Supplementary Information: A theory of mind: Best responses to memory-one strategies. The limitations of extortion and restricted memory

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1 Theorem 1 Proof

The utility of a memory one player p against an opponent q, $u_q(p)$, can be written as a ratio of two quadratic forms on R^4.

Proof. It was discussed that $u_q(p)$ it is the product of the steady state vector v and the PD payoffs,

$$u_q(p) = v \cdot (R, S, T, P).$$

The steady state vector which is the solution to $vM = v$ is given by

$$v = \Bigg[\frac{p_2p_3(q_2q_4 - q_3q_4) + p_2p_4(q_2q_3 - q_2q_4 - q_3 + q_4) + p_3p_4(-q_2q_3 + q_3q_4) - p_3q_2q_4 + p_4q_4(q_2 - 1)}{\bar{v}},$$
$$\frac{p_1p_3(q_1q_4 - q_2q_4) + p_1p_4(-q_1q_2 + q_1 + q_2q_4 - q_4) + p_3p_4(q_1q_2 - q_1q_4 - q_2 + q_4) + p_3q_4(q_2 - 1) - p_4q_2(q_4 + 1) + p_4(q_4 - 1)}{\bar{v}},$$
$$\frac{-p_1p_2(q_1q_4 - q_3q_4) - p_1p_4(-q_1q_3 + q_3q_4) + p_1q_1q_4 - p_2p_4(q_1q_3 - q_1q_4 - q_3 + q_4) - p_2q_4(q_3 + 1) - p_4q_4(q_1 + q_3) - p_4(q_3 + q_4) - q_4}{\bar{v}},$$
$$\frac{p_1p_2(q_1q_2 - q_1 - q_2q_3 + q_3) + p_1p_3(-q_1q_3 + q_2q_3) - p_1q_1(q_2 + 1) + p_2p_3(-q_1q_2 + q_1q_3 + q_2 - q_3) + p_2(q_3q_2 - q_2 - q_3 - 1) + p_3(q_1q_2 - q_3q_2 - q_2 - q_3) + q_2 - 1}{\bar{v}}\Bigg],$$

where,

$$\begin{aligned}\bar{v} = \; & p_1p_2(q_1q_2 - q_1q_4 - q_1 - q_2q_3 + q_3q_4 + q_3) - p_1p_3(q_1q_3 - q_1q_4 - q_2q_3 + q_2q_4) - p_1p_4(q_1q_2 - q_1q_3 - q_1 - q_2q_4 + q_3q_4 + q_4) - \\ & p_1q_1(q_2 + q_4 + 1) + p_2p_3(-q_1q_2 + q_1q_3 + q_2q_4 + q_2 - q_3q_4 - q_3) + p_2p_4(-q_1q_3 + q_1q_4 + q_2q_3 - q_2q_4) + p_2q_2(q_3 - 1) - p_2q_3(q_4 - 1) + \\ & p_2(q_4 + 1) + p_3p_4(q_1q_2 - q_1q_4 - q_2q_3 - q_2 + q_3q_4 + q_4) + p_3q_2q_1(-p_3 - 1) + p_3(q_3 - q_4) - p_4(q_1q_4 + q_2 + q_3q_4 - q_3 + q_4 - 1) + \\ & q_2 - q_4 - 1\end{aligned}$$

The dot product of $v \cdot (R, S, T, P)$ gives,

$$
\begin{aligned}
u_q(p) = & \frac{R\left(p_2p_3(q_2q_4 - q_3q_4) + p_2p_4(q_2q_3 - q_2q_4 - q_3 + q_4) + p_3p_4(-q_2q_3 + q_3q_4) - p_3q_2q_4 + p_4q_4(q_2 - 1)\right)}{\bar{v}} + \\
& \frac{S\left(p_1p_3(q_1q_4 - q_2q_4) + p_1p_4(-q_1q_2 + q_1 + q_2q_4 - q_4) + p_3p_4(q_1q_2 - q_1q_4 - q_2 + q_4) + p_3q_4(q_2 - 1) - p_4q_2(q_4 + 1) + p_4(q_4 - 1)\right)}{\bar{v}} + \\
& \frac{T\left(-p_1p_2(q_1q_4 - q_3q_4) - p_1p_4(-q_1q_3 + q_3q_4) + p_1q_1q_4 - p_2p_4(q_1q_3 - q_1q_4 - q_3 + q_4) - p_2q_4(q_3 + 1) - p_4q_4(q_1 + q_3) - p_4(q_3 + q_4) - q_4\right)}{\bar{v}} + \\
& \frac{P\left(p_1(p_2(q_1q_2 - q_1 - q_2q_3 + q_3) + p_3(-q_1q_3 + q_2q_3) - q_1(q_2 + 1)) + p_2p_3((-q_1q_2 + q_1q_3 + q_2 - q_3) + (q_3q_2 - q_2 - q_3 - 1))\right)}{\bar{v}} + \\
& \frac{P\left(p_3(q_1q_2 - q_3q_2 - q_2 - q_3) + q_2 - 1\right)}{\bar{v}} \implies
\end{aligned}
$$

