Stability of defection, optimisation of strategies and the limits of memory in the Prisoner's Dilemma.

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Abstract

1 Introduction

The Prisoner's Dilemma (PD) is a two player game used in understanding the evolution of co-operative behaviour, formally introduced in [9]. Each player has two options, to cooperate (C) or to defect (D). The decisions are made simultaneously and independently. The normal form representation of the game is given by:

$$S_p = \begin{pmatrix} R & S \\ T & P \end{pmatrix} \quad S_q = \begin{pmatrix} R & T \\ S & P \end{pmatrix} \tag{1}$$

where S_p represents the utilities of the row player and S_q the utilities of the column player. The payoffs, (R, P, S, T), are constrained by equations (2) and (3). Constraint (2) ensures that defection dominates cooperation and constraint (3) ensures that there is a dilemma; the sum of the utilities for both players is better when both choose to cooperate. The most common values used in the literature are $(3, 1, 0, 5)$ [3].

$$T > R > P > S \tag{2}$$

$$2R > T + S \tag{3}$$

The PD is a one shot game, however it is commonly studied in a manner where the history of the interactions matters. The repeated form of the game is called the Iterated Prisoner's Dilemma (IPD) and in the 1980s, following the work of [4, 5] it attracted the attention of the scientific community. In [4] and [5], the first well known computer tournaments of the IPD were performed. A total of 13 and 63 strategies were submitted respectively in the form of computer code. The contestants competed against each other, a copy of themselves and a random strategy. The winner was then decided on the average score a strategy achieved (not the total number of wins). The contestants were given access to the entire history of a match, however, how many turns of history a strategy would incorporate, refereed to as the *memory size* of a strategy, was a result of the particular strategic decisions made by the author.

The winning strategy of both tournaments was the strategy called Tit for Tat. Tit for Tat starts by cooperating and then mimics the last move of it's opponent, more specifically, it is a strategy that considers only the previous move of the opponent. These type of strategies are called *reactive* [19] and are a subset of so called *memory one* strategies. Memory one strategies similarly only consider the previous turn, however, they incorporate both players' recent moves. As the name suggests memory one strategies have a memory of size 1.

Several successful reactive strategies and memory one are found in the literature, such as Generous Tit For Tat [20] and Pavlov [16]. However, memory one strategies generated a small shock in the game theoretic community ([23] stated that "Press and Dyson have fundamentally changed the viewpoint on the Prisoner's Dilemma") when a cernent set of memory one strategies was introduced in [22]. These strategies are called zero determinate (ZD) and they chose their actions so that a linear relationship is forced between their score and that of the opponent. ZD strategies are indeed mathematically unique and are proven to be robust in pairwise interactions.

The purpose of this work is to consider a given memory one strategy in a similar fashion to [22], however whilst [22] found a way for a player to manipulate a given opponent, this work will consider a multidimensional optimisation approach to identify the best response to a group of opponents. The main findings are:

- A compact method of identifying the best memory one strategy against a given set of opponents.
- A well designed framework that allows the comparison of an optimal memory one strategy, and a more complex strategy that has a larger memory and was obtained through contemporary reinforcement learning techniques.
- An identification of conditions for which defection is known to be a best response; thus identifying environments where cooperation can not occur.

2 The utility

One specific advantage of memory one strategies is their mathematical tractability. They can be represented completely as a vector of $\mathbb{R}^4$. This originates from [19] where it is stated that if a strategy is concerned with only the outcome of a single turn then there are four possible 'states' the strategy could be in; CC, CD, DC, CC. Therefore, a memory one strategy can be denoted by the probability vector of cooperating after each of these states; $p = (p_1, p_2, p_3, p_4) \in \mathbb{R}^4_{[0,1]}$. In an IPD match two memory one strategies are moving from state to state, at each turn with a given probability. This exact behaviour can be modelled as a stochastic process, and more specifically as a Markov chain (Figure 1). The corresponding transition matrix M of Figure 1 is given below,

The long run steady state probability v is the solution to $vM = v$. The stationary vector v can be combined with the payoff matrices of equation (1) and the expected payoffs for each player can be estimated without simulating the actual interactions. More specifically, the utility for a memory one strategy p against an opponent q, denoted as $u_q(p)$, is defined by,

$$u_q(p) = v \times (R, P, S, T). \tag{4}$$

In Theorem 1, the first theoretical results of the manuscript is presented, that is that $u_q(p)$ is given by a ratio of two quadratic forms [12]. To the authors knowledge this is the first work that has been done on the form of $u_q(p)$.

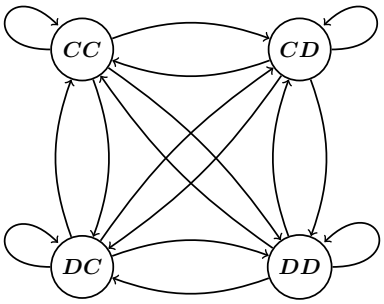


Figure 1: markov

$$M = \begin{bmatrix} p_1q_1 & p_1\left(-q_1+1\right) & q_1\left(-p_1+1\right) & \left(-p_1+1\right)\left(-q_1+1\right) \\ p_2q_3 & p_2\left(-q_3+1\right) & q_3\left(-p_2+1\right) & \left(-p_2+1\right)\left(-q_3+1\right) \\ p_3q_2 & p_3\left(-q_2+1\right) & q_2\left(-p_3+1\right) & \left(-p_3+1\right)\left(-q_2+1\right) \\ p_4q_4 & p_4\left(-q_4+1\right) & q_4\left(-p_4+1\right) & \left(-p_4+1\right)\left(-q_4+1\right) \end{bmatrix}$$

