Handling Analogical Proportions in Classical Logic and Fuzzy Logics Settings

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Abstract. Analogical proportions are statements of the form "A is to B as C is to D" which play a key role in analogical reasoning. We propose a logical encoding of analogical proportions in a propositional setting, which is then extended to different fuzzy logics. Being in an analogical proportion is viewed as a quaternary connective relating four propositional variables. Interestingly enough, the fuzzy formalizations that are thus obtained parallel numerical models of analogical proportions. Potential applications to case-based reasoning and learning are outlined.

1 Introduction

Although analogical reasoning is largely used by humans in creative thinking (e.g. [17]) or for assessing day life situations, its place w. r. t. the other forms of reasoning has remained singular. Indeed while deductive reasoning uses sound and correct inferences, the conclusions obtained by analogical reasoning are provisional in nature and are plausible at the best. Deduction, but also abduction, or induction, have received rigorous logical formalizations, while it does not seem that it is really the case for analogical reasoning. Deduction and analogy are two very different forms of reasoning: Deductive entailment is based on the inclusion of classes, while analogy parallels particular situations. The latter form of reasoning applies when the former does not, and jumps to conclusions that may be more creative since they are not implicitly contained in the premises as in deduction. Although analogical reasoning has remained much less formalized, it has been considered early in artificial intelligence, e. g. ([6], [12], [22]), and case-based reasoning [1], a special form of it, has become a subfield in itself, while more general forms of analogical reasoning continue to be investigated (e.g. in conceptual graphs structures, [18], or in logic programming settings [8],[21]).

Analogical reasoning equates the way two pairs of situations differ, by stating analogical proportions of the form "A is to B as C is to D". It expresses that A and B are similar, and differ, in the same way as C and D are similar, and differ. The name "analogical proportion" comes from a quantitative view of this correspondence as an equality of ratios between numerical features. In this paper, we are interested in looking for a logical modeling that provides a symbolic

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and qualitative representation of analogical proportion, and may be extended to fuzzy logic in order to obtain a logical graded counterpart of numerical models (defined in terms of ratios, or of differences). Then situations A, B, C and D are supposed to be described in terms of a family of properties, and to be represented by vectors of degrees of truth. Each vector component is the degree to which the corresponding property is true for the situation. In case of binary properties, the vector components belong to $\{0, 1\}$, and is equal to 1 if and only if the property holds in the considered situation.

The paper is organized in the following way. Section 2, after introducing the notations, discusses existing postulates for analogical proportions in relation with a set theoretic point of view. Section 3 proposes a classical logic representation of analogical proportions, and then studies its properties. Section 4 presents some fuzzy logic extensions of the logical modeling of analogical proportions, and compare them to numerical models. Section 5 and Section 6 point out potential applications in case-based reasoning and learning respectively.

2 Towards a Formalization of the Analogical Proportion

An analogical proportion is a statement of the form "A is to B as C is to D". This will be denoted by $(A : B :: C : D)$. In this particular form of analogy, the objects A, B, C and D usually correspond to descriptions of items under the form of objects such as sets, multisets, vectors, strings, or trees (see [20]). In the following, we are mainly interested in the basic cases where A, B, C and D may be binary values in $\{0, 1\}$, or "fuzzy values" in the unit interval $[0, 1]$, and more generally, vectors of such values (which may be used for the logical encoding of compound cases). These values can be thought in practice as degrees of truth of statements pertaining respectively to A, B, C and D. In the following if the objects A, B, C, and D are vectors having n components, i.e., $A = (a_1, \ldots, a_n)$, $\ldots$, $D = (d_1, \ldots, d_n)$, we shall say that A, B, C, and D are in analogical proportion if and only if for each component i an analogical proportion "a_i is to b_i as c_i is to d_i" holds. If there is no need to specify one particular component, we shall simply write $(a : b :: c : d)$ for stating that the 4-tuple (a, b, c, d) satisfies a relation of analogical proportion.

2.1 Postulates

We have now to specify what kind of relation an analogical proportion may mean. Intuitively speaking, we have to understand how to interpret "is to" and "as" in "A is to B as C is to D". A may be similar (or identical) to B in some respects, and differ in other respects. The way C differs from D should be the same as A differs from B, while C and D may be similar in some other respects, if we want the analogical proportion to hold. More formally, let us denote by U the features that both A and B have, by V the features possessed by A and not by B, and by W the features possessed by B and not by A, which can be symbolically written by $A = (U, V)$, and $B = (U, W)$. If C and D differ in the same way as A and B, this forces to have $C = (Z, V)$, and $D = (Z, W)$, where Z denotes the features that both C and D have. Note that U and Z may be different. This view is enough for justifying the following three postulates that date back to Aristotle's time. See, e.g. [13].

