Memory size in the Prisoner's Dilemma

Nikoleta E. Glynatsi   Vincent Knight

Abstract

In this manuscript we build upon a framework provided in 1989 for the study of these strategies and identify the best responses of memory one players. The aim of this work is to show the limitations of memory one strategies in multi-opponent interactions. A number of theoretic results are presented.

1 Introduction

The Prisoner's Dilemma (PD) is a two player person game used in understanding the evolution of co-operative behaviour. Each player can choose between cooperation (C) and defection (D). The decisions are made simultaneously and independently. The normal form representation of the game is given by:

$$S_p = \begin{pmatrix} R & S \\ T & P \end{pmatrix} \quad S_q = \begin{pmatrix} R & T \\ S & P \end{pmatrix} \tag{1}$$

where S_p represents the utilities of the first player and S_q the utilities of the second player. The payoffs, (R, P, S, T), are constrained by equations (2) and (3). Constrain (2) ensures that defection dominates cooperation and constrain (3) ensures that there is a dilemma. Because the sum of the utilities for both players is better when both choose cooperation. The most common values used in the literature are $(3, 1, 0, 5)$ [2].

$$T > R > P > S \tag{2}$$

$$2R > T + S \tag{3}$$

The PD is a one shot game, however it is commonly studied in a manner where the history of the interactions matters. The repeated form of the game is called the Iterated Prisoner's Dilemma (IPD) and in the 1980s following the work of [3, 4] it attracted the attention of the scientific community.

In [3] a computer tournament of the IPD was performed. A tournament is a series of rounds of the PD between pairs of strategies. The topology commonly used, [3, 4], is that of a round robin where all contestants compete against each other. The winner of these tournaments was decided on the average score and not in the number of wins.

These tournaments were the milestones of an era which to today is using computer tournaments to explore the robustness of strategies of IPD. The robustness can also be checked through evolutionary process [10]. However, this aspect will not be considered here, instead the focus is on performance in tournaments.

In Axelrod's original tournaments [3, 4], strategies were allowed access to the history and in the first tournament they also knew the number of total turns in each interaction. The history included the previous moves of both the player and the opponent. How many turns of history that a strategy would use, the memory size, was left to the creator of the strategy to decide. For example the winning strategy of the first tournaments, Tit for Tat was a strategy that made use of the previous move of the opponent only. Tit for Tat is a strategy that starts by cooperating and then mimics the previous action of it's opponent. Strategies like Tit for Tat are called memory one strategies. A framework for studying memory one strategies was introduced in [8] and further used in [7, 9].

In [11] Press and Dyson, introduced a new set of memory one strategies called zero determinant (ZD) strategies. The ZD strategies, manage to force a linear relationship between the score of the strategy and the opponent. Press and Dyson, prove their concept of the ZD strategies and claim that a ZD strategy can outperform any given opponent.

The ZD strategies have tracked a lot of attention. It was stated that "Press and Dyson have fundamentally changed the viewpoint on the Prisoner's Dilemma" [12]. In [12], the Axelrod's tournament have been re-run including ZD strategies and a new set of ZD strategies the Generous ZD. Even so, ZD and memory one strategies have also received criticism. In [6], the 'memory of a strategy does not matter' statement was questioned. A set of more complex strategies, strategies that take in account the entire history set of the game, were trained and proven to be more robust than ZD strategies.

2 Problem

The purpose of this work is to consider a given memory one strategy in a similar fashion to [11]. However whilst [11] found a way for a player to manipulate an opponent, this work will consider an optimisation approach to identify the best response to that opponent. In essence the aim is to produce a compact method of identifying the best memory one strategy against a given opponent.

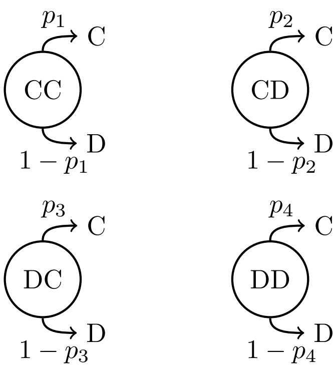
The second part of this manuscript we explore the limitation of the best response memory one strategies by comparing them to more complex strategies with a larger memory.

2.1 Background

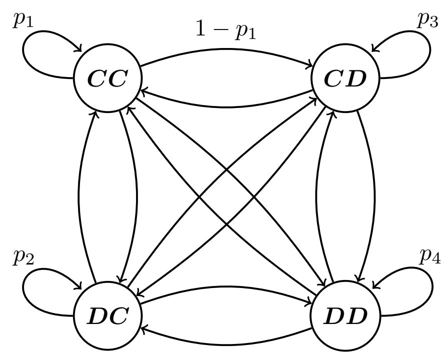
In this manuscript we explore the robustness of memory one strategies. A memory one strategy is defined as a strategy that decides it's action in turn m based on what occurred in turn $m-1$. If a strategy is concerned with only the outcome of a single turn then there are four possible 'states' the strategy could be in. These are CC, CD, DC, CC. A memory one strategy is denoted by the probabilities of cooperating after each of these states, $p = p_1, p_2, p_3, p_4 \in \mathbb{R}^4_{[0,1]}$. A diagrammatic representation of such as strategy is given in Figure 1a.

