Stability of defection, optimisation of strategies and the limits of memory in the Prisoner's Dilemma RESPONSE TO REVIEWS

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We would like to open this response by thanking the reviewers for their thoughtful comments and suggestions. We have fully taken their comments on board and made significant modifications to both the mathematical arguments and the narrative. We feel this has greatly improved the work.

Both reviewers commented on different aspects of the paper and so our modifications naturally fit in to two categories (which we will discuss in detail comment by comment):

- The first is an improvement of the theoretic results both in terms of their proofs but also the result themselves. For example, we omitted to include the derivative of a quadratic form in one of our proofs and also had only given that quadratic form for the specific case of $(R, P, T, S) = (3, 0, 5, 1)$. The reviewer was very right to point out issues like this and we feel that we have addressed them all.

- The second reviewer questioned the general narrative of the paper as well as the evolutionary work. This was very helpful and we took time to reflect on the narrative of the paper. As a result of this we have changed the title to "A theory of mind: Best responses to memory-one strategies. The limitations of extortion and restricted memory" and reorganised some of the content. We have also clarified the nature of the evolutionary process we're discussing.

We will now take each comment of the reviewers in turn and highlight our efforts to improve the work.

1 Response

Regarding the comments of Reviewer 1:

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''Theorem 2: First, it is not clear how equation 17 is obtained.''
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This comment points out that we had skipped a step in obtaining the dot product of the state vector v and the payoff matrix (R, S, T, P). This has been added in the proof of what used to be Theorem 2 but is now Theorem 1.

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''Theorem 2: Moreover, even if that's correct, it is only for a
specific PD payoff matrix. One cannot state the general results in (4) from
just a specific case. To obtain (4), authors need to establish it for the
general PD payoff matrix.''
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We agree with the comment of the reviewer. The utility was specific only for a given payoff matrix. We have generalised our result by considering the generalised payoffs (R, S, T, P) and agree that this a more useful result and whilst we were worried that this closed form could be messy it is in fact relatively elegant.

> ``Theorem 3: The proof in the Appendix did not show how to obtain equations
> (22) - (24), the key results of the Theorem''

Equations (22) - (24) relay on the derivative of the utility $\sum_{i=1}^{N} u_q^{(i)}(p)$ which is a sum of quadratic form ratios. The derivative is obtained by using the property $\frac{dxAx^T}{dx} = 2Ax$. We have added these details and a reference text in the proof.

> ``Theorem 3: and what was stated in (5)''

We have added a sentence clarifying that this is obtained by taking the average over the opponents of the previously obtained expression for the utility.

Regarding the comments of Reviewer 2:

> ``what exactly is the evolutionary process used, what are the parameters?
> This is not mentioned anywhere in the paper or appendix, and makes it then
> hard to evaluate the impact of the findings, or give any hope for reproducing
> the results. I realize that there is a repository containing raw data, but
> the lack of explanation of the setup really harms any expectation of the
> paper being self-contained. The issue of the evolutionary process not being
> explained also prevented me from fully following the argument the authors
> make for why best responses in the evolutionary process would be less
> extortionate. ''

We agree with the comment of the reviewer. We are not running a evolutionary process. We study best responses to memory-one strategies that include self interactions which in a sense are approximating evolutionary stable best response strategies. We have reworded this part of the work to clarify that we are only considering the possibility of including self interactions and our results in an evolutionary process. We also explain how this could be used in a Moran process setting.

> ``Another related problem is that the authors don't go into more detail about
> their SSE method. A more expansive explanation (also of Table 1) would be of
> critical importance, seeing that this is one of the central points of the
> results section.''

We have included details of the SSE method and a better discussion of the results based on Table 1. We have also included plots of the SSE distributions that accompany Table 1.

> ``Another point I want to raise is how this work relates to the findings in Hilbe
> et al.: Partners and rivals in direct reciprocity (Nature Human Behaviour,
> 2018). That previous review article shows, among other results, the conditions
> under which "partner" (aiming for mutual cooperation, fighting back against
> defection) or "rival" (defecting/extortionate) strategies evolve in the IPD. It
> seems that the authors of the current work investigate a similar distinction

(however, they should explain what exactly they mean with "adaptable", as it
features as a contrast to "extortionate"), and thus would profit from discussing
the Hilbe et al. article as well.
''

We thank the reviewer for their article recommendation. The findings of Hilbe et al and how our results relate to them, are discussed in the "Results" section.

''As a last point, I think that the flow of the paper could be improved. As it is
now, the paper reads a bit like a list of results that are more or less loosely
tied together (especially Section 1.2 suffers from this problem in comparison to
the others). It would be good to motivate single sections in the broader context
of the message the paper wants to tell. ''

Thank you for this comment which encouraged us to reflect on the narrative of the paper. As a result we have modified the title of the manuscript and the titles of the sections. Moreover, the results of Section 1.2 have been moved to the Appendix, and Theorem 1 (old version Theorem 2) has been moved in the main body. We feel that these changes improved our manuscript significantly.

''The figures should be larger, such that the labels are easier to read''

We have made this suggested change.

''Please make a reference to the Appendix when you mention the Theorems, and
maybe summarize their core statement in the main text, which would make
the paper flow better.''

We have made this suggested change.

2   Marked changes made to the manuscript

We are also attaching a copy of the revised manuscript where we have highlighted the changes made to the originally submitted manuscript.

A theory of mind: Best responses to memory-one strategies. The limitations of extortion and restricted memory. ~~Stability of defection, optimisation of strategies and the limits of memory in the Prisoner's Dilemma COVER LETTER~~

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Abstract

Memory-one strategies are a set of Iterated Prisoner's Dilemma strategies that have been praised for their mathematical tractability and performance against single opponents. This manuscript investigates a theory of mind: *best response* memory-one strategies, as a multidimensional optimisation problem. ~~Though extortionate memory-one strategies have gained much attention, we demonstrate that best response memory-one strategies do not behave in an extortionate way, and moreover, for memory-one strategies to be evolutionary robust they need to be able to behave in a forgiving way~~ We add to the literature that has shown that extortionate play is not always optimal by showing that optimal play is often not extortionate. We also provide evidence that memory-one strategies suffer from their limited memory in multi agent interactions and can be out performed by ~~longer memory strategies~~ optimised strategies with longer memory.

The Prisoner's Dilemma (PD) is a two player game used in understanding the evolution of cooperative behaviour, formally introduced in [6]. Each player has two options, to cooperate (C) or to defect (D). The decisions are made simultaneously and independently. The normal form representation of the game is given by:

$$S_p = \begin{pmatrix} R & S \\ T & P \end{pmatrix} \quad S_q = \begin{pmatrix} R & T \\ S & P \end{pmatrix} \tag{1}$$

where S_p represents the utilities of the row player and S_q the utilities of the column player. The payoffs, (R, P, S, T), are constrained by $T > R > P > S$ and $2R > T + S$, and the most common values used in the literature are $(R, P, S, T) = (3, 1, 0, 5)$ [4]. The numerical experiments of our manuscript are carried out using these payoff values. The PD is a one shot game, however, it is commonly studied in a manner where the history of the interactions matters. The repeated form of the game is called the Iterated Prisoner's Dilemma (IPD).

Memory-one strategies are a set of IPD strategies that have been studied thoroughly in the literature [25, 26], however, they have gained most of their attention when a certain subset of memory-one strategies was introduced in [28], the zero-determinant strategies (ZDs). In [29] it was stated that "Press and Dyson have fundamentally changed the viewpoint on the Prisoner's Dilemma". A special case of ZDs are extortionate strategies that choose their actions so that a linear relationship is forced between the players' score ensuring that they will always receive at least as much as their opponents. ZDs are indeed mathematically unique and are proven to be robust in pairwise interactions, however, their true effectiveness in tournaments and evolutionary dynamics has been questioned [3, 13, 14, 15, 19, 21].

In ~~a similar fashion to~~ [28] the authors stated that "Only a player with a theory of mind about his opponent can do better, in which case Iterated Prisoner's Dilemma is an Ultimatum Game". The purpose of this work is to ~~consider a given memory-one strategy; however, whilst [28] found a way for a player to manipulate a given opponent, this work will consider a multidimensional optimisation approach~~ investigate the first part of this sentence, more specifically, to identify the best response ~~to a~~ strategy with a theory of mind of a given group of opponents. The outcomes of our work reinforce known results, namely that memory-one strategies must be forgiving to be evolutionarily stable [30, 31] and that longer-memory strategies have a certain form of advantage over short memory strategies [12, 27].

