Vrije Universiteit Amsterdam

E_EORM_TSM

Assignment I

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1 Introduction

State Space Models are widely used to represent univariate, multivariate stationary and non stationary time series. In particular, modern programming software (e.g. R or Python) can be used for parameter estimation via a State Space model and the Kalman filter, respectively. In state space analysis we apply techniques such as filtering, smoothing, initialisation and forecasting. This approach is more flexible compared to the main analytical system as widely discussed in the Econometric literature. In this paper we implement and analyse the methods for the Local Level model in accordance with the monograph by Durbin and Koopman (2001). For this research we used two different data sets. Firstly, the Nile data, for a more detailed elaboration on this data set we refer to page 10 of Durbin and Koopman (2001). Secondly, we used a flight data set from FlightRadar24 which tracks the total number of flights from May 11th 2020 till February 1st 2021. We reproduced the figures from chapter 2 of the DK-book which can be reviewed in the Appendix. Thereafter, we applied the same methodology on the flight data set and analysed the obtained results.

2 Local Level Model

In Chapter 2 of the DK-book, there are 8 figures, each consisting of 4 sub-figures for the Nile data file. We apply the same chronological ordering to represent our results. Firstly, we take the local level model (1), which can be seen as the simplest specification of a State Space Model.

$$\begin{aligned}&y_t = a_t + \epsilon_t, \quad \epsilon \sim N(0, \sigma_\epsilon^2)\\&\alpha_{t+1} = \alpha_t + \nu_t, \quad \eta_t \sim N(0, \sigma_\eta^2),\ \alpha_1 \sim N(\alpha_1, P_1)\end{aligned} \tag{1}$$

2.1 Kalman Filter

Given the local level model (1), we compute the distribution in a multivariate form. Then, we apply the Kalman Filter with the purpose of smoothing and filtering the time series in order to update our knowledge of the system each time a new observation y_t is brought in. Subsequently, we derive the filtering recursive equations (2) for the local level model to obtain the Kalman Filter.

$$\begin{aligned}&v_t = y_t - \alpha_t\\&F_t = P_t + \sigma_\epsilon^2\\&K_t = \tfrac{P_t}{F_t}\\&\alpha_{t+1} = a_t + K_t v_t\\&P_{t+1} = P_t(1 - K_t) + \sigma_\eta^2\end{aligned} \tag{2}$$

where $\alpha_t = E(\alpha_t|Y_{t\text{-}1})$ and $P_t = Var(\alpha_t|Y_{t1})$. It should be noted that P_t depends only on σ_ϵ^2 and σ_η^2 and does not depend on Y_{t-1} (Durbin and Koopman (2001)).

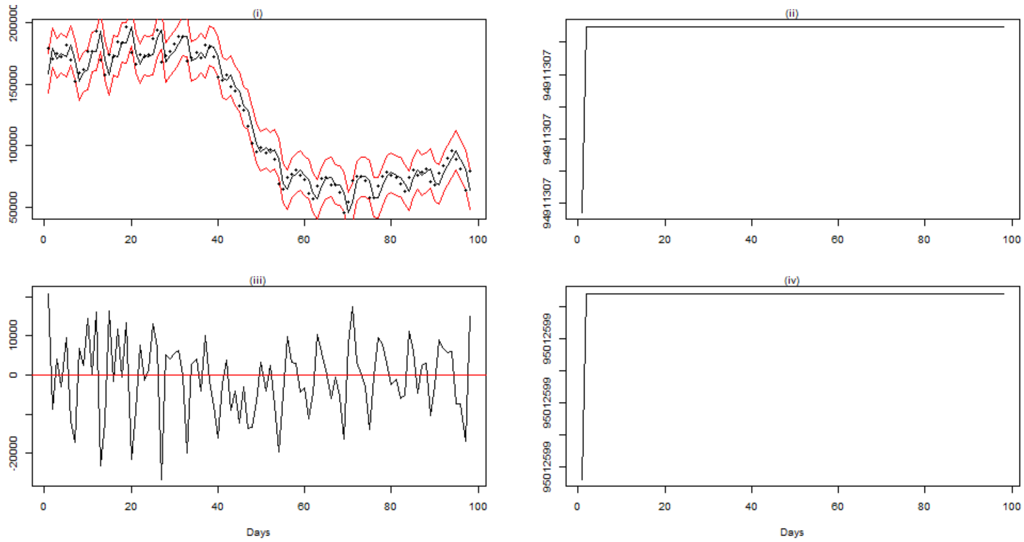

Figure 1: Output Kalman filter: (i) filtered state and its 90% confidence intervals; (ii) filtered state variance; (iii) prediction errors; (iv) prediction variance.

