Assignment 1: Black-Scholes Model and Binomial Tree Methods

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***Abstract*— In this paper the option pricing of European and American options are evualated and compared for the binomial tree method and the Black-Scholes equation. The convergence rate of the binomial tree towards the Black-Scholes value is shown and the computational complexity of the tree is studied. The delta hedge parameter is computed using the binomial tree method and compared to the analytical solution. Finally, a hedging simulation is performed to measure the effectiveness of weekly versus daily hedging and the effect of using differing volatility values for option pricing than that of the underlying stock price movement.**

I. Introduction

Options are financial instruments whose value depend on the value of some underlying asset. In this paper only stock options are considered such that the underlying value is the stock price itself. The value of an option of an option therefore relies on several factors, including the stock price, stock volatility and the risk-free rate. Two of the most common approaches in option valuation are the use of binomial trees and the Black-Scholes equation. The binomial tree method involves numerical approximation of the option price by backward induction whereas the Black-Scholes equation provides an analytical solution.

Managing the risk of a portfolio containing a short position on an option can be done by delta hedging, such that a long position is taken in the underlying asset (the stock) as to replicate the claim at the time of expiry. This is a dynamic process that determines the amount of long position stocks one should have in order to maintain a risk-free portfolio.

In this paper a method is provided for computing an (American and European) option's price using the binomial tree model for stock volatilities, and is subsequently compared to the values given by the Black-Scholes equation. The delta hedge parameter is computed using the binomial tree method and is also compared to the analytical solution for varying stock volatilities. Finally, a hedging simulation is performed by considering a portfolio of one short call option and long delta shares of a stock over time. For this the dynamics of the stock price is given by geometric Brownian motion and the option price is determined by the Black-Scholes equation.

II. Methods

In the following subsections the mathematical and numerical tools are explained. However, it is important to make a few assumptions about the market. Only European and American options are considered and are either call or put options. Additionally, the market is assumed to be arbitrage-free with no transaction costs, there are no dividend payments on stocks and the interest rate is constant. Furthermore, below are the frequently used mathematical notations.

- S_t = stock price at time t
- K = strike price of the option
- σ = volatility of the stock
- r = risk-free interest rate per annum
- T = time of expiration
- c = price of a European call option
- C = price of an American call option
- p = price of a European put option
- P = price of an American put option

A. Binomial Trees

For the binomial tree model a simple two-state economy is considered such that the stock price right now, denoted by S_0, can go into one of two directions: up or down at some later time $t = 1$. Thus, from S_0, the stock will become either S_0u with probability p or S_0d with probability $(1-p)$ at $t = 1$. p can be found by simply considering that

$$E[S_1] = pS_0u + (1-p)S_0d = S_0 \cdot e^{r\Delta T}$$

and solving for p gives

$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{1}$$

and

$$1 - p = \frac{u - e^{r\Delta t}}{u - d} \tag{2}$$

Deriving u and d is a bit more time consuming however. Given that $E[S_1] = S_0 \cdot e^{r\Delta T}$ in a risk-free economy, the variance is then defined as

$$\begin{aligned} V[S_1] &= E[S_1^2] - E[S_1]^2 \\ &= p \cdot (S_0u)^2 + (1-p) \cdot (S_0d)^2 - (S_0 \cdot e^{r\Delta t})^2 \end{aligned}$$

for which can p and $1-p$ can be substituted with Eq.1 and Eq. 2 respectively, such that

$$\begin{aligned} V[S_1] &= S_0^2\Big(\frac{e^{r\Delta t} - d}{u - d} \cdot u^2 + \frac{u - e^{r\Delta t}}{u - d} \cdot d^2 - e^{2r\Delta t}\Big) \\ &= S_0^2(e^{r\Delta t}(u + d) - ud - e^{2r\Delta t}) \end{aligned}$$