In particular, this work presents a ~~compact method of identifying the best response~~ closed form algebraic expression for the utility of a memory-one strategy against a given set of opponents, ~~and evaluates whether it~~ a compact method of identifying it's best response to that given set of opponents essentially: a theory of mind. The aim is to evaluate whether a best response memory-one strategy behaves in a zero-determinant way which in turn indicates whether it can be extortionate. ~~This is also done in evolutionary settings~~ We do

this using a linear algebraic approach presented in [20]. This is done in tournaments with and without self interactions. Moreover, we introduce a ~~well designed~~ framework that allows the comparison of an optimal ~~memory one strategy and a more complex~~ memory-one strategy and an optimised strategy which has a larger memory. ~~This is used to identify conditions for which defection is stable; thus identifying environments where cooperation will not occur.~~

To illustrate the analytical results obtained in this manuscript a number of numerical experiments are run. The source code for these experiments has been written in a sustainable manner [5]. It is open source (`https://github.com/Nikoleta-v3/Memory-size-in-the-prisoners-dilemma`) and tested which ensures the validity of the results. It has also been archived and can be found at [8].

1 Methods~~and Results~~

1.1 ~~Utility~~

One specific advantage of memory-one strategies is their mathematical tractability. They can be represented completely as an element of $\mathbb{R}^4_{[0,1]}$. This originates from [24] where it is stated that if a strategy is concerned with only the outcome of a single turn then there are four possible 'states' the strategy could be in; both players cooperated (CC), the first player cooperated whilst the second player defected (CD), the first player defected whilst the second player cooperated (DC) and both players defected (DD). Therefore, a memory-one strategy can be denoted by the probability vector of cooperating after each of these states; $p = (p_1, p_2, p_3, p_4) \in \mathbb{R}^4_{[0,1]}$.

In [24] it was shown that it is not necessary to simulate the play of a strategy p against a memory-one opponent q. Rather this exact behaviour can be modeled as a stochastic process, and more specifically as a Markov chain whose corresponding transition matrix M is given by Eq. 2. The long run steady state probability vector v, which is the solution to $vM = v$, can be combined with the payoff matrices of Eq. 1 to give the expected payoffs for each player. More specifically, the utility for a memory-one strategy p against an opponent q, denoted as $u_q(p)$, is given by Eq. 3.

$$M = \begin{bmatrix} p_1 q_1 & p_1 \left(-q_1 + 1\right) & q_1 \left(-p_1 + 1\right) & \left(-p_1 + 1\right)\left(-q_1 + 1\right) \\ p_2 q_3 & p_2 \left(-q_3 + 1\right) & q_3 \left(-p_2 + 1\right) & \left(-p_2 + 1\right)\left(-q_3 + 1\right) \\ p_3 q_2 & p_3 \left(-q_2 + 1\right) & q_2 \left(-p_3 + 1\right) & \left(-p_3 + 1\right)\left(-q_2 + 1\right) \\ p_4 q_4 & p_4 \left(-q_4 + 1\right) & q_4 \left(-p_4 + 1\right) & \left(-p_4 + 1\right)\left(-q_4 + 1\right) \end{bmatrix} \tag{2}$$

$$u_q(p) = v \cdot (R, S, T, P). \tag{3}$$

This manuscript has explored the form of $u_q(p)$, to the authors knowledge no previous work has done this, and ~~it proves~~ Theorem 1 states that $u_q(p)$ is given by a ratio of two quadratic forms [18]~~, (Theorem 1).~~

Theorem 1. *The expected utility of a memory-one strategy $p \in \mathbb{R}^4_{[0,1]}$ against a memory-one opponent $q \in \mathbb{R}^4_{[0,1]}$, denoted as $u_q(p)$, can be written as a ratio of two quadratic forms:*

$$u_q(p) = \frac{\frac{1}{2}pQp^T + cp + a}{\frac{1}{2}p\bar{Q}p^T + \bar{c}p + \bar{a}}, \tag{4}$$

3

~~where $Q = Q(q), \bar{Q} = \bar{Q}(q)$~~ where $Q, \bar{Q} \in \mathbb{R}^{4\times4}$ ~~, $c = c(q)$ and $\bar{c} = \bar{c}(q)$~~ are square matrices defined by the transition probabilities of the opponent q_1, q_2, q_3, q_4 as follows:

$$Q = \begin{bmatrix} 0 & -(q_1-q_3)(Pq_2-P-Tq_4) & (q_1-q_2)(Pq_3-Sq_4) & (q_1-q_4)(Sq_2-S-Tq_3) \\ -(q_1-q_3)(Pq_2-P-Tq_4) & 0 & (q_2-q_3)(Pq_1-P-Rq_4) & -(q_3-q_4)(Rq_2-R-Tq_1+T) \\ (q_1-q_2)(Pq_3-Sq_4) & (q_2-q_3)(Pq_1-P-Rq_4) & 0 & (q_2-q_4)(Rq_3-Sq_1+S) \\ (q_1-q_4)(Sq_2-S-Tq_3) & -(q_3-q_4)(Rq_2-R-Tq_1+T) & (q_2-q_4)(Rq_3-Sq_1+S) & 0 \end{bmatrix}, \quad (5)$$

$$\bar{Q} = \begin{bmatrix} 0 & -(q_1-q_3)(q_2-q_4-1) & (q_1-q_2)(q_3-q_4) & (q_1-q_4)(q_2-q_3-1) \\ -(q_1-q_3)(q_2-q_4-1) & 0 & (q_2-q_3)(q_1-q_4-1) & (q_1-q_2)(q_3-q_4) \\ (q_1-q_2)(q_3-q_4) & (q_2-q_3)(q_1-q_4-1) & 0 & -(q_2-q_4)(q_1-q_3-1) \\ (q_1-q_4)(q_2-q_3-1) & (q_1-q_2)(q_3-q_4) & -(q_2-q_4)(q_1-q_3-1) & 0 \end{bmatrix}. \quad (6)$$

c and $\bar{c} \in \mathbb{R}^{4\times1}$ ~~, $a = a(q)$ and $\bar{a} = \bar{a}(q) \in \mathbb{R}$.~~ are similarly defined by:

~~This~~

$$c = \begin{bmatrix} q_1(Pq_2-P-Tq_4) \\ -(q_3-1)(Pq_2-P-Tq_4) \\ -Pq_1q_2+Pq_2q_3+Pq_2-Pq_3+Rq_2q_4-Sq_2q_4+Sq_4 \\ -Rq_2q_4+Rq_4+Sq_2q_4-Sq_2-Sq_4+S+Tq_1q_4-Tq_3q_4+Tq_3-Tq_4 \end{bmatrix}, \quad (7)$$

$$\bar{c} = \begin{bmatrix} q_1(q_2-q_4-1) \\ -(q_3-1)(q_2-q_4-1) \\ -q_1q_2+q_2q_3+q_2-q_3+q_4 \\ q_1q_4-q_2-q_3q_4+q_3-q_4+1 \end{bmatrix}, \quad (8)$$

and the constant terms $a, \bar{a}$ are defined as $a = -Pq_2 + P + Tq_4$ and $\bar{a} = -q_2 + q_4 + 1$.

The proof of Theorem 1 is given in Appendix 5.1. Theorem 1 can be extended to consider multiple opponents. The IPD is commonly studied in tournaments and/or Moran Processes where a strategy interacts with a number of opponents. The payoff of a player in such interactions is given by the average payoff the player received against each opponent. More specifically the expected utility of a memory-one strategy against ~~a~~ N ~~number of~~ opponents is given by:

$$\frac{1}{N}\sum_{i=1}^{N} u_q^{(i)}(p) = \frac{\frac{1}{N}\sum_{i=1}^{N}(\frac{1}{2}pQ^{(i)}p^T + c^{(i)}p + a^{(i)}) \prod_{\substack{j=1 \\ j\neq i}}^{N}(\frac{1}{2}p\bar{Q}^{(j)}p^T + \bar{c}^{(j)}p + \bar{a}^{(j)})}{\prod_{i=1}^{N}(\frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)}p + \bar{a}^{(i)})}. \quad (9)$$

Eq. (9) is the average score (using Eq. (4)) against the set of opponents.

Estimating the utility of a memory-one strategy against any number of opponents without simulating the interactions is the main result used in the rest of this manuscript. It will be used to obtain best response memory-one strategies ~~, in tournaments and evolutionary dynamics, and~~ in tournaments with and without self interactions in order to explore the ~~conditions under which defection dominates cooperation.~~

4

1.1 ~~Stability of defection~~

~~An immediate result from our formulation can be obtained by evaluating the sign of Eq. 9's derivative at $p = (0,0,0,0)$. If at that point the derivative is negative, then the utility of a player only decreases if they were to change their behaviour, and thus **defection at that point is stable**~~limitations of extortion and restricted memory.

