Memory size in the Prisoner's Dilemma

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Abstract

In this manuscript we build upon a framework provided in 1989 for the study of these strategies and identify the best responses of memory one players. The aim of this work is to show the limitations of memory one strategies in multi-opponent interactions. A number of theoretic results are presented.

1 Introduction

The Prisoner's Dilemma (PD) is a two player person game used in understanding the evolution of co-operative behaviour. Each player can choose between cooperation (C) and defection (D). The decisions are made simultaneously and independently. The normal form representation of the game is given by:

$$S_p = \begin{pmatrix} R & S \\ T & P \end{pmatrix} \quad S_q = \begin{pmatrix} R & T \\ S & P \end{pmatrix} \tag{1}$$

where S_p represents the utilities of the first player and S_q the utilities of the second player. The payoffs, (R, P, S, T), are constrained by equations (2) and (3). Constrain (2) ensures that defection dominates cooperation and constrain (3) ensures that there is a dilemma. Because the sum of the utilities for both players is better when both choose cooperation. The most common values used in the literature are $(3, 1, 0, 5)$ [3].

$$T > R > P > S \tag{2}$$

$$2R > T + S \tag{3}$$

The PD is a one shot game, however it is commonly studied in a manner where the history of the interactions matters. The repeated form of the game is called the Iterated Prisoner's Dilemma (IPD) and in the 1980s following the work of [4, 5] it attracted the attention of the scientific community.

In [4] a computer tournament of the IPD was performed. A tournament is a series of rounds of the PD between pairs of strategies. The topology commonly used, [4, 5], is that of a round robin where all contestants compete against each other. The winner of these tournaments was decided on the average score and not in the number of wins.

These tournaments were the milestones of an era which to today is using computer tournaments to explore the robustness of strategies of IPD. The robustness can also be checked through evolutionary process [14]. However, this aspect will not be considered here, instead the focus is on performance in tournaments.

In Axelrod's original tournaments [4, 5], strategies were allowed access to the history and in the first tournament they also knew the number of total turns in each interaction. The history included the previous moves of both the player and the opponent. How many turns of history that a strategy would use, the memory size, was left to the creator of the strategy to decide. For example the winning strategy of the first tournaments, Tit for Tat was a strategy that made use of the previous move of the opponent only. Tit for Tat is a strategy that starts by cooperating and then mimics the previous action of it's opponent. Strategies like Tit for Tat are called memory one strategies. A framework for studying memory one strategies was introduced in [12] and further used in [11, 13].

In [15] Press and Dyson, introduced a new set of memory one strategies called zero determinant (ZD) strategies. The ZD strategies, manage to force a linear relationship between the score of the strategy and the opponent. Press and Dyson, prove their concept of the ZD strategies and claim that a ZD strategy can outperform any given opponent.

The ZD strategies have tracked a lot of attention. It was stated that "Press and Dyson have fundamentally changed the viewpoint on the Prisoner's Dilemma" [16]. In [16], the Axelrod's tournament have been re-run including ZD strategies and a new set of ZD strategies the Generous ZD. Even so, ZD and memory one strategies have also received criticism. In [10], the 'memory of a strategy does not matter' statement was questioned. A set of more complex strategies, strategies that take in account the entire history set of the game, were trained and proven to be more robust than ZD strategies.

2 Problem

The purpose of this work is to consider a given memory one strategy in a similar fashion to [15]. However whilst [15] found a way for a player to manipulate an opponent, this work will consider an optimisation approach to identify the best response to that opponent. In essence the aim is to produce a compact method of identifying the best memory one strategy against a given opponent.

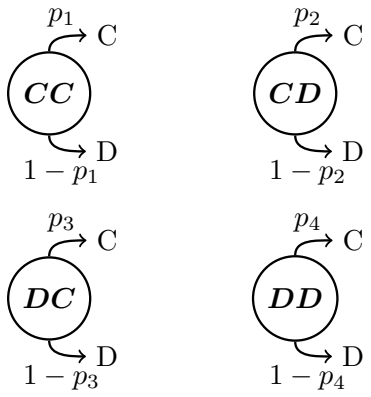
The second part of this manuscript we explore the limitation of the best response memory one strategies by comparing them to more complex strategies with a larger memory.