Figure 1, shows the output of the Kalman Filter (2) on daily flight data. When observing the above illustrations we notice that F_t and P_t converge rapidly to constant values which confirms that the local level model has a steady state solution.

2.2 State Smoothing Recursion

The basic behind the recursion computations is that the forecast errors $v_1, \ldots, v_n$ are mutually independent and linear transformation of $y_1, \ldots, y_n$, and $v_t, \ldots, \mu_n$ are independent of $y_1, \ldots, y_{t-1}$ with mean equal to zero. In section 2.1, $t+1$ was dependent on past information, we will now estimate the state using the entire sample. Then, we obtain the smoothed state $\hat{\alpha}_t$ by backwards recursion.

$$r_{t-1} = \frac{v_t}{F_t} + L_t r_t \hat{\alpha}_t = \alpha_t + P_t r_{t-1} \tag{3}$$

The error variance of the smoothed state V_t and the smoothing variance cumumulant are obtained using the state variance smoothing recursion 3.

$$V_t = P_t - P_t^2 N_{t-1} N_{t-1} = \frac{1}{F_t} + L_t^2 N_t \tag{4}$$

We refer to pages 19, 20 and 21 of Durbin and Koopman (2001), "Time Series Analysis by State Space methods" for a more detailed elaboration.

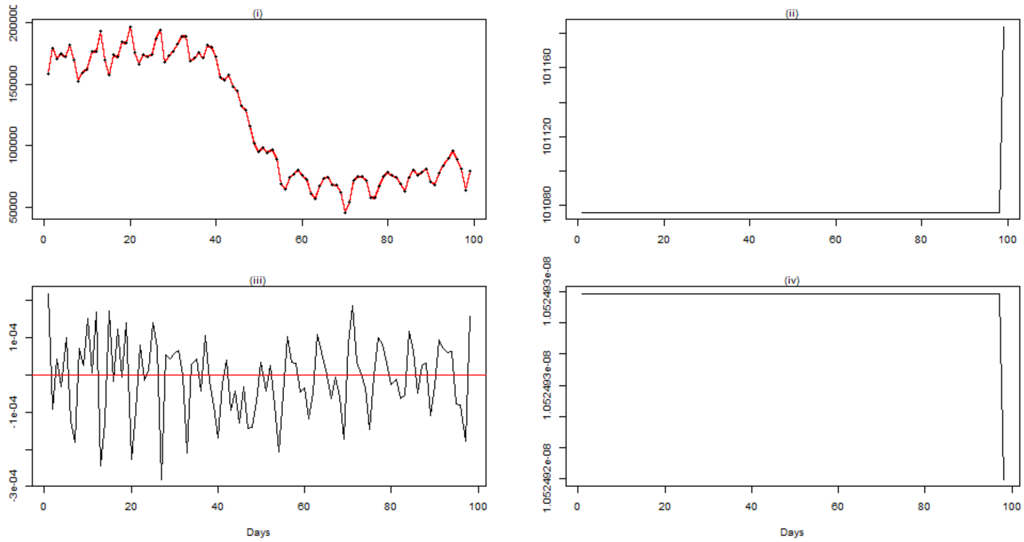

Figure 2: Output smoothing recursion: (i) smoothed state and its 90% confidence intervals; (ii) smoothed state variance; (iii) smoothing cummulant; (iv) smoothing variance cummulant.

Figure 2, shows the backwards smoothing recursion output. The estimated state $\hat{\alpha}_t$ is smoother compared to the estimated state in Figure 1. Furthermore, the confidence intervals are narrower. However, at the end of the series the smoothness of $\hat{\alpha}$ and a_t become much alike, as it should be. For the simple reason that for the last couple of estimations for the Kalman filter the data set Y_t to predict α_{t+1} almost consists of the entire data set on which $\hat{\alpha}_t$ is also based.

2.3 Disturbance Smoothing

Disturbance smoothing are useful in order to make the MLE computations more efficient. The smoothed observation disturbances $E(\epsilon_t|y)$, $E(\eta_t|y)$ and their error variances are computed through the following recursive relations.

$$\begin{aligned}
&\hat{\epsilon}_t = \sigma_\epsilon^2 U_t \\
&u_t = F_t^{-1} v_t - K_t r_t \\
&Var(\epsilon_t|y) = \sigma_\epsilon^2 - \sigma_\epsilon^4 D_t \\
&D_t = F_t^{-1} + K_t^2 N_t \\
&\hat{\eta}_t = \sigma_\eta^2 r_t \\
&u_t = F_t^{-1} v_t - K_t r_t \\
&Var(\eta_t|y) = \sigma_\eta^2 - \sigma_\eta^4 N_t
\end{aligned} \tag{5}$$