Definition 1. *An* analogical proportion *is a quaternary relation on a set X that verifies, for all A, B, C and D in X the three* postulates of analogy*:*

(ID) $(A : B :: A : B)$; (S) $(A : B :: C : D) \Leftrightarrow (C : D :: A : B)$
(CP) $(A : B :: C : D) \Leftrightarrow (A : C :: B : D)$

(ID) and (S) express reflexivity and symmetry for the comparison "as", while (CP) allows for a central permutation. These postulates are natural requirements, if we keep in mind that $A = (U, V)$, $B = (U, W)$, $C = (Z, V)$, and $D = (Z, W)$. Indeed the first two are particularly obvious; concerning the third, let us notice that V (resp. W) is the common part of A and C (resp. B and D) and when going from A to C, we leave U and get Z, as when going from B to D. The third postulate is peculiar to analogical proportions (and is reminiscent of numerical proportions). Immediate consequences of the postulates are:

(I)**:** $(A : B :: C : D) \Leftrightarrow (B : A :: D : C)$,
(EP)**:** $(A : B :: C : D) \Leftrightarrow (D : B :: C : A)$,
$(SR1)$**:** $(A : B :: C : D) \Leftrightarrow (D : C :: B : A)$,
$(SR2)$**:** $(A : B :: C : D) \Leftrightarrow (B : D :: A : C)$,
$(SR3)$**:** $(A : B :: C : D) \Leftrightarrow (C : A :: D : B)$,

where (I) allows for the inversion of the relations (obtained by applying (CP), (S) and (CP)), (EP) allows for external permutation (obtained by applying (S), (CP) and (S)), $(SR1)$, $(SR2)$ and $(SR3)$ expressing symmetries for the reading (and can be respectively obtained by (I) and (S), (CP) and (S), and (S) and (CP)). Note also that $(A : A :: B : B)$ is obtained from (ID) and (CP).

It has been noticed that starting with (A : B :: C : D), the repeated application of (S) and (CP) generate only 8 of the 24 possible permutations of the 4-element set $\{A, B, C, D\}$. Indeed, if (A : B :: C : D) is an analogical proportion, it is not expected that (B : A :: C : D), or (C : B :: A : D) be analogical proportions also. For instance, the statement "a calf is to a bull what a kitten is to a tomcat" does not mean that "a bull is to a calf what a kitten is to a tomcat", or that "a kitten is to a bull what a calf is to a tomcat", while the statement "a calf is to a kitten what a bull is to a tomcat", obtained by (CP), sounds more acceptable.

2.2 The Set Theoretic Point of View

When the objects are finite sets, they can be seen as subsets of some universal set $\mathcal{P}$. The relation "as" is simply chosen as the equality between sets. In this framework, Lepage has given in [13] the following informal definition: four subsets of $\mathcal{P}$ are in analogical proportion $(A : B :: C : D)$ if A is transformed into B and C is transformed into D by adding and deleting the same elements. For example, the four sets $A = \{t_1, t_2, t_3, t_4, \}$, $B = \{t_1, t_2, t_3, t_5\}$ and $C = \{t_1, t_4, t_6, t_7\}$, $D = \{t_1, t_5, t_6, t_7, \}$ are in analogical proportion: t_4 is deleted from A (resp. C) and t_5 is added to A (resp. C) in order to obtain B (resp. D).

More formally, a definition has been first proposed by Lepage in [13] in computational linguistics a few years ago, and further developed in [19]. We restate it in a different way here. We denote $\overline{A}$ the complementary set of A in $\mathcal{P}$ and $A - B = A \cap \overline{B}$. We notice that "$A : B$" stands for the set operation that transforms A into B by deleting the elements of $A - B$ and adding the elements of $B - A$. The analogical proportion states the identity of the operations that transform A into B and C into D. This leads to the following definition:

Definition 2. *Let A, B, C, D be subsets of a referential $\mathcal{P}$.*
$(A : B :: C : D) \Leftrightarrow (A - B = C - D)$ *and* $(B - A = D - C)$.