2.1 Background

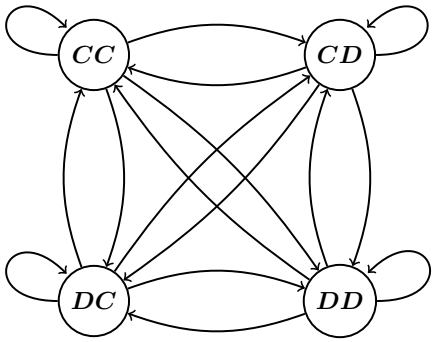
In this manuscript we explore the robustness of memory one strategies. A memory one strategy is defined as a strategy that decides it's action in turn m based on what occurred in turn $m-1$. If a strategy is concerned with only the outcome of a single turn then there are four possible 'states' the strategy could be in. These are CC, CD, DC, CC. A memory one strategy is denoted by the probabilities of cooperating after each of these states, $p = p_1, p_2, p_3, p_4 \in \mathbb{R}^4_{[0,1]}$. A diagrammatic representation of such as strategy is given in Figure 1a.

In [13] a framework was introduced to study the interactions of memory one strategies modelled as a stochastic process, where the players move from one of the states CC, CD, DC, CC to another. More specifically, it can be modelled by the use of a Markov process of four states, shown by Figure 1b.

The transition matrix of the markov chain in Figure 1b is defined as M and is given by,



(a) Diagrammatic representation of a memory one strategy.



(b) Markov chain on a PD game.

$$M = \begin{bmatrix} p_1q_1 & p_1\left(-q_1+1\right) & q_1\left(-p_1+1\right) & \left(-p_1+1\right)\left(-q_1+1\right) \\ p_2q_3 & p_2\left(-q_3+1\right) & q_3\left(-p_2+1\right) & \left(-p_2+1\right)\left(-q_3+1\right) \\ p_3q_2 & p_3\left(-q_2+1\right) & q_2\left(-p_3+1\right) & \left(-p_3+1\right)\left(-q_2+1\right) \\ p_4q_4 & p_4\left(-q_4+1\right) & q_4\left(-p_4+1\right) & \left(-p_4+1\right)\left(-q_4+1\right) \end{bmatrix}. \tag{4}$$

Let the vector of the stationary probabilities of M be defined as v. Vector v are given in the Appendix. The scores of each player can be retrieved by multiplying the probabilities of each state, at the stationary state, with the equivalent payoff. Thus, the utility for player p against q, denoted as $u_q(p)$, is defined by,

$$u_q(p) = v \times S_p. \tag{5}$$

2.2 Utility

The analytical formulation gives the advantage of time. That is because the payoffs of a match between two opponents are now retrievable without simulating the actual match itself.

Note though that $u_q(p)$ is a function of 4 variables which is also affected by the transition probabilities of the opponent q. The first theoretical result that we introduce in this work is a compact way of writing $u_q(p)$. This is given by the Theorem 1.

Theorem 1 *For a given memory one strategy $p \in \mathbb{R}^4_{[0,1]}$ playing another memory one strategy $q \in \mathbb{R}^4_{[0,1]}$, the utility of the player $u_q(p)$ can be re written as a ratio of two quadratic forms:*

$$u_q(p) = \frac{\frac{1}{2}p^TQp + c^Tp + a}{\frac{1}{2}p^T\bar{Q}p + \bar{c}^Tp + \bar{a}}, \tag{6}$$

where $Q, \bar{Q}$ are matrices of 4×4 defined with the transition probabilities of the opponent's transition probabilities q_1, q_2, q_3, q_4.

$$Q = \begin{bmatrix} \mathbf{0} & \mathbf{-(q_1-q_3)(q_2-5q_4-1)} & \mathbf{q_3(q_1-q_2)} & \mathbf{-5q_3(q_1-q_4)} \\ \mathbf{-(q_1-q_3)(q_2-5q_4-1)} & \mathbf{0} & \mathbf{(q_2-q_3)(q_1-3q_4-1)} & \mathbf{(q_3-q_4)(5q_1-3q_2-2)} \\ \mathbf{q_3(q_1-q_2)} & \mathbf{(q_2-q_3)(q_1-3q_4-1)} & \mathbf{0} & \mathbf{3q_3(q_2-q_4)} \\ \mathbf{-5q_3(q_1-q_4)} & \mathbf{(q_3-q_4)(5q_1-3q_2-2)} & \mathbf{3q_3(q_2-q_4)} & \mathbf{0} \end{bmatrix}, \tag{7}$$