For small Δt it is shown that $V[S_1] = S_0^2\sigma^2\Delta t$. By dividing both sides by S_0^2 the equation then reads

$$\sigma^2\Delta t = e^{r\Delta t}(u + \frac{1}{u}) - 1 - e^{2r\Delta t}$$

$$u + \frac{1}{u} = e^{-r\Delta t}(\sigma^2 \Delta t + 1 + e^{2r\Delta t})$$

$$= e^{-r\Delta t}\sigma^2 \Delta t + e^{-r\Delta t} + e^{r\Delta t}t$$

By Taylor expansion $e^{-r\Delta t} \approx 1 - r\Delta t$ is approximated by ignoring $(\Delta t)^2$ and higher order terms to arrive at

$$u + \frac{1}{u} \approx (1 - e^{-r\Delta t})\sigma^2 \Delta t + (1 - r\Delta t) + (1 + r\Delta t)$$

$$\approx \sigma^2 \Delta t + 2$$

which can be rewritten as

$$u^2 - (\sigma^2 \Delta t + 2)u + 1 = 0$$

Using the quadratic formula yields

$$u \approx \frac{1}{2}\sigma^2 \Delta t \pm 1 \pm \sigma\sqrt{\Delta t}$$

Finally a second-order Taylor expansion of $f(x) = e^{\sigma x}$ around 0 yields $f(x) \approx 1 + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2 x^2$ and therefore it holds that

$$u \approx 1 \pm \sigma\sqrt{\Delta t} \pm \frac{1}{2}\sigma^2 \Delta t$$

$$\approx f(\sqrt{\Delta t}) = e^{\sigma \Delta t}$$

and conclude that

$$u = e^{\sigma \Delta t} \tag{3}$$

$$d = e^{-\sigma \Delta t} = \frac{1}{u} \tag{4}$$

With Eq. 3 and Eq. 4 it is now possible to solve p for u and d.

Now consider a European call option with price c and a one step binomial tree (containing only $t = 0$ and $t = T$). The option's value at T is known to be $S_0 e^{r\Delta t} u$ or $S_0 e^{r\Delta t} d$ in a risk-free economy. The option price at the time of expiry T for these two possibilities is given by $S_0 u \Delta$ or $S_0 d \Delta$, where Δ is the hedge parameter that determines the amount of stocks needed to maintain a risk-free portfolio. The portfolio is risk-free if $S_0 u \Delta - f_u = S_0 d \Delta - f_d$ is satisfied, with f denoting the the payoff of the option. The hedge parameter Δ is therefore given by

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d} = \frac{\Delta f}{\Delta S_0} \tag{5}$$

Starting at T in the binomial tree by going back to the previous (parent) node the option price today can be found using

$$f = e^{-rT}[p f_u + (1 - p) f_d] \tag{6}$$

which is the backward movement of the risk-free rate of the option payoffs at T scaled by their respective probabilities.

For each additional step in the tree $n+1$ nodes are added. The size of a binomial heap as a function of increasing order is $O(n^2)$. However since the branches binomial tree method converge at every time step, this is not the case. An numerical approximation found that the time complexity seems to lie around $\approx O(n^{1.8})$.

B. Black-Scholes formula

The Black-Scholes formula (BS) is an analytical method to determine the price of a European call option [1] at time t and is given by

$$c_t = N(d_1)S_t - N(d_1)Ke^{r(T-t)} \tag{7}$$

with

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\Big[ln\Big(\frac{S_t}{K}\Big) + \Big(r + \frac{\sigma^2}{2}\Big)(T-t)\Big]$$

$$d_2 = \frac{1}{\sigma\sqrt{T-t}}\Big[ln\Big(\frac{S_t}{K}\Big) + \Big(r - \frac{\sigma^2}{2}\Big)(T-t)\Big]$$

and $N(x)$ denoting the cumulative probability function of the standard normal distribution

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}z^2} dz$$

The Black-Scholes formula thus requires S_t, K, σ, r and T to be known. Moreover, the hedge parameter Δ in Eq. 5 at time $t \leq T$ can be rewritten in such a way that

$$\Delta_t = \frac{\partial c_t}{\partial S_t} = N(d_1) \tag{8}$$

by partially differentiating the call price with respect to the stock price.

