CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

Consumption-Saving

Mini-Course: Heterogenous Agent Macro

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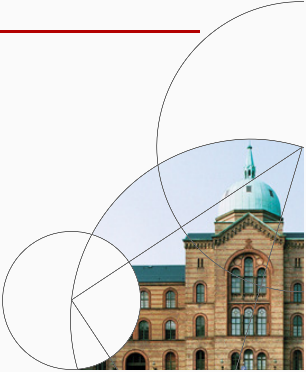

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Introduction

Introduction

- **Generations of models:**
  1. **Permanent income hypothesis (PIH)** (Friedman, 1957) or life-cycle model (Modigliani and Brumburg, 1954)
  2. **Buffer-stock consumption model** Deaton (1991, 1992); Carroll (1992, 1997, 2019)
  3. **Multiple-asset buffer-stock consumption models** e.g. Kaplan and Violante (2014); Harmenberg and Öberg (2021)
- **Consumption-and-saving over the life-cycle dynamic** e.g. Gourinchas and Parker (2002); Druedahl and Martinello (2022)
- **Empirical MPCs and income risk** e.g. Fagereng et. al. (2021); Guvenen et. al. (2021)

Book: The Economics of Consumption, Jappelli and Pistaferri (2017)

Plan

PIH

Consumption-saving

$$
v_0 = \max_{\{c_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t)
$$

$$
\text{s.t.}
$$

$$
a_t = (1+r)a_{t-1} + wz_t - c_t
$$

$$
a_{T-1} \geq 0
$$

- **Variables:**
  Consumption: c_t
  Productivity: z_t
  End-of-period savings: a_t (*no debt at death*)
- **Parameters:**
  Discount factor: β
  Wage: w
  Interest rate: r (define $R \equiv 1 + r$ as interest factor)

It is a *static* problem

$$v_0 = \max_{\{c_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t)$$

$$\text{s.t.}$$

$$a_t = (1+r)a_{t-1} + wz_t - c_t$$

$$a_{T-1} \geq 0$$

- **It is a *static* problem:**
  1. **Information:** z_t is known for all t at $t = 0$
  2. **Target:** Discounted utility, $\sum_{t=0}^{T-1} \beta^t u(c_t)$
  3. **Behavior:** Choose $c_0, c_1, \ldots, c_{T-1}$ *simultaneously*
  4. **Solution:** Sequence of consumption *choices* $c_0^*, c_1^*, \ldots, c_{T-1}^*$

IBC

- **Substitution** implies *Intertemporal Budget Constraint* (IBC)

$$\begin{aligned} a_{T-1} &= Ra_{T-2} + wz_{T-1} - c_{T-1} \\ &= R^2 a_{T-3} + Rwz_{T-2} - Rc_{T-2} + wz_{T-1} - c_{T-1} \\ &= R^T a_{-1} + \sum_{t=0}^{T-1} R^{T-1-t}(wz_t - c_t) \end{aligned}$$

- Use **terminal condition** $a_{T-1} = 0$ (equality due utility max.)

$$R^{-(T-1)} a_{T-1} = 0 \quad \Leftrightarrow \quad s_0 + h_0 - \sum_{t=0}^{T-1} R^{-t} c_t = 0$$

where $s_0 \equiv Ra_{-1}$ (after-interest assets)
and $h_0 \equiv \sum_{t=0}^{T-1} R^{-t} wz_t$ (human capital)

FOC and Euler-equation

$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t u(c_t) + \lambda \left[\sum_{t=0}^{T-1} R^{-t} c_t - s_0 - h_0 \right]$$

- **First order conditions**:

$$\forall t : 0 = \beta^t u'(c_t) - \lambda (1+r)^{-t} \Leftrightarrow u'(c_t) = -\lambda \left(\beta R\right)^{-t}$$

- **Euler-equation** for $k \in \{1, 2, \dots\}$:

$$\frac{u'(c_t)}{u'(c_{t+k})} = \frac{-\lambda \left(\beta R\right)^{-t}}{-\lambda \left(\beta R\right)^{-(t+k)}} = \left(\beta R\right)^k$$

Consumption choice

- **CRRA:** $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$ imply Euler-equation

$$\frac{c_0^{-\sigma}}{c_t^{-\sigma}} = (\beta R)^t \Leftrightarrow c_t = (\beta R)^{\frac{t}{\sigma}} c_0$$