~~In a tournament of N players $\{q^{(1)}, q^{(2)}, \ldots, q^{(N)}\}$ for $q^{(i)} \in \mathbb{R}^4_{[0,1]}$ defection is stable if the transition probabilities of the opponents satisfy conditions Eq. 32 and Eq. 33.~~

2 Results

~~while,~~

~~For defection to be stable the derivative of the utility at the point $p = (0,0,0,0)$ must be negative.~~

~~Substituting $p = (0,0,0,0)$ in Eq. 34 gives:~~

~~The sign of the numerator $\sum_{i=1}^{N}(c^{(i)T}\bar{a}^{(i)} - \bar{c}^{(i)T}a^{(i)})$ can vary based on the transition probabilities of the opponents. The denominator can not be negative, and otherwise is always positive. Thus the sign of the derivative is negative if and only if $\sum_{i=1}^{N}(c^{(i)T}\bar{a}^{(i)} - \bar{c}^{(i)T}a^{(i)}) \leq 0$.~~

~~Consider a population for which defection is known to be stable. In that population all the members will over time adopt the same behaviour; thus in such population cooperation will never take over. This is demonstrated in Fig. 4. These have been simulated using [1] an open source research framework for the study of the IPD.~~

~~A. For $q_1 = (0.22199, 0.87073, 0.20672, 0.91861)$, $q_2 = (0.48841, 0.61174, 0.76591, 0.51842)$ and $q_3 = (0.2968, 0.18772, 0.08074, 0.$ Eq. 32 and Eq. 33 hold and Defector takes over the population. B. For $q_1 = (0.96703, 0.54723, 0.97268, 0.71482)$, $q_2 = (0.69773, 0.21609, 0.97627, 0.0062)$ and $q_3 = (0.25298, 0.43479, 0.77938, 0.19769)$, Eq. 32 fails and Defector does not take over the population.~~

2.1 ~~Best response memory-one strategies~~

~~As discussed ZDs have been acclaimed for their robustness against a single opponent. ZDs are evidence that extortion works in pairwise interactions, their behaviour ensures that the strategies will never lose a game. However, this paper argues that in multi opponent interactions, where the payoffs matter, strategies trying to exploit their opponents will suffer. Compared to ZDs, best response memory-one strategies, which have a theory of mind of their opponents, utilise their behaviour in order to gain the most from their interactions. The question that arises then is whether best response strategies are optimal because they behave in an extortionate way.~~

~~To answer this question, we initially~~ Here we define *memory-one best response* strategies as a multi dimensional optimisation problem given by:

$$\max_{p} : \sum_{i=1}^{N} u_q^{(i)}(p) \tag{10}$$

$$\text{such that} : p \in \mathbb{R}_{[0,1]}$$

Optimising this particular ratio of quadratic forms is not trivial. It can be verified empirically for the case of a single opponent that there exists at least one point for which the definition of concavity does not hold. The non concavity of $u(p)$ indicates multiple local optimal points. This is also intuitive. The best response against a cooperator, $q = (1,1,1,1)$, is a defector $p^* = (0,0,0,0)$. The strategies $p = (\frac{1}{2},0,0,0)$ and $p = (\frac{1}{2},0,0,\frac{1}{2})$ are also best responses. The approach taken here is to introduce a compact way of constructing the discrete candidate set of all local optimal points, and evaluating the objective function Eq. 9. This gives the best response memory-one strategy. The approach is given in Theorem 2.

Theorem 2. *The optimal behaviour of a memory-one strategy player* $p^* \in \mathbb{R}_{[0,1]}^4$ *against a set of N opponents* $\{q^{(1)}, q^{(2)}, \ldots, q^{(N)}\}$ *for* $q^{(i)} \in \mathbb{R}_{[0,1]}^4$ *is given by:*

$$p^* = \operatorname{argmax} \sum_{i=1}^{N} u_q(p), \ \ p \in S_q.$$

The set S_q is defined as all the possible combinations of:

$$S_q = \left\{ p \in \mathbb{R}^4 \ \middle| \ \begin{array}{l} \bullet \quad p_j \in \{0,1\} \quad and \quad \dfrac{d}{dp_k} \sum_{i=1}^{N} u_q^{(i)}(p) = 0 \\[2mm] \qquad for \ all \quad j \in J \quad \& \quad k \in K \quad for \ all \quad J, K \\ \qquad where \quad J \cap K = \varnothing \quad and \quad J \cup K = \{1,2,3,4\}. \\[2mm] \bullet \quad p \in \{0,1\}^4 \end{array} \right\}. \tag{11}$$

Note that there is no immediate way to find the zeros of $\frac{d}{dp} \sum_{i=1}^{N} u_q(p)$ where,

$$\frac{d}{dp} \sum_{i=1}^{N} u_q^{(i)}(p) = \sum_{i=1}^{N} \frac{\left(pQ^{(i)} + c^{(i)} \right)\left(\frac{1}{2} p\bar{Q}^{(i)} p^T + \bar{c}^{(i)} p + \bar{a}^{(i)} \right)}{\left(\frac{1}{2} p\bar{Q}^{(i)} p^T + \bar{c}^{(i)} p + \bar{a}^{(i)} \right)^2} - \frac{\left(p\bar{Q}^{(i)} + \bar{c}^{(i)} \right)\left(\frac{1}{2} pQ^{(i)} p^T + c^{(i)} p + a^{(i)} \right)}{\left(\frac{1}{2} p\bar{Q}^{(i)} p^T + \bar{c}^{(i)} p + \bar{a}^{(i)} \right)^2} \tag{12}$$

For $\frac{d}{dp} \sum_{i=1}^{N} u_q(p)$ to equal zero then:

$$\sum_{i=1}^{N} \left(pQ^{(i)} + c^{(i)} \right)\left(\frac{1}{2} p\bar{Q}^{(i)} p^T + \bar{c}^{(i)} p + \bar{a}^{(i)} \right) - \left(p\bar{Q}^{(i)} + \bar{c}^{(i)} \right)\left(\frac{1}{2} pQ^{(i)} p^T + c^{(i)} p + a^{(i)} \right) = 0, \quad while \tag{13}$$

$$\sum_{i=1}^{N} \frac{1}{2} p\bar{Q}^{(i)} p^T + \bar{c}^{(i)} p + \bar{a}^{(i)} \neq 0. \tag{14}$$

The proof of Theorem 2 is given in Appendix 5.2. Finding best response memory-one strategies is analytically feasible using the formulation of Theorem 2 and resultant theory [17]. However, for large systems building the resultant becomes intractable. As a result, best responses will be estimated heuristically using a numerical

method, suitable for problems with local optima, called Bayesian optimisation [23].

~~This is extended to evolutionary settings . In these settings~~ In several evolutionary settings such as Moran Processes self interactions are key. ~~Self interactions can be incorporated in the formulation that has been used so far. More specifically, the optimisation problem of~~ Previous work has identified interesting results such as the appearance of self recognition mechanisms when training strategies using evolutionary algorithms in Moran processes [19]. This aspect of reinforcement learning can be done for best response memory-one strategies by incorporating the strategy itself in the objective function as shown in Eq. 26~~is extended to include self interactions :~~. K is the number of self interactions that will take place.

$$\max_{p} : \frac{1}{N} \sum_{i=1}^{N} u_q^{(i)}(p) + K u_p(p) \tag{15}$$
$$\text{such that} : p \in \mathbb{R}_{[0,1]}$$

For determining the memory-one best response ~~in an evolutionary setting~~with self interactions, an algorithmic approach is considered, called *best response dynamics*. The best response dynamics approach used in this manuscript is given by Algorithm 1.

Algorithm 1: Best response dynamics Algorithm

$p^{(t)} \leftarrow (1, 1, 1, 1)$;
while $p^{(t)} \neq p^{(t-1)}$ **do**
$\quad p^{(t+1)} = \text{argmax} \frac{1}{N} \sum_{i=1}^{N} u_q^{(i)}(p^{(t)}) + K u_{p^{(t)}}(p^{(t)})$;
end

Using this approach it would be possible to create a memory-one best response strategy that updates on every generation of a Moran process to recalculate the optimal behaviour given the population. This extension of the "theory of mind" is an interesting avenue for future work.

In multi opponent settings, where the payoffs matter, strategies trying to exploit their opponents will suffer. Compared to ZDs, best response memory-one strategies, which have a theory of mind of their opponents, utilise their behaviour in order to gain the most from their interactions. The question that arises then is whether best response strategies are optimal because they behave in an extortionate way.

The results of this section use Bayesian optimisation to generate a data set of best response memory-one strategies ~~, in tournaments and evolutionary dynamics whilst~~ for $N = 2$ opponents. The data set is available at [7]. It contains a total of 1000 trials corresponding to 1000 different instances of a best response strategy in tournaments ~~and evolutionary dynamics~~with and without self interactions. For each trial a set of 2 opponents is randomly generated and the memory-one best ~~responses~~ response against them is found.