C. Geometric Brownian Motion

The movement of a stock price can be generalized as a form of Geometric Brownian Motion (GBM) and is also referred as being a Wiener Process [2]. The dynamics of the stock price can therefore be written as a GBM of the form

$$dS = rSdt + \sigma dZ \tag{9}$$

and satisfies the Markov property of being memoryless, meaning that the past behavior of the stock is irrelevant for determining it's future trajectory. The first termin in Eq. 9 is the rate of growth of the stock price given the risk-free rate r. The second term denotes the stochastic movement of the stock price given the volatility σ and a random number drawn from the standard normal distribution such that

$$dZ = \epsilon\sqrt{\delta t} = \phi(0,1)\sqrt{\delta t}$$

Therefore the stochastic movement of the stock price depends only δt steps taken in time.

III. Results

A. European call option price

Consider a European call option with $S_0 = 100$, $K = 99$, $\sigma = 20\%$, $r = 0.06$ and $T = 1$ (year). Figure 1 shows the price of the option using the binomial tree method using 50 time steps ($\delta t = T/N$, with $N = 50$) and the Black-Scholes formula. The binomial tree method performs close to the BS formula. The option price increases as the volatility increases, because a higher payoff is possible, and settles at $c \approx 100$. Interestingly, the residual between these two increases as the volatility rises and reaches a peak at $\sigma \approx 3$ before eventually dropping towards zero.

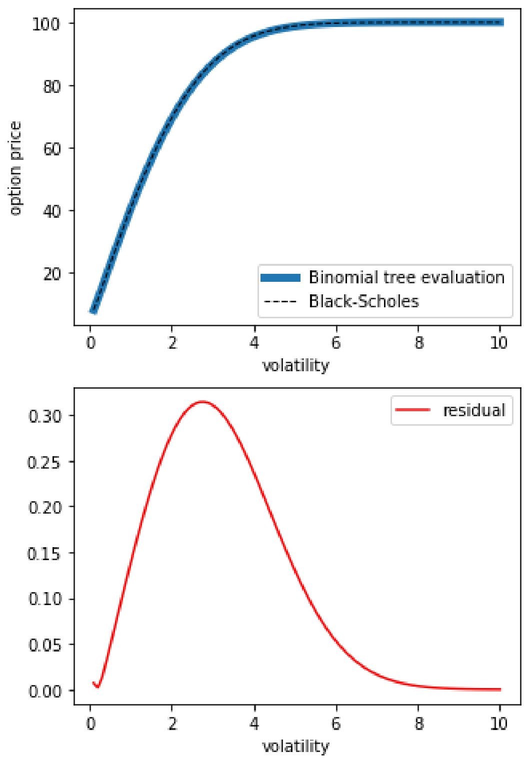

Fig. 1. Comparison of a European call option price using the binomial tree method with 50 time steps and the Black-Scholes formula. Parameters: $S_0 = 100$, $K = 99$, $r = 0.06$ and $T = 1$ (year).

The relatively small residual is achieved by having a relatively large amount of steps in the binomial tree. Investigating the convergence of the binomial tree method towards the BS formula reveals that the amount of steps in the tree leads to oscillatory behavior, as shown in Figure 2. The smaller values in the oscillation correspond to an odd number steps (and an even number of nodes at T) and the larger values correspond to an even amount of steps (and an odd number of nodes at T) in the tree. Having either an even or odd number of nodes at T makes it so that the binomial tree method over- or underestimates the option price when compared to the BS formula. Another interesting phenomenon occurs at 100 steps where the oscillations have decreased close to a halt, but eventually lead to larger oscillation soon thereafter. Naturally, the residual shows similar behavior and converges in a similar fashion.