Definition 2 clearly satisfies the three postulates. Stroppa and Yvon [19] have given an equivalent set-theoretic characterization of the analogical proportion:

Definition 3. *Let A, B, C, D be subsets of $\mathcal{P}$. $(A : B :: C : D)$ holds if and only if there exist four subsets U, V, W and Z of $\mathcal{P}$, such that $A = U \cup V$, $B = U \cup W$, $C = Z \cup V$, $D = Z \cup W$.*

This decomposition is not unique, and the sets U, V, W, Z do not need to be disjoint. When they are, this provides a constructive description of the analogical process: X (resp. Z) is the elements that are untouched when going from A and B (resp. from C and D), while the elements in V go out, and those in W go in.

3 Proposal for a Classical Logic Embedding

Attempts at providing a logical embedding of analogical proportion at least dates back to the proposal made by a computer scientist, Klein, working in anthropology, more than twenty-five years ago in [10]. Klein used an operator (called by him ATO for "Appositional Transformation Operator") on binary truth-like tables, which is nothing but the logical equivalence connective: $a \equiv b = 1$ if $(a = b)$ and $a \equiv b = 0$ otherwise. His view amounts to define an analogical proportion semantically as a logical connective having the truth table of the logical expression $(a \equiv b) \equiv (c \equiv d)$. It partially agrees with the idea that A differs from B as C w. r. t. D, since it can be also written $(a \Delta b) \equiv (c \Delta d)$ where Δ denotes XOR. It can still be rewritten as well as $(a \wedge \neg b) \vee (\neg a \wedge b) \equiv (c \wedge \neg d) \vee (\neg c \wedge d)$.

However, this latter expression remains symmetrical, since it makes no difference between the way A differs from B and the way B differs from A. It is clearly weaker than stating the two equivalences $(a \wedge \neg b) \equiv (c \wedge \neg d)$ and $(b \wedge \neg a) \equiv (d \wedge \neg c)$ separately. This is the logical counterpart of Definition 2 given from the set theoretic point of view. Klein's view of analogy was indeed too permissive, since $(a \equiv b) \equiv (c \equiv d)$ is still equivalent to $(b \equiv a) \equiv (c \equiv d)$, thus making no difference between "A is to B" and "B is to A".

The 8 cases where Klein's expression, $(a \equiv b) \equiv (c \equiv d)$, takes truth value 1 are listed in Table 1. For the 8 other possible combinations of values of a, b, c,

Table 1. Contrasting Klein's definition of analogy with ours

	a b c d	$(a \equiv b) \equiv (c \equiv d)$	$(a : b :: c : d)$
1	1 1 1 1	1	1
2	1 1 0 0	1	1
3	1 0 1 0	1	1
4	1 0 0 1	1	0
5	0 1 1 0	1	0
6	0 1 0 1	1	1
7	0 0 1 1	1	1
8	0 0 0 0	1	1

and d that are not in Table 1, $(a \equiv b) \equiv (c \equiv d)$ has truth value 0. Cases 1, 2, 7, and 8 correspond to situations where a and b are identical as well as c and d. Cases 3 and 6 correspond to changes from a to b, and from c to d, that go in the same sense. All this fits the semantics of the analogical proportion. The two other cases, namely 4 and 5, do not fit the idea that a is to b as c is to d, since the changes from a to b and from c to d are not in the same sense. They in fact correspond to cases of maximal analogical dissimilarity, where "d is not at all to c what b is to a", but rather "c is to d what b is to a". It emphasizes the non symmetry of the relations between b and a, and between d and c.

Our definition is equivalent to stating the two equivalences $(a \wedge \neg b) \equiv (c \wedge \neg d)$ and $(b \wedge \neg a) \equiv (d \wedge \neg c)$ separately, which is coherent with the definition of analogical proportion between finite sets given at section 2.2.

