappa} \mathcal{M}$ is a monomorphism as $\mathcal{K}(U) \hookrightarrow \mathcal{M}(U)$ is injective for every open subset $U \subseteq X$.

If $\mathcal{T} \xrightarrow{\tau} \mathcal{M}$ is a morphism of $\mathcal{R}$-modules such that $f\tau = 0$, then, for every $t \in \mathcal{T}(U)$, we have $f(\tau(t)) = 0$, hence $\widetilde{\tau}(t) := \tau(t) \in \ker\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right) = \mathcal{K}(U)$ and τ factors over

$$\mathcal{T} \xrightarrow{\quad \tau \quad} \mathcal{M}$$
$$\exists! \, \widetilde{\tau} \searrow \qquad \nearrow \kappa$$
$$\mathcal{K}$$

This proves that $\mathcal{K}$ is indeed a kernel of f in the category $\mathcal{R}$-Mod. *q.e.d.*

Remark 2. (*a*) It is a consequence of the exactness of the $\varinjlim$ functor (for filtered systems of abelian groups; exactness of $\varinjlim$ does *not* hold in general, not even for filtered colimits in abelian categories), that

$$\mathcal{K}_x = \varinjlim_{U \ni x} \ker\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right) \simeq \ker\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x\right).$$

This isomorphism can also be seen in a straightforward way.

(b) One may check that in any additive category (with kernels), a morphism i is a monomorphism iff $\ker(i) = 0$. Thus, in our example we have the equivalent conditions

(α) $\mathcal{M} \xrightarrow{f} \mathcal{N}$ is a monomorphism.

(β) $\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)$ is injective for all open subsets $U \subseteq X$.

(γ) $\ker(f) = 0$ (the zero sheaf).

(δ) $\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x$ is injective for all $x \in X$.

The construction of cokernels won't be that straightforward (duh!), related to the fact that epimorphisms in categories of sheaves aren't as simple as you might think. If $\mathcal{G}$ and $\mathcal{H}$ are sheaves on some topological space X and f is a morphism between them such that $\mathcal{G}(U) \xrightarrow{f} \mathcal{H}(U)$ is surjective for all open U, then f is an epimorphism, but there are epimorphisms f for which this fails.

However, it follows from the fact that a sheaf $\mathcal{G}$ is canonically isomorphic to its sheafification $\mathcal{G}^{\mathrm{sh}}$ (cf. [1, Proposition 1.2.1(d)]) that a morphism between sheaves (of sets, groups, ...) is uniquely determined by the maps it induces on stalks. Thus, $\mathcal{G} \to \mathcal{H}$ is an epimorphism if $\mathcal{G}_x \to \mathcal{H}_x$ is an epimorphism in the respective target category for all $x \in X$.

Lemma 3. *The category $\mathcal{R}$-Mod has cokernels.*

Proof. For a morphism $\mathcal{M} \xrightarrow{f} \mathcal{N}$ of sheaves of $\mathcal{R}$-modules, the map

$$U \mapsto \operatorname{coker}\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right) = \mathcal{M}(U)/\mathcal{N}(U) \quad \text{for } U \subseteq X \text{ open}$$

defines a presheaf $\mathcal{F}$ of $\mathcal{R}$-modules, but in general, $\mathcal{F}$ will fail to be a sheaf. We put $\mathcal{C} = \mathcal{F}^{\mathrm{sh}}$ (the *sheafification* of $\mathcal{F}$, cf. [1, Definition 1.2.3]) and claim that $\mathcal{N} \to \mathcal{C}$ is a cokernel of f.

Our first goal is to show that

$$\mathcal{C}_x \simeq \operatorname{coker}\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x\right). \tag{$*$}$$

In the lecture, we did a direct proof, which was somewhat ugly and (in my opinion) lacking the essential step. From [1, Proposition 1.2.1(a)], we get that $\mathcal{C}_x \simeq \mathcal{F}_x$ (which is basically what we proved in the lecture for this particular special case), so we need to show that

$$\mathcal{F}_x = \varinjlim_{U \ni x} \operatorname{coker}\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right) \simeq \operatorname{coker}\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x\right).$$

Since $\mathcal{M}_x = \varinjlim_{U \ni x} \mathcal{M}(U)$ and similar for $\mathcal{N}_x$, this amounts to showing that cokernels and certain colimits commute. But by Remark 1, cokernels are just a special case of colimits, so what we are actually going to show is that colimits commute with colimits – in the following sense.