$$
u_q(p) = \left(\frac{\begin{array}{c} -p_1p_2(q_1 - q_3)(Pq_2 - P - Tq_4) + p_1p_3(q_1 - q_2)(Pq_3 - Sq_4) + p_1p_4(q_1 - q_4)(Sq_2 - S - Tq_3) + p_2p_3(q_2 - q_3)(Pq_1 - P - Rq_4) - \\ p_2p_4(q_3 - q_4)(Rq_2 - R - Tq_1 + T) + p_3p_4(q_2 - q_4)(Rq_3 - Sq_1 + S) + p_1q_1(Pq_2 - P - Tq_4) - p_2(q_3 - 1)(Pq_2 - P - Tq_4) + \\ p_3(-Pq_1q_2 + Pq_2q_3 + Pq_2 - Pq_3 + Rq_2q_4 - Sq_2q_4 + Sq_4) + p_4(-Rq_2q_4 + Rq_4 + Sq_2q_4 - Sq_2 - Sq_4 + S + Tq_1q_4 - Tq_3q_4 + Tq_3 - Tq_4) \\ -Pq_2 + P + Tq_4 \end{array}}{\begin{array}{c} p_1p_2(q_1q_2 - q_1q_4 - q_1 - q_2q_3 + q_3q_4 + q_3) + p_1p_3(-q_1q_3 + q_1q_4 + q_2q_3 - q_2q_4) + p_1p_4(-q_1q_2 + q_1q_3 + q_1 + q_2q_4 - q_3q_4 - q_4) + \\ p_2p_3(-q_1q_2 + q_1q_3 + q_2q_4 + q_2 - q_3q_4 - q_3) + p_2p_4(-q_1q_3 + q_1q_4 + q_2q_3 - q_2q_4) + p_3p_4(q_1q_2 - q_1q_4 - q_2q_3 - q_2 + q_3q_4 + q_4) + \\ p_1(-q_1q_2 + q_1q_4 + q_1) + p_2(q_2q_3 - q_2 - q_3q_4 - q_3 + q_4 + 1) + p_3(q_1q_2 - q_2q_3 - q_2 + q_3 - q_4) + p_4(-q_1q_4 + q_2 + q_3q_4 - q_3 + q_4 - 1) + \\ q_2 - q_4 - 1 \end{array}} \right).
$$

Let us consider the numerator of $u_q(p)$. The cross product terms p_ip_j are given by,

$$
\begin{gathered}
-p_1p_2(q_1 - q_3)(Pq_2 - P - Tq_4) + p_1p_3(q_1 - q_2)(Pq_3 - Sq_4) + p_1p_4(q_1 - q_4)(Sq_2 - S - Tq_3) + \\
p_2p_3(q_2 - q_3)(Pq_1 - P - Rq_4) - p_2p_4(q_3 - q_4)(Rq_2 - R - Tq_1 + T) + p_3p_4(q_2 - q_4)(Rq_3 - Sq_1 + S)
\end{gathered}
$$

This can be re written in a matrix format given by Eq. 1.

$$
(p_1, p_2, p_3, p_4)\frac{1}{2}\begin{bmatrix} 0 & -(q_1 - q_3)(Pq_2 - P - Tq_4) & (q_1 - q_2)(Pq_3 - Sq_4) & (q_1 - q_4)(Sq_2 - S - Tq_3) \\ -(q_1 - q_3)(Pq_2 - P - Tq_4) & 0 & (q_2 - q_3)(Pq_1 - P - Rq_4) & -(q_3 - q_4)(Rq_2 - R - Tq_1 + T) \\ (q_1 - q_2)(Pq_3 - Sq_4) & (q_2 - q_3)(Pq_1 - P - Rq_4) & 0 & (q_2 - q_4)(Rq_3 - Sq_1 + S) \\ (q_1 - q_4)(Sq_2 - S - Tq_3) & -(q_3 - q_4)(Rq_2 - R - Tq_1 + T) & (q_2 - q_4)(Rq_3 - Sq_1 + S) & 0 \end{bmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} \tag{1}
$$

