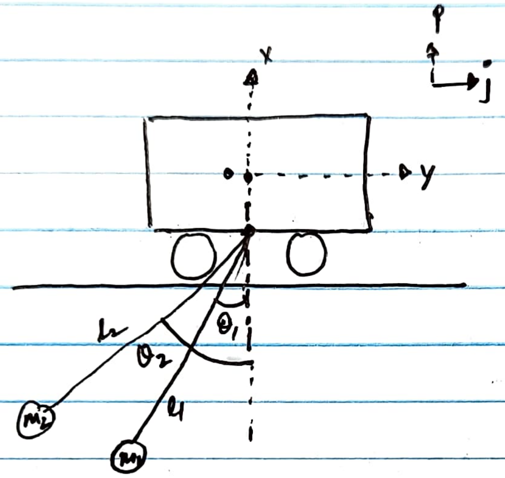
(a) Equations of motion for Non-linear State representation.

→ States of the system are linear velocity, linear acc. of the crane along with angular vel & angular acc. of the masses of pendulum m_1 & m_2.



Position of m_1 w.r.t the reference frame

$$r_1(t) = (x - l_1\sin\theta_1)\hat{i} - l_1\cos\theta_1\hat{j}$$
$$r_2(t) = (x - l_2\sin\theta_2)\hat{i} - l_2\cos\theta_2\hat{j}$$

where x, θ_1 & θ_2 are funct^n of time.

$$\dot{r}_1(t) = (\dot{x} - l_1\dot{\theta}_1\cos\theta_1)\hat{i} + l_1\dot{\theta}_1\sin\theta_1\hat{j}$$
$$\dot{r}_2(t) = (\dot{x} - l_2\dot{\theta}_2\cos\theta_2)\hat{i} + l_2\dot{\theta}_2\sin\theta_2\hat{j}$$

$$KE = \frac{1}{2}M\ddot{x}^2 + \frac{1}{2}m_1(\dot{x} - l_1\dot{\theta}_1\cos\theta_1)^2 + \frac{1}{2}m_2(\dot{x} - l_2\dot{\theta}_2\cos\theta_2)^2 + \frac{1}{2}m_1(l_1\dot{\theta}_1\sin\theta_1)^2 + \frac{1}{2}m_2(l_2\dot{\theta}_2\sin\theta_2)^2$$

$$PE = -mgl_1\cos\theta_1 - mgl_2\cos\theta_2$$

from Lagrange's Equation

$$L = K \cdot E - P \cdot E$$

$$L = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m_1 (\dot{x} - l_1\dot{\theta}_1 \cos\theta_1)^2 + \frac{1}{2} m_2 (\dot{x} - l_2\dot{\theta}_2 \cos\theta_2)^2 + \frac{1}{2} m_1 (l_1 \dot{\theta}_1 \sin\theta_1)^2 + \frac{1}{2} m_2 (l_2\dot{\theta}_2 \sin\theta_2)^2$$
$$+ mgl_1 \cos\theta_1 + mgl_2 \cos\theta_2 \quad \text{— (1)}$$

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + m_1(\dot{x} - l_1\dot{\theta}_1 \cos\theta_1) + m_2(\dot{x} - l_2\dot{\theta}_2 \cos\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M\ddot{x} + m_1 (\ddot{x} - l_1\ddot{\theta}_1 \cos\theta_1 + l_1\dot{\theta}_1^2 \sin\theta_1) + m_2 (\ddot{x} - l_2\ddot{\theta}_2 \cos\theta_2 + l_2\dot{\theta}_2^2 \sin\theta_2)$$

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = M\ddot{x} + m_1 (\ddot{x} - l_1\ddot{\theta}_1 \cos\theta_1 + l_1\dot{\theta}_1^2 \sin\theta_1) + m_2(\ddot{x} - l_2\ddot{\theta}_2 \cos\theta_2 + l_2\dot{\theta}_2^2 \sin\theta_2)$$
$$= F \quad \text{— (2)}$$

diff wrt 'θ_1'

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 (\dot{x} - l_1\dot{\theta}_1 \cos\theta_1)(-l_1 \cos\theta_1) + m_1 (l_1\dot{\theta}_1 \sin\theta_1)(l_1 \sin\theta_1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = -m_1\ddot{x} l_1 \cos\theta_1 + m_1 l_1^2 \ddot{\theta}_1 + m_1 \dot{x} l_1 \dot{\theta}_1 \sin\theta_1$$

$$\frac{\partial L}{\partial \theta_1} = m_1 l_1^2 \dot{\theta}_1^2 - m_1 \dot{x} l_1 \cos\theta_1$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = -m_1 \ddot{x} l_1 \cos\theta_1 + m_1 l_1^2 \ddot{\theta}_1 + m_1 \dot{x} l_1 \dot{\theta}_1 \sin\theta_1 - m_1 l_1^2 \dot{\theta}_1 + m_1 \dot{x} l_1 \cos\theta_1 = 0$$
$$\hookrightarrow (3)$$

diff wrt 'θ_2'

