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Supplementary Information for

Stability of defection, optimisation of strategies and the limits of memory in the Prisoner's Dilemma.

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Supplementary text

Supporting Information Text

Proof of Theorem 2. The utility of a memory one player p against an opponent q, $u_q(p)$, can be written as a ratio of two quadratic forms on R^4.

Proof. It was discussed that $u_q(p)$ it's the product of the steady states v and the PD payoffs,

$$u_q(p) = v \cdot (R, S, T, P).$$

More specifically, with $(R, P, S, T) = (3, 1, 0, 5)$

$$u_q(p) = \left(\frac{\begin{gathered} p_1p_2(q_1q_2 - 5q_1q_4 - q_1 - q_2q_3 + 5q_3q_4 + q_3) + p_1p_3(-q_1q_3 + q_2q_3) + p_1p_4(5q_1q_3 - 5q_3q_4) + p_3p_4(-3q_2q_3 + 3q_3q_4)+ \\ p_2p_3(-q_1q_2 + q_1q_3 + 3q_2q_4 + q_2 - 3q_3q_4 - q_3) + p_2p_4(-5q_1q_3 + 5q_1q_4 + 3q_2q_3 - 3q_2q_4 + 2q_3 - 2q_4)+ \\ p_1(-q_1q_2 + 5q_1q_4 + q_1) + p_2(q_2q_3 - q_2 - 5q_3q_4 - q_3 + 5q_4 + 1) + p_3(q_1q_2 - q_2q_3 - 3q_2q_4 - q_2 + q_3)+ \\ p_4(-5q_1q_4 + 3q_2q_4 + 5q_3q_4 - 5q_3 + 2q_4) + q_2 - 5q_4 - 1 \end{gathered}}{\begin{gathered} p_1p_2(q_1q_2 - q_1q_4 - q_1 - q_2q_3 + q_3q_4 + q_3) + p_1p_3(-q_1q_3 + q_1q_4 + q_2q_3 - q_2q_4) + p_1p_4(-q_1q_2 + q_1q_3 + q_1 + q_2q_4 - q_3q_4 - q_4)+ \\ p_2p_3(-q_1q_2 + q_1q_3 + q_2q_4 + q_2 - q_3q_4 - q_3) + p_2p_4(-q_1q_3 + q_1q_4 + q_2q_3 - q_2q_4) + p_3p_4(q_1q_2 - q_1q_4 - q_2q_3 - q_2 + q_3q_4 + q_4)+ \\ p_1(-q_1q_2 + q_1q_4 + q_1) + p_2(q_2q_3 - q_2 - q_3q_4 - q_3 + q_4 + 1) + p_3(q_1q_2 - q_2q_3 - q_2 + q_3 - q_4) + p_4(-q_1q_4 + q_2 + q_3q_4 - q_3 + q_4 - 1)+ \\ q_2 - q_4 - 1 \end{gathered}} \right). \quad [1]$$

Let us consider the numerator of the $u_q(p)$. The cross product terms p_ip_j are given by,

$$\begin{gathered} p_1p_2(q_1q_2 - 5q_1q_4 - q_1 - q_2q_3 + 5q_3q_4 + q_3) + p_1p_3(-q_1q_3 + q_2q_3) + p_1p_4(5q_1q_3 - 5q_3q_4) + p_3p_4(-3q_2q_3 + 3q_3q_4)+ \\ p_2p_3(-q_1q_2 + q_1q_3 + 3q_2q_4 + q_2 - 3q_3q_4 - q_3) + p_2p_4(-5q_1q_3 + 5q_1q_4 + 3q_2q_3 - 3q_2q_4 + 2q_3 - 2q_4). \end{gathered}$$

This can be re written in a matrix format given by Eq. 2.

$$(p_1, p_2, p_3, p_4)\frac{1}{2}\begin{bmatrix} 0 & -(q_1 - q_3)(q_2 - 5q_4 - 1) & q_3(q_1 - q_2) & -5q_3(q_1 - q_4) \\ -(q_1 - q_3)(q_2 - 5q_4 - 1) & 0 & (q_2 - q_3)(q_1 - 3q_4 - 1) & (q_3 - q_4)(5q_1 - 3q_2 - 2) \\ q_3(q_1 - q_2) & (q_2 - q_3)(q_1 - 3q_4 - 1) & 0 & 3q_3(q_2 - q_4) \\ -5q_3(q_1 - q_4) & (q_3 - q_4)(5q_1 - 3q_2 - 2) & 3q_3(q_2 - q_4) & 0 \end{bmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} \quad [2]$$