Similarly, the linear terms are given by,

$$
\begin{gathered}
p_1q_1(Pq_2 - P - Tq_4) - p_2(q_3 - 1)(Pq_2 - P - Tq_4) + p_3(-Pq_1q_2 + Pq_2q_3 + Pq_2 - Pq_3 + Rq_2q_4 - Sq_2q_4 + Sq_4) + \\
p_4(-Rq_2q_4 + Rq_4 + Sq_2q_4 - Sq_2 - Sq_4 + S + Tq_1q_4 - Tq_3q_4 + Tq_3 - Tq_4)
\end{gathered}
$$

and the expression can be written using a matrix format as Eq. 2.

$$
(p_1, p_2, p_3, p_4)\begin{bmatrix} q_1(Pq_2 - P - Tq_4) \\ -(q_3 - 1)(Pq_2 - P - Tq_4) \\ -Pq_1q_2 + Pq_2q_3 + Pq_2 - Pq_3 + Rq_2q_4 - Sq_2q_4 + Sq_4 \\ -Rq_2q_4 + Rq_4 + Sq_2q_4 - Sq_2 - Sq_4 + S + Tq_1q_4 - Tq_3q_4 + Tq_3 - Tq_4 \end{bmatrix} \tag{2}
$$

Finally, the constant term of the numerator, which is obtained by substituting $p = (0, 0, 0, 0)$, is given by Eq. 3.

$$-Pq_2 + P + Tq_4 \tag{3}$$

Combining Eq. 1, Eq. 2 and Eq. 3 gives that the numerator of $u_q(p)$ can be written as,

$$\frac{1}{2}p\begin{bmatrix} 0 & -(q_1-q_3)(Pq_2-P-Tq_4) & (q_1-q_2)(Pq_3-Sq_4) & (q_1-q_4)(Sq_2-S-Tq_3) \\ -(q_1-q_3)(Pq_2-P-Tq_4) & 0 & (q_2-q_3)(Pq_1-P-Rq_4) & -(q_3-q_4)(Rq_2-R-Tq_1+T) \\ (q_1-q_2)(Pq_3-Sq_4) & (q_2-q_3)(Pq_1-P-Rq_4) & 0 & (q_2-q_4)(Rq_3-Sq_1+S) \\ (q_1-q_4)(Sq_2-S-Tq_3) & -(q_3-q_4)(Rq_2-R-Tq_1+T) & (q_2-q_4)(Rq_3-Sq_1+S) & 0 \end{bmatrix}p^T +$$
$$\begin{bmatrix} q_1(Pq_2-P-Tq_4) \\ -(q_3-1)(Pq_2-P-Tq_4) \\ -Pq_1q_2+Pq_2q_3+Pq_2-Pq_3+Rq_2q_4-Sq_2q_4+Sq_4 \\ -Rq_2q_4+Rq_4+Sq_2q_4-Sq_2-Sq_4+S+Tq_1q_4-Tq_3q_4+Tq_3-Tq_4 \end{bmatrix}p - Pq_2 + P + Tq_4$$

and equivalently as,

$$\frac{1}{2}pQp^T + cp + a$$

where $Q \in \mathbb{R}^{4\times4}$ is a square matrix defined by the transition probabilities of the opponent q_1, q_2, q_3, q_4 as follows:

$$Q = \begin{bmatrix} 0 & -(q_1-q_3)(Pq_2-P-Tq_4) & (q_1-q_2)(Pq_3-Sq_4) & (q_1-q_4)(Sq_2-S-Tq_3) \\ -(q_1-q_3)(Pq_2-P-Tq_4) & 0 & (q_2-q_3)(Pq_1-P-Rq_4) & -(q_3-q_4)(Rq_2-R-Tq_1+T) \\ (q_1-q_2)(Pq_3-Sq_4) & (q_2-q_3)(Pq_1-P-Rq_4) & 0 & (q_2-q_4)(Rq_3-Sq_1+S) \\ (q_1-q_4)(Sq_2-S-Tq_3) & -(q_3-q_4)(Rq_2-R-Tq_1+T) & (q_2-q_4)(Rq_3-Sq_1+S) & 0 \end{bmatrix},$$

$c \in \mathbb{R}^{4\times1}$ is similarly defined by:

$$c = \begin{bmatrix} q_1(Pq_2-P-Tq_4) \\ -(q_3-1)(Pq_2-P-Tq_4) \\ -Pq_1q_2+Pq_2q_3+Pq_2-Pq_3+Rq_2q_4-Sq_2q_4+Sq_4 \\ -Rq_2q_4+Rq_4+Sq_2q_4-Sq_2-Sq_4+S+Tq_1q_4-Tq_3q_4+Tq_3-Tq_4 \end{bmatrix},$$

and $a = -Pq_2 + P + Tq_4$.