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 (\dot{x} - l_2\dot{\theta}_2 \cos\theta_2)(-l_2 \cos\theta_2) + m_2 (l_2\dot{\theta}_2 \sin\theta_2)(l_2 \sin\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) = -m_2\ddot{x}l_2\cos\theta_2 + m_2 l_2^2\ddot{\theta}_2 + m_2\dot{x}l_2\dot{\theta}_2\sin\theta_2$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_2^2\dot{\theta}_2 - m_2\dot{x}l_2\cos\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) - \frac{\partial L}{\partial \theta_2} = -m_2\ddot{x}l_2\cos\theta_2 + m_2 l_2^2\ddot{\theta}_2 + m_2\dot{x}l_2\dot{\theta}_2\sin\theta_2 - m_2 l_2^2\dot{\theta}_2 + m_2\dot{x}l_2\cos\theta_2 = 0$$

$\hookrightarrow$ 4

From the above eqns:

$$l_1\ddot{\theta}_1 = \ddot{x}\cos\theta_1 - g\sin\theta_1 \quad -(5)$$

$$l_2\ddot{\theta}_2 = \ddot{x}\cos\theta_2 - g\sin\theta_2 \quad -(6)$$

Substituting in eqn 2

$$(M+m_1+m_2)\ddot{x} = m_1(\ddot{x}\cos\theta_1 - g\sin\theta_1)\cos\theta_1 + m_2(\ddot{x}\cos\theta_2 - g\sin\theta_2)\cos\theta_2 - m_1 l_1\dot{\theta}_1^2\sin\theta_1 - m_2 l_2\dot{\theta}_2^2\sin\theta_2 + F$$

$$\ddot{x}(M + m_1\sin^2\theta_1 + m_2\sin^2\theta_2) = F - m_1 g\cos\theta_1\sin\theta_1 - m_2 g\cos\theta_2\sin\theta_2 - m_1 l_1\dot{\theta}_1^2\sin\theta_1 - m_2 l_2\dot{\theta}_2^2\sin\theta_2$$

$\hookrightarrow$ 7

Substituting in 5 & 6

$$l_1\ddot{\theta}_1 = \cos\theta_1\frac{(F - m_1 g\cos\theta_1\sin\theta_1 - m_2 g\cos\theta_2\sin\theta_2 - m_1 l_1\dot{\theta}_1^2\sin\theta_1 - m_2 l_2\dot{\theta}_2^2\sin\theta_2)}{(M + m_1\sin^2\theta_1 + m_2\sin^2\theta_2)} - g\sin\theta_1$$

8

$$l_2\ddot{\theta}_2 = \cos\theta_2\frac{(F - m_1 g\cos\theta_1\sin\theta_1 - m_2 g\cos\theta_2\sin\theta_2 - m_1 l_1\dot{\theta}_1^2\sin\theta_1 - m_2 l_2\dot{\theta}_2^2\sin\theta_2)}{(M + m_1\sin^2\theta_1 + m_2\sin^2\theta_2)} - g\sin\theta_2$$

$\hookrightarrow$ 9

Non-linear State space representation.

$$\begin{bmatrix} \ddot{x}(t) \\ \ddot{\theta}_1(t) \\ \ddot{\theta}_2(t) \end{bmatrix} = f\left(t, x(t), \dot{x}(t), \theta_1(t), \dot{\theta}_1(t), \theta_2(t), \dot{\theta}_2(t)\right)$$

$$\begin{bmatrix} \ddot{x}(t) \\ \ddot{\theta}_1(t) \\ \ddot{\theta}_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{(F - m_1 g\cos\theta_1\sin\theta_1 - m_2 g\cos\theta_2\sin\theta_2 - m_1 l_1\dot{\theta}_1^2\sin\theta_1 - m_2 l_2\dot{\theta}_2\sin\theta_2)}{(M + m_1\sin^2\theta_1 + m_2\sin^2\theta_2)} \\ \dfrac{\cos\theta_1(F - m_1 g\cos\theta_1\sin\theta_1 - m_2 g\cos\theta_2\sin\theta_2 - m_1 l_1\dot{\theta}_1^2\sin\theta_1 - m_2 l_2\dot{\theta}_2\sin\theta_2)}{l_1(M + m_1\sin^2\theta_1 + m_2\sin^2\theta_2} - \dfrac{g\sin\theta_1}{l_1} \\ \dfrac{\cos\theta_2(F - m_1 g\cos\theta_1\sin\theta_1 - m_2 g\cos\theta_2\sin\theta_2 - m_1 l_1\dot{\theta}_1^2\sin\theta_1 - m_2 l_2\dot{\theta}_2^2\sin\theta_2)}{l_2(M + m_1\sin^2\theta_1 + m_2\sin^2\theta_2)} - \dfrac{g\sin\theta_2}{l_2} \end{bmatrix}$$