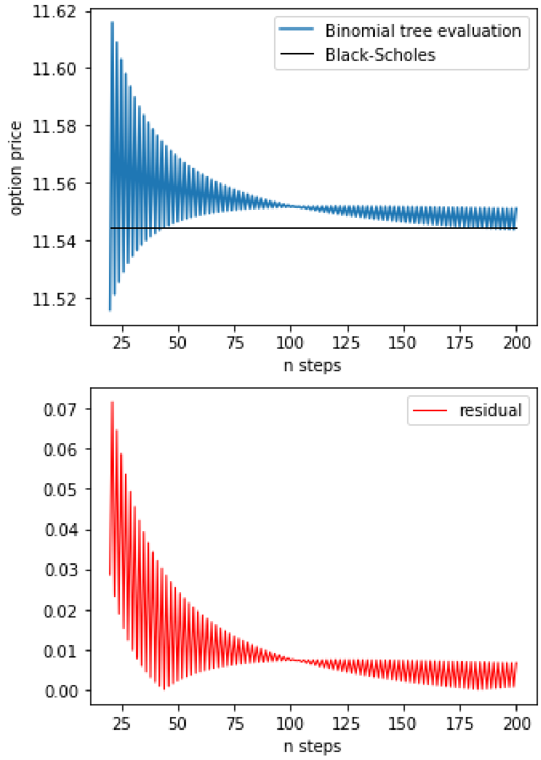

Fig. 2. Comparison of a European call option price using the binomial tree method over varying time steps and the Black-Scholes formula. Parameters: $S_0 = 100$, $K = 99$, $\sigma = 20\%$, $r = 0.06$ and $T = 1$ (year).

A similar experiment was done for the hedge parameter Δ using both the binomial tree method to compute Δ using Eq. 5 and by analytical solution with Eq. 8 for $t = 0$. The results are shown in Figure 3. The residual, while small, follows a similar trend to that found for option price as a function of the volatility σ. In addition to the peak found at $\sigma \approx 3$, another peak is shown to arise near zero (likely because of the underlying function it is derived from).

B. American call and put option price

A key difference when considering an American option is that the price for calls and puts follow different trends and that they are more sensitive the underlying asset's volatility. Consider both an American call and put option with $S_0 = 100$, $K = 99$, $\sigma = 20\%$, $r = 0.06$ and $T = 1$ (year). Figure 4 shows that the American call option's price shows a tremendous growth as the volatility increases due to the option's early exercise opportunity and does not stop increasing, unlike a European option. An American put option on the other hand stagnates far quicker than it's European counterpart.

C. Hedging Simulation

For the hedging simulation consider a portfolio consisting of:

- short 1 European call option

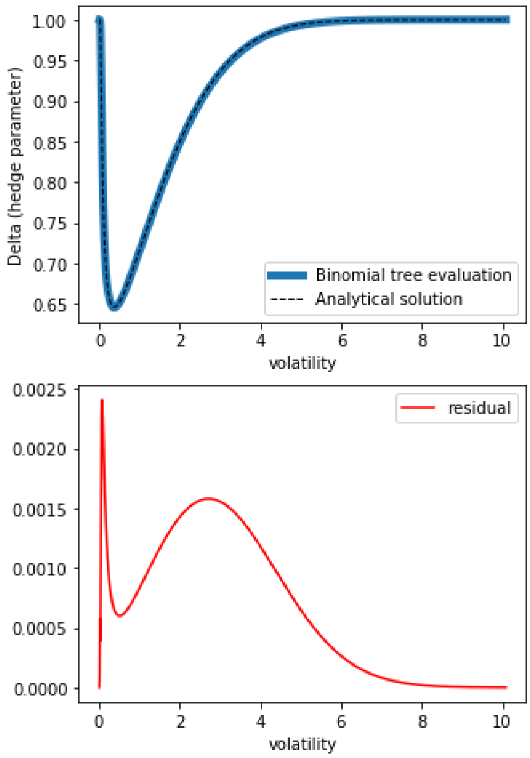

Fig. 3. Comparison of a European call option hedge parameter using the binomial tree method for 50 time steps and the Black-Scholes formula. Parameters: $S_0 = 100$, $K = 99$, $r = 0.06$ and $T = 1$ (year).