$$\bar{Q} = \begin{bmatrix} \mathbf{0} & \mathbf{-(q_1-q_3)(q_2-q_4-1)} & \mathbf{(q_1-q_2)(q_3-q_4)} & \mathbf{(q_1-q_4)(q_2-q_3-1)} \\ \mathbf{-(q_1-q_3)(q_2-q_4-1)} & \mathbf{0} & \mathbf{(q_2-q_3)(q_1-q_4-1)} & \mathbf{(q_1-q_2)(q_3-q_4)} \\ \mathbf{(q_1-q_2)(q_3-q_4)} & \mathbf{(q_2-q_3)(q_1-q_4-1)} & \mathbf{0} & \mathbf{-(q_2-q_4)(q_1-q_3-1)} \\ \mathbf{(q_1-q_4)(q_2-q_3-1)} & \mathbf{(q_1-q_2)(q_3-q_4)} & \mathbf{-(q_2-q_4)(q_1-q_3-1)} & \mathbf{0} \end{bmatrix}. \tag{8}$$

c and $\bar{c}$, are 4×1 vectors defined by the transition rates q_1, q_2, q_3, q_4.

$$c = \begin{bmatrix} q_1(q_2-5q_4-1) \\ -(q_3-1)(q_2-5q_4-1) \\ -q_1q_2+q_2q_3+3q_2q_4+q_2-q_3 \\ 5q_1q_4-3q_2q_4-5q_3q_4+5q_3-2q_4 \end{bmatrix}, \tag{9}$$

$$\bar{c} = \begin{bmatrix} q_1(q_2-q_4-1) \\ -(q_3-1)(q_2-q_4-1) \\ -q_1q_2+q_2q_3+q_2-q_3+q_4 \\ q_1q_4-q_2-q_3q_4+q_3-q_4+1 \end{bmatrix}. \tag{10}$$

Lastly, $a = -q_2 + 5q_4 + 1$ and $\bar{a} = -q_2 + q_4 + 1$.

2.3 Validation

In this section we validate the formulation of Theorem 1 using numerical experiments. All the simulated results of this work are done using [1] which is an open research framework for the study of the IPD. This package is described in [9].

To validate the formulation of $u_q(p)$ several memory one players were matched against 20 opponents.The simulated value of $u_q(p)$ has been calculated using [1] and the theoretical by substituting in equation (6).

In Figure 2, both the simulated and the theoretical value of $u_q(p)$, against each opponent, are plotted for three different memory one strategies. Figure 2 indicates that the formulation of $u_q(p)$ as a quadratic ratio successfully captures the simulated behaviour.

3 Best responses Analytically

In the introduction a question was raised: which memory one strategy is the **best response** against another memory one? This will be considered as an optimisation problem, where a memory one strategy p wants to optimise it's utility $u_q(p)$ against an opponent q. The decision variable is the vector p and the solitary constrains $p \in \mathbb{R}^4_{[0,1]}$. The optimisation problem is given by (11).

$$\begin{aligned} \max_{p} : \quad & \frac{\frac{1}{2}pQp^T + c^Tp + a}{\frac{1}{2}p\bar{Q}p^T + \bar{c}^Tp + \bar{a}} \\ \text{such that} : \quad & p \in \mathbb{R}^4_{[0,1]}. \end{aligned} \tag{11}$$