Similarly, the linear terms are given by,

$$\begin{gathered} p_1(-q_1q_2 + 5q_1q_4 + q_1) + p_2(q_2q_3 - q_2 - 5q_3q_4 - q_3 + 5q_4 + 1) + p_3(q_1q_2 - q_2q_3 - 3q_2q_4 - q_2 + q_3)+ \\ p_4(-5q_1q_4 + 3q_2q_4 + 5q_3q_4 - 5q_3 + 2q_4). \end{gathered}$$

and the expression can be written using a matrix format as Eq. 3.

$$(p_1, p_2, p_3, p_4)\begin{bmatrix} q_1(q_2 - 5q_4 - 1) \\ -(q_3 - 1)(q_2 - 5q_4 - 1) \\ -q_1q_2 + q_2q_3 + 3q_2q_4 + q_2 - q_3 \\ 5q_1q_4 - 3q_2q_4 - 5q_3q_4 + 5q_3 - 2q_4 \end{bmatrix} \quad [3]$$

Finally, the constant term of the numerator, which is obtained by substituting $p = (0, 0, 0, 0)$, is given by Eq. 4.

$$q_2 - 5q_4 - 1 \quad [4]$$

Combining Eq. 2, Eq. 3 and Eq. 4 gives that the numerator of $u_q(p)$ can be written as,

$$\begin{gathered} \frac{1}{2}p\begin{bmatrix} 0 & -(q_1 - q_3)(q_2 - 5q_4 - 1) & q_3(q_1 - q_2) & -5q_3(q_1 - q_4) \\ -(q_1 - q_3)(q_2 - 5q_4 - 1) & 0 & (q_2 - q_3)(q_1 - 3q_4 - 1) & (q_3 - q_4)(5q_1 - 3q_2 - 2) \\ q_3(q_1 - q_2) & (q_2 - q_3)(q_1 - 3q_4 - 1) & 0 & 3q_3(q_2 - q_4) \\ -5q_3(q_1 - q_4) & (q_3 - q_4)(5q_1 - 3q_2 - 2) & 3q_3(q_2 - q_4) & 0 \end{bmatrix}p^T + \\ \begin{bmatrix} 0 & -(q_1 - q_3)(q_2 - 5q_4 - 1) & q_3(q_1 - q_2) & -5q_3(q_1 - q_4) \\ -(q_1 - q_3)(q_2 - 5q_4 - 1) & 0 & (q_2 - q_3)(q_1 - 3q_4 - 1) & (q_3 - q_4)(5q_1 - 3q_2 - 2) \\ q_3(q_1 - q_2) & (q_2 - q_3)(q_1 - 3q_4 - 1) & 0 & 3q_3(q_2 - q_4) \\ -5q_3(q_1 - q_4) & (q_3 - q_4)(5q_1 - 3q_2 - 2) & 3q_3(q_2 - q_4) & 0 \end{bmatrix}p + q_2 - 5q_4 - 1 \end{gathered}$$

and equivalently as,

$$\frac{1}{2}pQp^T + cp + a$$

where $Q \in \mathbb{R}^{4\times4}$ is a square matrix defined by the transition probabilities of the opponent q_1, q_2, q_3, q_4 as follows:

$$Q = \begin{bmatrix} 0 & -(q_1 - q_3)(q_2 - 5q_4 - 1) & q_3(q_1 - q_2) & -5q_3(q_1 - q_4) \\ -(q_1 - q_3)(q_2 - 5q_4 - 1) & 0 & (q_2 - q_3)(q_1 - 3q_4 - 1) & (q_3 - q_4)(5q_1 - 3q_2 - 2) \\ q_3(q_1 - q_2) & (q_2 - q_3)(q_1 - 3q_4 - 1) & 0 & 3q_3(q_2 - q_4) \\ -5q_3(q_1 - q_4) & (q_3 - q_4)(5q_1 - 3q_2 - 2) & 3q_3(q_2 - q_4) & 0 \end{bmatrix},$$

$c \in \mathbb{R}^{4\times1}$ is similarly defined by:

$$c = \begin{bmatrix} q_1(q_2 - 5q_4 - 1) \\ -(q_3 - 1)(q_2 - 5q_4 - 1) \\ -q_1q_2 + q_2q_3 + 3q_2q_4 + q_2 - q_3 \\ 5q_1q_4 - 3q_2q_4 - 5q_3q_4 + 5q_3 - 2q_4 \end{bmatrix},$$

and $a = -q_2 + 5q_4 + 1$.