~~The source code for the experiments presented in this manuscript has been written in a sustainable manner [5] . It is open source () and tested which ensures the validity of the results. It has also been archived and can be found at [8].~~

In order to investigate whether best responses behave in an extortionate matter the SSE method described in [20] is used. ~~In [20] it is proven that all extortionate ZDs reside on a triangular plane. For a given~~ More specifically, in [20] the point x^*, in the space of memory-one strategies, that is the nearest extortionate strategy to a given strategy p ~~, a strategy x^* is defined as the nearest ZDs. The distance between the two strategies is explicitly calculated and~~ is given by.

$$x^* = \left(C^T C\right)^{-1} C^T \bar{p} \tag{16}$$

where $\bar{p} = (p_1 - 1, p_2 - 1, p_3, p_4)$ and

$$C = \begin{bmatrix} R - P & R - P \\ S - P & T - P \\ T - P & S - P \\ 0 & 0 \end{bmatrix}. \tag{17}$$

Once this closest ZDs is found, the squared norm of the remaining error is referred to as ~~the~~ sum of squared errors of prediction (SSE)~~; which corresponds to how far p~~:

$$\text{SSE} = \bar{p}^T \bar{p} - \bar{p} C \left(C^T C\right)^{-1} C^T \bar{p} = \bar{p}^T \bar{p} - \bar{p} C x^* \tag{18}$$

Thus, SSE is defined as how far a strategy is from behaving as a ~~ZDs. Thus, a~~ ZD. A high SSE implies ~~non ZD, which in turn implies~~ a non extortionate behaviour. The ~~SSE method has been applied to the data set. A statistics summary of the SSE distribution~~ distributions of SSE for the best response in tournaments ~~and evolutionary dynamics~~ $(N = 2)$ with and without self interactions (with $K = 1$) are given in Figure 1. Moreover, a statistical summary of the SSE distributions is given in Table 1.

	mean	std	5%	50%	95%	max	median	skew	kurt
Tournament without self interactions	0.34	0.40	0.028	0.17	1.05	2.47	0.17	1.87	3.60
Tournament with self interactions	0.17	0.23	0.01	0.12	0.67	1.53	0.12	3.42	1.92

Table 1: SSE of best response memory-one when $N = 2$
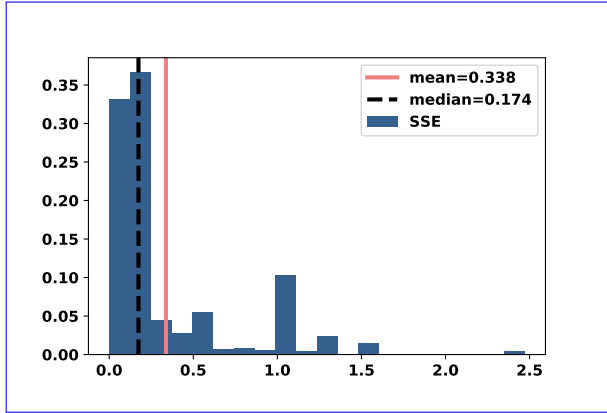
(a) SEE distribution for best response in tournaments without self interactions.
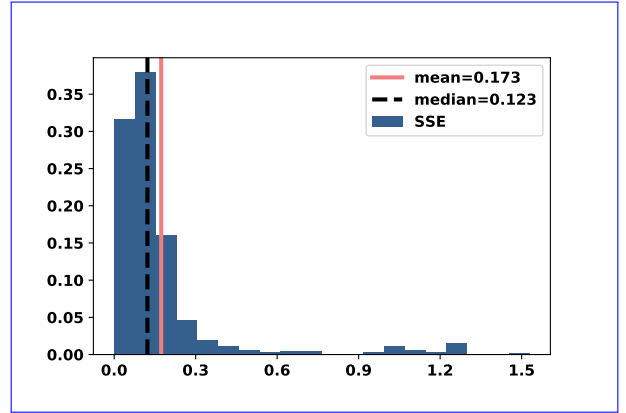
(b) SEE distribution for best response in tournaments with self interactions.

Figure 1: SEE distributions for best response in tournaments without and with self interactions.

For the best response in tournaments ~~that do not include self interactions~~ the distribution of SSE is skewed to the left, indicating that the best response does exhibit ZDs behaviour and so could be extortionate, however, the best response is not uniformly a ZDs. A positive measure of skewness and kurtosis ~~indicates a~~, and a mean of 0.34 indicate a heavy tail to the right. Therefore, in several cases the strategy is not trying to extort its opponents. ~~Similarly the evolutionary best response strategy does not behave uniformly extortionately. A larger value of both the kurtosis and the skewness of the SSE distribution indicates that in evolutionary settings a~~ In [11] a similar behaviour is refereed to as the *partner strategy*. The partner strategy aims to share the payoff for mutual cooperation, but it is ready to fight back when being exploited. The partner strategy was designed, but the best responses which are defined by their opponents seem to exhibit the same behaviour.

Similarly, when considering self interactions, the distribution of SSE for the best response strategy has skewness and kurtosis that indicate a heavy tail to the right. This indicates that evolutionary stable memory-one ~~best response is even more adaptable .~~ strategies need to more adaptable than a ZDs, and aim for mutual cooperation as well as exploitation which is in line with the results of [11] where their strategy was designed to adapt and was shown to be evolutionary stable. The findings of this work show that an optimal strategy acts in the same way.

The difference between best responses in tournaments ~~and in evolutionary settings~~ without and with self interactions is further explored by Fig. 2. Though, no statistically significant differences have been found, from Fig. 2, it seems that ~~evolutionary best response~~ the best response that incorporate self interactions has a higher median p_2; which corresponds to the probability of cooperating after receiving a defection. Thus, they are more likely to forgive after being tricked. This is due to the fact that they could be playing against themselves, and they need to be able to forgive so that future cooperation can occur.

~~SSE of best response memory-one when $N = 2$~~

2.1 ~~Longer memory best responses~~

~~This section focuses on the memory size of strategies. The~~ The other main finding presented in [**?**] was that short memory of the strategies was all that was needed. We argue that the second limitation of ZDs in multi opponent interactions is that of their restricted memory. To demonstrate the effectiveness of memory in the IPD ~~has been previously explored in the literature, however, no one has compared the performance of~~ we explore a best response longer-memory ~~strategies to~~ strategy against a given set of memory-one opponents, and compare it's performance to that of a memory-one best ~~responses~~ response.

In [9], a strategy called *Gambler* which makes probabilistic decisions based on the opponent's n_1 first moves, the opponent's m_1 last moves and the player's m_2 last moves was introduced. In this manuscript Gambler with parameters: $n_1 = 2, m_1 = 1$ and $m_2 = 1$ is used as a longer-memory strategy. By considering the opponent's first two moves, the opponents last move and the player's last move, there are only 16 ($4 \times 2 \times 2$) possible outcomes that can occur, furthermore, Gambler also makes a probabilistic decision of cooperating in the opening move. Thus, Gambler is a function $f : \{C, D\} \to [0, 1]_{\mathbb{R}}$. This can be hard coded as an element of $[0, 1]_{\mathbb{R}}^{16+1}$, one probability for each outcome plus the opening move. Hence, compared to Eq. 26, finding an optimal Gambler is a 17 dimensional problem given by:

