# Homework1 solution

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所有结论正确，过程合理的解答均可得到全部分数，结论错误，过程正确也可以得到相应的步骤分。

## 1 2.13

*For what value of d does the volume, $V(d)$, of a d-dimensional unit ball take on its maximum?*

**Solution:** （25 分）

由

$$V(d)=\frac{2\pi^{\frac{d}{2}}}{d\Gamma(\frac{d}{2})}$$

代入有

$$\frac{V(d)}{V(d-2)}=\frac{2\pi^{\frac{d}{2}}}{d\Gamma(\frac{d}{2})}\cdot\frac{(d-2)\Gamma(\frac{d}{2}-1)}{2\pi^{\frac{d}{2}-1}}=\frac{2\pi}{d}$$

…………………… $(10')$

于是对 $d\leq 6$，有 $V(d)>V(d-2)$；对 $d\geq 7$，有 $V(d)<V(d-2)$.

因此只需比较 $V(5)$ 和 $V(6)$. …………………… $(10')$

同样利用体积公式:

$$V(5)=\frac{2\pi^{\frac{5}{2}}}{5\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\pi^{\frac{1}{2}}}=\frac{8\pi^2}{15}$$

$$V(6)=\frac{2\pi^3}{6\cdot 2\cdot 1\cdot 1}=\frac{\pi^3}{6}$$

$d=5$ 时单位球体的体积最大. …………………… $(5')$

## 2 2.14

（如果没有提到归纳法或者递推的想法，按每问 8 分批改）

*A 3-dimensional cube has vertices, edges, and edges. In a d-dimensional cube, these components are called faces. A vertex is a 0-dimensional face, an edge a 1-dimensional face, etc.*

*1. For $0\leq k\leq d$, how many k-dimensional faces does a d-dimensional cube have?*

*2. What is the total number of faces of all dimensions? The d-dimensional face is the cube itself which you can include in your count.*

*3. What is the surface area of a unit cube in d-dimensions(a unit cube has side-length one in each dimension)?*

*4. What is the surface area of the cube if the length of each side was 2?*

*5. Prove that the volume of a unit cube is close to its surface.*

**Solution:** (40 分)
(可以参考http://www.physicsinsights.org/hypercubes_1.html)

> ”To construct an n cube, we start with an n-1 cube, and sweep it through space perpendicular to the hyperplane in which it lies......(n-1)-cube forms two images of itself, which become two hyperfaces of the n-cube. In other words, the n-cube is the volume swept out by one of its hyperfaces. In addition, each hyperface of the (n-1)-cube sweeps out a new (n-1)-cube, which in turns becomes a face of the n-cube.”

$d$ 维立方体可以看成由两个 $d-1$ 维的立方体“连起来”。记 $(k,d)$ 为 $d$ 维立方体的 *k-dimensional face* 数，有 $(k,d)=2(k,d-1)+(k-1,d-1)$. ·········(30′)
(能理解 $d-1$ 维到 $d$ 维或者写出递推式就可以得分)

| d \ k | 0 | 1 | 2 | 3 | 4 | sum |
|---|---|---|---|---|---|---|
| 0 | 1 | | | | | 1 |
| 1 | 2 | 1 | | | | 3 |
| 2 | 4 | 4 | 1 | | | 9 |
| 3 | 8 | 12 | 6 | 1 | | 27 |
| 4 | 16 | 32 | 24 | 8 | 1 | 81 |

1. (从上表中看出总数为 $3^d$，又因为格点数为 $2^d$，很自然的猜测把 $(2+1)^d$ 展开即为所需结果) 猜测 $(k,d)=\binom{d}{k}\cdot 2^{d-k}$, 可以用数学归纳法证明. ·········(2′)
2. 根据第一问，由二项式展开可以得到总面数为 $3^d$. ·········(2′)
3. $d$ 维立方体有 $(d-1,d)=2d$ 个 *d-dimensional face*，其中每个是一个 $d-1$ 维立方体。表面积为 $2d\cdot 1^{d-1}=2d$. ·········(2′)
4. 表面积为 $2d\cdot 2^{d-1}=d\cdot 2^d$. ·········(2′)
5. 书上 2.3 节部分：将立方体向内缩 $\epsilon$，$\frac{V'}{V}=(1-\epsilon)^d\xrightarrow{d\to\infty}0$. ·········(2′)