The same process is done for the denominator. $\square$

Proof of Theorem 3.

Proof. The optimal behaviour of a memory-one strategy player $p^* \in \mathbb{R}^4_{[0,1]}$ against a set of N opponents $\{q^{(1)}, q^{(2)}, \dots, q^{(N)}\}$ for $q^{(i)} \in \mathbb{R}^4_{[0,1]}$ is established by:

$$p^* = \text{argmax}\left(\sum_{i=1}^{N} u_q(p)\right), \ p \in S_q,$$

where S_q is given by:

$$S_q = \left\{ p \in \mathbb{R}^4 \left| \begin{array}{l} \bullet \quad p_j \in \{0,1\} \quad \text{and} \quad \dfrac{d}{dp_k}\displaystyle\sum_{i=1}^{N} u_q^{(i)}(p) = 0 \\ \qquad\qquad\quad \text{for all} \quad j \in J \quad \& \quad k \in K \quad \text{for all} \quad J, K \\ \qquad\qquad\quad \text{where} \quad J \cap K = \text{‰} \quad \text{and} \quad J \cup K = \{1,2,3,4\}. \\ \bullet \quad p \in \{0,1\}^4 \end{array} \right. \right\}. \quad [5]$$

The optimisation problem of Eq. 6

$$\begin{aligned} \max_p : & \sum_{i=1}^{N} {u_q}^{(i)}(p) \\ \text{such that} : & \ p \in \mathbb{R}_{[0,1]} \end{aligned} \quad [6]$$

can be written as:

$$\begin{aligned} \max_p : & \sum_{i=1}^{N} {u_q}^{(i)}(p) \\ \text{such that} : & \ p_i \leq 1 \text{ for } \ \in \{1,2,3,4\} \\ & -p_i \leq 0 \text{ for } \ \in \{1,2,3,4\} \end{aligned} \quad [7]$$

The optimisation problem has two inequality constraints and regarding the optimality this means that:

- either the optimum is away from the boundary of the optimization domain, and so the constraints plays no role;
- or the optimum is on the constraint boundary.

Thus, the following three cases must be considered:

Case 1: The solution is on the boundary and any of the possible combinations for $p_i \in \{0,1\}$ for $i \in \{1,2,3,4\}$ are candidate optimal solutions.

Case 2: The optimum is away from the boundary of the optimization domain and the interior solution p^* necessarily satisfies the condition $\frac{d}{dp}\sum\limits_{i=1}^{N} u_q(p^*) = 0$.

Case 3: The optimum is away from the boundary of the optimization domain but some constraints are equalities. The candidate solutions in this case are any combinations of $p_j \in \{0,1\}$ and $\frac{d}{dp_k}\sum\limits_{i=1}^{N} u_q^{(i)}(p) = 0$ forall $j \in J$ & $k \in K$ forall J, K where $J \cap K = \text{‰}$ and $J \cup K = \{1,2,3,4\}$.

Combining cases 1-3 a set of candidate solution is constructed as:

$$S_q = \left\{ p \in \mathbb{R}^4 \left| \begin{array}{l} \bullet \quad p_j \in \{0,1\} \quad \text{and} \quad \dfrac{d}{dp_k}\displaystyle\sum_{i=1}^{N} u_q^{(i)}(p) = 0 \quad \text{for all} \quad j \in J \quad \& \quad k \in K \quad \text{for all} \quad J, K \\ \qquad\qquad\quad \text{where} \quad J \cap K = \text{‰} \quad \text{and} \quad J \cup K = \{1,2,3,4\}. \\ \bullet \quad p \in \{0,1\}^4 \end{array} \right. \right\}.$$

This set is denoted as S_q and the optimal solution to Eq. 6 is the point from S_q for which the utility is maximised. $\square$