3.1 Logical Expressions for the Analogical Proportion

We are now looking for logical expressions corresponding to our definition, i.e. only covering cases 1, 2, 3, 6, 7, 8 in Table 1. Viewing $(a : b :: c : d)$ as a logical connective that reflects the analogical process, it can be checked that, taking

$$(a : b :: c : d) = ((a \equiv b) \equiv (c \equiv d)) \wedge ((a \Delta b) \rightarrow (a \equiv c)) \tag{1}$$

this expression is true only for the 6 cases required in Table 1. Indeed, it expresses in its second component that a and c should be identical where a and b differs, which is a natural constraint for making sure that the change from c to d will be in the same sense as the one from a to b. There exist equivalent expressions whose structures well reflect the meaning of analogical proportion:

$$(a : b :: c : d) = ((a \equiv b) \wedge (c \equiv d)) \vee ((a \equiv c) \wedge (b \equiv d)) \tag{2}$$

$$(a : b :: c : d) = ((a \rightarrow b) \equiv (c \rightarrow d)) \wedge ((b \rightarrow a) \equiv (d \rightarrow c)) \tag{3}$$

$$(a : b :: c : d) = ((a \wedge d) \equiv (b \wedge c)) \wedge ((a \vee d) \equiv (b \vee c)) \tag{4}$$

$$(a : b :: c : d) = ((a \wedge d) \equiv (b \wedge c)) \wedge ((a \vee d) \equiv (b \vee c))^1 \tag{5}$$

[1] Pointed out by Didier Dubois to the authors.

These expressions help to understand the structure of analogical proportion. For instance, expression (3) at the logical level parallels the difference-based view of the analogical proportion, expressed by the condition $(a - b) = (c - d)$. When a and b are equal to 0 or 1, $a - b \in \{-1, 0, 1\}$. It is why expression (3), which works in $\{0, 1\}$, states the equivalences in each sense (remember that $a \to b = 1$ if $a \leq b$ and $a \to b = 0$ if $a > b$, and observe that the condition $a \leq b$ covers two situations: $a = b$ (no change) or $a < b$ (change)).

Clearly, since $a - b \in \{-1, 0, 1\}$, $(a - b)$ is not a connective, but keeps track of the sense of the change if any. It is the basis of the notion of *analogical dissimilarity* $AD(a, b, c, d)$ [15] that measures how far objects a, b, c and d are from being in an analogical proportion. $AD(a, b, c, d)$ must be equal to 0 if they are in such a relation, and positive otherwise. Required properties are: i) *Coherence with analogy*: $AD(a, b, c, d) = 0 \Leftrightarrow (a : b :: c : d)$ ii) *Symmetry of "as"*: $AD(a, b, c, d) = AD(c, d, a, b)$ iii) *Triangle inequality*: $AD(a, b, c, d) \leq AD(a, b, e, f) + AD(e, f, c, d)$ iv) *Central permutation*: $AD(a, b, c, d) = AD(a, c, b, d)$ v) *Dissymmetry of "is to"*: $AD(a, b, c, d) \neq AD(b, a, c, d)$, in general. Defining $AD(b, a, c, d) = |(a - b) - (c - d)|$ agrees with the required properties.

3.2 Some Properties

We now state some results (symbolic propositional expressions and their semantical counterparts in terms of truth degrees are denoted in the same way):

Proposition 1. $(a : b :: \neg b : \neg a) = 1$.

This looks similar to the logical equivalence between $a \to b$ and $\neg b \to \neg a$.

Proposition 2. *If* $(a \to b) = 1$ *and* $(a : b :: c : d) = 1$ *then* $(c \to d) = 1$. *Similar results hold if* $\to$ *is replaced by* $\leftarrow$, $\equiv$ *or* Δ. *It does not hold for* $\vee$ *nor* $\wedge$.

It ensures that if "A is to B as C is to D", and B is more general than A, then D should be more general than C, as expected. More generally, it expresses a form of agreement with connectives related to entailment.

Proposition 3. $(a : b :: c : d) = 1, (c : d :: e : f) = 1 \Longrightarrow (a : b :: e : f) = 1$

It expresses that transitivity holds for analogical proportion. A less obvious result is about the behavior of analogical proportion w.r.t. conjunction and disjunction.

Proposition 4. $(a \wedge b : a \wedge c :: d \wedge b : d \wedge c) = 1$ *if and only if* $(a \vee b : a \vee c :: d \vee b : d \vee c) = 1$.

It can be seen as resulting from the combination of the two universal analogical proportions $(a : a :: d : d) = 1$ and $(b : c :: b : c) = 1$.

3.3 Solving an Analogical Proportion Equation

In its basic form, analogical reasoning amounts to solving an analogical proportion equation that is supposed to hold for some characteristics of objects, the assumption being generally based on the observation that the analogical proportion already holds for other known characteristics. Solving an analogical proportion equation consists in finding x s.t. $(a : b :: c : x) = 1$. When it exists, the value of x is unique in the classical logic setting, but does not always exist.