- Insert **Euler** into **IBC** to get consumption choice

$$\begin{aligned}\sum_{t=0}^{T-1} \left((\beta R)^{1/\sigma} R^{-1} \right)^t c_0 &= s_0 + h_0 \Leftrightarrow \\ c_0^* &= \frac{1-(\beta R)^{1/\sigma} R^{-1}}{1-\left((\beta R)^{1/\sigma} R^{-1} \right)^T} (s_0 + h_0)\end{aligned}$$

Infinite horizon

- **Infinite horizon** for $(\beta R)^{1/\sigma} R^{-1} < 1$: Let $T \to \infty$ to get

$$c_0^* = \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right)(s_0 + h_0)$$

$$\text{if } \forall z_t = 1: \; c_0^* = \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right)\left(Ra_{-1} + \frac{R}{R-1} w\right)$$

- **Consume annuity value:** $\beta R = 1, z_t = 1 \Rightarrow c_0^* = ra_{-1} + w$
- **Intertemporal elasticity of substitution** (IES $= \frac{1}{\sigma}$)**:**

$$\log c_{t+1} - \log c_t = \frac{1}{\sigma} \log \beta R$$

Constant consumption if:

1. $\beta R = 1$
2. $\sigma \to \infty$ (zero elasticity of substitution)

Propensities to consume ($\beta R \approx 1, z_t \approx 1$)

$$c_0^* \approx \frac{r}{1+r}\left((1+r)a_{-1} + \sum_{t=0}^{\infty}\frac{wz_t}{(1+r)^t}\right) \approx ra_{-1} + w$$

Different types of shocks:

1. MPC of *windfall* income: $\frac{\partial c_0}{\partial s_0} \approx \frac{r}{1+r}$
2. MPC of *future* income change: $\frac{\partial c_0}{\partial wz_t} \approx \frac{r}{1+r}(1+r)^{-t}$
3. MPC of *permanent* income change: $\frac{\partial c_0}{\partial w} \approx \frac{r}{1+r}\frac{1}{1-(1+r)^{-1}} = 1$

Dynamic affects: The same when $\beta R = 1$, for all $k > 0$

$$\frac{\partial c_k}{\partial s_0} = \frac{\partial c_0}{\partial s_0}$$
$$\frac{\partial c_k}{\partial wz_t} = \frac{\partial c_0}{\partial wz_t}$$
$$\frac{\partial c_k}{\partial w} = \frac{\partial c_0}{\partial w}$$

Savings ($\beta R = 1$)

- **Constant savings** $z_t = 1$:

$$c_t = ra_{t-1} + w \Rightarrow a_t = Ra_{t-1} + w - c_t = a_{t-1}$$

1. Decreasing savings with $\beta R < 1$: $c_t \uparrow \Rightarrow a_t < a_{t-1}$
2. Increasing savings with $\beta R > 1$: $c_t \downarrow \Rightarrow a_t > a_{t-1}$

- **Same consumption if NPV of** wz_t **is unchanged**

$$\frac{r}{1+r} \sum_{t=0}^{\infty} \frac{z_t}{(1+r)^t} = 1$$

$\Rightarrow$ *savings change with income*

Initial liquidity/borrowing constraint

- Implied period 0 **savings** are: $a_0 = s_0 + wz_0 - c_0$
- Hard **borrowing constraint:** $a_0 \geq -wb$
- **Maximum consumption:** $\overline{c}_0 = s_0 + wz_0 + wb$
- **Optimal consumption:** Constrained or unconstrained.

$$c_0^* = \min\left\{\overline{c}_0, \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right)(s_0 + h_0)\right\}$$

- **Empirical realism.** MPC of constrained is one

$$c_0^* = \overline{c}_0 \Rightarrow \frac{\partial c_0^*}{\partial s_0} = \frac{\partial \overline{c}_0}{\partial s_0} = 1$$

- **Technical issue:** *Borrowing constraints further in the future complicates the analytical solution considerably.*