Theorem 1. *The expected utility of a memory one strategy $p \in \mathbb{R}^4_{[0,1]}$ against a memory one opponent $q \in \mathbb{R}^4_{[0,1]}$, denoted as $u_q(p)$, can be written as a ratio of two quadratic forms:*

$$u_q(p) = \frac{\frac{1}{2}pQp^T + cp + a}{\frac{1}{2}p\bar{Q}p^T + \bar{c}p + \bar{a}}, \tag{5}$$

where $Q, \bar{Q} \in \mathbb{R}^{4\times4}$ are hollow matrices defined by the transition probabilities of the opponent q_1, q_2, q_3, q_4 as follows:

$$\boldsymbol{Q} = \begin{bmatrix} \mathbf{0} & \mathbf{-\left(q_1-q_3\right)\left(q_2-5q_4-1\right)} & \mathbf{q_3\left(q_1-q_2\right)} & \mathbf{-5q_3\left(q_1-q_4\right)} \\ \mathbf{-\left(q_1-q_3\right)\left(q_2-5q_4-1\right)} & \mathbf{0} & \mathbf{\left(q_2-q_3\right)\left(q_1-3q_4-1\right)} & \mathbf{\left(q_3-q_4\right)\left(5q_1-3q_2-2\right)} \\ \mathbf{q_3\left(q_1-q_2\right)} & \mathbf{\left(q_2-q_3\right)\left(q_1-3q_4-1\right)} & \mathbf{0} & \mathbf{3q_3\left(q_2-q_4\right)} \\ \mathbf{-5q_3\left(q_1-q_4\right)} & \mathbf{\left(q_3-q_4\right)\left(5q_1-3q_2-2\right)} & \mathbf{3q_3\left(q_2-q_4\right)} & \mathbf{0} \end{bmatrix}, \tag{6}$$

$$\bar{\boldsymbol{Q}} = \begin{bmatrix} \mathbf{0} & \mathbf{-\left(q_1-q_3\right)\left(q_2-q_4-1\right)} & \mathbf{\left(q_1-q_2\right)\left(q_3-q_4\right)} & \mathbf{\left(q_1-q_4\right)\left(q_2-q_3-1\right)} \\ \mathbf{-\left(q_1-q_3\right)\left(q_2-q_4-1\right)} & \mathbf{0} & \mathbf{\left(q_2-q_3\right)\left(q_1-q_4-1\right)} & \mathbf{\left(q_1-q_2\right)\left(q_3-q_4\right)} \\ \mathbf{\left(q_1-q_2\right)\left(q_3-q_4\right)} & \mathbf{\left(q_2-q_3\right)\left(q_1-q_4-1\right)} & \mathbf{0} & \mathbf{-\left(q_2-q_4\right)\left(q_1-q_3-1\right)} \\ \mathbf{\left(q_1-q_4\right)\left(q_2-q_3-1\right)} & \mathbf{\left(q_1-q_2\right)\left(q_3-q_4\right)} & \mathbf{-\left(q_2-q_4\right)\left(q_1-q_3-1\right)} & \mathbf{0} \end{bmatrix}. \tag{7}$$

c and $\bar{c} \in \mathbb{R}^{4\times1}$ are similarly defined by:

$$\boldsymbol{c} = \begin{bmatrix} \mathbf{q_1\left(q_2-5q_4-1\right)} \\ \mathbf{-\left(q_3-1\right)\left(q_2-5q_4-1\right)} \\ \mathbf{-q_1q_2+q_2q_3+3q_2q_4+q_2-q_3} \\ \mathbf{5q_1q_4-3q_2q_4-5q_3q_4+5q_3-2q_4} \end{bmatrix}, \tag{8}$$

$$\bar{\boldsymbol{c}} = \begin{bmatrix} \mathbf{q_1\left(q_2-q_4-1\right)} \\ \mathbf{-\left(q_3-1\right)\left(q_2-q_4-1\right)} \\ \mathbf{-q_1q_2+q_2q_3+q_2-q_3+q_4} \\ \mathbf{q_1q_4-q_2-q_3q_4+q_3-q_4+1} \end{bmatrix}. \tag{9}$$

and $a = -q_2 + 5q_4 + 1$ and $\bar{a} = -q_2 + q_4 + 1$.

The proof of Theorem 1 is given in Appendix.

Numerical simulations have been carried out to validate the formulation of $u_q(p)$ as a quadratic ratio, a data set is available at. Two examples are graphically represented in Figure 2 and show that the formulation successfully captures the simulated behaviour. The simulated utility, which is denoted as $U_q(p)$, has been calculated using [1], an open source research framework for the study of the IPD. The project is described in [13].

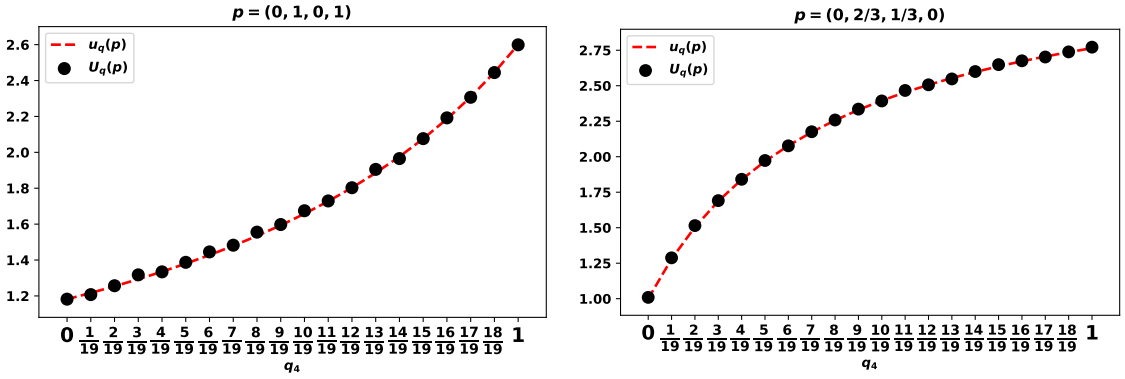


Figure 2: Differences between simulated and analytical results for $q = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, q_4)$.

Theorem 1 can be extended to consider multiple opponents. The IPD is commonly studied in tournaments and/or Moran Processes where a strategy interacts with a number of opponents. The payoff of a player in such interactions is given by the average payoff the player received against each opponent. More specifically the expected utility of a memory one strategy against a N number of opponents is given by Theorem 2.