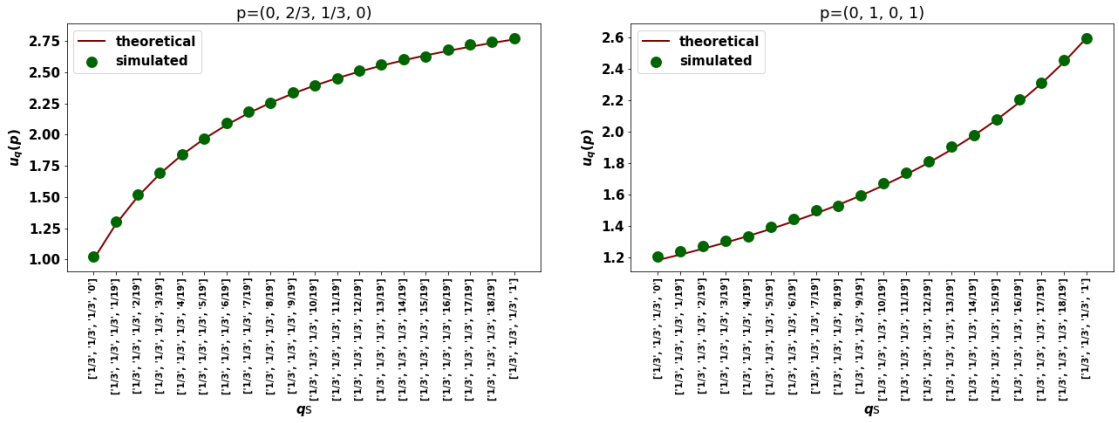


Figure 2: Differences between simulated and analytical results.

3.1 Convexity

This work is concerned with a fractional optimisation problem of quadratic forms. Initially, the convexity, whether or not $u_q(p)$ is concave [8], will be checked (concave because is a maximisation problem).

To test the hypothesis that $u_q(p)$ is concave an empirical analysis was performed using computer code. It was shown that there exists at least one point for which the definition of concavity does not hold. Optimising a non concave function is rather tricky.

Several articles in fractional optimisation of quadratic forms that was non concave can be found [6, 7]. Though in these works both the numerator and denominator of the fractional problem were concave. In [2] it is stated that a quadratic form will be concave if and only if it's symmetric matrix is negative semi definite.

In Appendix, it is proved that neither the numerator or the denominator of equation (6) are concave.

3.2 Proof

The non concavity of $u(p)$ indicates multiple local optimal points. Thus a compact way of searching the candidate optimal points needs to be introduced. Once the method is defined then the utility of each point is compared to the rest. The optimal point is the point that has the highest value of $u(p)$.

The problem considered is a bounded, this mean that the we know that the candinate solutions will be either on the bounds of our feasable solution in a point in the center. Note that the points are the roots of the derivate $\frac{du}{dp}$. Bound case are all the possible combinations of $p_1, p_2, p_3, p_4 \in \{0, 1\}$. Figure 3 illustrates an example of possible locations of candidate solutions for a 3 dimensional problem.

The derivate $\frac{du}{dp}$ is given by,

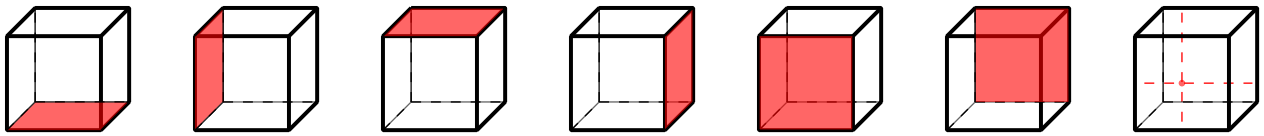


Figure 3: Candidate solution of a 3 dimensional problem.

$$\begin{aligned}\frac{du}{dp} &= \frac{(\frac{1}{2}pQp^T + c^Tp + a)'(\frac{1}{2}p\bar{Q}p^T + \bar{c}^Tp + \bar{a}) - (\frac{1}{2}p\bar{Q}p^T + \bar{c}^Tp + \bar{a})'(\frac{1}{2}pQp^T + c^Tp + a)}{(\frac{1}{2}p\bar{Q}p^T + \bar{c}^Tp + \bar{a})^2} \\ &= \frac{(pQ + c^T)(\frac{1}{2}p\bar{Q}p^T + \bar{c}^Tp + \bar{a}) - (p\bar{Q} + \bar{c}^T)(\frac{1}{2}pQp^T + c^Tp + a)}{(\frac{1}{2}p\bar{Q}p^T + \bar{c}^Tp + \bar{a})^2}\end{aligned} \tag{12}$$

Thus we conclude that the best response of a memory one strategy in matchs is given by Lemma 2.