Empirical MPCs

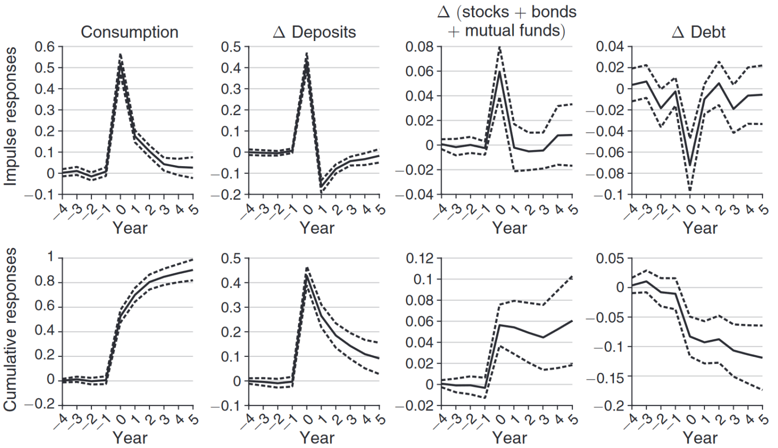


FIGURE 2. DYNAMIC HOUSEHOLD RESPONSES TO LOTTERY PRIZES

Source: Fagereng et. al. (2021)

Buffer-stock

Uncertainty and always borrowing constraint

$$v_0(z_0, a_{-1}) = \max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

s.t.

$$a_t = (1+r)a_{t-1} + wz_t - c_t$$

$$z_{t+1} \sim \mathcal{Z}(z_t)$$

$$a_t \geq -wb$$

$$\lim_{t \to \infty} (1+r)^{-t} a_t \geq 0 \quad \text{[No-Ponzi game]}$$

- **Stochastic income** from 1st order Markov-process, $\mathcal{Z}$
- ***A true dynamic problem:***
  1. **Information:** z_t is revealed period-by-period
  2. **Target:** *Expected* discounted utility, $\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$
  3. **Behavior:** Choose c_t *sequentially* as information is revealed
  4. **Solution:** Sequence of consumption *functions*, $c_t^*(z_t, a_{t-1})$

IBC

- **Substitution** still implies:

$$R^{-(T-1)}a_{T-1} = 0 \quad \Leftrightarrow \quad s_0 + h_0 - \sum_{t=0}^{T-1} R^{-t} c_t = 0$$

- **What if $T \to \infty$?** We must have $\lim_{T\to\infty} R^{-(T-1)}a_{T-1} = 0$
  1. $\lim_{T\to\infty} R^{-(T-1)}a_{T-1} > 0$: Consumption can be increased
  2. $\lim_{T\to\infty} R^{-(T-1)}a_{T-1} < 0$: Violates No-Ponzi game condition
- For $T \to \infty$ we have the **IBC**:

$$\sum_{t=0}^{\infty} R^{-t} c_t = Ra_{-1} + \sum_{t=0}^{\infty} R^{-t} w z_t$$

Natural borrowing limit

- Denote **minimum possible productivity** by $\underline{z}$
- **Consumption must be non-negative** $\Rightarrow$
  interest payments must be less than minimum income

$$c_t \geq 0 \Rightarrow r(-a_t) \leq w\underline{z} \Leftrightarrow a_t \geq -\frac{w\underline{z}}{r}$$

  If debt was larger it would in the worst case ($\forall z_t = \underline{z}$) grow without bound even with zero consumption ($\forall c_t = 0$)

$$\begin{aligned}
a_0 &= -\frac{w\underline{z}}{r} - \Delta \\
a_1 &= (1+r)a_0 + w\underline{z} = a_0 - (1+r)\Delta \\
a_2 &= (1+r)a_1 + w\underline{z} = a_0 - (1+r)^2\Delta \\
&\vdots
\end{aligned}$$

- **Natural borrowing constraint:** $a_t \geq \underline{a} = -w \min\left\{b, \frac{\underline{z}}{r}\right\}$

Euler-equation from variation argument

- **Case I:** If $u'(c_t) > \beta R \mathbb{E}_t [u'(c_{t+1})]$:
  Increase c_t by marginal $\Delta > 0$, and lower c_{t+1} by $R\Delta$
  1. **Feasible:** Yes, if $a_t > \underline{a}$
  2. **Utility change:** $u'(c_t) + \beta (-R) \mathbb{E}_t [u'(c_{t+1})] > 0$
- **Case II:** If $u'(c_t) < \beta R \mathbb{E}_t [u'(c_{t+1})]$:
  Lower c_t by marginal $\Delta > 0$, and increase c_{t+1} by $R\Delta$
  1. **Feasible:** Yes (always)
  2. **Utility change:** $u'(c_t) + \beta R \mathbb{E}_t [u'(c_{t+1})] > 0$
- **Conclusion:** By contradiction
  1. **Constrained:** $a_t = \underline{a}$ and $u'(c_t) \geq \beta R \mathbb{E}_t [u'(c_{t+1})]$, or
  2. **Unconstrained:** $a_t > \underline{a}$ and $u'(c_t) = \beta R \mathbb{E}_t [u'(c_{t+1})]$
- **Sufficiency:** From concavity of value function
  FOC: $c_t^{-\sigma} = \beta \mathbb{E}_t [v_a(z_{t+1}, a_t)]$
  Envelope: $v_a(z_t, a_{t-1}) = R c_t^{-\sigma}$