Lemma 4. *Let $(X_{i,j})_{i\in I, j\in J}$ be objects of a category $\mathcal{A}$. For each $i_1, i_2 \in I$ let there be an indexing set I_{i_1,i_2} and for each $\alpha \in I_{i_1,i_2}$ and $j \in J$ a morphism*

$$f_\alpha^j \colon X_{i_1,j} \longrightarrow X_{i_2,j} .$$

Similarly, for each $j_1, j_2 \in J$ let there be an indexing set J_{j_1,j_2} and for each $\beta \in J_{j_1,j_2}$ and $i \in I$ a morphism

$$g_\beta^i \colon X_{i,j_1} \longrightarrow X_{i,j_2} .$$

Moreover, suppose that for each $i_1, i_2 \in I$ and $j_1, j_2 \in J$ and $\alpha \in I_{i_1,i_2}$ and $\beta \in J_{j_1,j_2}$ the diagram

$$\begin{array}{ccc}
X_{i_1,j_1} & \xrightarrow{\ f_\alpha^{j_1}\ } & X_{i_2,j_1} \\
{\scriptstyle g_\beta^{i_1}}\downarrow & & \downarrow{\scriptstyle g_\beta^{i_2}} \\
X_{i_1,j_2} & \xrightarrow{\ f_\alpha^{j_2}\ } & X_{i_2,j_2}
\end{array} \qquad\qquad (\#)$$

commutes. Then there is an isomorphism

$$\varinjlim_{i\in I} \varinjlim_{j\in J} X_{i,j} \simeq \varinjlim_{j\in J} \varinjlim_{i\in I} X_{i,j} \simeq \varinjlim_{(i,j)\in I\times J} X_{i,j} .$$

Proof. Clearly, it is enough to show the rightmost isomorphism. What we need to show is that $L := \varinjlim_j \varinjlim_i X_{i,j}$ satisfies the universal property of $L' := \varinjlim_{(i,j)} X_{i,j}$.

Let T be an object of $\mathcal{A}$ and $\left(X_{i,j} \xrightarrow{\tau_{i,j}} T\right)_{i\in I, j\in J}$ be a cocone below the diagram $(X_{i,j})_{i,j}$. That is, for every $\alpha \in I_{i_1,i_2}$ and $j \in J$ the diagram
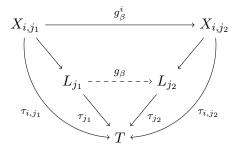
$$\begin{array}{ccc}
X_{i_1,j} & \xrightarrow{\ f_\alpha^{j}\ } & X_{i_2,j} \\
& {\scriptstyle \tau_{i_1,j}}\searrow \quad \swarrow{\scriptstyle \tau_{i_2,j}} & \\
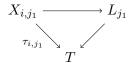& T &
\end{array}$$

commutes. By the universal properties of the $L_j := \varinjlim_i X_{i,j}$, the $\tau_{i,j}$ factor over some maps $L_j \xrightarrow{\tau_j} T$. Moreover, for each $j_1, j_2 \in J$ and $\beta \in J_{j_1,j_2}$, the compositions

$$X_{i,j_1} \xrightarrow{\ g_\beta^i\ } X_{i,j_2} \longrightarrow L_{j_2}$$

induce a map $L_{j_1} \xrightarrow{g_\beta} L_{j_2}$ by the universal property of L_{j_1} (here, we silently used the commutativity of $(\#)$, otherwise the above compositions wouldn't be a cocone below $(X_{i,j_1})_{i\in I}$). We thus get a diagram