In [9] a framework was introduced to study the interactions of memory one strategies modelled as a stochastic process, where the players move from one of the states CC, CD, DC, CC to another. More specifically, it can be modelled by the use of a Markov process of four states, shown by Figure 1b.

The transition matrix of the markov chain in Figure 1b is defined as M and is given by,



(a) Diagrammatic representation of a memory one strategy.



(b) Markov chain on a PD game.

$$M = \begin{bmatrix} p_1q_1 & p_1(-q_1+1) & q_1(-p_1+1) & (-p_1+1)(-q_1+1) \\ p_2q_3 & p_2(-q_3+1) & q_3(-p_2+1) & (-p_2+1)(-q_3+1) \\ p_3q_2 & p_3(-q_2+1) & q_2(-p_3+1) & (-p_3+1)(-q_2+1) \\ p_4q_4 & p_4(-q_4+1) & q_4(-p_4+1) & (-p_4+1)(-q_4+1) \end{bmatrix}. \tag{4}$$

Let the vector of the stationary probabilities of M be defined as v. Vector v are given in the Appendix. The scores of each player can be retrieved by multiplying the probabilities of each state, at the stationary state, with the equivalent payoff. Thus, the utility for player p against q, denoted as $u_q(p)$, is defined by,

$$u_q(p) = v \times S_p. \tag{5}$$

2.2 Utility

The analytical formulation gives the advantage of time. That is because the payoffs of a match between two opponents are now retrievable without simulating the actual match itself.

Note though that $u_q(p)$ is a function of 4 variables which is also affected by the transition probabilities of the opponent q. The first theoretical result that we introduce in this work is a compact way of writing $u_q(p)$. This is given by the Theorem 1.

Theorem 1 *For a given memory one strategy $p \in \mathbb{R}^4_{[0,1]}$ playing another memory one strategy $q \in \mathbb{R}^4_{[0,1]}$, the utility of the player $u_q(p)$ can be re written as a ratio of two quadratic forms:*

$$u_q(p) = \frac{\frac{1}{2}p^TQp + c^Tp + a}{\frac{1}{2}p^T\bar{Q}p + \bar{c}^Tp + \bar{a}}, \tag{6}$$

where $Q, \bar{Q}$ are matrices of 4×4 defined with the transition probabilities of the opponent's transition probabilities q_1, q_2, q_3, q_4.

$$Q = \begin{bmatrix} 0 & -(q_1-q_3)(q_2-5q_4-1) & q_3(q_1-q_2) & -5q_3(q_1-q_4) \\ -(q_1-q_3)(q_2-5q_4-1) & 0 & (q_2-q_3)(q_1-3q_4-1) & (q_3-q_4)(5q_1-3q_2-2) \\ q_3(q_1-q_2) & (q_2-q_3)(q_1-3q_4-1) & 0 & 3q_3(q_2-q_4) \\ -5q_3(q_1-q_4) & (q_3-q_4)(5q_1-3q_2-2) & 3q_3(q_2-q_4) & 0 \end{bmatrix}, \quad (7)$$

$$\bar{Q} = \begin{bmatrix} 0 & -(q_1-q_3)(q_2-q_4-1) & (q_1-q_2)(q_3-q_4) & (q_1-q_4)(q_2-q_3-1) \\ -(q_1-q_3)(q_2-q_4-1) & 0 & (q_2-q_3)(q_1-q_4-1) & (q_1-q_2)(q_3-q_4) \\ (q_1-q_2)(q_3-q_4) & (q_2-q_3)(q_1-q_4-1) & 0 & -(q_2-q_4)(q_1-q_3-1) \\ (q_1-q_4)(q_2-q_3-1) & (q_1-q_2)(q_3-q_4) & -(q_2-q_4)(q_1-q_3-1) & 0 \end{bmatrix}. \quad (8)$$

c and $\bar{c}$, are 4×1 vectors defined by the transition rates q_1, q_2, q_3, q_4.

$$c = \begin{bmatrix} q_1(q_2-5q_4-1) \\ -(q_3-1)(q_2-5q_4-1) \\ -q_1q_2+q_2q_3+3q_2q_4+q_2-q_3 \\ 5q_1q_4-3q_2q_4-5q_3q_4+5q_3-2q_4 \end{bmatrix}, \quad (9)$$

$$\bar{c} = \begin{bmatrix} q_1(q_2-q_4-1) \\ -(q_3-1)(q_2-q_4-1) \\ -q_1q_2+q_2q_3+q_2-q_3+q_4 \\ q_1q_4-q_2-q_3q_4+q_3-q_4+1 \end{bmatrix}. \quad (10)$$

Lastly, $a = -q_2 + 5q_4 + 1$ and $\bar{a} = -q_2 + q_4 + 1$.

2.3 Validation

In this section we validate the formulation of Theorem 1 using numerical experiments. All the simulated results of this work are done using [1] which is an open research framework for the study of the IPD. This package is described in [5].