$$\max_p : \sum_{i=1}^{N} U_q{}^{(i)}(f)$$
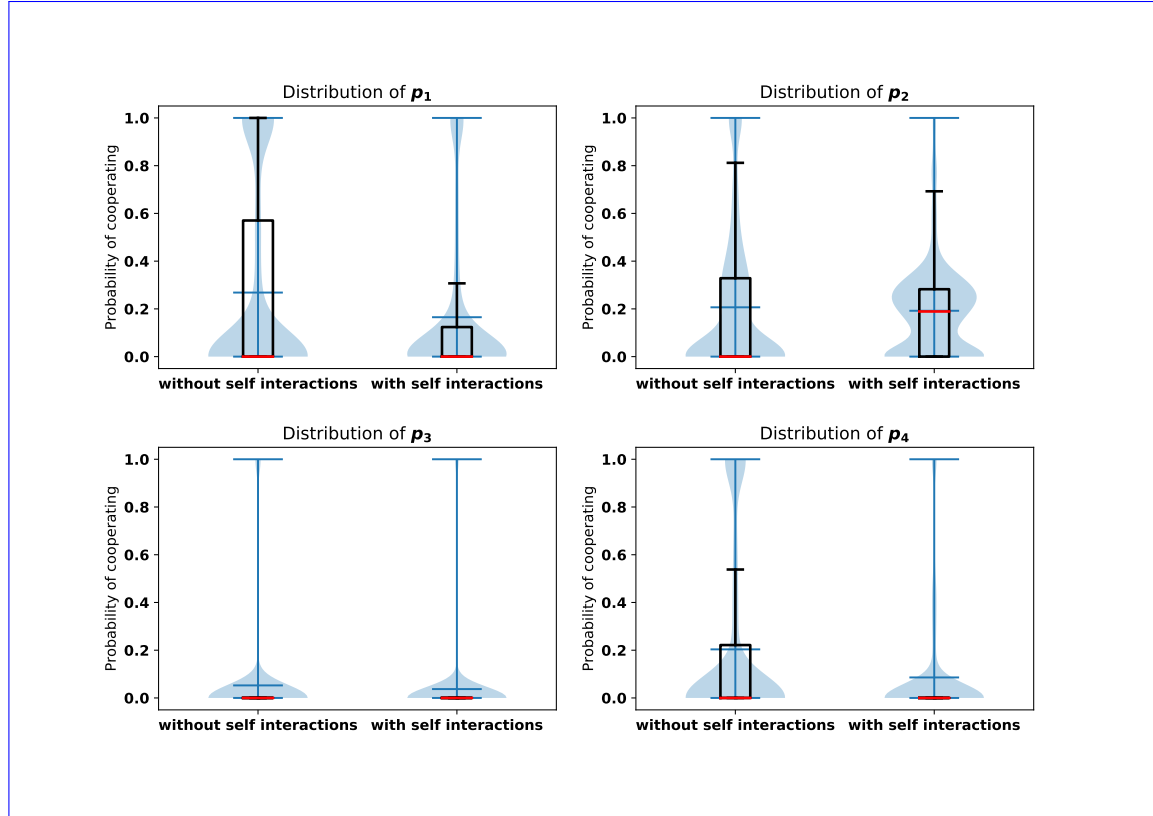$$\text{such that} : f \in \mathbb{R}_{[0,1]}^{17}$$
(19)

Figure 2: Distributions of p^* for best responses in tournaments and evolutionary settings. The medians, denoted as $\bar{p}^*$, for tournaments are $\bar{p}^* = (0, 0, 0, 0)$, and for evolutionary settings $\bar{p}^* = (0, 0.19, 0, 0)$.

Note that Eq. 9 can not be used here for the utility of Gambler, and actual simulated players are used. This is done using [1] with 500 turns and 200 repetitions, moreover, Eq. 19 is solved numerically using Bayesian optimisation.

Similarly to previous sections, a large data set has been generated with instances of an optimal Gambler and a memory-one best response, available at [7]. Estimating a best response Gambler (17 dimensions) is computational more expensive compared to a best response memory-one (4 dimensions). As a result, the analysis of this section is based on a total of 152 trials. ~~For each trial two~~ As before, for each trial $N = 2$ random opponents have been selected. ~~The 152 pair of opponents are a sub set of the opponents used in section ??.~~

The ratio between Gambler's utility and the best response memory-one strategy's utility has been calculated and its distribution in given in Fig. 3. It is evident from Fig. 3 that Gambler always performs as well as the best response memory-one strategy and often performs better. There are no points where the ratio value is less than 1, thus Gambler never performed less than the best response memory-one strategy and in places outperforms it. ~~This seems to be at odd with the result of [28] that against a memory-one opponent having a longer memory will not give a strategy any advantage.~~ However, against two memory-one opponents Gambler's performance is better than the optimal memory-one strategy. This is evidence that in the case of ~~two opponents~~ multiple opponents, having a shorter memory is limiting.
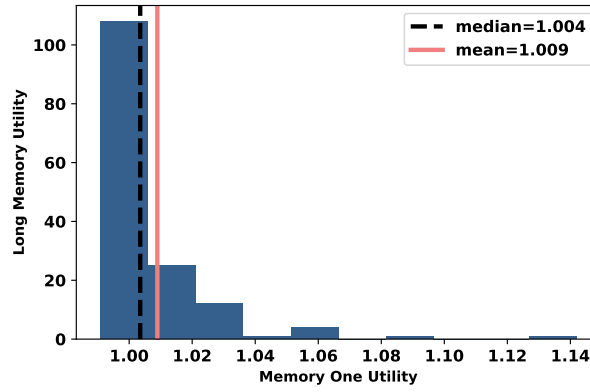

Figure 3: The ratio between the utilities of Gambler and best response memory-one strategy for 152 different pair of opponents.

3 Discussion

This manuscript has considered *best response* strategies in the IPD game, and more specifically, *memory-one best responses*. It has proven that ~~there~~ :

- The utility of a memory-one strategy against a set of memory-one opponents can be written as a sum of ratios of quadratic forms (Theorem 1).
- There is a compact way of identifying a memory-one best response to a group of opponents ~~, and moreover it obtained~~ through a search over a discrete set (Theorem 2).

Note that Theorem 2 which does not only have game theoretic novelty, but also the mathematical novelty of solving quadratic ratio optimisation problems where the quadratics are non concave.

Moreover Theorem 1, allows us to obtain a condition for which in an environment of memory-one opponents defection is the stable choice, based only on the coefficients of the opponents. ~~The later parts of this paper focused on a series of empirical results , where it was~~, as stated in Lemma 3.

Lemma 3. *In a tournament of N players $\{q^{(1)}, q^{(2)}, \ldots, q^{(N)}\}$ for $q^{(i)} \in \mathbb{R}^4_{[0,1]}$ defection is stable if the transition probabilities of the opponents satisfy conditions Eq. 32 and Eq. 33.*

$$\sum_{i=1}^{N} (c^{(i)T}\bar{a}^{(i)} - \bar{c}^{(i)T}a^{(i)}) \leq 0 \tag{20}$$

while,

$$\sum_{i=1}^{N} \bar{a}^{(i)} \neq 0 \tag{21}$$

The proof of Lemma 3 is given in Appendix **??** and a numerical simulation demonstrating the result is given in Fig. 4.
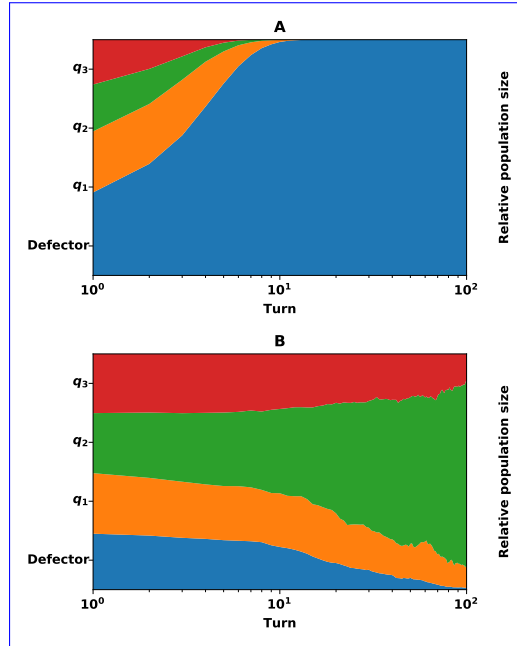

Figure 4: A. For $q_1 = (0.22199, 0.87073, 0.20672, 0.91861)$, $q_2 = (0.48841, 0.61174, 0.76591, 0.51842)$ and $q_3 = (0.2968, 0.18772, 0.08074, 0.73844)$, Eq. 32 and Eq. 33 hold and Defector takes over the population. B. For $q_1 = (0.96703, 0.54723, 0.97268, 0.71482)$, $q_2 = (0.69773, 0.21609, 0.97627, 0.0062)$ and $q_3 = (0.25298, 0.43479, 0.77938, 0.19769)$, Eq. 32 fails and Defector does not take over the population. These results have been obtained by using [1] an open source research framework for the study of the IPD.

The empirical results have shown that the performance and the evolutionary stability of memory-one strategies rely on adaptability and not on extortion. ~~Finally, it was shown~~, and that memory-one strategies' performance is limited by their memory in cases where they interact with multiple opponents.