Theorem 2. *The expected utility of a memory one strategy $p \in \mathbb{R}^4_{[0,1]}$ against a group of opponents $q^{(1)}, q^{(2)}, \ldots, q^{(N)}$, denoted as $\frac{1}{N}\sum_{i=1}^{N} {u_q}^{(i)}(p)$ is given by:*

$$\frac{1}{N}\sum_{i=1}^{N} {u_q}^{(i)}(p) = \frac{1}{N} \frac{\sum_{i=1}^{N} (\frac{1}{2} p Q^{(i)} p^T + c^{(i)} p + a^{(i)}) \prod_{\substack{j=1 \\ j \neq i}}^{N} (\frac{1}{2} p \bar{Q}^{(i)} p^T + \bar{c}^{(i)} p + \bar{a}^{(i)})}{\prod_{i=1}^{N} (\frac{1}{2} p \bar{Q}^{(i)} p^T + \bar{c}^{(i)} p + \bar{a}^{(i)})}. \tag{10}$$

As an illustration, Theorem 2 is used to calculate the theoretical payoffs of several memory one strategies against a set of 10 opponents. The opponents used are the memory one strategies for the tournament conducted in [23]; the names of the strategic rules are given by Table 1. Figure 3 provides evidence that the values of $\frac{1}{N}\sum_{i=1}^{N} u_{q^{(i)}}(p)$ and $\frac{1}{N}\sum_{i=1}^{N} U_{q^{(i)}}(p)$ match.

Note that it was explored whether the the utility against a group of strategies could be captured by the utility against the mean opponent. Thus, finding the best response to a given group of opponents correspond to finding the best response to a single player, the mean player formed by the probabilities of that given group. The hypothesis however fails, as:

$$\frac{1}{N}\sum_{i=1}^{N} {u_q}^{(i)}(p) \neq u_{\frac{1}{N}\sum_{i=1}^{N} q^{(i)}}(p). \tag{11}$$

	Name	Memory one representation	Reference
1	Cooperator	$(1, 1, 1, 1)$	[3]
2	Defector	$(0, 0, 0, 0)$	[3]
3	Random	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	[3]
4	Tit for Tat	$(1, 0, 1, 0)$	[3]
5	Grudger	$(1, 0, 0, 0)$	[15]
6	Generous Tit for Tat	$(1, \frac{1}{3}, 1, \frac{1}{3})$	[20]
7	Win Stay Lose Shift	$(1, 0, 0, 1)$	[16]
8	ZDGTFT2	$(1, \frac{1}{8}, 1, \frac{1}{4})$	[23]
9	ZDExtort2	$(\frac{8}{9}, \frac{1}{2}, \frac{1}{3}, 0)$	[23]
10	Hard Joss	$(\frac{9}{10}, 0, \frac{9}{10}, 0)$	[23]

Table 1: List of strategies used in the tournament described in [23].

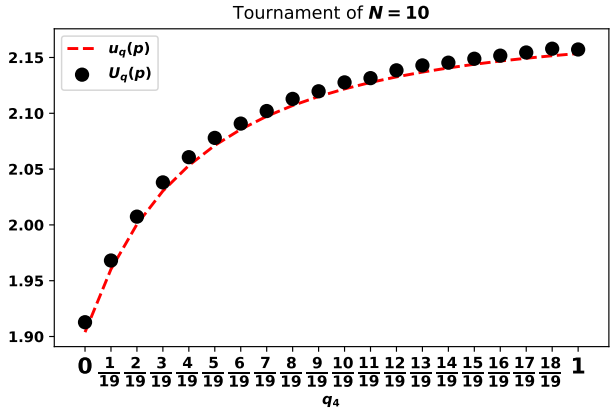


Figure 3: Results of memory one strategies against the strategies in Table 1.

which is captured by Figure 4.

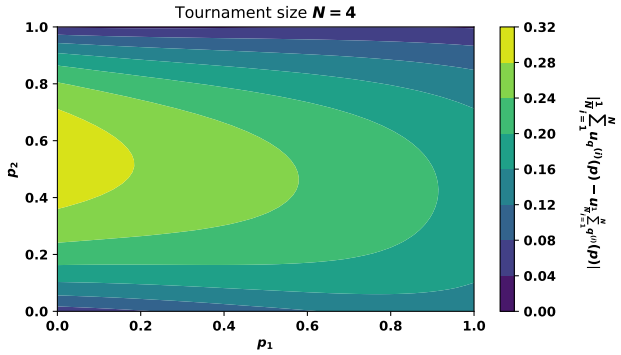


Figure 4: The difference between the average utility against 10 opponents and the utility against the average player of these 10 opponents is plotted. It is clear that the hypothesis fails for this case as the absolute difference is manly positive for the example. The hypothesis fails fro at least one case, thus it can not be assumed to be true.

In the following section best responses are introduced and explored for the case of memory one strategies. Moreover, in the following sections several other theoretical results and presented and the advantages of analytical formulation of Theorem 2 become evident.

3 Best responses to memory one players

In game theory a *best response* is the strategy which corresponds to the most favourable outcome. In this manuscript a best response memory one strategy corresponds to the p^* for which $\sum u_{q^{(i)}}(p^*)$ for $i \in \{1, \ldots, N\}$ is maximized. This is considered as a multi dimensional optimisation problem where the decision variable is the vector p, the solitary constraint is that $p \in \mathbb{R}^4_{[0,1]}$ and the objective function is a sum of quadratic ratios. The optimisation problem is formally given by (12).

$$\begin{aligned} &\max_p : \sum_{i=1}^{N} {u_q}^{(i)}(p) \\ &\text{such that} : p \in \mathbb{R}_{[0,1]} \end{aligned} \tag{12}$$

Optimising this particular ratio of quadratic forms is not trivial. It can be verified empirically for the case of a single opponent that there exist at least one point for which the definition of concavity does not hold. There is some work on the optimisation on non concave ratios of of quadratic forms [6, 8], in these both the numerator and the denominator of the fractional problem were concave or that the denominator was greater than zero. Both assumptions fails for (12). These results are established in Theorem 3.

Theorem 3. *The utility of a player p against an opponent q, $u_q(p)$ given by (5), is not concave. Furthermore neither the numeration or the denominator of (5), are concave.*

Proof. A function $f(x)$ is said to be concave on an interval $[a, b]$ if, for any points x_1 and $x_2 \in [a, b]$, the function $-f(x)$ is convex on that interval.

A function $f(x)$ is convex on an interval $[a, b]$ if for any two points x_1 and x_2 in $[a, b]$ and any λ where $0 < \lambda < 1$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2). \tag{13}$$

Let f be $u_{(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3})}$ it can be shown that for $x_1 = (\frac{1}{4}, \frac{1}{2}, \frac{1}{5}, \frac{1}{2}), x_2 = (\frac{8}{10}, \frac{1}{2}, \frac{9}{10}, \frac{7}{10})$ and $\lambda = \frac{1}{10}$ condition (13) does not hold as: $1.49 \geq 1.48$.