The same process is done for the denominator. $\square$

2 Theorem 2 Proof

The optimal behaviour of a memory-one strategy player $p^* \in \mathbb{R}^4_{[0,1]}$ against a set of N opponents $\{q^{(1)}, q^{(2)}, \dots, q^{(N)}\}$ for $q^{(i)} \in \mathbb{R}^4_{[0,1]}$ is given by:

$$p^* = \text{argmax} \sum_{i=1}^{N} u_q(p), \ p \in S_q.$$

The set S_q is defined as all the possible combinations of:

$$S_q = \left\{ p \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} \bullet \quad p_j \in \{0,1\} \quad \text{and} \quad \dfrac{d}{dp_k} \displaystyle\sum_{i=1}^{N} u_q^{(i)}(p) = 0 \\ \qquad\qquad\qquad \text{for all} \quad j \in J \quad \& \quad k \in K \quad \text{for all} \quad J, K \\ \qquad\qquad\qquad \text{where} \quad J \cap K = \quad \text{and} \quad J \cup K = \{1,2,3,4\}. \\ \bullet \quad p \in \{0,1\}^4 \end{array} \right\}. \tag{4}$$

Proof. The optimisation problem of Eq. 5

$$\begin{aligned} &\max_p : \sum_{i=1}^{N} {u_q}^{(i)}(p) \\ &\text{such that} : p \in \mathbb{R}_{[0,1]} \end{aligned} \tag{5}$$

can be written as:

$$\begin{aligned} &\max_p : \sum_{i=1}^{N} {u_q}^{(i)}(p) \\ &\text{such that} : p_i \leq 1 \text{ for } \in \{1,2,3,4\} \\ &\qquad\qquad -p_i \leq 0 \text{ for } \in \{1,2,3,4\} \end{aligned} \tag{6}$$

The optimisation problem has two inequality constraints and regarding the optimality this means that:

- either the optimum is away from the boundary of the optimization domain, and so the constraints plays no role;
- or the optimum is on the constraint boundary.

Thus, the following three cases must be considered:

Case 1: The solution is on the boundary and any of the possible combinations for $p_i \in \{0,1\}$ for $i \in \{1,2,3,4\}$ are candidate optimal solutions.

Case 2: The optimum is away from the boundary of the optimization domain and the interior solution p^* necessarily satisfies the condition $\frac{d}{dp} \sum\limits_{i=1}^{N} u_q(p^*) = 0$.

Case 3: The optimum is away from the boundary of the optimization domain but some constraints are equalities. The candidate solutions in this case are any combinations of $p_j \in \{0,1\}$ and $\frac{d}{dp_k} \sum\limits_{i=1}^{N} u_q^{(i)}(p) = 0$ forall $j \in J$ & $k \in K$ forall J, K where $J \cap K =$ and $J \cup K = \{1,2,3,4\}$.

Combining cases 1-3 a set of candidate solutions, denoted as S_q, is constructed as:

$$S_q = \left\{ p \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} \bullet \quad p_j \in \{0,1\} \quad \text{and} \quad \dfrac{d}{dp_k} \displaystyle\sum_{i=1}^{N} u_q^{(i)}(p) = 0 \quad \text{for all} \quad j \in J \quad \& \quad k \in K \quad \text{for all} \quad J, K \\ \qquad\qquad\qquad\quad \text{where} \quad J \cap K = \quad \text{and} \quad J \cup K = \{1,2,3,4\}. \\ \bullet \quad p \in \{0,1\}^4 \end{array} \right\}.$$

The derivative of $\sum\limits_{i=1}^{N} u_q^{(i)}(p)$ calculated using the following property (see [1] for details):

$$\frac{dxAx^T}{dx} = 2Ax. \tag{7}$$

Using property (7):

$$\frac{d}{dp}\frac{1}{2}pQp^T + cp + a = pQ + c \text{ and } \frac{d}{dp}\frac{1}{2}p\bar{Q}p^T + \bar{c}p + \bar{a} = p\bar{Q} + \bar{c}. \tag{8}$$