Proposition 5. *A triple* $(a\ b\ c)$ *can be completed by* d *in such a way that* $(a : b :: c : d) = 1$ *if and only if* $((a \equiv b) \vee (a \equiv c)) = 1$.

Proposition 6. *When it exists, the unique solution of the equation* $(a : b :: c : x) = 1$ *is logically expressed by* $x = (a \equiv (b \equiv c))$.

Proposition 5 points out that a should be equivalent to b or to c, in order to get rid of the two triples $(a\ b\ c) = (1\ 0\ 0)$ and $(a\ b\ c) = (0\ 1\ 1)$ that cannot be completed analogically (see Table 1). In the set-theoretic view, a, b, c, and d stand for the membership degrees of an element in a referential $\mathcal{P}$ to A, B, C, and D respectively. Then the impossibility of $(a\ b\ c) = (1\ 0\ 0)$ and $(a\ b\ c) = (0\ 1\ 1)$ translates respectively into $A \cap (\overline{B} \cap \overline{C}) = \emptyset$ and $\overline{A} \cap (B \cap C) = \emptyset$, i.e. the logical condition for analogical completion $(a \equiv b) \vee (a \equiv c) = 1$ can be written in set terms as $(B \cap C) \subset A \subset (B \cap C)$, a condition already given in Section 3.

Proposition 6 provides a compact writing of the solution of an analogical proportion. This is the solution first suggested by Klein [10] who noticed that the repeated use of what he called the ATO operator enables him to compute the solution of analogical proportions, according to the equality $d = (c \equiv (a \equiv b))$.

Other expressions of x under the requirement of Proposition 5 exist, e.g.:

Property 1. When it exists, the unique solution of the analogical equation $(a : b :: c : x) = 1$ is logically expressed by

$$x = ((b \vee c) \wedge \neg a) \vee (b \wedge c) = (b \wedge \neg a) \vee (c \wedge \neg a) \vee (a \wedge b \wedge c)$$

$$x = (a \rightarrow (b \wedge c)) \wedge (b \vee c) = (a \rightarrow b) \wedge (a \rightarrow c) \wedge (b \vee c)$$

Both can be easily checked on a truth table. The first one is nothing but the logical counterpart of expressions recently proposed in [15] in the "set element-interpretation". Both Proposition 6 and Proposition 1 could be applied when $(a\ b\ c)$ cannot be analogically completed, i.e. when $(a \equiv b) \vee (a \equiv c) = 0$. Mind that while Proposition 6 applied to the two "undesirable cases" $(a\ b\ c) = (1\ 0\ 0)$ and $(a\ b\ c) = (0\ 1\ 1)$ yields $x = 1$ and $x = 0$ respectively, the two expressions of Proposition 1 give the converse, namely $x = 0$ and $x = 1$ respectively in these two cases. This means that the expression $x = (a \equiv (b \equiv c))$ is logically equivalent to the two expressions in Proposition 1, only under the condition $(a \equiv b) \vee (a \equiv c) = 1$ (which is equivalent to conditions $((b \wedge c) \rightarrow a) = 1$ and $(a \rightarrow (b \vee c)) = 1$). It can be seen from Proposition 1 that x is also such that $((b \wedge c) \rightarrow x = 1)$ and $(x \rightarrow (b \vee c)) = 1$.

4 Extensions to Fuzzy Logic

When moving from the binary case to the graded (or fuzzy) case where truth values now belong to the continuous interval $[0, 1]$, many choices are possible for defining the connectives, and it should be clear that some of the equivalences previously found may now fail to hold since, whatever the choices, we shall be no longer in a Boolean algebra. Here we only consider choices that seem to be especially worth of interest, due to their resemblance with numerical models.

4.1 Construction of a Fuzzy Analogical Proportion

Let us recall that in fuzzy logic [11] there are three main choices for the conjunction, namely $a \wedge b = min(a, b)$, $a \wedge b = a \cdot b$, or $a \wedge b = max(0, a + b - 1)$, associated with the three disjunctions $a \vee b = max(a, b)$, $a \vee b = a + b - a \cdot b$, or $a \vee b = min(1, a + b)$ respectively. Then there are two main ways for defining implications, either as $a \to b = \neg a \vee b$, or by residuation: $a \to b = sup(x | a \wedge x \leq b)$.