In [2] it is stated that a quadratic form will be concave if and only if it's symmetric matrix is negative semi definite. A matrix$A$ is semi-negative definite if:

$$|A|_i \leq 0 \text{ for } i \text{ is odd and } |A|_i \geq 0 \text{ for } i \text{ is even.} \tag{14}$$

For both Q and $\bar{Q}$ it is exhibited that for $i = 2$ (odd):

$$|Q|_2 = -\left(q_1 - q_3\right)^2 \left(q_2 - 5q_4 - 1\right)^2,$$
$$|\bar{Q}|_2 = -\left(q_1 - q_3\right)^2 \left(q_2 - q_4 - 1\right)^2$$

both determinants are negative, thus the concavity condition (14) fails for both quadratic forms. □

The non concavity of $u(p)$ indicates multiple local optimal points. The approach taken here is to introduce a compact way of constructing the candidate set of all local optimal points. Once the set is defined the point that maximises (10) corresponds to the best response strategy, this approach transforms the continuous optimisation problem in to a discrete problem. The problem considered is a bounded because $p \in \mathbb{R}^4_{[0,1]}$. The candidate solutions will exist either at the boundaries of the feasible solution space, or within that space. The method of Lagrange Multipliers [7] and Karush-Kuhn-Tucker conditions [10] are based on this. The Karush-Kuhn-Tucker conditions are used here because the constraints are inequalities. These lead to Lemma 4 which presents the best response memory one strategy to a group of opponents.

Lemma 4. *The optimal behaviour of a memory one strategy player $p^* \in \mathbb{R}^4_{[0,1]}$ against a set of N opponents $\{q^{(1)}, q^{(2)}, \ldots, q^{(N)}\}$ for $q^{(i)} \in \mathbb{R}^4_{[0,1]}$ is established by:*

$$p^* = \text{argmax}(\sum_{i=1}^{N} u_q(p)),\ p \in S_q,$$

where the set S_q is defined as

$$S_q = \left\{ \bar{p} \in \mathbb{R}^4 \;\middle|\; \begin{array}{c} \bar{p}_k \in \{0,1\} \text{ for all } k \in \{1,2,3,4\} \text{ or } F(\bar{p}) = 0 \\ \prod_{i=1}^{N} Q_D^{(i)} \neq 0 \end{array} \right\}$$

where,

$$F(p) = (\sum_{i=1}^{N} Q_N^{(i)'} \prod_{\substack{j=1 \\ j\neq i}}^{N} Q_D^{(i)} + \sum_{i=1}^{N} Q_D^{(i)'} \sum_{\substack{j=1 \\ j\neq i}}^{N} Q_N^{(i)} \prod_{\substack{j=1 \\ j\neq\{i,j\}}}^{N} Q_D^{(i)}) \times \prod_{i=1}^{N} Q_D^{(i)} - (\sum_{i=1}^{N} Q_D^{(i)'} \prod_{\substack{j=1 \\ j\neq i}}^{N} Q_D^{(i)}) \times (\sum_{i=1}^{N} Q_N^{(i)} \prod_{\substack{j=1 \\ j\neq i}}^{N} Q_D^{(i)}) \tag{15}$$

and,

$$\begin{aligned} Q_N^{(i)} &= \frac{1}{2} p Q^{(i)} p^T + c^{(i)} p + a^{(i)}, \\ Q_N^{(i)'} &= p Q^{(i)} + c^{(i)}, \\ Q_D^{(i)} &= \frac{1}{2} p \bar{Q}^{(i)} p^T + \bar{c}^{(i)} p + \bar{a}^{(i)}, \\ Q_D^{(i)'} &= p \bar{Q}^{(i)} + \bar{c}^{(i)}. \end{aligned}$$

Proof. The best response of a memory one strategy against a group of memory one strategies can captured by a candidate set of behaviours. This candidate set is constructed by considering behaviours where any or all of p_1, p_2, p_3, p_4 are $\in \{0, 1\}$ and the rest or all of p_1, p_2, p_3, p_4 are given by roots of the partial derivatives.

Note that for $p_i \in \{0, 1\}$ we consider the roots of the partial derivatives for $p_j \neq p_i$ for $i, j \in [1, 4]$. The derivatives, $\frac{d \sum u}{dp}$, are given by,

$$\frac{d}{dp} \sum_{i=1}^{N} {u_q}^{(i)}(p) = \\ = \frac{(\sum_{i=1}^{N} Q_N^{(i)'} \prod_{\substack{j=1 \\ j\neq i}}^{N} Q_D^{(i)} + \sum_{i=1}^{N} Q_D^{(i)'} \sum_{\substack{j=1 \\ j\neq i}}^{N} Q_N^{(i)} \prod_{\substack{j=1 \\ j\neq\{i,j\}}}^{N} Q_D^{(i)}) \times \prod_{i=1}^{N} Q_D^{(i)} - (\sum_{i=1}^{N} Q_D^{(i)'} y - vk \prod_{\substack{j=1 \\ j\neq i}}^{N} Q_D^{(i)}) \times (\sum_{i=1}^{N} Q_N^{(i)} \prod_{\substack{j=1 \\ j\neq i}}^{N} Q_D^{(i)})}{(\prod_{i=1}^{N} Q_D^{(i)})^2} \tag{16}$$

For equation 16 to be zero, the numerator must fall to zero and the denominator can not nullified. One the candidate set is constructed each point is evaluated using equation (10). The point with the maximum utility is selected. □

A special case of Lemma 4 is for $N = 1$, thus when a strategy plays against a single opponent. In this case the formulation of Theorem 1 is used and the best response is captured by Lemma 5.