## 3 2.16

*Consider a unit radius, circular cylinder in 3-dimensions of height one. The top of the cylinder could be an horizontal plane or half of a circular ball. Consider these two possibilities for a unit radius, circular cylinder in 4-dimensions. In 4-dimensions the horizontal plane is 3-dimensional and the half circular ball is 4-dimensional. In each of the two cases, what is the surface area of the top face of the cylinder? You can use V(d) for the volume of a unit radius, d-dimensional ball and A(d) for the surface area of a unit radius, d-dimensional ball. An infinite length, unit radius, circular cylinder in 4-dimensions would be the set* $\{(x_1,x_2,x_3,x_4)|x_2^2+x_3^2+x_4^2\le 1\}$ *where the coordinate* $x_1$ *is the axis.*

**Solution:** (15 分) (只需用 $V(d),A(d)$ 表示出结果，写成含 $d$ 表达式没有代入 $d=4$ 也算对，如果答案中认为上下两面都要算，扣 5 分，因为题目中明确指出 the top 只是一个平面或者半球)
Plane: $V(3)=\frac{4}{3}\pi$ ·········(7.5′)

Ball: $\frac{A(4)}{2} = \pi^2$ $\cdots\cdots\cdots(7.5')$

## 4 2.22

*Consider the upper hemisphere of a unit-radius ball in d-dimensions. What is the height of the maximum volume cylinder that can be placed entirely inside the hemisphere? As you increase the height of the cylinder, you need to reduce the cylinder's radius so that it will lie entirely within the hemisphere.*

**Solution:**（20 分）

（只需考虑 $d$ 维的 cylinder，因为高维空间中的低维物体体积为 0。$d \le 1$ 无意义，可以不写。$d = 2$ 时横着放与竖着放体积是一样的，这时候应该长方形（退化的圆柱体）的宽和高都是结果，没有考虑到这点扣 2 分）

因为当圆柱体是斜着放的时候，我们总可以使其绕着球心旋转使得其底面或者高与半球的底面平行，所以我们只需要考虑竖着放和横着放两种情况 $\cdots\cdots\cdots(5')$

第一种情况：圆柱体是竖着放的。

设圆柱底面所在“圆”半径为 $r$, 于是有：

$x_1^2 + ... + x_{d-1}^2 \le r^2$

$x_1^2 + ... + x_d^2 = 1$

圆柱的高 $h = \sqrt{1-r^2}$.

圆柱的体积 $V = \sqrt{1-r^2} \cdot V(d-1) \cdot r^{d-1}$，其中 $V(d-1)$ 表示 $d-1$ 维的单位球的体积。$V(d-1)$ 与 $r$ 无关。

设 $f(r) = (1-r^2)r^{2d-2}$，需要求出使 $f(r)$ 取最大值的 $r$.

$f'(r) = (2d-2)r^{2d-3} - 2d \cdot r^{2d-1} = 0$ 时，$r = \sqrt{\frac{d-1}{d}}$.

可验证此时 $f(r)$ 取到最大值，此时 $V = \frac{(d-1)^{\frac{d-1}{2}}}{d^{\frac{d}{2}}} V(d-1)$，圆柱的高为 $\sqrt{\frac{1}{d}}$ $\cdots\cdots\cdots(5')$

第二种情况：圆柱体是横着放的。

设圆柱底面所在“圆”半径为 $r$, 于是有：

圆柱的高 $h = 2\sqrt{1-4r^2}$.

圆柱的体积 $V = 2\sqrt{1-4r^2} \cdot V(d-1) \cdot r^{d-1}$，其中 $V(d-1)$ 表示 $d-1$ 维的单位球的体积。$V(d-1)$ 与 $r$ 无关。

设 $f(r) = (1-4r^2)r^{2d-2}$，需要求出使 $f(r)$ 取最大值的 $r$.

$f'(r) = (2d-2)r^{2d-3} - 8d \cdot r^{2d-1} = 0$ 时，$r = \sqrt{\frac{d-1}{4d}}$.

可验证此时 $f(r)$ 取到最大值，此时 $V = \frac{(d-1)^{\frac{d-1}{2}}}{d^{\frac{d}{2}}} 2^{2-d} V(d-1)$，圆柱的高为 $2\sqrt{\frac{1}{d}}$ $\cdots\cdots\cdots(5')$

因为 $d > 2$ 时第一种情况体积更大，此时结果为 $h = \sqrt{\frac{1}{d}}$ $\cdots\cdots\cdots(3')$

$d = 2$ 时一样大，结果为 $h = \frac{\sqrt{2}}{2}$ 或 $\sqrt{2}$ $\cdots\cdots\cdots(2')$