$\hookrightarrow$ (10)

(B) eqm point specified by $[x = 0$ & $x_1 = x_2 = 0]$

State is represented by

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \\ \theta_1(t) \\ \dot{\theta}_1(t) \\ \theta_2(t) \\ \dot{\theta}_2(t) \end{bmatrix} \rightarrow (11)$$

Linearizing at pts. $x = 0,\ \theta_1 = 0$ & $\theta_2 = 0$ with $\sin\theta \approx \theta$ & $\cos\theta \approx 1$

$$\begin{bmatrix} \ddot{x}(t) \\ \ddot{\theta}_1(t) \\ \ddot{\theta}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{F}{M} - \frac{m_1 g \theta_1}{M} - \frac{m_2 g \theta_2}{M} \\ \frac{F}{Ml_1} - \frac{m_1 g \theta_1}{Ml_1} - \frac{g\theta_1}{l_1} - \frac{m_2 g \theta_2}{Ml_1} \\ \frac{F}{Ml_2} - \frac{m_2 g \theta_2}{Ml_2} - \frac{g\theta_2}{l_2} - \frac{m_1 g \theta_1}{Ml_2} \end{bmatrix} \rightarrow (12)$$

$$\begin{bmatrix} \ddot{x}(t) \\ \ddot{\theta}_1(t) \\ \ddot{\theta}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{-m_1 g}{M} & \frac{-m_2 g}{M} \\ 0 & \frac{-m_1 g}{Ml_1} & \frac{-m_2 g}{Ml_1} \\ 0 & \frac{-m_1 g}{Ml_2} & \frac{-m_2 g}{Ml_2} - g/l_2 \end{bmatrix} \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{M} \\ \frac{1}{Ml_1} \\ \frac{1}{Ml_2} \end{bmatrix} F \rightarrow (13)$$

$$
\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \\ \dot{\theta}_1(t) \\ \ddot{\theta}_1(t) \\ \dot{\theta}_2(t) \\ \ddot{\theta}_2(t) \end{bmatrix}
=
\underbrace{\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{m_1 g}{M} & 0 & -\frac{m_2 g}{M} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -\frac{m_1 g}{M l_1} - \frac{g}{l_1} & 0 & -\frac{m_2 g}{M l_1} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -\frac{m_1 g}{M l_2} & 0 & -\frac{m_2 g}{M l_2} - \frac{g}{l_2} & 0
\end{bmatrix}}_{A \text{ matrix}}
\begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}
+
\begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{M l_1} \\ 0 \\ \frac{1}{M l_2} \end{bmatrix} F \rightarrow (14)
$$

Using Lyapunov's Indirect Method.

$$
J = \begin{bmatrix}
\frac{\partial \ddot{x}}{\partial x} & \frac{\partial \ddot{x}}{\partial \dot{x}} & \frac{\partial \ddot{x}}{\partial \theta_1} & \frac{\partial \ddot{x}}{\partial \dot{\theta}_1} & \frac{\partial \ddot{x}}{\partial \theta_2} & \frac{\partial \ddot{x}}{\partial \dot{\theta}_2} \\
\frac{\partial \ddot{\theta}_1}{\partial x} & \frac{\partial \ddot{\theta}_1}{\partial \dot{x}} & \frac{\partial \ddot{\theta}_1}{\partial \theta_1} & \frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_1} & \frac{\partial \ddot{\theta}_1}{\partial \theta_2} & \frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_2} \\
\frac{\partial \ddot{\theta}_2}{\partial x} & \frac{\partial \ddot{\theta}_2}{\partial \dot{x}} & \frac{\partial \ddot{\theta}_2}{\partial \theta_1} & \frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_1} & \frac{\partial \ddot{\theta}_2}{\partial \theta_2} & \frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_2}
\end{bmatrix}
$$

(c)

From code we have; rank = 6

By comparing the ~~rows~~ columns for linear independence, we observe that controllability decreases if $\boxed{l_1 = l_2}$

$\therefore$ For the system to be controllable, $\boxed{l_1 \neq l_2}$

d) LQR control design.

Cost function $J = \int_0^{\infty} (x^T Q x + u^T R u)\, dt$

Riccatti Eqn. is given by

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

The controller gain is found using $u = -Kx$. where the the K matrix

$$K = R^{-1} B P$$

LQR function; Initial cond^{ns} in degrees $X = [0 \ 0 \ 15 \ 0 \ 20 \ 0]$

'Q' matrix is shown in code.

Using Lyapunov's method we compute stability.

If we have a system that has N eigen values & all the values have no positive real part, then the system is locally stable.