Lemma 5. *The optimal behaviour of a memory one strategy player $p^* \in \mathbb{R}^4_{[0,1]}$ against a given opponent $q \in \mathbb{R}^4_{[0,1]}$ is given by:*

$$p^* = \text{argmax}(u_q(p)),\ p \in S_q,$$

where the set S_q is defined as

$$S_q = \{0, \bar{p}_i, 1\}^4 \text{ for } i \in \mathbb{R},$$

where any $\bar{p}$ satisfy conditions:

$$(\bar{p}Q + c)(\frac{1}{2}\bar{p}\bar{Q}\bar{p}^T + \bar{c}\bar{p} + \bar{a}) - (\bar{p}\bar{Q} + \bar{c})(\frac{1}{2}\bar{p}Q\bar{p}^T + c\bar{p} + a) = 0 \tag{17}$$

and

$$\frac{1}{2}\bar{p}\bar{Q}\bar{p}^T + \bar{c}\bar{p} + \bar{a} \neq 0 \tag{18}$$

Proof. The best response of a memory one strategy against another memory one strategy can captured by a candidate set of behaviours. This candidate set is constructed by considering behaviours where any or all of p_1, p_2, p_3, p_4 are $\in \{0, 1\}$ and the rest or all of p_1, p_2, p_3, p_4 are given by roots of the partial derivatives.

Note that for $p_i \in \{0, 1\}$ we consider the roots of the partial derivatives for $p_j \neq p_i$ for $i, j \in [1, 4]$. The derivatives, $\frac{du}{dp}$, are given by,

$$\frac{du_q(p)}{dp} = \frac{(pQ + c)(\frac{1}{2}p\bar{Q}p^T + \bar{c}p + \bar{a}) - (p\bar{Q} + \bar{c})(\frac{1}{2}pQp^T + cp + a)}{(\frac{1}{2}p\bar{Q}p^T + \bar{c}p + \bar{a})^2} \tag{19}$$

For equation 16 to be zero, the numerator must fall to zero and the denominator can not be zero. □

Equation (17) is systems of at most 4 polynomials and the degree of the polynomials is gradually increasing every time an extra opponent is taken into account. Solving system of polynomials corresponds to the calculation of a resultant and for large systems these quickly become intractable. Because of that no further analytical consideration is given to problems described here

4 Stability of defection

Defection is known to be the dominant action in the PD and it can be proven to be the dominant strategy for the IPD for given environments. Even so, several works have proven that cooperation emerges in the IPD and many studies focus on the emergence of cooperation. This manuscript provides an identification of conditions for which defection is known to be a best response; thus identifying environments where cooperation can not occur, Lemma 6.

Lemma 6. *In a tournament of N players where $q^{(i)} = (q_1^{(i)}, q_2^{(i)}, q_3^{(i)}, q_4^{(i)})$ defection is a best response if the transition probabilities of the opponents satisfy the condition:*

$$\sum_{i=1}^{N} (c^{(i)T}\bar{a}^{(i)} - \bar{c}^{(i)T}a^{(i)}) \leq 0 \tag{20}$$

Proof. For defection to be evolutionary stable the derivative of the utility at the point $p = (0, 0, 0, 0)$ must be negative. This would indicate that the utility function is only declining from that point onwards.

Substituting $p = (0, 0, 0, 0)$ in equation (16) which gives:

$$\sum_{i=1}^{N}(c^{(i)T}\bar{a}^{(i)} - \bar{c}^{(i)T}a^{(i)}) \prod_{\substack{j=1 \\ j\neq i}}^{N} (\bar{a}^{(i)})^2 \tag{21}$$

The second term $\prod_{\substack{j=1 \\ j\neq i}}^{N} (\bar{a}^{(i)})^2$ is always positive,however, the sign of the first term $\sum_{i=1}^{N}(c^{(i)T}\bar{a}^{(i)} - \bar{c}^{(i)T}a^{(i)})$ can vary based on the transition probabilities of the opponents. Thus the sign of the derivative is negative if and only if $\sum_{i=1}^{N}(c^{(i)T}\bar{a}^{(i)} - \bar{c}^{(i)T}a^{(i)}) \leq 0$. □

A further constrained version of Lemma 6, is for single interactions while the opponent is a reactive player. Defection is known to be stable in such interactions by the condition given in Lemma 7.

Lemma 7. *Defection is the best responses of a memory one player p against a reactive player q if the transition probabilities of the opponent satisfy the condition:*

$$4q_1 - 5q_2 - 1 > 0 \tag{22}$$

Proof. Initially, consider equation (19) for $p = (0, 0, 0, 0)$,

$$\frac{du}{dp_{|p=(0,0,0,0)}} = \quad \frac{c\bar{a} - \bar{c}a}{\bar{a}^2}. \tag{23}$$

The numerator $\bar{c}a - c\bar{a}$ is given by,

$$\begin{bmatrix} 0 \\ 0 \\ q_4\left(4q_1q_2 - 3q_2^2 - 4q_2q_3 + 3q_2q_4 + 4q_3 - 5q_4 - 1\right) \\ -\left(q_2 - 1\right)\left(4q_1q_4 - 3q_2q_4 + q_2 - 4q_3q_4 + 4q_3 + 3q_4^2 - 6q_4 - 1\right) \end{bmatrix}$$

and the denominator $\bar{a}^2 = (-q_2 + q_4 + 1)^2$, which is always positive. In order for defection to be the best response the derivative must have a negative sign at the point $p = (0, 0, 0, 0)$. That means that the utility is only decreasing after $p = (0, 0, 0, 0)$.

Because $\bar{a}^2$ is always positive the sign of the derivative is given by $\bar{c}a - c\bar{a}$. More specifically from equations,

$$q_4\left(4q_1q_2 - 3q_2^2 - 4q_2q_3 + 3q_2q_4 + 4q_3 - 5q_4 - 1\right) \tag{24}$$

$$-\left(q_2 - 1\right)\left(4q_1q_4 - 3q_2q_4 + q_2 - 4q_3q_4 + 4q_3 + 3q_4^2 - 6q_4 - 1\right) \tag{25}$$

Both signs of the partial derivatives must be negative in order for the overall function to be decreasing ensuring defection is a best response. The signs of equations (24) and (25) vary. There are cases that they have the same sign and cases that they do not,

Moreover lets us consider a constrained version of the problem once again. Lets us assume that in an pairwise interaction the opponent is a reactive player $q = (q_1, q_2, q_1, q_2)$. By substituting $q_3 = q_1$ and $q_4 = q_2$ equations (24) and (25) are now re written as follow,

$$\begin{bmatrix} -q_2\left(4q_1 - 5q_2 - 1\right) \\ \left(q_2 - 1\right)\left(4q_1 - 5q_2 - 1\right) \end{bmatrix}$$

□

5 Numerical experiments and Results

In this section best responses are explored numerically. Best responses are estimated using the Bayesian optimisation algorithm. Bayesian optimisation is a global optimisation algorithm, introduced in [17], which has proven to outperform many other popular algorithms [11]. Differential evolution had also been considered, however it was not chosen due to Bayesian being computationally faster.