~~Following the work described in [24], where it was shown that the utility between two memory-one strategies~~

~~can be estimated by a Markov stationary state, we proved that the utilities can be written as a ration of two quadratic forms in R^4, Theorem 1. This was extended to include multiple opponents, as the IPD is commonly studied in such situations. This formulation allowed us to introduce an approach for identifying memory-one best responses to any number of opponents; Theorem 2. This does not only have game theoretic novelty, but also a mathematical novelty of solving quadratic ratio optimisation problems where the quadratics are non concave. The results were used to define a condition for which defection is known to be stable.~~

~~This manuscript presented several experimental results. All data for the results is archived in [7].~~ These results were mainly to investigate the behaviour of memory-one strategies and their limitations. A large data set which contained best responses in tournaments ~~and in evolutionary settings for $N = 2$ was generated~~ whilst including or not self interactions for $N = 2$ was generated and is archived in [7]. This allowed us to investigate their respective behaviours, and whether it was extortionate acts that made them the most favorable strategies. ~~However, it~~ It was shown that it was not extortion but adaptability that allowed the strategies to gain the most from their interactions. In ~~evolutionary settings it was shown that the best response strategy was even more adaptable, and~~ settings with self interactions there is some evidence that it is more likely to forgive after being tricked. ~~Moreover, the performance of memory-one strategies was put against the performance of a longer memory strategy called Gambler. There were several cases where Gambler would outperform the memory-one strategy, however, a memory-one strategy never managed to outperform a Gambler. This result occurred whilst considering a Gambler with a sufficiently larger memory but not a sufficiently larger amount of information regarding the game.~~

All the empirical results presented in this manuscript have been for the case of $N = 2$. In future work we would consider larger values of N, however, we believe that for larger values of N the results that have been presented here would only be more evident. In addition, we would investigate potential theoretical results for the ~~evolutionary~~ best responses dynamics algorithm discussed. Another interesting avenue would be to study the Moran process with a dynamically updating best response.

By specifically exploring the entire ~~memory space-one~~ space of memory-one strategies to identify the ~~optimal~~ best strategy for a variety of situations, this work ~~casts~~ adds to the literature casting doubt on the effectiveness of ZDs, highlights the importance of adaptability and provides a framework for the continued understanding of these important questions.

4   Acknowledgements

A variety of software libraries have been used in this work:

- The Axelrod library for IPD simulations [1].
- The Scikit-optimize library for an implementation of Bayesian optimisation [10].
- The Matplotlib library for visualisation [16].
- The SymPy library for symbolic mathematics [22].
- The Numpy library for data manipulation [32].

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5 Appendix

5.1 Theorem 1 Proof

The ~~expected~~ utility of a ~~memory-one strategy $p \in \mathbb{R}^4_{[0,1]}$ against a memory-one opponent $q \in \mathbb{R}^4_{[0,1]}$, denoted as~~ memory one player p against an opponent q, $u_q(p)$, can be written as a ratio of two quadratic forms ~~: on R^4.~~

~~where $Q, \bar{Q} \in \mathbb{R}^{4 \times 4}$ are square matrices defined by the transition probabilities of the opponent q_1, q_2, q_3, q_4 as follows:~~

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Proof. It was discussed that $u_q(p)$ it is the product of the steady ~~states~~ state vector v and the PD payoffs,

$$u_q(p) = v \cdot (R, S, T, P).$$

~~More specifically, with $(R, P, S, T) = (3, 1, 0, 5)$~~ The steady state vector which is the solution to $vM = v$ is given by

$$v = \left[\frac{p_2 p_3 (q_2 q_4 - q_3 q_4) + p_2 p_4 (q_2 q_3 - q_2 q_4 - q_3 + q_4) + p_3 p_4 (-q_2 q_3 + q_3 q_4) - p_3 q_2 q_4 + p_4 q_4 (q_2 - 1)}{\bar{v}} , \right.$$
$$\frac{p_1 p_3 (q_1 q_4 - q_2 q_4) + p_1 p_4 (-q_1 q_2 + q_1 + q_2 q_4 - q_4) + p_3 p_4 (q_1 q_2 - q_1 q_4 - q_2 + q_4) + p_3 q_4 (q_2 - 1) - p_4 q_2 (q_4 + 1) + p_4 (q_4 - 1)}{\bar{v}} ,$$
$$\frac{-p_1 p_2 (q_1 q_4 - q_3 q_4) - p_1 p_4 (-q_1 q_3 + q_3 q_4) + p_1 q_1 q_4 - p_2 p_4 (q_1 q_3 - q_1 q_4 - q_3 + q_4) - p_2 q_4 (q_3 + 1) - p_4 q_4 (q_1 + q_3) - p_4 (q_3 + q_4) - q_4}{\bar{v}} ,$$
$$\left. \frac{p_1 p_2 (q_1 q_2 - q_1 - q_2 q_3 + q_3) + p_1 p_3 (-q_1 q_3 + q_2 q_3) - p_1 q_1 (q_2 + 1) + p_2 p_3 (-q_1 q_2 + q_1 q_3 + q_2 - q_3) + p_2 (q_3 q_2 - q_2 - q_3 - 1) + p_3 (q_1 q_2 - q_3 q_2 - q_2 - q_3) + q_2 - 1}{\bar{v}} \right],$$

where,

$$\begin{aligned}
\bar{v} =\ & p_1 p_2 (q_1 q_2 - q_1 q_4 - q_1 - q_2 q_3 + q_3 q_4 + q_3) - p_1 p_3 (q_1 q_3 - q_1 q_4 - q_2 q_3 + q_2 q_4) - p_1 p_4 (q_1 q_2 - q_1 q_3 - q_1 - q_2 q_4 + q_3 q_4 + q_4) - \\
& p_1 q_1 (q_2 + q_4 + 1) + p_2 p_3 (-q_1 q_2 + q_1 q_3 + q_2 q_4 + q_2 - q_3 q_4 - q_3) + p_2 p_4 (-q_1 q_3 + q_1 q_4 + q_2 q_3 - q_2 q_4) + p_2 q_2 (q_3 - 1) - p_2 q_3 (q_4 - 1) + \\
& p_2 (q_4 + 1) + p_3 p_4 (q_1 q_2 - q_1 q_4 - q_2 q_3 - q_2 + q_3 q_4 + q_4) + p_3 q_2 q_1 (-p_3 - 1) + p_3 (q_3 - q_4) - p_4 (q_1 q_4 + q_2 + q_3 q_4 - q_3 + q_4 - 1) + \\
& q_2 - q_4 - 1
\end{aligned}$$

The dot product of $v \cdot (R, S, T, P)$ gives,

$$\begin{aligned}
u_q(p) =\ & \frac{R \left(p_2 p_3 (q_2 q_4 - q_3 q_4) + p_2 p_4 (q_2 q_3 - q_2 q_4 - q_3 + q_4) + p_3 p_4 (-q_2 q_3 + q_3 q_4) - p_3 q_2 q_4 + p_4 q_4 (q_2 - 1)\right)}{\bar{v}} + \\
& \frac{S \left(p_1 p_3 (q_1 q_4 - q_2 q_4) + p_1 p_4 (-q_1 q_2 + q_1 + q_2 q_4 - q_4) + p_3 p_4 (q_1 q_2 - q_1 q_4 - q_2 + q_4) + p_3 q_4 (q_2 - 1) - p_4 q_2 (q_4 + 1) + p_4 (q_4 - 1)\right)}{\bar{v}} + \\
& \frac{T \left(-p_1 p_2 (q_1 q_4 - q_3 q_4) - p_1 p_4 (-q_1 q_3 + q_3 q_4) + p_1 q_1 q_4 - p_2 p_4 (q_1 q_3 - q_1 q_4 - q_3 + q_4) - p_2 q_4 (q_3 + 1) - p_4 q_4 (q_1 + q_3) - p_4 (q_3 + q_4) - q_4\right)}{\bar{v}} + \\
& \frac{P \left(p_1 (p_2 (q_1 q_2 - q_1 - q_2 q_3 + q_3) + p_3 (-q_1 q_3 + q_2 q_3) - q_1 (q_2 + 1)) + p_2 p_3 ((-q_1 q_2 + q_1 q_3 + q_2 - q_3) + (q_3 q_2 - q_2 - q_3 - 1))\right)}{\bar{v}} + \\
& \frac{P \left(p_3 (q_1 q_2 - q_3 q_2 - q_2 - q_3) + q_2 - 1\right)}{\bar{v}} \implies
\end{aligned}$$