From our code we found eigen values of A have no positive real part ∴ our system is locally stable.

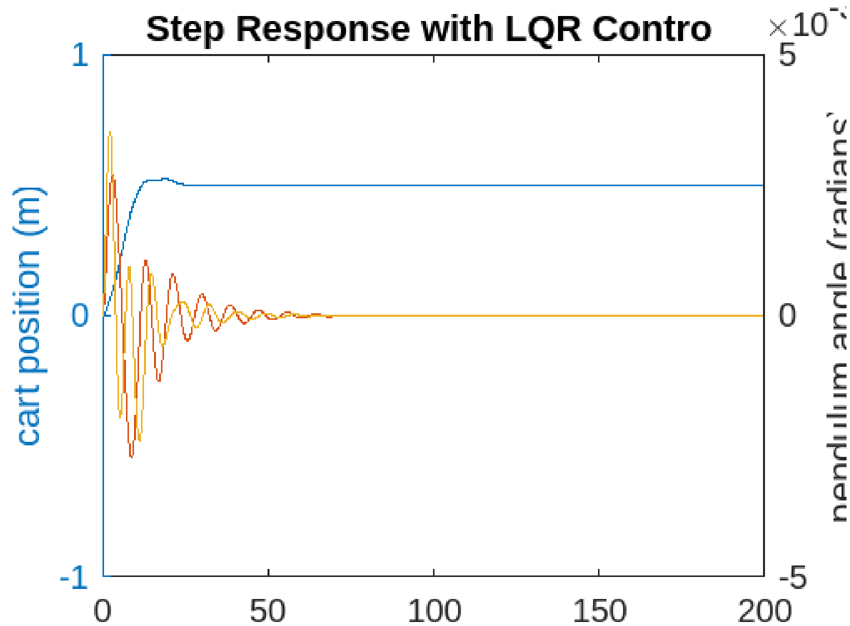


Figure 1: Control on Linearized model

Initial conditions response

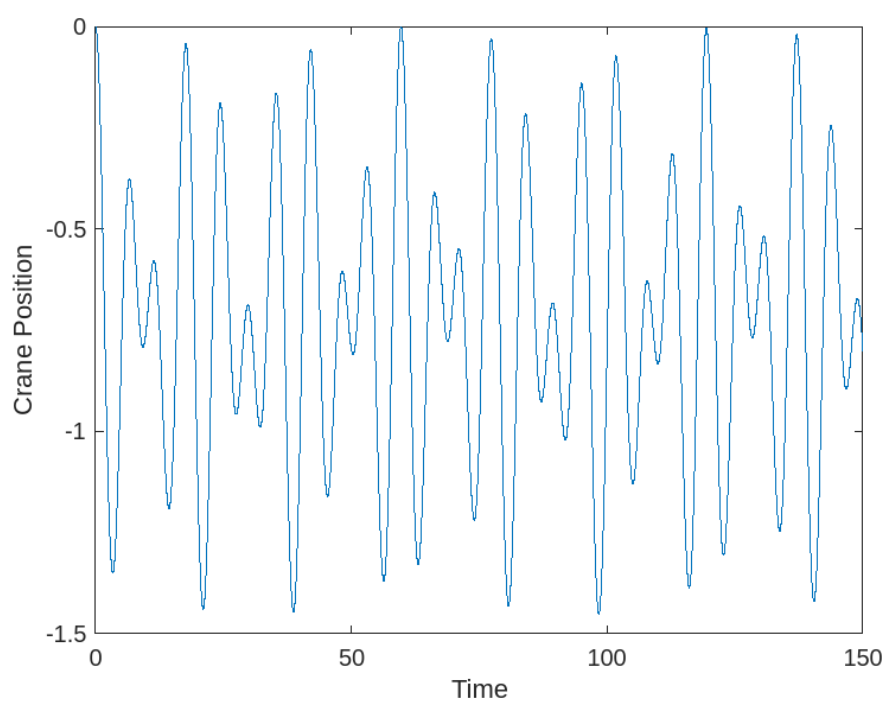


Figure 2: Crane position vs time (initial condition)