$$\begin{array}{ccccc}
X_{i,j_1} & & \xrightarrow{\quad\quad g_\beta^i \quad\quad} & & X_{i,j_2} \\
& \searrow & & \swarrow & \\
& L_{j_1} & \xdashrightarrow{\ g_\beta\ } & L_{j_2} & \\
{\scriptstyle \tau_{i,j_1}}\downarrow & {\scriptstyle \tau_{j_1}}\searrow & & \swarrow{\scriptstyle \tau_{j_2}} & \downarrow{\scriptstyle \tau_{i,j_2}} \\
& & T & &
\end{array}$$

in which everything but the bottom-middle triangle commutes. We show that this triangle commutes as well. Indeed, by the universal property of L_{j_1}, τ_{j_1} is the unique morphism $L_{j_1} \to T$ making each

$$X_{i,j_1} \longrightarrow L_{j_1}$$
$$\tau_{i,j_1} \searrow \qquad \swarrow$$
$$T$$

commute. But apparently, $\tau_{j_2} g_\beta$ has this property as well, proving $\tau_{j_1} = \tau_{j_2} g_\beta$. Then the morphisms $\left(L_j \xrightarrow{\tau_j} T \right)_{j \in J}$ form a cocone below the diagram $(L_j)_{j \in J}$, hence factor uniquely over some $L \xrightarrow{\tau} T$ by the universal property of L.

It remains to prove uniqueness of τ. If $L \xrightarrow{\tau} T$ is a morphism over which each $X_{i,j} \xrightarrow{\tau_{i,j}} T$ factors, then the composition $L_j \to L \xrightarrow{\tau} T$ must equal τ_j since τ_j is uniquely determined by the universal property of L_j. But τ is uniquely determined by the τ_j, proving uniqueness. $\qquad$ *q.e.d.*

Having thus proved $(*)$, we now proceed with the proof of Lemma 3. We have a morphism $\mathcal{N} \to \mathcal{C}$ sending $n \in \mathcal{N}(U)$ to
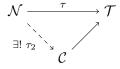
$$\left(\text{image of } n \text{ under } \mathcal{N}(U) \longrightarrow \mathcal{N}_x \longrightarrow \mathrm{coker}\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x \right) \right)_{x \in U} .$$

Since $\mathcal{C}_x \simeq \mathrm{coker}\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x \right)$, this morphism $\mathcal{N} \to \mathcal{C}$ induces surjections on stalks, hence is an epimorphism of sheaves. We show that the morphism $\mathcal{N} \to \mathcal{C}$ satisfies the universal property of the cokernel.

Let $\mathcal{N} \xrightarrow{\tau} \mathcal{T}$ be a morphism of sheaves of $\mathcal{R}$-modules such that $\tau f = 0$. Let $U \subseteq X$ be open. For

$$\nu = (\nu_x)_{x \in U} \in \mathcal{C}(U) \subseteq \prod_{x \in U} \mathrm{coker}\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x \right)$$

we define $\tau_1(\nu) \in \prod_{x \in U} \mathcal{T}_x$ by selecting $n \in \mathcal{N}_x$ whose image in $\mathrm{coker}\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x \right)$ equals ν_x, then put $\tau_1(\nu)_x = \tau(n)_x$ which is independent of the choice of n as $\tau f = 0$. It follows from the coherence condition for $\mathcal{C}$ that $\tau_1(\nu)$ satisfies the coherence condition for $\mathcal{T}^{\mathrm{sh}}$, i.e. $\tau_1(\nu) \in \mathcal{T}^{\mathrm{sh}}(U) \subseteq \prod_{x \in U} \mathcal{T}_x$. Hence there is $\mathcal{C} \xrightarrow{\tau_2} \mathcal{T}$ such that $\tau_1 = \left(\mathcal{T} \xrightarrow{\sim} \mathcal{T}^{\mathrm{sh}} \right) \circ \tau_2$ and τ_2 makes

$$\mathcal{N} \xrightarrow{\quad \tau \quad} \mathcal{T}$$
$$\exists! \, \tau_2 \searrow \qquad \nearrow$$
$$\mathcal{C}$$

commutative. Uniqueness of τ_2 is easy to see stalk-wise. It follows that $\mathcal{N} \to \mathcal{C}$ is ineed a cokernel of f. $\qquad$ *q.e.d.*

Remark 3. One may check that in any additive category (with cokernels) a morphism f is an epimorphism if $\operatorname{coker}(f) = 0$. By our previous construction of cokernels and the description of stalks, we have equivalent conditions

(a) $\mathcal{M} \xrightarrow{f} \mathcal{N}$ is an epimorphism of sheaves of $\mathcal{R}$-modules.