Lemma 2 *The optimal behaviour of a memory one strategy player* (p^*) *against a given opponent* q *is given by:*

$$p^* = \text{argmax}(u_q(p)),\ p \in S_q,$$

where the set S_q *is defined as*

$$S_q = \{0, \bar{p}, 1\}^4$$

where the vector $\bar{p}$ *is the vector for which the following condition is true:*

$$(pQ + c^T)(\frac{1}{2}p\bar{Q}p^T + \bar{c}^Tp + \bar{a}) - (p\bar{Q} + \bar{c}^T)(\frac{1}{2}pQp^T + c^Tp + a) = 0$$

Note that this is a $4-$ *polynomial system of* 4 *variables. Each polynomial corresponds to a partial derivative of* $u_q(p)$*.*

The method of Lagrange Multipliers and Karush-Kuhn-Tucker conditions has also been applied and reached the same conclusion. The Karush-Kuhn-Tucker conditions are used becase ourconstrains are inequalities.

A question that arises immediately after the work that has been carried out to this point is the following: What is the optimal memory player against multiple opponents, in a tournament environment.

Let us consider a collection of opponents: $\{q^{(1)}, q^{(2)}, \ldots, q^{(N)}\}$, finding the optimal behaviour of a strategy can now be written as:

$$\begin{aligned} \max_{p} : & \frac{1}{N}\sum_{i=1}^{N} {u_q}^{(i)}(p) \\ st : & p_1 = p_2 = p_3 = p_4 = p \\ & p \in \mathbb{R}_{[0,1]} \end{aligned} \tag{13}$$

Thus, the average utility against a set of opponents is maximised and the average utility is given by,

$$\frac{1}{N}\sum_{i=1}^{N} {u_q}^{(i)}(p) = \frac{1}{N}\frac{\sum_{i=1}^{N}(\frac{1}{2}pQ^{(i)}p^T + c^{(i)T}p + a^{(i)}) \prod_{\substack{j=1 \\ j\neq i}}^{N}(\frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)T}p + \bar{a}^{(i)})}{\prod_{i=1}^{N}(\frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)T}p + \bar{a}^{(i)})}. \tag{14}$$

If we were to follow a similar approach to that of pairwise interactions the derivative of the utility would be calculated and set to zero. Furthemore, the bound cases would we explored. The derivate of the average utility is given by,

$$\begin{aligned} &\frac{d}{dp}\frac{1}{N}\sum_{i=1}^{N} {u_q}^{(i)}(p) = \\ &= \frac{(\sum_{i=1}^{N} Q_N^{(i)'} \prod_{\substack{j=1 \\ j\neq i}}^{N} Q_D^{(i)} + \sum_{i=1}^{N} Q_D^{(i)'} \sum_{\substack{j=1 \\ j\neq i}}^{N} Q_N^{(i)} \prod_{\substack{l=1 \\ l\neq i \\ l\neq j}}^{N} Q_D^{(i)}) \times \prod_{i=1}^{N} Q_D^{(i)} - (\sum_{i=1}^{N} Q_D^{(i)'} \prod_{\substack{j=1 \\ j\neq i}}^{N} Q_D^{(i)}) \times (\sum_{i=1}^{N} Q_N^{(i)} \prod_{\substack{j=1 \\ j\neq i}}^{N} Q_D^{(i)})}{(\prod_{i=1}^{N} Q_D^{(i)})^2} \end{aligned}$$

where,

$$\begin{aligned} Q_N^{(i)} &= \frac{1}{2}pQ^{(i)}p^T + c^{(i)T}p + a^{(i)}, \\ Q_N^{(i)'} &= pQ^{(i)} + c^{(i)T}, \\ Q_D^{(i)} &= \frac{1}{2}p\bar{Q}^{(i)}p^T + \bar{c}^{(i)T}p + \bar{a}^{(i)}, \\ Q_D^{(i)'} &= p\bar{Q}^{(i)} + \bar{c}^{(i)T}. \end{aligned}$$

Note that neither problems can be further handled analytically. For pairwise interactions, the roots of the derivate are given by solving a multivariare $4-$ system of $4-$ variables. The problem's size is gradually increasing every time an extra opponent is taken into account.

Thus no further analytical consideration is given to these problem in this work. In the following sections numerical methods are introduced which will be used to discover best responses. Moreover, several constrain versions of our problem and their exact solutions are discussed.

4 Numerical Experiments

In this section we introcude several numerical methods to supplement our analytical approach.

Initally, several insights and exact methods considered in this work to identify best responses in further constained problems are discussed. These further constrained problems taken into account subsets of memeory one strategies, which are

- purely random strategies
- reactive strategies.