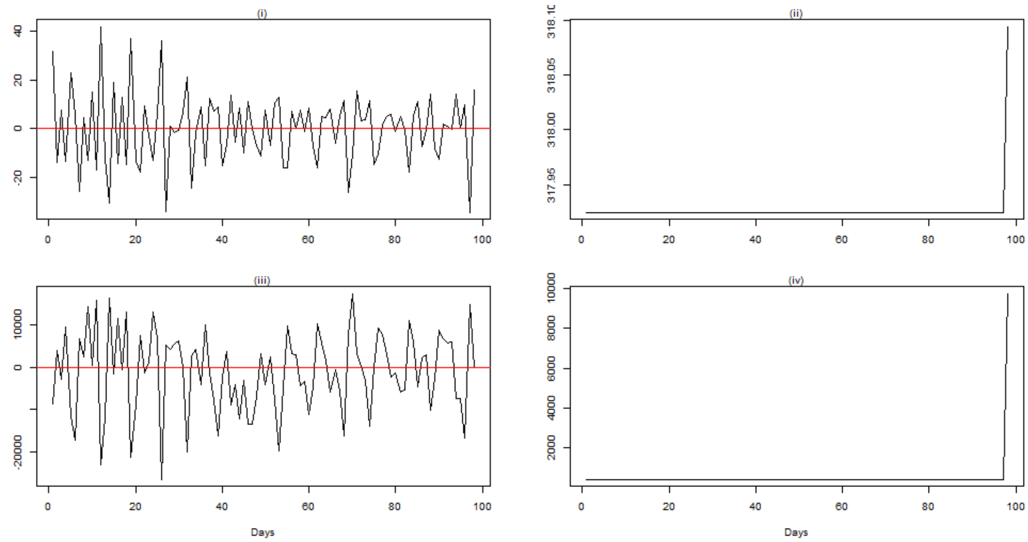

Figure 3: Output disturbance smoothing recursion: (i) observation error; (ii) observation error standard deviation; (iii) state error; (iv) state error standard deviation.

Figure 3, shows the results of the disturbance smoothing recursion. Firstly, we note that conditional variances, $Var(\epsilon_t|Y_n)$ and $Var(\eta_t|Y_n)$, are larger at the beginning and end of the series. As expected, the plot of r_t in figure 2 and the plot of $\hat{\eta}_T$ in figure 3 are the same apart from a different scale.

2.4 Missing observations

Missing observations are easily handled in the state space setting where all the recursive equations hold with the respective innovation term for the missing observations. To demonstrate this, we omit points 21,...,40 and 60,...,80. We then started by computing the Kalman Filter (2) and stored the output. Then, we applied the state smoothing recursion as mentioned in 2.2. Since we have missing observation, we used the following formulas to obtain r_t and $\hat{\alpha_t}$.

$$\begin{aligned} r_{t-1} &= r_t \\ \hat{\alpha_t} &= \alpha_t + P_t r_{t-1} \end{aligned} \tag{6}$$

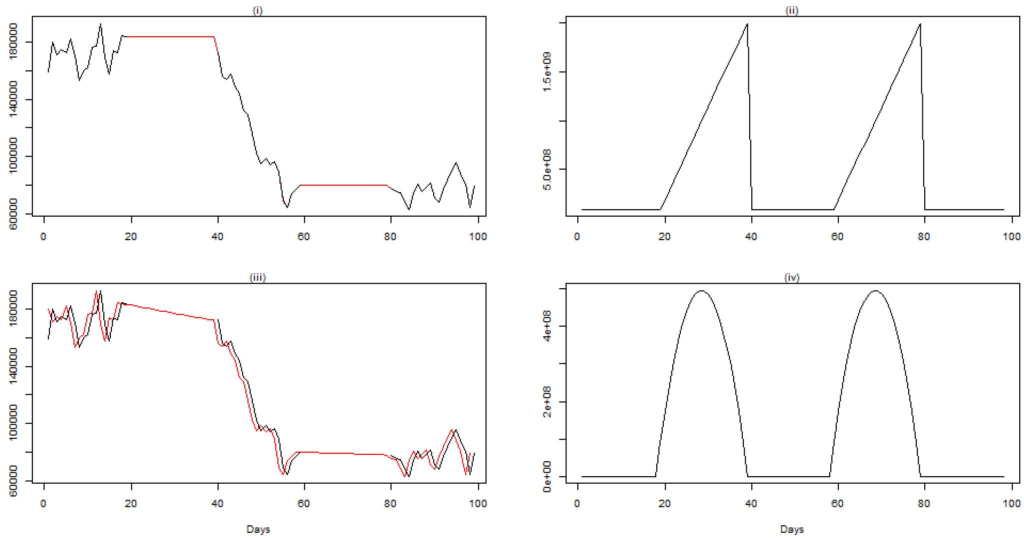

Figure 4: Output filtering and smoothing when observations are missing: (i) filtered state; (ii) filtered state variance; (iii) smoothed state; (iv) smoothed state variance.