The algorithm can be used to solve the problem of (12). Consider an example where $N = 2$, for a given number of calls the algorithm tries to find p^* such that the utility is maximized. Figure 5 illustrates the change of the utility function over number of calls.

The default number of iterations that have been used in this work is 60. After the 60 calls the convergence of the utility is checked. If it is not then the calls are increased by 20, this step is then repeated until the utility reaches convergence.

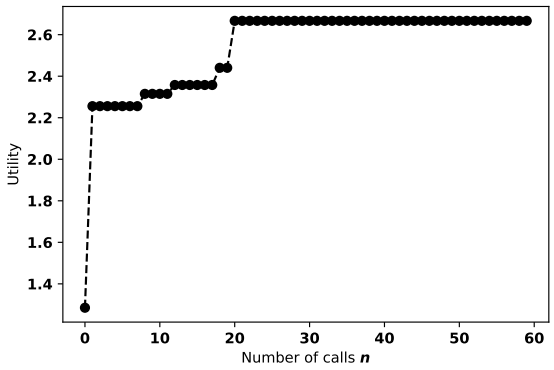


Figure 5: Utility over time of calls using Bayesian optimisation. The opponents are $q^{(1)} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $q^{(2)} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

A series of experiments are carried out and Bayesian optimisation is used to estimate the best responses for different settings. Such as:

- Memory one best responses to $N = 2$ opponents.
- Memory one best responses in evolutionary dynamics.

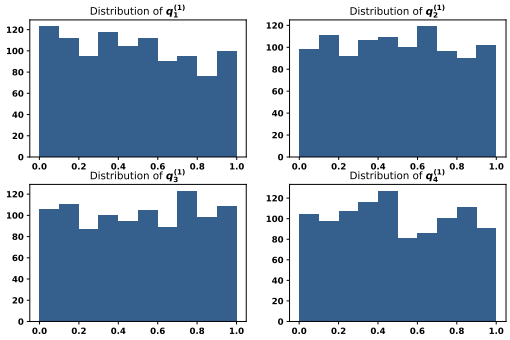
- Longer memory best responses.

The details of the experiments and their results are presented in the following subsections.

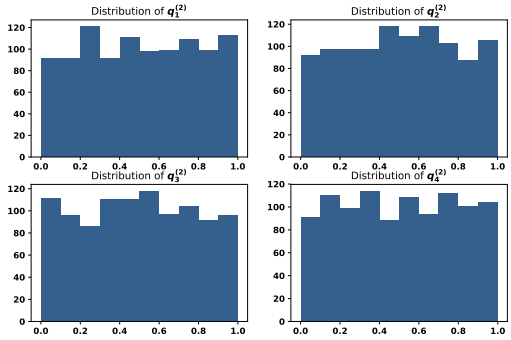
5.1 Memory one best responses for $N = 2$

The first experiment that has been carried out is estimating the memory one best responses in a tournament with two opponents. $N = 2$ has been chosen as it's the smallest size of a tournament. For each trial of the experiment a set of 2 opponents is randomly generated, the memory one best response against them is calculated and it's behaviour is being recorded. A total of 1000 trials have been recorded. The data have archived and found at.

Though the probabilities q_i of the opponents are randomly generated, Figure 6b shows that they are uniformly distributed. Thus, the space full space of possible opponents has been covered.



(a) Distributions of opponents' probabilities.



(b) Distributions of opponents' probabilities.

Section 1 discussed ZD strategies and their robustness against a single opponent. ZD strategies control their opponent's score and thus are able to always receive a higher payoff than them. In tournament settings, where more than 2 strategies interact, winning at each single interaction does not guaranty a strategy's overall win. This manuscript argues that ZD do not receive the most favoured payoff from their interactions and thus come sort in a tournament setting environment.

Best responses are strategies that aim to gain the most of their entire opponent set. In [Knight 2019] a method of measuring the extortionate behaviour of a strategy, based on it's estimated probabilities, has been defined. The method allows to estimate the error of behaving as a ZD strategy, SSerror. The SSerror method is applied on the data set of this work. The distribution of the error is shown in Figure 9 and a statistics summary by Table 3 (outliers have been removed).

The mean and median of the SSerror are estimated at 1.5. With a tolerance of 0.1, only 25 percent of the memory one best responses appear to be behaving in an extortionate way. Moreover, a positive measure of skewness ($= 0.94$), indicates that the distribution is skewed to the right.

Overall only a very small percentage of the best responses seem to behaving in an extortionate way, confirming our original hypothesis. That ZD strategies do not gain the maximum outcome from their interactions. The following section the second experiment and the result of memory one best responses in evolutionary dynamics are presented.

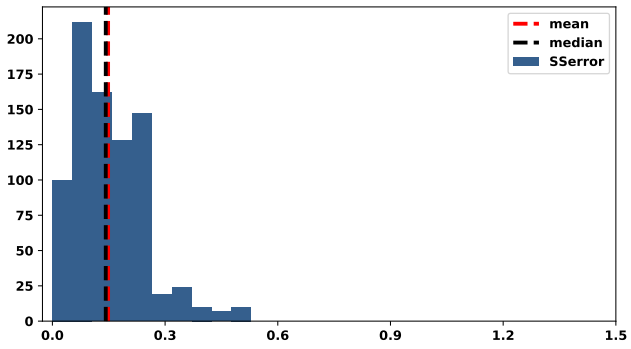


Figure 7: Distribution of sserrors for memory one best responses, when $N = 2$.

	SSerror
count	819.000000
mean	0.148246
std	0.100002
min	0.000000
25%	0.058824
35%	0.093944
50%	0.142050
75%	0.217360
85%	0.235294
max	0.529412

Table 2: Summary statistics SSerror

5.2 Memory one best responses in evolutionary dynamics

As briefly discussed in Section 2, the IPD is commonly studied in Moran Processes, and generally in evolutionary processes. In evolutionary processes, a finite population is assumed where the strategies that compose the population can adapt and change their behaviour based on the outcomes of their interactions at each turn. A key in successfully being an evolution stable strategy (ESS) is self interactions. An ESS must be a best response not only to the opponents in the population, but also it has to be a best response to it's self.