- long Δ shares of a stock

As the before, the parameters are $S_0 = 100$, $K = 99$, $\sigma = 20\%$, $r = 0.06$ and $T = 1$ (year). A dynamic hedging strategy involves adjusting the Δ shares through time in order to minimize the risk of losing money. Thus at the start of the simulation ($t = 0$) the trader would have to borrow money in order to buy Δ shares. Then, at some later time, the trader re-evaluates the Δ hedge parameter and does one of two things:

- If Δ has increased, borrow more money to increase the amount of shares held.
- If Δ has decreased, sell the surplus shares $(\Delta_{t-1} - \Delta_t)$ at market price. The extra money can be now be used to pay off some of the debt at the risk-free rate.

This process is done continuously to ensure that a correct amount of Δ shares are held at any time, and that the debt is minimized. At the end of the simulation (1 year) the total portfolio value is computed, selling the shares at S_T if $S_T < K$ and exercising the option at K if $S_T > K$, and by clearing all debt with available money. The time at which the option is sold is done stochastically such that $0 \leq c_t \leq T$ and therefore the option price varies between simulations. The results summarized in Figure 5 shows the portfolio at time T of hedging daily (252 trading days in a year), weekly hedging (52 weeks for a year) and quarterly hedging (every 3 months). Both daily and weekly hedging have similar means and standard deviations for the payoff at

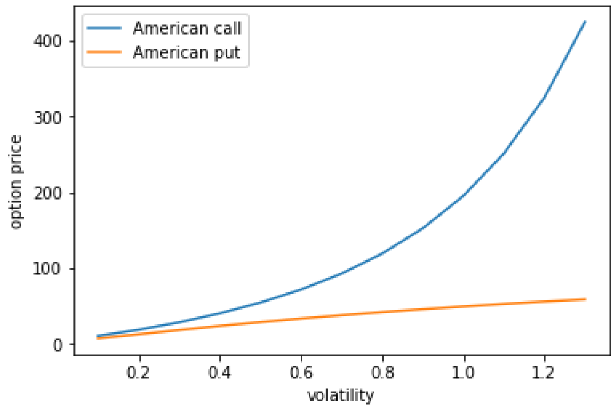

Fig. 4. Comparison of an American call and put option price using the binomial tree method for 50 time steps. Parameters: $S_0 = 100$, $K = 99$, $r = 0.06$ and $T = 1$ (year).

T and seem to follow a lognormal distribution with a long right tail. Quarterly hedging however has a reduced mean with less variance, meaning that more frequent hedging can alter the risk of the portfolio by increasing both the mean and the variance of the payoff. The frequency of hedging does seem to hit a point of diminishing returns as shown by the similarity between daily and weekly hedging.

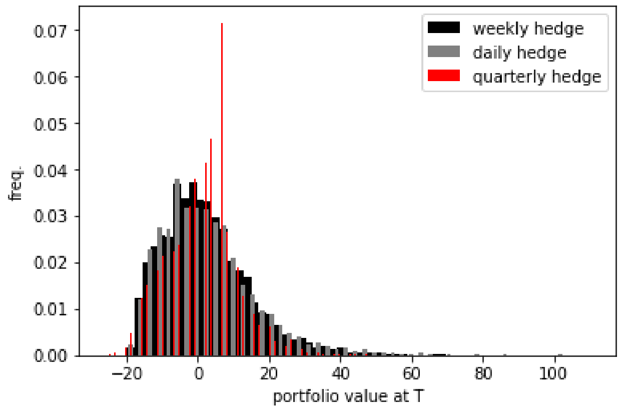

Fig. 5. Comparison of hedging strategy using the Black-Scholes formula over 5000 simulations per hedge strategy. Parameters: $S_0 = 100$, $K = 99$, $r = 0.06$ and $T = 1$ (year). For daily hedging mean $\mu = 2.87$ and stand deviation $\sigma = 13.93$. For weekly hedging $\mu = 2.15$ and $\sigma = 12.63$. For quarterly hedging $\mu = 1.43$ and $\sigma = 9.8$.