(b) $\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x$ is surjective for all $x \in X$.

(c) For every open $U \subseteq X$ and $n \in \mathcal{N}(U)$ there are an open covering $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ and $m_\lambda \in \mathcal{M}(U_\lambda)$ such that $n|_{U_\lambda} = f(m_\lambda)$

. . . but (c) does *not* imply the surjectivity of $\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)$, unless, e.g., f is also a monomorphism.

Proof of Proposition 1. We verify the rest of the abelianness conditions. First, let $\mathcal{M} \xrightarrow{f} \mathcal{N}$ be a mono- and epimorphism. Then it induces isomorphisms on stalks (by Remark 2(b) and Remark 3), hence is an isomorphism itself.

Let $\mathcal{M} \xrightarrow{i} \mathcal{N}$ be a monomorphism and $\mathcal{N} \to \mathcal{C}$ be its cokernel. Then

$$\ker\left(\mathcal{N} \longrightarrow \mathcal{C}\right)_x = \ker\left(\mathcal{N}_x \longrightarrow \mathcal{C}_x\right) = \ker\left(\mathcal{N}_x \longrightarrow \operatorname{coker}\left(\mathcal{M}_x \xrightarrow{i} \mathcal{N}_x\right)\right) \simeq \mathcal{M}_x$$

as $\mathcal{M}_x \xrightarrow{i} \mathcal{N}_x$ is injective. Hence $\mathcal{M} \to \ker\left(\mathcal{N} \to \mathcal{C}\right)$ induces isomorphisms on stalks and thus is an isomorphism itself. It follows by Definition 3(d) that any monomorphism is an effective monomorphism.

Similar arguments apply to epimorphisms. *q.e.d.*

Recall the definition of an adjoint pair of functors.

Definition 5 ([1, Definition A.2.3])**.** Let $\mathcal{A}, \mathcal{B}$ be categories. A pair $\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{B}$ of (covariant) functors is called **adjoint**, if there is a canonical bijection

$$\operatorname{Hom}_{\mathcal{A}}(X, RY) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{B}}(LX, Y)$$

which is functorial in both $X \in \operatorname{Ob}(\mathcal{A})$ and $Y \in \operatorname{Ob}(\mathcal{B})$.

Remark 4. It can be easily seen that L preserves colimits (in particular, coproducts, and in particular again, initial objects) and R preserves limits (in particular, products, and in particular again, final objects). When $\mathcal{A}$ and $\mathcal{B}$ are additive, it follows that both L and R map 0 to 0 and are compatible with finite direct sums. Moreover, L preserves cokernels and R preserves kernels since these are special cases of colimits and limits, respectively (in particular, I have no idea what the purpose of Franke's extra calculation was).

Bibliography

[1] Nicholas Schwab; Ferdinand Wagner. *Algebraic Geometry I by Jens Franke (lecture notes)*. GitHub: https://github.com/Nicholas42/AlgebraFranke/tree/master/AlgGeoI.

[2] Nicholas Schwab; Ferdinand Wagner. *Algebra I by Jens Franke (lecture notes)*. GitHub: https://github.com/Nicholas42/AlgebraFranke/tree/master/AlgebraI.

[3] D. Eisenbud. *Commutative Algebra: With a View Toward Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1995. ISBN: 978-0-387-94269-8. URL: http://xavirivas.com/cloud/Commutative%20Algebra/Eisenbud%20-%20Commutative%20algebra,%20with%20a%20view%20toward%20algebraic%20geometry%20(Springer,%20GTM150)(T)(778s).pdf.