$$u_q(p) = \frac{\begin{pmatrix} -p_1 p_2 (q_1 - q_3)(Pq_2 - P - Tq_4) + p_1 p_3 (q_1 - q_2)(Pq_3 - Sq_4) + p_1 p_4 (q_1 - q_4)(Sq_2 - S - Tq_3) + p_2 p_3 (q_2 - q_3)(Pq_1 - P - Rq_4) - \\ p_2 p_4 (q_3 - q_4)(Rq_2 - R - Tq_1 + T) + p_3 p_4 (q_2 - q_4)(Rq_3 - Sq_1 + S) + p_1 q_1 (Pq_2 - P - Tq_4) - p_2 (q_3 - 1)(Pq_2 - P - Tq_4) + \\ p_3 (-Pq_1 q_2 + Pq_2 q_3 + Pq_2 - Pq_3 + Rq_2 q_4 - Sq_2 q_4 + Sq_4) + p_4 (-Rq_2 q_4 + Rq_4 + Sq_2 q_4 - Sq_2 - Sq_4 + S + Tq_1 q_4 - Tq_3 q_4 + Tq_3 - Tq_4) \\ -Pq_2 + P + Tq_4 \end{pmatrix}}{\begin{pmatrix} p_1 p_2 (q_1 q_2 - q_1 q_4 - q_1 - q_2 q_3 + q_3 q_4 + q_3) + p_1 p_3 (-q_1 q_3 + q_1 q_4 + q_2 q_3 - q_2 q_4) + p_1 p_4 (-q_1 q_2 + q_1 q_3 + q_1 + q_2 q_4 - q_3 q_4 - q_4) + \\ p_2 p_3 (-q_1 q_2 + q_1 q_3 + q_2 q_4 + q_2 - q_3 q_4 - q_3) + p_2 p_4 (-q_1 q_3 + q_1 q_4 + q_2 q_3 - q_2 q_4) + p_3 p_4 (q_1 q_2 - q_1 q_4 - q_2 q_3 - q_2 + q_3 q_4 + q_4) + \\ p_1 (-q_1 q_2 + q_1 q_4 + q_1) + p_2 (q_2 q_3 - q_2 - q_3 q_4 - q_3 + q_4 + 1) + p_3 (q_1 q_2 - q_2 q_3 - q_2 + q_3 - q_4) + p_4 (-q_1 q_4 + q_2 + q_3 q_4 - q_3 + q_4 - 1) + \\ q_2 - q_4 - 1 \end{pmatrix}}.$$

Let us consider the numerator of $u_q(p)$. The cross product terms $p_i p_j$ are given by,

$$-p_1p_2(q_1 - q_3)(Pq_2 - P - Tq_4) + p_1p_3(q_1 - q_2)(Pq_3 - Sq_4) + p_1p_4(q_1 - q_4)(Sq_2 - S - Tq_3)+$$
$$p_2p_3(q_2 - q_3)(Pq_1 - P - Rq_4) - p_2p_4(q_3 - q_4)(Rq_2 - R - Tq_1 + T) + p_3p_4(q_2 - q_4)(Rq_3 - Sq_1 + S)$$

This can be re written in a matrix format given by Eq. 22.

$$(p_1, p_2, p_3, p_4)\frac{1}{2}\begin{bmatrix} 0 & -(q_1-q_3)(Pq_2-P-Tq_4) & (q_1-q_2)(Pq_3-Sq_4) & (q_1-q_4)(Sq_2-S-Tq_3) \\ -(q_1-q_3)(Pq_2-P-Tq_4) & 0 & (q_2-q_3)(Pq_1-P-Rq_4) & -(q_3-q_4)(Rq_2-R-Tq_1+T) \\ (q_1-q_2)(Pq_3-Sq_4) & (q_2-q_3)(Pq_1-P-Rq_4) & 0 & (q_2-q_4)(Rq_3-Sq_1+S) \\ (q_1-q_4)(Sq_2-S-Tq_3) & -(q_3-q_4)(Rq_2-R-Tq_1+T) & (q_2-q_4)(Rq_3-Sq_1+S) & 0 \end{bmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} \tag{22}$$

Similarly, the linear terms are given by,

$$p_1q_1(Pq_2 - P - Tq_4) - p_2(q_3 - 1)(Pq_2 - P - Tq_4) + p_3(-Pq_1q_2 + Pq_2q_3 + Pq_2 - Pq_3 + Rq_2q_4 - Sq_2q_4 + Sq_4)+$$
$$p_4(-Rq_2q_4 + Rq_4 + Sq_2q_4 - Sq_2 - Sq_4 + S + Tq_1q_4 - Tq_3q_4 + Tq_3 - Tq_4)$$

and the expression can be written using a matrix format as Eq. 23.

$$(p_1, p_2, p_3, p_4)\begin{bmatrix} q_1(Pq_2 - P - Tq_4) \\ -(q_3 - 1)(Pq_2 - P - Tq_4) \\ -Pq_1q_2 + Pq_2q_3 + Pq_2 - Pq_3 + Rq_2q_4 - Sq_2q_4 + Sq_4 \\ -Rq_2q_4 + Rq_4 + Sq_2q_4 - Sq_2 - Sq_4 + S + Tq_1q_4 - Tq_3q_4 + Tq_3 - Tq_4 \end{bmatrix} \tag{23}$$

Finally, the constant term of the numerator, which is obtained by substituting $p = (0, 0, 0, 0)$, is given by Eq. 24.

$$-Pq_2 + P + Tq_4 \tag{24}$$

Combining Eq. 22, Eq. 23 and Eq. 24 gives that the numerator of $u_q(p)$ can be written as,

$$\frac{1}{2}p\begin{bmatrix} 0 & -(q_1-q_3)(Pq_2-P-Tq_4) & (q_1-q_2)(Pq_3-Sq_4) & (q_1-q_4)(Sq_2-S-Tq_3) \\ -(q_1-q_3)(Pq_2-P-Tq_4) & 0 & (q_2-q_3)(Pq_1-P-Rq_4) & -(q_3-q_4)(Rq_2-R-Tq_1+T) \\ (q_1-q_2)(Pq_3-Sq_4) & (q_2-q_3)(Pq_1-P-Rq_4) & 0 & (q_2-q_4)(Rq_3-Sq_1+S) \\ (q_1-q_4)(Sq_2-S-Tq_3) & -(q_3-q_4)(Rq_2-R-Tq_1+T) & (q_2-q_4)(Rq_3-Sq_1+S) & 0 \end{bmatrix}p^T +$$

$$\begin{bmatrix} q_1(Pq_2 - P - Tq_4) \\ -(q_3 - 1)(Pq_2 - P - Tq_4) \\ -Pq_1q_2 + Pq_2q_3 + Pq_2 - Pq_3 + Rq_2q_4 - Sq_2q_4 + Sq_4 \\ -Rq_2q_4 + Rq_4 + Sq_2q_4 - Sq_2 - Sq_4 + S + Tq_1q_4 - Tq_3q_4 + Tq_3 - Tq_4 \end{bmatrix}p - Pq_2 + P + Tq_4$$

and equivalently as,

$$\frac{1}{2}pQp^T + cp + a$$

where $Q \in \mathbb{R}^{4 \times 4}$ is a square matrix defined by the transition probabilities of the opponent q_1, q_2, q_3, q_4 as follows:

$$Q = \begin{bmatrix} 0 & -(q_1 - q_3)(Pq_2 - P - Tq_4) & (q_1 - q_2)(Pq_3 - Sq_4) & (q_1 - q_4)(Sq_2 - S - Tq_3) \\ -(q_1 - q_3)(Pq_2 - P - Tq_4) & 0 & (q_2 - q_3)(Pq_1 - P - Rq_4) & -(q_3 - q_4)(Rq_2 - R - Tq_1 + T) \\ (q_1 - q_2)(Pq_3 - Sq_4) & (q_2 - q_3)(Pq_1 - P - Rq_4) & 0 & (q_2 - q_4)(Rq_3 - Sq_1 + S) \\ (q_1 - q_4)(Sq_2 - S - Tq_3) & -(q_3 - q_4)(Rq_2 - R - Tq_1 + T) & (q_2 - q_4)(Rq_3 - Sq_1 + S) & 0 \end{bmatrix},$$

$c \in \mathbb{R}^{4 \times 1}$ is similarly defined by:

$$c = \begin{bmatrix} q_1(Pq_2 - P - Tq_4) \\ -(q_3 - 1)(Pq_2 - P - Tq_4) \\ -Pq_1 q_2 + Pq_2 q_3 + Pq_2 - Pq_3 + Rq_2 q_4 - Sq_2 q_4 + Sq_4 \\ -Rq_2 q_4 + Rq_4 + Sq_2 q_4 - Sq_2 - Sq_4 + S + Tq_1 q_4 - Tq_3 q_4 + Tq_3 - Tq_4 \end{bmatrix},$$

and $a = -Pq_2 + P + Tq_4$.