Self interactions can easily be incorporated in the formulation that has been used in this paper. The utility of a memory one strategy in an evolutionary setting is given by,

$$\frac{1}{N}\sum_{i=1}^{N} {u_q}^{(i)}(p) + u_p(p). \tag{26}$$

and respectively the optimisation problem of (12) is now re written as,

$$\begin{aligned} &\max_p : \frac{1}{N}\sum_{i=1}^{N} {u_q}^{(i)}(p) + u_p(p) \\ &\text{such that} : p \in \mathbb{R}_{[0,1]} \end{aligned} \tag{27}$$

Due to the new term being added to the utility, the assumption that has been made so far regarding the form of the utility now fails. The utility is not a ratio of two quadratic forms any more. Furthermore, a new method for identifying an evolutionary best response is composed in this Section. The method considered is called *best response dynamics*, and the algorithm describing the method is given by Algorithm **??**.

Best response dynamics are commonly used in evolutionary game theory. Best response dynamics represent a class of strategy updating rules, where players strategies in the next round are determined by their best responses to some subset of the population, whether this might be in a large population model such as Moran Processes [14] or in a spatial model [21]. Moreover, in the theory of potential games, best response dynamics refers to a way of finding a pure Nash equilibrium by computing the best response for every player [18]. Here

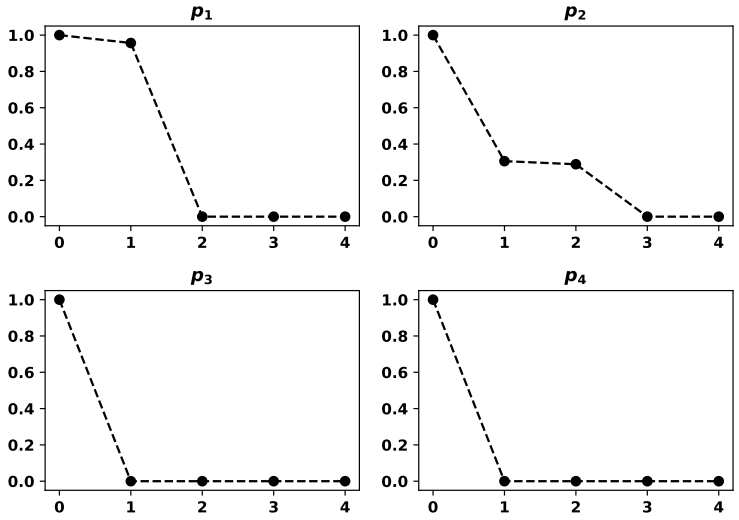


Figure 8: Best response dynamics with $N = 2$. More specifically, for $q^{(1)} = (0.2360, 0.1031, 0.3960, 0.1549)$ and $q^{(2)} = (0.0665, 0.4015, 0.9179, 0.8004)$.

we defined a combination of the two methods.

$p^{(t)} \leftarrow (1, 1, 1, 1)$;
while $p^{(t)}$ *not converged* **do**
  $p^{(t+1)} = \text{argmax} \frac{1}{N} \sum_{i=1}^{N} {u_q}^{(i)}(p+1) + u_p(p+1)$;
end

Algorithm 1: Best response dynamics Algorithm

Numerical simulations have been carried out to visualise how the algorithm convergences after a few iterations. In Figure 8, the results are illustrated. The algorithm have been set to start from the point $(1, 1, 1, 1)$. A more optimal point could be considered, but it has been shown that the algorithm converges to same optimal solution for different initial starts. Moreover, in Figure 8 it is shown that the algorithm stops once the shame point has been evaluated.

Using the same opponents space as described in Section 5.1, memory one best evolutionary responses have also been estimated for each trial. The SSerror for the EVOs is shown in figure 9 and a statistical summary is given by Table 3 (outliers have been removed).

The mean and median of the EVO SSerror are estimated at 1.5, same to the memory one best responses. It's confirm using a ttest that there is no significant difference between the means (p-value = 0). A higher value of skewness indicates a slightly bigger shift to the right.

Similarly to the previous results EVOs do not appear to behave in an extortionate way. An overview of the difference in behaviour of best responses and EVOs can be studied by examining the vector probabilities. In Figure **??** the distributions of each of the cooperating probabilities after each of the four states is given. The plot has also been evaluated for the 95% percentile of the distributions.

The difference betwee the

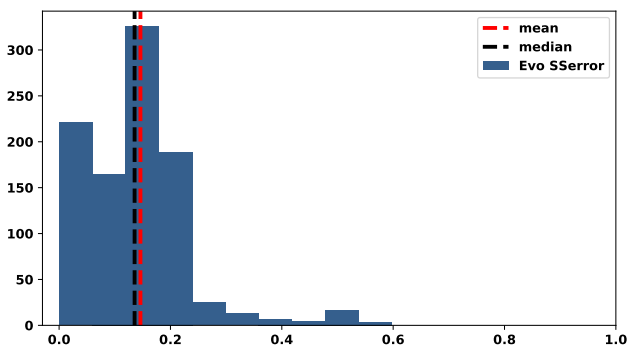
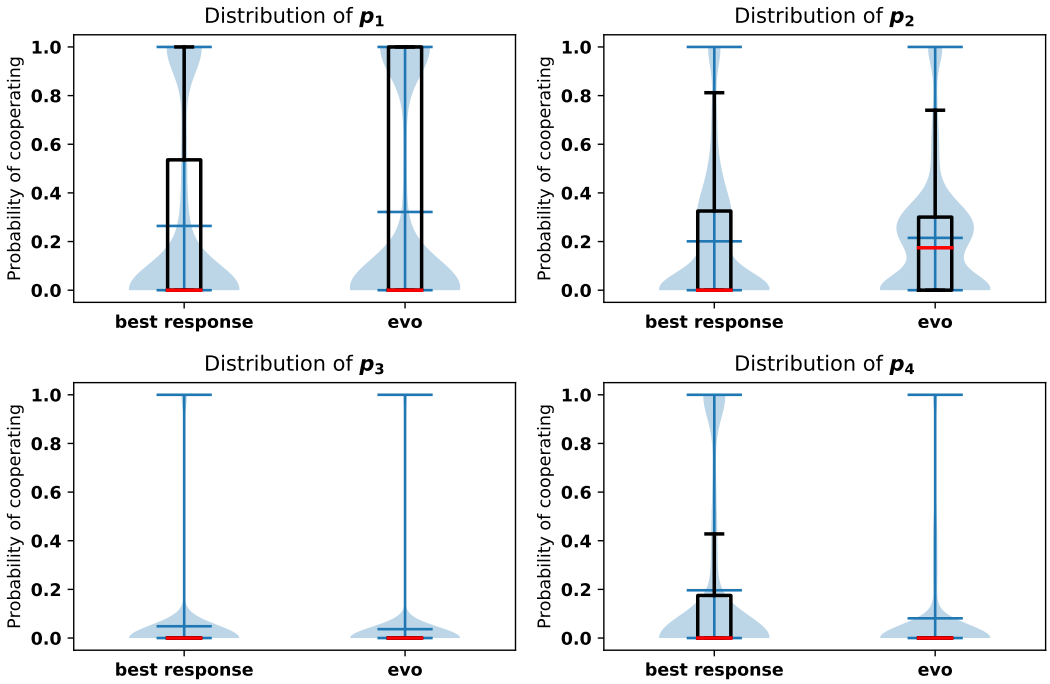