To validate the formulation of $u_q(p)$ several memory one players were matched against 20 opponents.The simulated value of $u_q(p)$ has been calculated using [1] and the theoretical by substituting in equation (6).

In Figure 2, both the simulated and the theoretical value of $u_q(p)$, against each opponent, are plotted for three different memory one strategies. Figure 2 indicates that the formulation of $u_q(p)$ as a quadratic ratio successfully captures the simulated behaviour.

3 Best responses Analytically

In the introduction a question was raised: which memory one strategy is the **best response** against another memory one? This will be considered as an optimisation problem, where a memory one strategy p wants to optimise it's utility $u_q(p)$ against an opponent q. The decision variable is the vector p and the solitary constrains $p \in \mathbb{R}^4_{[0,1]}$. The optimisation problem is given by (11).

$$\begin{aligned} \max_p : \quad & \frac{\frac{1}{2}pQp^T + c^Tp + a}{\frac{1}{2}p\bar{Q}p^T + \bar{c}^Tp + \bar{a}} \\ \text{such that} : \quad & p \in \mathbb{R}^4_{[0,1]}. \end{aligned} \quad (11)$$

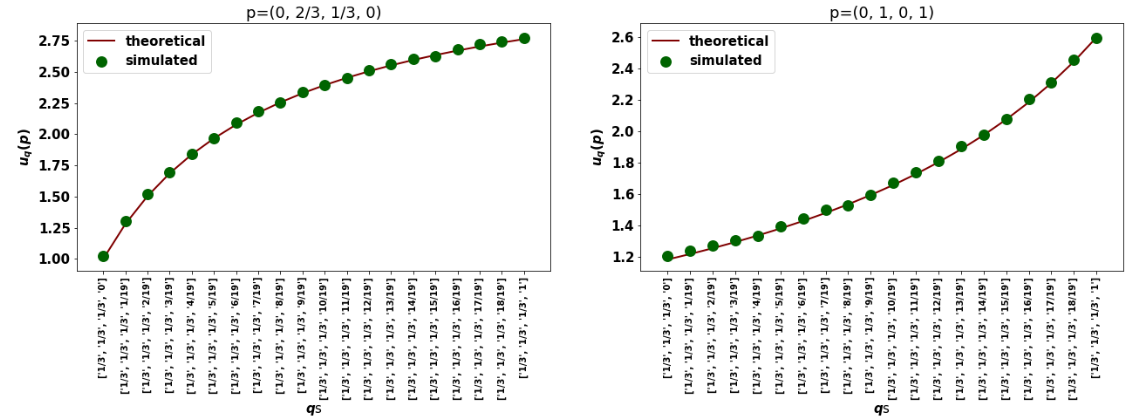


Figure 2: Differences between simulated and analytical results.

4 Best responses Numerically

5 Limitation of memory

6 Stability of defection

7 Numerical experiments

References

[1] The Axelrod project developers . Axelrod: ¡release title¿, April 2016.

[2] R Axelrod and WD Hamilton. The evolution of cooperation. *Science*, 211(4489):1390–1396, 1981.

[3] Robert Axelrod. Effective choice in the prisoner's dilemma. *The Journal of Conflict Resolution*, 24(1):3–25, 1980.

[4] Robert Axelrod. More effective choice in the prisoner's dilemma. *The Journal of Conflict Resolution*, 24(3):379–403, 1980.

[5] Vincent Knight, Owen Campbell, Marc Harper, Karol Langner, James Campbell, Thomas Campbell, Alex Carney, Martin Chorley, Cameron Davidson-Pilon, Kristian Glass, Tomáš Ehrlich, Martin Jones, Georgios Koutsovoulos, Holly Tibble, Müller Jochen, Geraint Palmer, Paul Slavin, Timothy Standen, Luis Visintini, and Karl Molden. An open reproducible framework for the study of the iterated prisoner's dilemma. 1(1), 2016.

[6] Christopher Lee, Marc Harper, and Dashiell Fryer. The art of war: Beyond memory-one strategies in population games. *PLOS ONE*, 10(3):1–16, 03 2015.

[7] Frederick A Matsen and Martin A Nowak. Win–stay, lose–shift in language learning from peers. *Proceedings of the National Academy of Sciences*, 101(52):18053–18057, 2004.

[8] Martin Nowak and Karl Sigmund. Game-dynamical aspects of the prisoner's dilemma. *Applied Mathematics and Computation*, 30(3):191–213, 1989.

[9] Martin Nowak and Karl Sigmund. The evolution of stochastic strategies in the prisoner's dilemma. *Acta Applicandae Mathematicae*, 20(3):247–265, 1990.

[10] Martin A Nowak. *Evolutionary Dynamics: Exploring the Equations of Life.* Cambridge: Harvard University Press.

[11] William H. Press and Freeman J. Dyson. Iterated prisoners dilemma contains strategies that dominate any evolutionary opponent. *Proceedings of the National Academy of Sciences*, 109(26):10409–10413, 2012.

[12] Alexander J. Stewart and Joshua B. Plotkin. Extortion and cooperation in the prisoners dilemma. *Proceedings of the National Academy of Sciences*, 109(26):10134–10135, 2012.