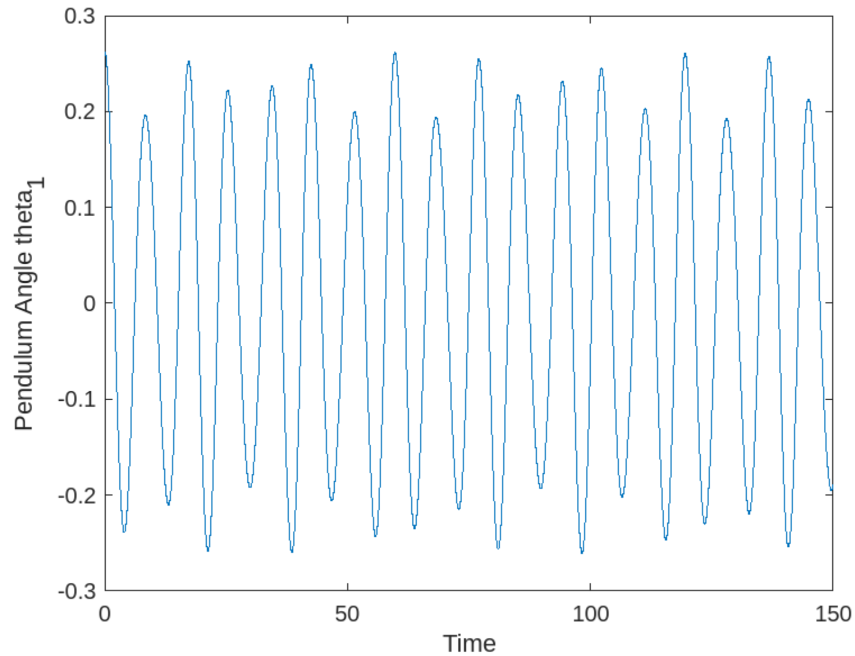


Figure 3: Theta_1 vs time(initial condition)

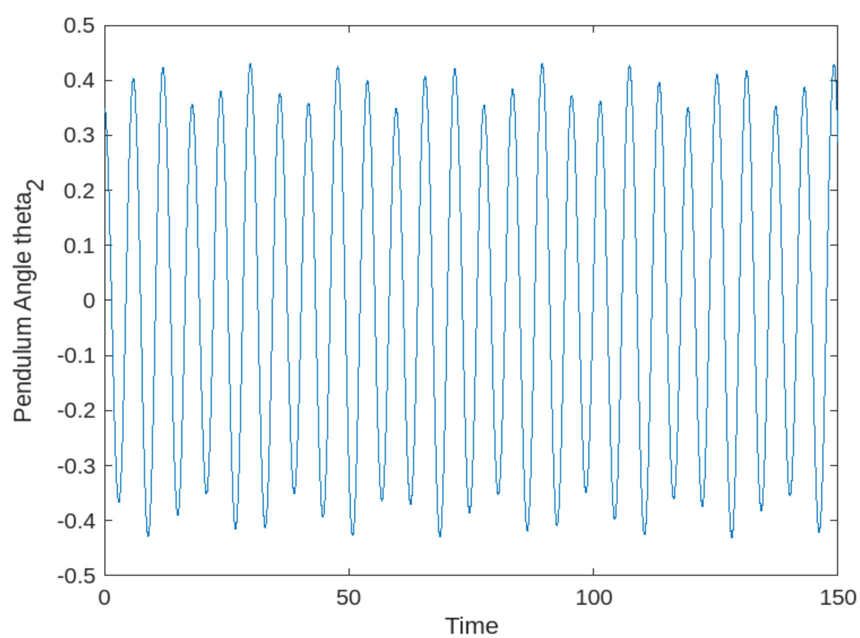


Figure 4: theta_2 vs time(initial condition)

Original Non-linear Conditions

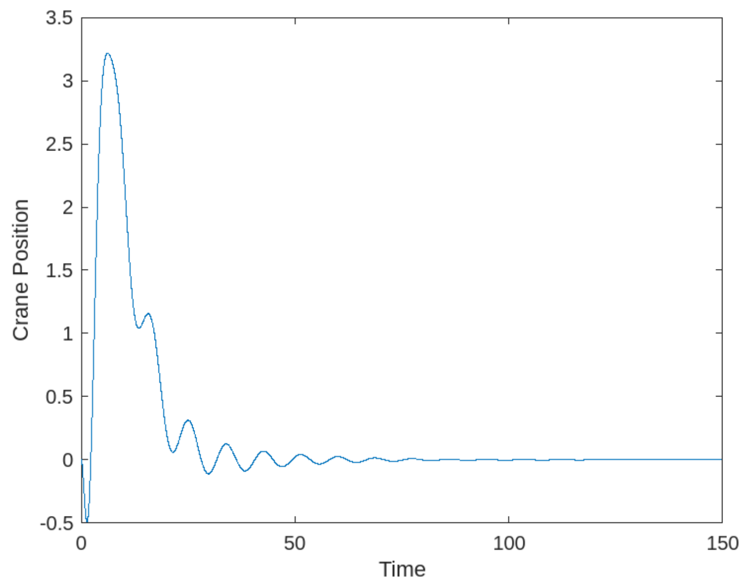


Figure 5: Crane vs time (original system)