It leads to distinct connectives for the first two pairs of conjunction/disjunction: $a \to b = max(1 - a, b)$ (Dienes implication) and $a \to b = 1$ if $a \leq b$ and $a \to b = b$ if $a > b$ (Gödel implication) for min/max, $a \to b = 1 - a + a \cdot b$ (Reinchenbach implication) and $a \to b = min(1, b/a)$ if $a > 0$, and $a \to b = 1$ if $a = 0$ (Goguen implication) with the second pair (using $\neg a = 1 - a$). For the last pair of conjunction/disjunction, Lukasiewicz implication $a \to b = min(1, 1 - a + b)$ is obtained in both cases.

The equivalence connective associated to Dienes implication is $(a \equiv b) = min(max(1 - a, b), max(1 - b, a)) = max(min(a, b), min(1 - a, 1 - b))$, to Gödel implication is $(a \equiv b) = 1$ if $a = b$, and $(a \equiv b) = min(a, b)$ otherwise (in a crisp version, one may take $(a \equiv b) = 0$ if $a \neq b$). Using min conjunction and Lukasiewicz implication, one gets $(a \equiv b) = min(min(1, 1 - a + b), min(1, 1 - b + a)) = 1 - |a - b|$. Using min or product conjunction and Goguen implication, one gets $(a \equiv b) = min(1, b/a, a/b) = min(b/a, a/b) = min(1, b/a) \cdot min(1, a/b)$ for $a \neq 0$, $b \neq 0$ (if $a = 0$ and $b \neq 0$, $(a \equiv b) = 0$; if $a = 0$ and $b = 0$, $(a \equiv b) = 1$).

In the following, we only discuss the fuzzification of equation 3 that clearly states the identity of the differences, using successively the two following choices:

1. $a \wedge b = min(a, b)$; $a \to b = min(1, 1 - a + b)$; $a \equiv b = 1 - |a - b|$
2. $a \wedge b = a \cdot b; a \to b = max(1, b/a); a \equiv b = min(b/a, a/b)$

It leads to the two formulas below for the value of the fuzzy analogical proportion

$$min \begin{cases} 1 - |min(1, 1 - a + b) - min(1, 1 - c + d)| \\ 1 - |min(1, 1 + a - b) - min(1, 1 + c - d)| \end{cases}$$

$$min\left(\frac{max(1, \frac{b}{a})}{max(1, \frac{d}{c})}, \frac{max(1, \frac{d}{c})}{max(1, \frac{b}{a})}\right) . \, min\left(\frac{max(1, \frac{a}{b})}{max(1, \frac{c}{d})}, \frac{max(1, \frac{c}{d})}{max(1, \frac{a}{b})}\right).$$

The first formula yields 1 iff $a - b :: c - d$, the values of a, b c and d being in the unit interval. The second formula yields 1 iff $a/b :: c/d$. Both are also consistent with the logical analogy defined above.

4.2 Coherence with Numerical Analogy

When defining an analogical relation between four fuzzy values, first one has to make sure that it remains consistent with the classical logical definition when fuzzy values reach the bounds of the interval $[0,1]$, as discussed in subsection 4.1. But it is also important to maintain a link with the definition of analogy between real numbers. Several definitions have been proposed for analogical proportions between real numbers, in particular: i) the additive analogy: $(a : b :: c : d) \Leftrightarrow (a + d = b + c)$, ii) the multiplicative analogy: $(a : b :: c : d) \Leftrightarrow (ad = bc)$.

The first fuzzification is rather coherent with the numerical additive case. We can analyse their difference in taking s and t as small positive numbers. We get in the fuzzy case $(a : a + s :: c : c - t) = 1 - min(s, t)$. In the numerical case, we would have obtained $(a : a+s :: c : c-t) = 1-AD(a, a+s, c, c-t) = 1-(s+t)$, still using the same definition of analogical dissimilarity, now applied to fuzzy values. However, note that in the fuzzy case we deal with truth values, while in the numerical case we deal with attribute values!