Figure 4, shows the filtering and smoothing series with missing observations. Graph I and II in figure 4 are the Kalman filter values of a_t and P_t, respectively. Graph III and IV are the smoothing output $\hat{\alpha_t}$ and V_t, respectively. We observe that the application of the Kalman filter to the missing observations can be regarded as extrapolation of the series to the missing time points, while smoothing at these points is effectively interpolation, as expected.

2.5 Forecasting

Forecasting for the local level model is done by filtering the observations $y_1, \ldots, y_n, y_{n+1}, \ldots, y_{n+J}$ using the recursion (2) and treating the last J observations $y_{n+1}, \ldots, y_{n+J}$ as missing. Hence, we assume the Kalman gain, K_t, and innovation for the period $t = n+1, \ldots, n+J$ to be 0. Figure 5 shows the Flight data extended by 30 missing observations, so $J = 30$.

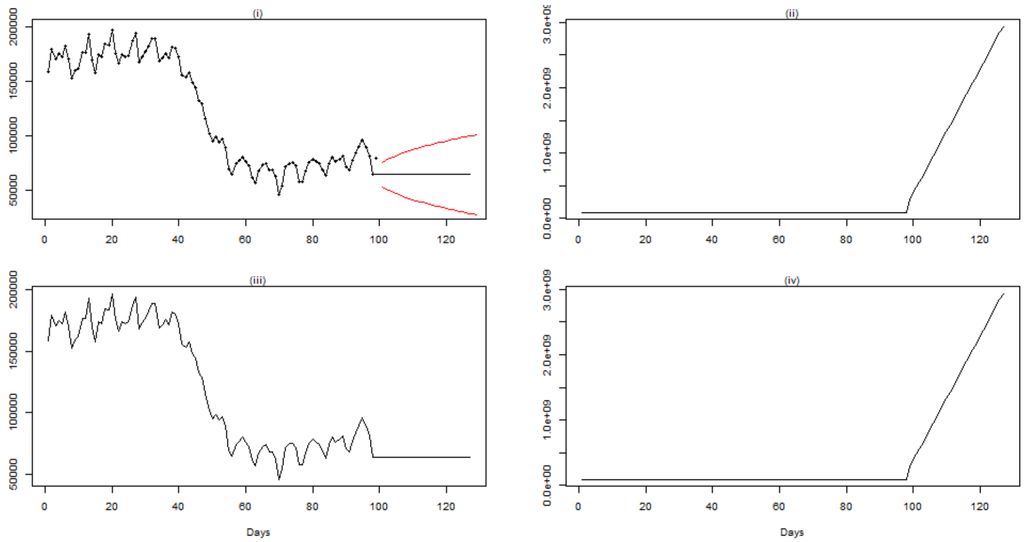

Figure 5: Output forecasting: (i) state forecast and its 50% confidence intervals; (ii) state variance; (iii) observation forecast; (iv) observation forecast variance.

2.6 Initialisation

This section explains about initialization of Kalman filter. When one does not know anything about the prior density of the initial state, it is reasonable to represent α_1 as having diffuse prior density (Durbin and Koopman, 2001).

$$v_1 = y_1 - \alpha_1, \quad F_1 = P_1 + \sigma_\epsilon^2$$

Then, we substitute the above values in to the equations for a_2 and P_2, and letting $P_1 \to \infty$, we get $a_2 = y_1$, $P_2 = \sigma_\epsilon^2 + \sigma_\eta^2$. Thereafter, we proceed normally with Kalman filter for $t = 2, \ldots, n$, following the same methodology as Durbin and Koopman (2001). Based on the diffuse prior, the state and smoothing equations also have to be changed. They are not affected for $t = 2, \ldots, n$ but for the smoothed state for initial state and smoothed conditional variance of initial state change. So do smoothed mean and variance of the disturbances.

$$\hat{\alpha_1} = y_1 + \sigma_\epsilon^2 r_1$$
$$V_1 = \sigma_\epsilon^2 - \sigma_\epsilon^4 N_1$$