The same process is done for the denominator. $\qquad\square$

5.2 Theorem 2 Proof

The optimal behaviour of a memory-one strategy player $p^* \in \mathbb{R}^4_{[0,1]}$ against a set of N opponents $\{q^{(1)}, q^{(2)}, \ldots, q^{(N)}\}$ for $q^{(i)} \in \mathbb{R}^4_{[0,1]}$ is given by:

$$p^* = \operatorname*{argmax} \sum_{i=1}^{N} u_q(p), \ p \in S_q.$$

The set S_q is defined as all the possible combinations of:

$$S_q = \left\{ p \in \mathbb{R}^4 \left| \begin{array}{l} \bullet \quad p_j \in \{0, 1\} \quad \text{and} \quad \dfrac{d}{dp_k} \sum_{i=1}^{N} u_q^{(i)}(p) = 0 \\[2mm] \qquad \text{for all} \quad j \in J \quad \& \quad k \in K \quad \text{for all} \quad J, K \\ \qquad \text{where} \quad J \cap K = \emptyset \quad \text{and} \quad J \cup K = \{1, 2, 3, 4\}. \\[2mm] \bullet \quad p \in \{0, 1\}^4 \end{array} \right. \right\}. \tag{25}$$

~~Note that there is no immediate way to find the zeros of $\frac{d}{dp} \sum_{i=1}^{N} u_q(p)$ where,~~

~~For $\frac{d}{dp} \sum_{i=1}^{N} u_q(p)$ to equal zero then:~~

Proof. The ~~optimal behaviour of a memory-one strategy player $p^* \in \mathbb{R}^4_{[0,1]}$ against a set of N opponents $\{q^{(1)}, q^{(2)}, \ldots, q^{(N)}\}$ for $q^{(i)} \in \mathbb{R}^4_{[0,1]}$ is established by:~~

~~where S_q is given by:~~

~~The~~ optimisation problem of Eq. 26

$$\max_p : \sum_{i=1}^{N} u_q^{(i)}(p) \tag{26}$$
$$\text{such that} : p \in \mathbb{R}_{[0,1]}$$

can be written as:

$$\max_p : \sum_{i=1}^{N} u_q^{(i)}(p)$$
$$\text{such that} : p_i \leq 1 \text{ for } \in \{1, 2, 3, 4\} \tag{27}$$
$$-p_i \leq 0 \text{ for } \in \{1, 2, 3, 4\}$$

The optimisation problem has two inequality constraints and regarding the optimality this means that:

- either the optimum is away from the boundary of the optimization domain, and so the constraints plays no role;

- or the optimum is on the constraint boundary.

Thus, the following three cases must be considered:

Case 1: The solution is on the boundary and any of the possible combinations for $p_i \in \{0, 1\}$ for $i \in \{1, 2, 3, 4\}$ are candidate optimal solutions.

Case 2: The optimum is away from the boundary of the optimization domain and the interior solution p^* necessarily satisfies the condition $\frac{d}{dp} \sum_{i=1}^{N} u_q(p^*) = 0$.

Case 3: The optimum is away from the boundary of the optimization domain but some constraints are equalities. The candidate solutions in this case are any combinations of $p_j \in \{0, 1\}$ and $\frac{d}{dp_k} \sum_{i=1}^{N} u_q^{(i)}(p) = 0$ forall $j \in J$ & $k \in K$ forall J, K where $J \cap K = \emptyset$ and $J \cup K = \{1, 2, 3, 4\}$.

Combining cases 1-3 a set of candidate ~~solution~~ solutions, denoted as S_q, is constructed as:

$$S_q = \left\{ p \in \mathbb{R}^4 \left|
\begin{array}{l}
\bullet \quad p_j \in \{0,1\} \quad \text{and} \quad \frac{d}{dp_k} \sum_{i=1}^{N} u_q^{(i)}(p) = 0 \quad \text{for all} \quad j \in J \quad \& \quad k \in K \quad \text{for all} \quad J, K \\
\qquad \text{where} \quad J \cap K = \emptyset \quad \text{and} \quad J \cup K = \{1, 2, 3, 4\}. \\
\bullet \quad p \in \{0,1\}^4
\end{array}
\right. \right\}.$$

~~This set is denoted as S_q and the~~ The derivative of $\sum_{i=1}^{N} u_q^{(i)}(p)$ calculated using the following property (see [2] for details):

$$\frac{dxAx^T}{dx} = 2Ax. \tag{28}$$

Using property (28):

$$\frac{d}{dp}\frac{1}{2}pQp^T + cp + a = pQ + c \text{ and } \frac{d}{dp}\frac{1}{2}p\bar{Q}p^T + \bar{c}p + \bar{a} = p\bar{Q} + \bar{c}. \tag{29}$$

Note that the derivative of cp is c and the constant disappears. Combining these it can be proven that:

$$\frac{d}{dp}\sum_{i=1}^{N} u_q^{(i)}(p) = \sum_{i=1}^{N} \frac{\frac{d}{dp}(\frac{1}{2}pQ^{(i)}p^T + c^{(i)}p + a^{(i)})(\frac{1}{2}p\bar{Q}^{(i)}p^T + c^{\bar{(i)}}p + a^{\bar{(i)}}) - \frac{d}{dp}(\frac{1}{2}p\bar{Q}^{(i)}p^T + c^{\bar{(i)}}p + a^{\bar{(i)}})(\frac{1}{2}pQ^{(i)}p^T + c^{(i)}p + a^{(i)})}{(\frac{1}{2}p\bar{Q}^{(i)}p^T + c^{\bar{(i)}}p + a^{\bar{(i)}})^2}$$

$$= \sum_{i=1}^{N} \frac{(pQ^{(i)} + c^{(i)}+)(\frac{1}{2}p\bar{Q}^{(i)}p^T + c^{\bar{(i)}}p + a^{\bar{(i)}})}{(\frac{1}{2}p\bar{Q}^{(i)}p^T + c^{\bar{(i)}}p + a^{\bar{(i)}})^2} - \frac{(p\bar{Q}^{(i)} + c^{\bar{(i)}})(\frac{1}{2}pQ^{(i)}p^T + c^{(i)}p + a^{(i)})}{(\frac{1}{2}p\bar{Q}^{(i)}p^T + c^{\bar{(i)}}p + a^{\bar{(i)}})^2}$$

For $\frac{d}{dp}\sum_{i=1}^{N} u_q(p)$ to equal zero then:

$$\sum_{i=1}^{N}\left(pQ^{(i)} + c^{(i)}\right)\left(\frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)}p + \bar{a}^{(i)}\right) - \left(p\bar{Q}^{(i)} + \bar{c}^{(i)}\right)\left(\frac{1}{2}pQ^{(i)}p^T + c^{(i)}p + a^{(i)}\right) = 0, \quad while \tag{30}$$

$$\sum_{i=1}^{N}\frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)}p + \bar{a}^{(i)} \neq 0. \tag{31}$$

The optimal solution to Eq. 26 is the point from S_q for which the utility is maximised. $\qquad\square$

6 Lemma 3 Proof

In a tournament of N players $\{q^{(1)}, q^{(2)}, \dots, q^{(N)}\}$ for $q^{(i)} \in \mathbb{R}^4_{[0,1]}$ defection is stable if the transition probabilities of the opponents satisfy conditions Equation (32) and Equation (33).

$$\sum_{i=1}^{N}(c^{(i)T}\bar{a}^{(i)} - \bar{c}^{(i)T}a^{(i)}) \leq 0 \tag{32}$$

while,

$$\sum_{i=1}^{N}\bar{a}^{(i)} \neq 0 \tag{33}$$

Proof. For defection to be stable the derivative of the utility at the point $p = (0, 0, 0, 0)$ must be negative.

Substituting $p = (0,0,0,0)$ in,

$$\frac{d}{dp} \sum_{i=1}^{N} u_q^{(i)}(p) = \sum_{i=1}^{N} \frac{\left(pQ^{(i)} + c^{(i)}\right)\left(\frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)}p + \bar{a}^{(i)}\right)}{\left(\frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)}p + \bar{a}^{(i)}\right)^2} - \frac{\left(p\bar{Q}^{(i)} + \bar{c}^{(i)}\right)\left(\frac{1}{2}pQ^{(i)}p^T + c^{(i)}p + a^{(i)}\right)}{\left(\frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)}p + \bar{a}^{(i)}\right)^2} \quad (34)$$

gives:

$$\frac{d \sum_{i=1}^{N} u_q^{(i)}(p)}{dp}\Bigg|_{p=(0,0,0,0)} = \sum_{i=1}^{N} \frac{\left(c^{(i)}\bar{a}^{(i)} - \bar{c}^{(i)}a^{(i)}\right)}{(\bar{a}^{(i)})^2} \quad (35)$$

The sign of the numerator $\sum_{i=1}^{N} (c^{(i)}\bar{a}^{(i)} - \bar{c}^{(i)}a^{(i)})$ can vary based on the transition probabilities of the opponents. The denominator can not be negative, and otherwise is always positive. Thus the sign of the derivative is negative if and only if $\sum_{i=1}^{N} (c^{(i)}\bar{a}^{(i)} - \bar{c}^{(i)}a^{(i)}) \leq 0$. $\qquad \square$