Purely random strategies are a set of memory one strategies where the transition probabilities of each state are the same. The optimisation problem of (11) now has an extra constraint and is re written as,

$$\begin{aligned} \max_{p} &: u_q(p) \\ \text{such that} &: p_1 = p_2 = p_3 = p_4 = p \\ &0 \leq p \leq 1. \end{aligned} \tag{15}$$

The utility for a random player is now a function of a signle unknown. For the random player in an N interactions enviroment the folliwing algorithm allow us to retrieve the exact best responose.

Algorithm 1 Best response algorithm for purely random strategies

1: **procedure** PURELY RANDOM SEARCH
2: $N \leftarrow$ number of opponets
3: $S_q \leftarrow \{0, 1\}$
4: *loop* $i = 1$ to $2N$:
5: $S_q \cup \bar{p}_i$ for $\frac{du}{d\bar{p}_i} = 0$.
6: $i \leftarrow i + 1$.
7: **goto** *loop*.
8: **close**;
9: $p^* \leftarrow \text{argmax}(u_q(p)), p \in S_q$.

Numerical experiments are perform and they suggest that algorithm 1 has managed to caputer the optimal behaviour. The results of the numerical experiments are given by Figure 4.

For the case of the purely random players two more theoretical results are discussed. These are the cases where:

- the oppponent has manage to make us indifferent
- the best behaviour is pure strategy.

The results are given equivilently by Lemmas 3 and 4 and they are respective to the actions of the opponent.

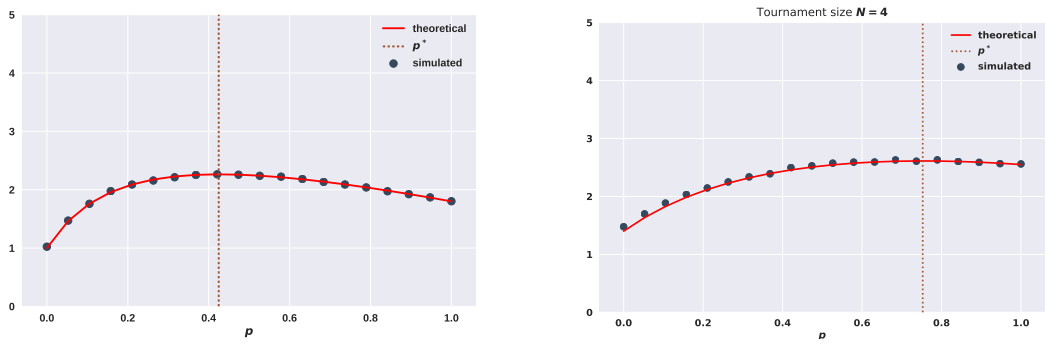


Figure 4: Numerical experiments for algorthim 1.

Lemma 3 *A given memory one player,* (q_1, q_2, q_3, q_4)*, makes a* ***purely random*** *player,* (p, p, p, p)*, indifferent if and only if,* $-q_1+q_2+2q_3-2q_4 = 0$ *and* $(q_2-q_4-1)(q_1-2q_2-5q_3+7q_4+1)-(q_2-5q_4-1)(q_1-q_2-q_3+q_4) = 0$.

Lemma 4 *Against a memory one player,* (q_1, q_2, q_3, q_4)*, a* ***purely random*** *player would always play a pure strategy if and only if* $(q_1q_4 - q_2q_3 + q_3 - q_4)(4q_1 - 3q_2 - 4q_3 + 3q_4 - 1) = 0$.

Reactive strategies are a set of memory one strategies where they only take into account the opponents's previous moves. The optimisation problem of (11) now has an extra constraint and is re written as,

$$\begin{aligned} \max_p &: u_q(p) \\ \text{such that} &: p_1 = p_3 \text{ and } p_2 = p_4 \\ &0 \leq p_1, p_2 \leq 1. \end{aligned} \tag{16}$$

The following algorithm is used.

Note that resultant theory is a field of algebraic geometry and Sylvester's formulation is a common resultant used for systems of 2 equations. We suggested that several other resultant can be used here but Sylberster; has been used for simplisiti and speed reasons.

A numerical experiment suggests that the best response behvaiour is captured by our algorithm.

The numerical methods results.

5 Limitation of memory

6 Stability of defection

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