2.7 Standardised prediction errors

It is assumed that the disturbance ϵ_t and η_t for the local level model are normally distributed and serially independent with the constant variances. The one-step ahead forecast errors are as follows;

$$e_t = \frac{v_t}{\sqrt{F_t}}.$$

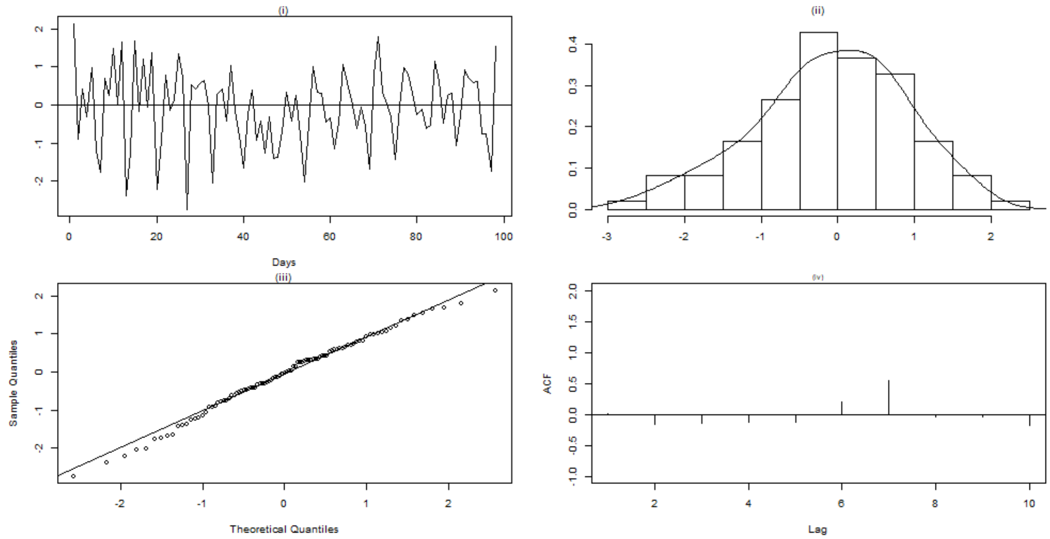

Figure 6: Output diagnostic plots standardised errors: (i) standardised residuals; (ii) histogram plus estimated density; (iii) ordered residuals; (iv) correlogram.

In figure 6 the errors are plotted. The density plot (ii) and the QQ plot (iii) support the normality assumption. Furthermore, plot (iv) displays the correlogram that supports the assumption that the errors are serially independent with unit variance.

2.8 Smoothed standardised prediction errors

The standardised smoothed residuals are as follows;

$$u_t^* = \frac{\hat{\epsilon}_t}{\sqrt{Var(\hat{\epsilon}_t)}} = D_t^{-\frac{1}{2}} u_t$$
$$r_t^* = \frac{\hat{\eta}_t}{\sqrt{Var(\hat{\eta}_t)}} = N_t^{-\frac{1}{2}} u_t$$

The residuals can be used to detect outliers and structural breaks in time series. An outlier in a postulated time series generated by the local level model is indicated by a large value for $\hat{\epsilon}_t$ or u_t^* and a break in the level α_{t-1} is indicated by a large value for $\hat{\eta}_t$ or r_t^*. Figure 8 displays the observation residuals u_t^* and the state residuals r_t^*, in (i) and (iii) respectively.. The estimated density histograms are both given for u_t^* and r_t^*, (ii) and (iv) respectively.