It is important to note here that the fuzzy counterpart of Proposition 1 cannot be straightforwardly applied for finding the solution of an analogical proportion in the graded case. Proper equivalent expressions have to be found. For instance, if we use $((b \to a) \to c)$ if $a \to b = 1$, and $\neg(c \to \neg(a \to b))$ if $b \to a = 1$, which is indeed equivalent to $(a \equiv (b \equiv c))$, we shall obtain with Lukasiewicz implication, $min(1, c+(b-a))$ if $a \le b$, and $max(0, c-(a-b))$ if $a \ge b$, which are normalized versions of the solution of the numerical equation $a - b = c - x$.

5 Analogical Proportion-Based Reasoning

We have already noticed that, for any pair of propositions p and q it holds that $(p : q :: p : q) = 1$ (consistently with the 1st postulate), and $(p : q :: \neg q : \neg p) = 1$. This shows a form of agreement between the analogical proportion and the *modus ponens*, and (which is less expected), with the *modus tollens*. Indeed from a non specified, hypothetical relation between p and q, denoted by $p : q$ and from p, one concludes q, since it can be checked that $x \equiv q$ is the only solution of the equation $(p : q :: p : x) = 1$. Similarly, from $(p : q :: \neg q : x) = 1$, one gets $x \equiv \neg p$. But $(p : q :: \neg p : \neg q) = 1$ does not hold (while it holds with Klein's definition).

An object, a situation, a problem may be described in terms of sets of features (resp. properties) that are present (resp. true) or absent (resp. false) in the binary case, or more generally that are present (resp. true) to some extent in the fuzzy case. In that respect, the classical logic equivalence $(p : q :: r : s) \equiv (p : q :: \neg r : \neg s)$ insures the neutrality of the encoding whatever the convention used for stating what is present and what is absent ($\neg p$ present is the same as p absent). It holds in the fuzzy case under the form $(a : b :: c : d) = (1-a : 1-b :: 1-c : 1-d)$ for degrees of truth, using Lukasiewicz implication, since $(1 - a) \to (1 - b) = min(1, 1-(1-a)+1-b) = min(1, 1-b+a) = b \to a$. It could be also preserved using a symmetrized form of Goguen implication: $min(1, b/a, (1 - a)/(1 - b))$.

The featured view can be applied to logical formulas themselves, e.g. "$\neg p \wedge q$ is to $p \wedge q$ as $\neg p \wedge \neg q$ is to $p \wedge \neg q$" can be encoded as the 2-component analogical proportion ((01) : (11) :: (00) : (10)) where 0/1 stands for the negative/positive presence of p and q in the clauses. This agrees with the use of Hamming distance for computing the amount of change from a formula to another. It suggests that analogical reasoning could be related to revision operations based on the idea of minimal change, as recently proposed in case-based reasoning [14].

Towards case-based reasoning. Let us illustrate a more direct and practical use of the featured view. Assume a base of cases describing houses to let. In the example, we consider four features: nature (villa (1) or apartment (0)), air conditioning (equipped (1) or not (0)), price (cheap (1) or expensive (0)), tax to pay (yes (1) or no (0)). Assume we know the three cases:

$A =$ (villa, equip., expen., tax) $= (1, 1, 0, 1)$
$B =$ (villa, not-eq., cheap, tax) $= (1, 0, 1, 1)$
$C =$ (apart., equip., expen., tax) $= (0, 1, 0, 1)$

Assume now a fourth house described by $D =$ (apart., not-eq., x, y) $= (0, 0, x, y)$ for which one has to guess a price and if there is a tax to pay. After checking that for the first two components we have an analogical proportion between A, B, C, and D (indeed in terms of truth values we have $(1 : 1 :: 0 : 0) = 1$, and $(1 : 0 :: 1 : 0) = 1$), one may assume that it also holds for the two other components and the unique solution of the equations $(0 : 1 :: 0 : x) = 1$ and $(1 : 1 :: 1 : y) = 1$ is $x = 1$ and $y = 1$, which means "cheap" and "tax to pay".

A refined version of the example can be described in a graded manner as e.g., $A = (1, 1, .2, .9)$, $B = (1, 0, .8, .8)$, $C = (0, 1, .3, .6)$, where the degrees respectively stand for the extent to which the price is cheap and the tax is high. Applying a difference-based approach, using Lukasiewicz implication, one gets $D = (0, 0, .9, .5)$, i.e. D should be quite cheap (.9) with a not too high tax (.5).