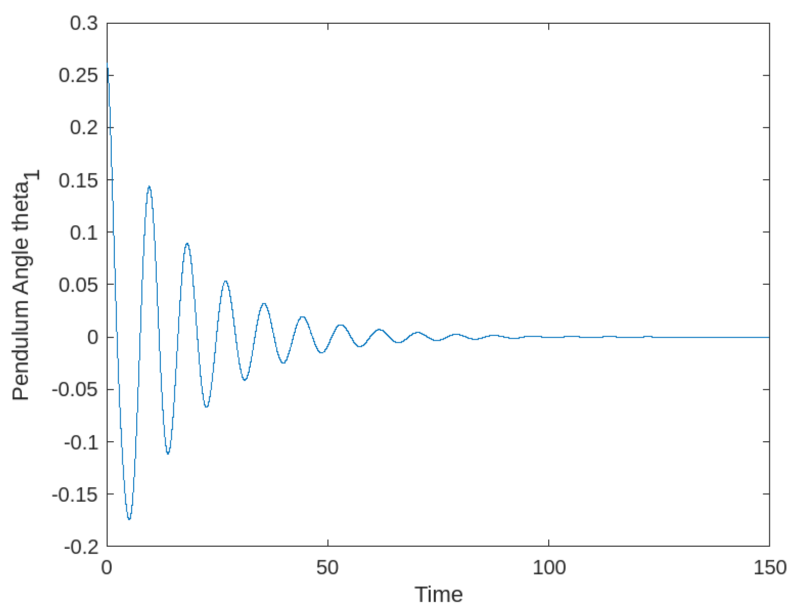


Figure 6 : Theta_1 vs Time (Original system)

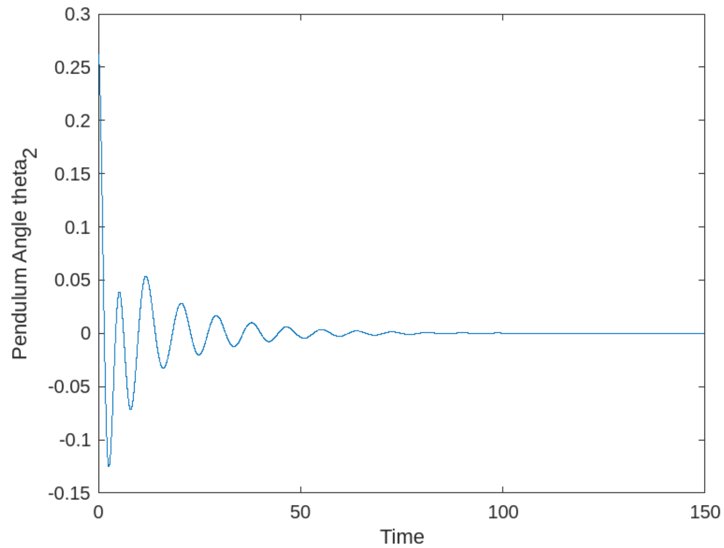


Figure 7: theta_2 vs Time (original system)

E) Suppose that you can select the following output vectors: x(t), ($\theta_1(t)$, $\theta_2(t)$), (x(t), $\theta_2(t)$) or (x(t), $\theta_1(t)$, $\theta_2(t)$).
Determine for which output vectors the linearized system is observable.

The state equation is observable if and only if the observability matrix satisfies:
rank[C^T $A^T C^T$ ... $(A^T)^{n-1}C^T$] = n

The observability matrix of rank calculation is shown below and more detail is in Observability.m file.

```
% Output vector x, (theta1, theta2), (x, theta2) or (x, theta1, theta2)
% Y = CX + DU
% The state equation is observable if and only if the observability matrix
% satisfies rank[CT ATCT ... (AT)^(n-1)CT ] = n

C1 = [1, 0, 0, 0, 0, 0];
C2 = [0, 0, 1, 0, 1, 0];
C3 = [1, 0, 0, 0, 1, 0];
C4 = [1, 0, 1, 0, 1, 0];

% Observability

fprintf('Obervability Matrix\n')
O1 = [C1.', A1.'*C1.', A1.'*A1.'*C1.', A1.'*A1.'*A1.'*C1.', A1.'*A1.'*A1.'*A1.'*C1.', A1.'*A1.'*A1.'*A1.'*A1.'*C1.' ];
fprintf('rank of C1(x) is %d\n', rank(O1))


O2 = [C2.', A1.'*C2.', A1.'*A1.'*C2.', A1.'*A1.'*A1.'*C2.', A1.'*A1.'*A1.'*A1.'*C2.', A1.'*A1.'*A1.'*A1.'*A1.'*C2.' ];
fprintf('rank of C2(theta1, theta2) is %d\n', rank(O2))

O3 = [C3.', A1.'*C3.', A1.'*A1.'*C3.', A1.'*A1.'*A1.'*C3.', A1.'*A1.'*A1.'*A1.'*C3.', A1.'*A1.'*A1.'*A1.'*A1.'*C3.' ];
fprintf('rank of C3(x, theta2) is %d\n', rank(O3))

O4 = [C4.', A1.'*C4.', A1.'*A1.'*C4.', A1.'*A1.'*A1.'*C4.', A1.'*A1.'*A1.'*A1.'*C4.', A1.'*A1.'*A1.'*A1.'*A1.'*C4.' ];
fprintf('rank of C4(x, theta1, theta2) is %d\n', rank(O4))
```

The output is

```
Obervability Matrix
rank of C1(x) is 6
rank of C2(theta1, theta2) is 4
rank of C3(x, theta2) is 6
rank of C4(x, theta1, theta2) is 6
```

Therefore, only the matrix C2 with the output vector ($\theta_1(t)$, $\theta_2(t)$) is not observable.