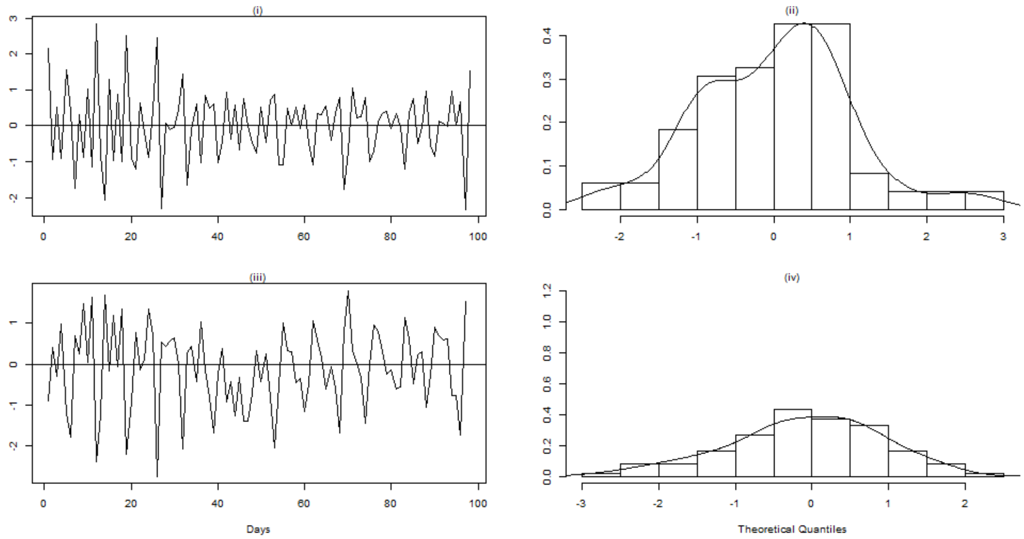

Figure 7: Output diagnostic plots for auxiliary residuals: (i) observation residuals; (ii) histogram and estimated density; (iii) state residuals; (iv) histogram and estimated density for r_t^*.

References

Durbin, J. and S. J. Koopman (2001). *Time series analysis by state space methods*. Oxford University Press.

3 Appendix

All the plots in the appendix are the results of the Nile data set.

3.1 Kalman filter

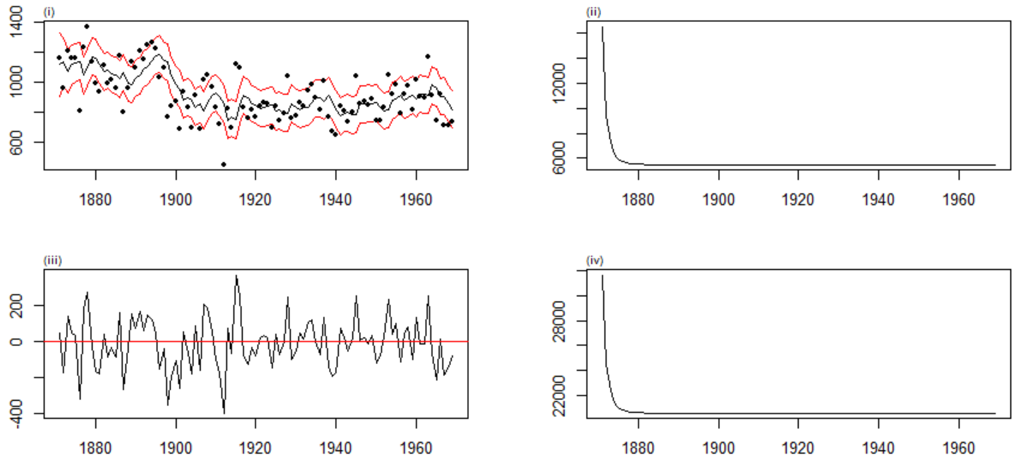

Figure 8: Output Kalman filter: (i) filtered state and its 90% confidence intervals; (ii) filtered state variance; (iii) prediction errors; (iv) prediction variance.

3.2 State smoothing recursion

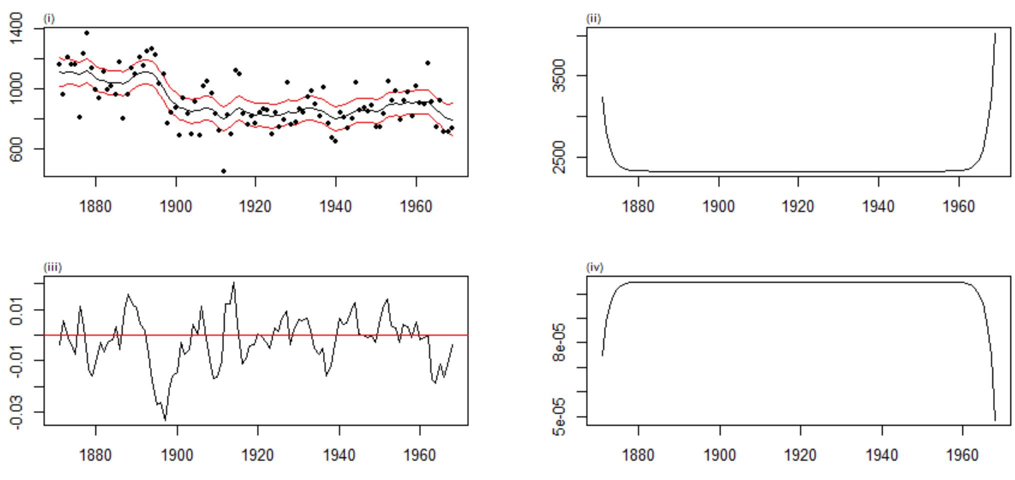

Figure 9: Output smoothing recursion: (i) smoothed state and its 90% confidence intervals; (ii) smoothed state variance; (iii) smoothing cummulant; (iv) smoothing variance cummulant.