This example suggests that one may apply such an approach to case-based reasoning. Then A, B and C are three problems with their respective solutions that are identical according to some features and that differ with respect to other features. Both problems and solutions (e.g., a disease and a medical treatment for it) are described in terms of feature values, D is a new problem for which one looks for a tentative solution. Then the solution for D will be computed as an adapted version of those for cases A, B and C, from the differences and similarities between them, as outlined in the above example where the role of the "problem" was played by the nature and the air conditioning availability of the house, and the problem was to guess the price of house D and if there is a tax to pay. It is clear that in general there may exist in a repertory of cases several triples A_i, B_i, C_i, from which the solution for D can be computed analogically as just explained. Then it would lead to aggregate the different solutions *disjunctively* into an imprecise solution, as already done in the simplest type of case-based decision [4] and in case-based reasoning [5]. Indeed, in these approaches, a solution is proposed for a new case on the basis of a formal principle that states that "the more similar two cases are in some respects, the more guaranteed the possibility that they are similar in the other respect for which a solution is looked for for the new case". Then, since the new case may resemble several cases, in the repertory of cases, which are different with respect to the feature to predict (or the solution to choose) in the new case, a disjunctive combination should take place for aggregating the solutions found. We are here in a similar situation, since several triples A_i, B_i, C_i may be found in analogical proportion with D. The details of the procedure are left for further research.

6 Analogical Proportion-Based Learning

Let $\mathcal{S} = \{(x, \omega(x))\}$ be a finite set of training examples, where x is the description of an example as a binary vector and $\omega(x)$ its label in a finite set. Given the binary vector y of a new pattern, we want to assign a label $\omega(y)$ to y, based only on knowledge in $\mathcal{S}$. Finding $\omega(y)$ amounts to the *inductive learning* of a classification rule from examples (e.g. [16]). The nearest neighbor (*1-nn*) method, the most simple lazy learning technique, merely finds in $\mathcal{S}$ *one* description $x^\star$ which minimizes some distance to y and hypothesizes $\omega(x^\star)$, the label of $x^\star$, for the label of y. Moving one step further, learning from *one* analogical proportion would consist in searching in $\mathcal{S}$ for one triple $(x^\star, z^\star, t^\star)$ such that $x^\star : z^\star :: t^\star : y$ or $AD(x^\star, z^\star, t^\star, y)$ is minimal and would predict for y the label $\hat{\omega}(y)$ solution of equation $\omega(x^\star) : \omega(z^\star) :: \omega(t^\star) : \hat{\omega}(y)$.

The *1-nn* method is easily extended to examine a larger neighbourhood, resulting in the *k-nn* method: in $\mathcal{S}$, find the k descriptions which minimize the distance to y and let vote the k corresponding labels to choose the winner as the label of y. Extending the learning from one analogical proportion to k ones can be designed similarly. First, we define an analogical dissimilarity AD between two binary vectors. It can straightforwardly be defined as the sum of the AD of their components (see section 3.1). Secondly, we follow the following procedure:

- Consider only trivial equations on classes, e.g. $(\omega_1 : \omega_2 : \omega_1 : \hat{\omega}(y))$, which produces the solution $\hat{\omega}(y) = \omega_2$.
- Use a weighting of the binary attributes (the weights are learned from $\mathcal{S}$).
- For an integer k, use all the triples in $\mathcal{S}$ that make AD less than k for y.

To experiment the efficiency of this technique (*learning from k analogical proportions*), we have made experiments on several classical datasets with binary or nominal attributes, the latter being straightforwardly binarised. The results, given in [2], show that it gives excellent results on all the data bases, including those with missing data or composed with more than two classes. This method can be seen as an extension of the k-nn method. It also suggests possible links with fuzzy instance-based learning [9], which also extends the k-nn method.

7 Conclusion

This paper is a first attempt towards a logical formalization of analogical reasoning based on analogical proportions. It offers a unified treatment of symbolic and numerical analogical proportions, thanks to the extension of the proposed classical logic formulation to different fuzzy logics keeping the additive or the multiplicative flavors of numerical modeling. Beyond the theoretical interest of such logical encodings, the paper has indicated how reasoning from several cases and learning can benefit from the resolution of analogical proportions. Another direction for further research would be to discuss the relation between fuzzy set-based approximate reasoning and analogical reasoning, already studied in [3] with another approach. In the long range, it would be also of interest to develop cognitive validation tests in order to study if the predictions that can be obtained with the approach are in agreement with human reasoning (as in e.g.[7]).

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