(f) Luenberger Observer

General eqn:-

$$\dot{\hat{X}} = AX + BU + LC(X - \hat{X})$$

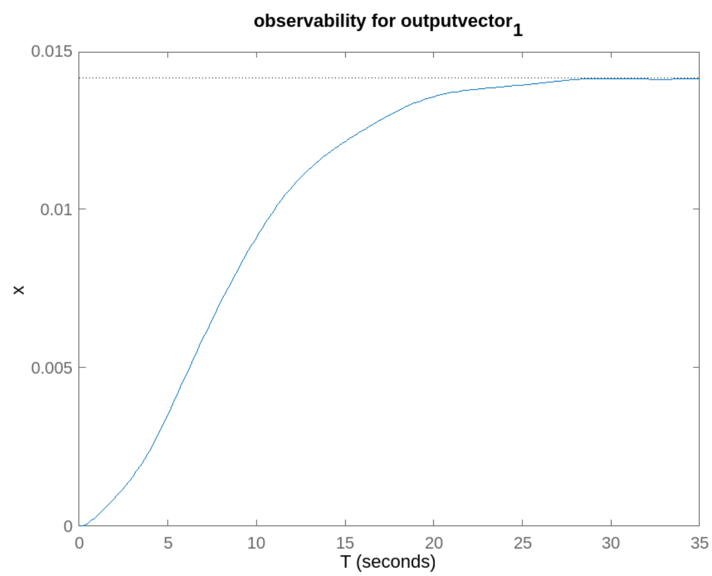
Accounting for error $X_e = X - \hat{X}$