Note that the derivative of cp is c and the constant disappears. Combining these it can be proven that:

$$\begin{aligned}\frac{d}{dp}\sum_{i=1}^{N} u_q^{(i)}(p) &= \sum_{i=1}^{N} \frac{\frac{d}{dp}(\frac{1}{2}pQ^{(i)}p^T + c^{(i)}p + a^{(i)})(\frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)}p + \bar{a}^{(i)}) - \frac{d}{dp}(\frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)}p + \bar{a}^{(i)})(\frac{1}{2}pQ^{(i)}p^T + c^{(i)}p + a^{(i)})}{(\frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)}p + \bar{a}^{(i)})^2} \\ &= \sum_{i=1}^{N} \frac{(pQ^{(i)} + c^{(i)}+)(\frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)}p + \bar{a}^{(i)})}{(\frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)}p + \bar{a}^{(i)})^2} - \frac{(p\bar{Q}^{(i)} + \bar{c}^{(i)})(\frac{1}{2}pQ^{(i)}p^T + c^{(i)}p + a^{(i)})}{(\frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)}p + \bar{a}^{(i)})^2}\end{aligned}$$

For $\frac{d}{dp}\sum\limits_{i=1}^{N} u_q(p)$ to equal zero then:

$$\sum_{i=1}^{N} \left(pQ^{(i)} + c^{(i)}\right)\left(\frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)}p + \bar{a}^{(i)}\right) - \left(p\bar{Q}^{(i)} + \bar{c}^{(i)}\right)\left(\frac{1}{2}pQ^{(i)}p^T + c^{(i)}p + a^{(i)}\right) = 0, \quad \textit{while} \tag{9}$$

$$\sum_{i=1}^{N} \frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)}p + \bar{a}^{(i)} \neq 0. \tag{10}$$

The optimal solution to Eq. 5 is the point from S_q for which the utility is maximised. $\square$

3 Lemma 3 Proof

In a tournament of N players $\{q^{(1)}, q^{(2)}, \ldots, q^{(N)}\}$ for $q^{(i)} \in \mathbb{R}^4_{[0,1]}$ defection is stable if the transition probabilities of the opponents satisfy conditions Eq. 11 and Eq. 12.

$$\sum_{i=1}^{N}(c^{(i)T}\bar{a}^{(i)} - \bar{c}^{(i)T}a^{(i)}) \leq 0 \tag{11}$$

while,

$$\sum_{i=1}^{N}\bar{a}^{(i)} \neq 0 \tag{12}$$

Proof. For defection to be stable the derivative of the utility at the point $p = (0,0,0,0)$ must be negative. Substituting $p = (0,0,0,0)$ in,

$$\frac{d}{dp}\sum_{i=1}^{N}{u_q}^{(i)}(p) = \sum_{i=1}^{N}\frac{\left(pQ^{(i)} + c^{(i)}\right)\left(\frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)}p + \bar{a}^{(i)}\right)}{\left(\frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)}p + \bar{a}^{(i)}\right)^2} - \frac{\left(p\bar{Q}^{(i)} + \bar{c}^{(i)}\right)\left(\frac{1}{2}pQ^{(i)}p^T + c^{(i)}p + a^{(i)}\right)}{\left(\frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)}p + \bar{a}^{(i)}\right)^2} \tag{13}$$

gives:

$$\left.\frac{d\sum\limits_{i=1}^{N}{u_q}^{(i)}(p)}{dp}\right|_{p=(0,0,0,0)} = \sum_{i=1}^{N}\frac{(c^{(i)}\bar{a}^{(i)} - \bar{c}^{(i)}a^{(i)})}{(\bar{a}^{(i)})^2} \tag{14}$$

The sign of the numerator $\sum\limits_{i=1}^{N}(c^{(i)}\bar{a}^{(i)} - \bar{c}^{(i)}a^{(i)})$ can vary based on the transition probabilities of the opponents. The denominator can not be negative, and otherwise is always positive. Thus the sign of the derivative is negative if and only if $\sum\limits_{i=1}^{N}(c^{(i)}\bar{a}^{(i)} - \bar{c}^{(i)}a^{(i)}) \leq 0$. □

References

[1] Karim M Abadir and Jan R Magnus. *Matrix algebra*, volume 1. Cambridge University Press, 2005.