The previous experiment used the same volatility σ ($= 20\%$) for determining the stock price, the hedge parameter Δ and the option price using Black-Scholes. Only considering the weekly hedging strategy (due to computational cost), Figure 6 shows the effect of using different σ values ($\in [5\%, 50\%]$) on the total portfolio value at T. Using the non-matching $\sigma = 50\%$ for the stock price it is shown that the mean portfolio value at T is reduced close to zero and the standard deviation drastically increases as shown by long tail of the distribution. If the stock price volatility is lower however (5%), the opposite is true.

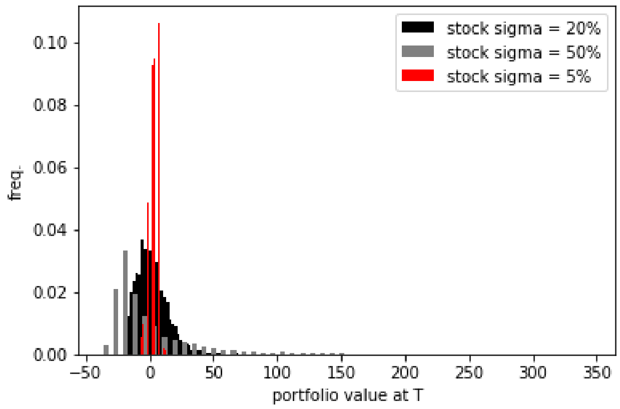

Fig. 6. Comparison of a weekly hedging strategy using the Black-Scholes formula for matching and non-matching volatilities over 5000 simulations per volatility. For non-matching volatility the stock price movement $\sigma \in [5\%, 50\%]$ and 20% otherwise. Parameters: $S_0 = 100$, $K = 99$, $r = 0.06$ and $T = 1$ (year). For using the same volatility: $\mu = 2.15$ and $\sigma = 12.63$. For using different 50% volatility: $\mu = -0.02$ and $\sigma = 35.12$. 5% volatility: $\mu = 3.44$ and $\sigma = 3.63$.

IV. Conclusion

In this paper two methods for pricing American and European call and put options were presented. The binomial tree method is a numerical approximation of an option's price using back propagation through time of the option's payoff at the time of expiration. The Black-Scholes formula on the other hand provides an analytical solution to the option price. It was shown that the binomial tree approximation closely follows a the BS formula for a range of stock price volatilities. The same holds true for the hedge parameter Δ as a function of the stock price volatility σ. The residual for both is nearly identical with strong peaks at $\sigma \approx 3$. Furthermore, the amount of steps used in the tree shows an oscillatory convergence towards the value given by the BS formula due to the discrete nature of the binomial tree method.

A hedging simulation showed that more frequent hedging can reduce risk with an increased mean payoff and is met with increased variance. Another experiment was performed where σ^{stock}, which the stock price motion, was different than σ^{BS}, which is the BS formula used for the hedge parameter Δ and S_t. For $\sigma^{stock} < \sigma^{BS}$ a reduced mean is observed with reduced variance compared to $\sigma^{stock} = \sigma^{BS}$. If $stock > \sigma^{BS}$ the opposite was shown.

References

[1] F. Black and M. Scholes, "The pricing of options and corporate liabilities," *Journal of political economy*, vol. 81, no. 3, pp. 637–654, 1973.

[2] P. Jäckel, "Monte carlo methods in finance," *West Sussex*, 2002.