Figure 9: Distribution of sserrors for memory one best responses, when $N = 2$

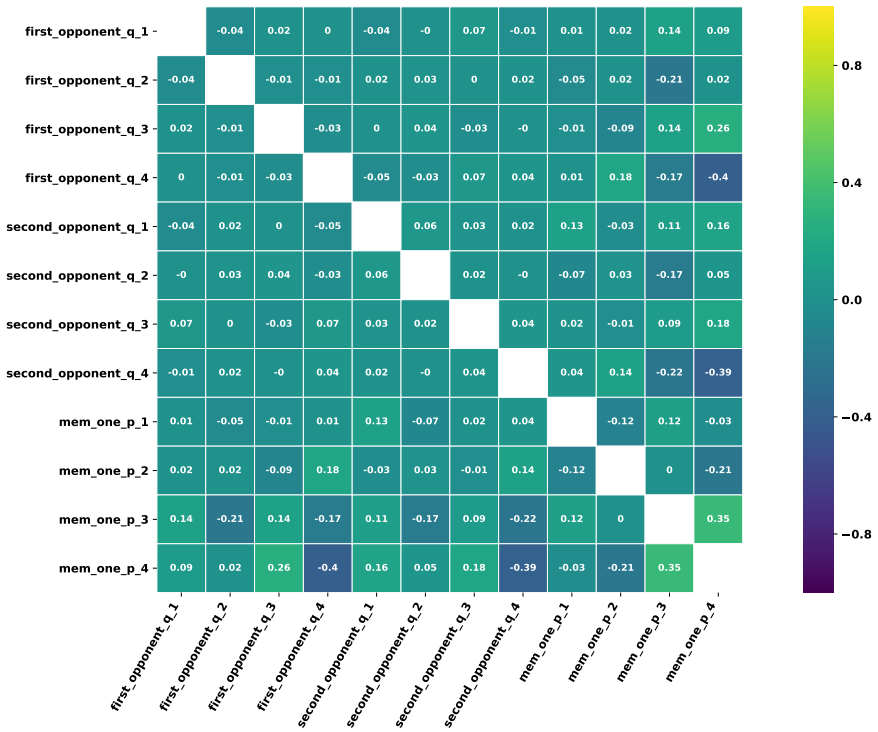
	Evo SSerror
count	972.000000
mean	0.146091
std	0.097928
min	0.000000
25%	0.069945
35%	0.107359
50%	0.135578
75%	0.188117
85%	0.235294
max	0.597936

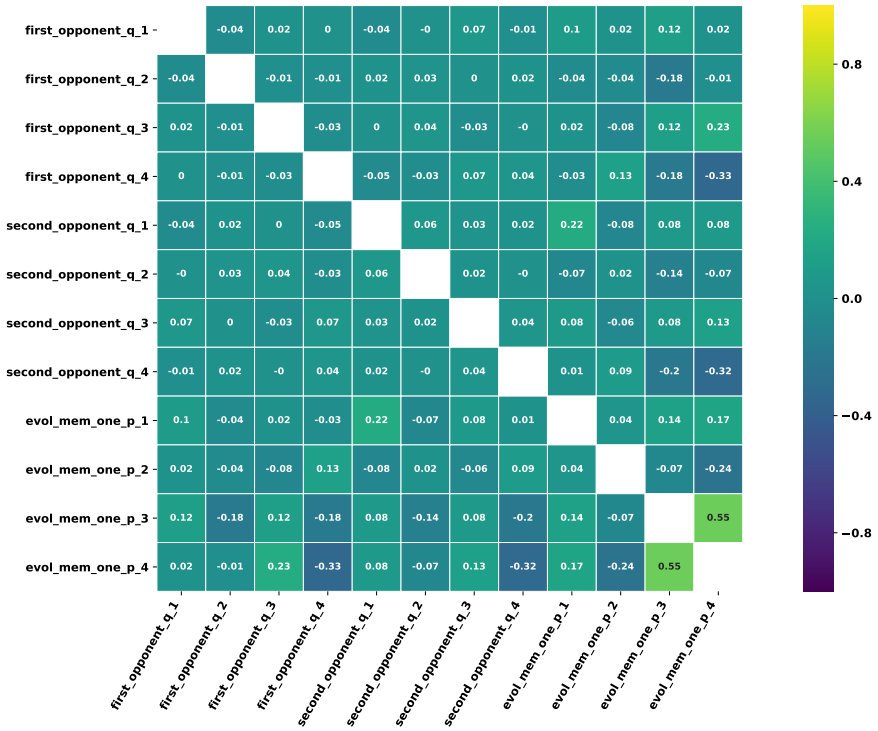
Table 3: Summary statistics SSerror



	Memory one Median	Evo Median	p-values
Distribution p_1	0.0	0.000000	0.0
Distribution p_2	0.0	0.174359	0.0
Distribution p_3	0.0	0.000000	0.0
Distribution p_4	0.0	0.000000	0.0

	Memory one Skew	Evo Skew
Distribution p_1	1.094933	0.797134
Distribution p_2	1.523980	1.607066
Distribution p_3	4.219578	4.881982
Distribution p_4	1.542853	2.990635





6 Longer memory best response

7 Conclusion

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