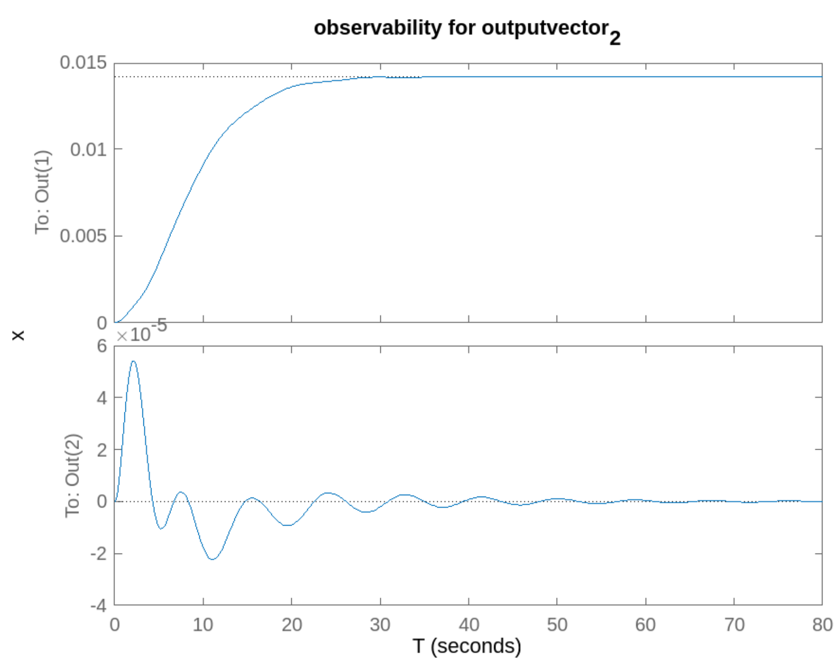
$$\dot{X}_e = (A - LC)(X_e)$$

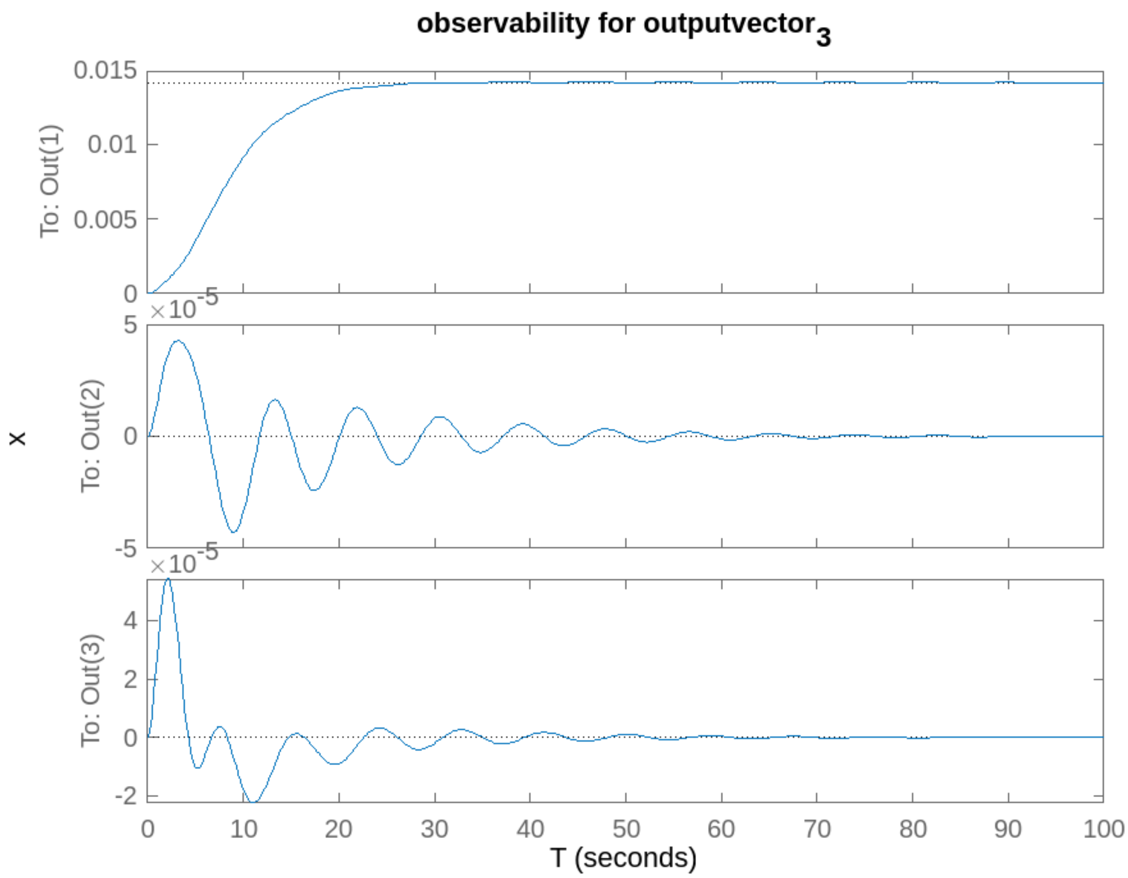
Considerin 2 cases; linearized & original non linear system

Matrix $(A - LC)^T$ must be stable for $A - LC$ to be stable.

Observability







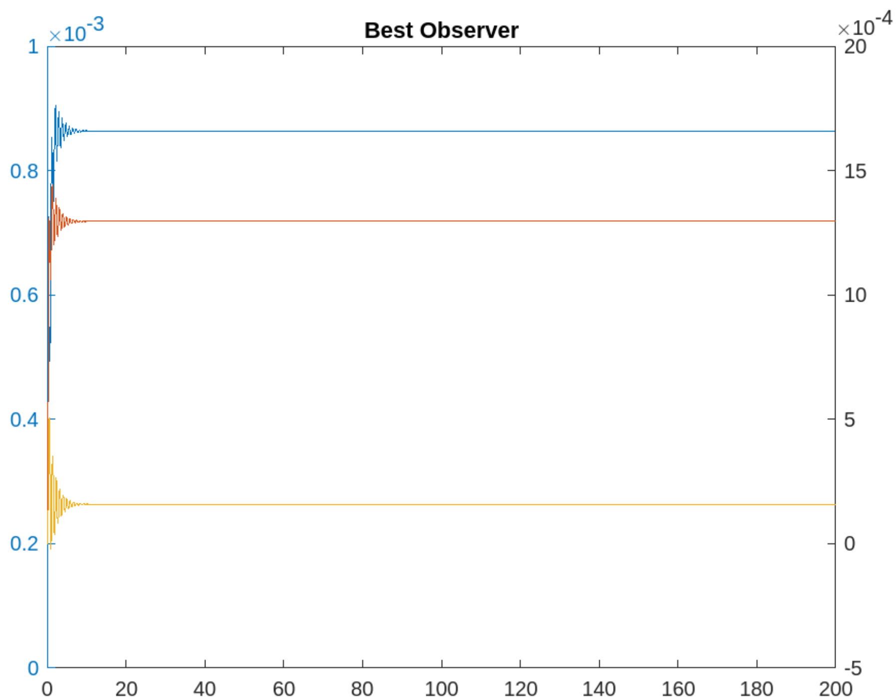


Figure 8: Best Luenberger observer for 3 vectors