3.3 Disturbance smoothing

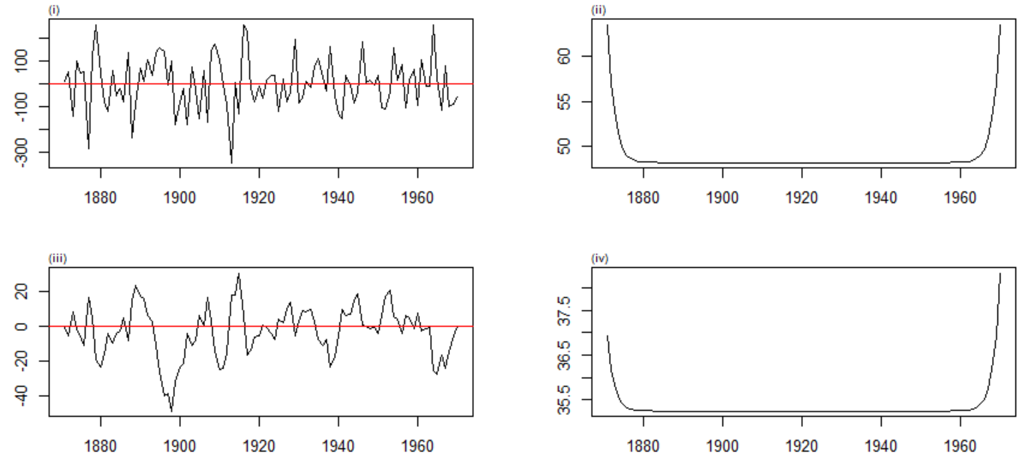

Figure 10: Output disturbance smoothing recursion: (i) observation error; (ii) observation error standard deviation; (iii) state error; (iv) state error standard deviation.

3.4 Missing observations

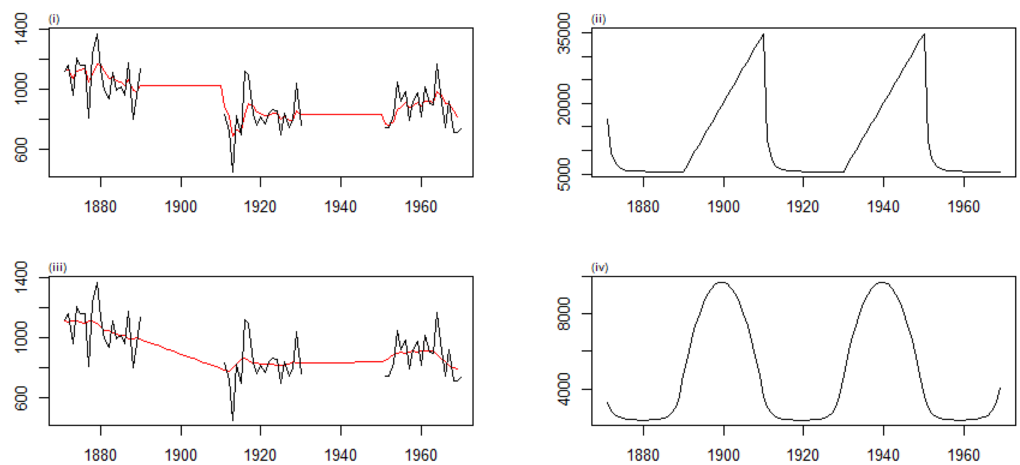

Figure 11: Output filtering and smoothing when observations are missing: (i) filtered state; (ii) filtered state variance; (iii) smoothed state; (iv) smoothed state variance.

3.5 Forecasting

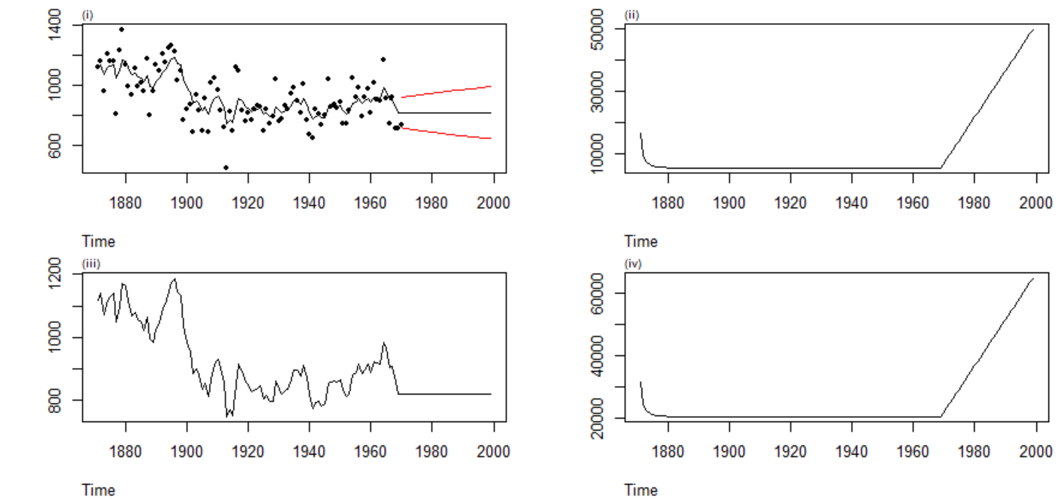

Figure 12: Output forecasting: (i) state forecast and its 50% confidence intervals; (ii) state variance; (iii) observation forecast; (iv) observation forecast variance.

3.6 Standardised prediction errors

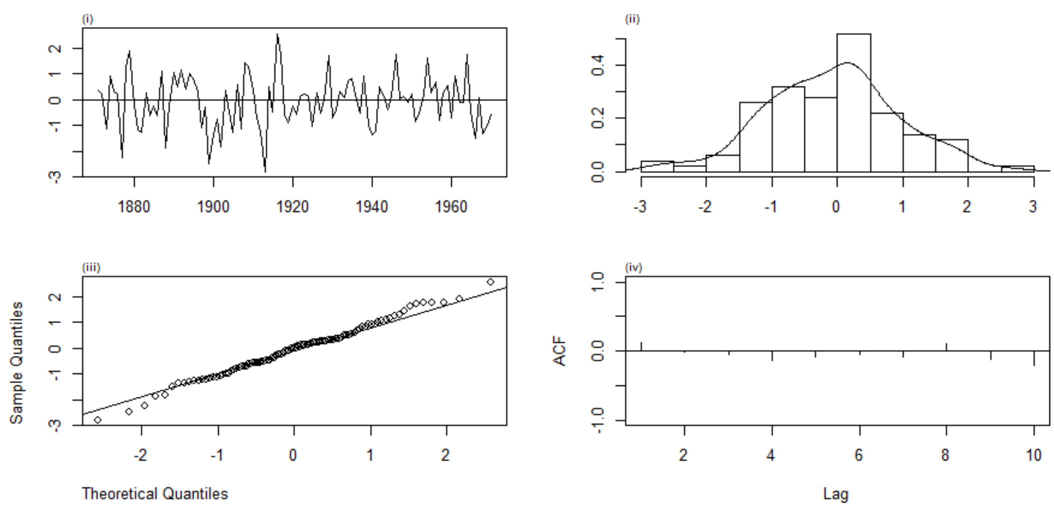

Figure 13: Output diagnostic plots standardised errors: (i) standardised residuals; (ii) histogram plus estimated density; (iii) ordered residuals; (iv) correlogram.

3.7 Smoothed standardised prediction errors

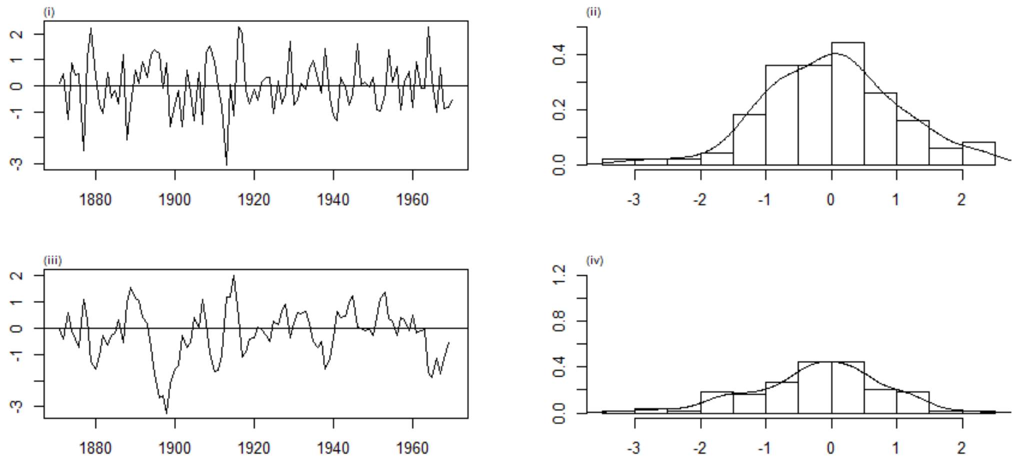

Figure 14: Output diagnostic plots for auxiliary residuals: (i) observation residuals; (ii) histogram and estimated density; (iii) state residuals; (iv) histogram and estimated density for r_t^*.