Special case I: Quadratic utility

- **Quadratic utility:** $u(c_t) = -\frac{1}{2}(\overline{c} - c)^2$ with $\beta R = 1$ and »large« $\overline{c}$
- **Euler-equation:** *Consumption = expected future consumption*

$$(\overline{c} - c_t) = \mathbb{E}_t\left[(\overline{c} - c_{t+k})\right] \Leftrightarrow c_t = \mathbb{E}_t\left[c_{t+k}\right]$$

- Use **IBC** in expectation to get **consumption function:**

$$\sum_{t=0}^{\infty} R^{-t}\mathbb{E}_0\left[c_t\right] = Ra_{-1} + \sum_{t=0}^{\infty} R^{-t} w\mathbb{E}_0\left[z_t\right] \Rightarrow$$

$$c^*(z_t, a_{t-1}) = c_0 = ra_{-1} + \frac{r}{R}\sum_{t=0}^{T} R^{-t} w\mathbb{E}_0\left[z_t\right]$$

where we formally disregard the borrowing constraint

- **Certainty equivalence:** *Only expected income matter.*

Special case II: CARA utility

- **CARA utility:** $u(c_t) = -\frac{1}{\alpha}e^{-\alpha c}$
- **Productivity is absolute random walk:**

$$z_t = z_{t-1} + \psi_t$$
$$\psi_t \sim \mathcal{N}(0, \sigma_\psi^2)$$

- **Consumption function** (see proof)**:**

$$c^*(a_{t-1}, z_t) = ra_{t-1} + wz_t - \frac{\log(\beta R)^{\frac{1}{\alpha}} + \alpha\frac{\sigma_\psi^2}{2}}{r^2}$$

where we formally disregard the borrowing constraint

- **Precautionary saving:** $\sigma_\psi^2 \uparrow$ implies $c_t^* \downarrow$ for given z_t and a_{t-1}
$\Rightarrow$ *accumulation of buffer-stock*

Dynamic solution: Bellman's Principle of Optimality

- **Origin:** Bellman, 1957, Chap. III.3.
- **Value function, v_t:** Defined *recursively* from

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

with $v_T(\bullet) = 0$.

- **Policy function, c_t^*:** Is the same as

$$c_t^*(z_t, a_{t-1}) = \arg\max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

Vocabulary

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$

$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

1. **State variables:** z_t and a_{t-1}
2. **Control** (choice) **variable:** c_t
3. **Continuation value:** $\beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$
4. **Parameters:** r, w, and stuff in $u(\bullet)$

Note: Straightforward to extend to more goods, more assets or other states, more complex risk, bounded rationality etc.

Infinite horizon: $T \to \infty$?

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

- **Contraction mapping result:** *If β is low enough (strong enough impatience) then the value and policy functions converge to $v(z_t, a_{t-1})$ and $c^*(z_t, a_{t-1})$ for large enough T*
- **In practice:**
  1. Make arbitrary initial guess (e.g. $v_{t+1} = 0$)
  2. Solve backwards until value and policy functions does not change anymore (given some tolerance)

3-periods

3-period model

- **Expected discounted utility:** $v(z_0, a_{-1}) = \mathbb{E}_0 \sum_{t=0}^{2} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$
- **Income = wage × productivity + transfer:**

$$y_t = wz_t + \chi_t$$

- **Cash-on-hand, savings and borrowing constraint:**

$$\begin{aligned} m_t &= (1+r)a_{t-1} + y_t \\ a_t &= m_t - c_t \\ a_t &\geq \underline{a} \end{aligned}$$

- **Stochastic transition:** $\Pr[z_{t+1} | z_t] = \pi_t(z_t, z_{t+1})$ such that

$$\begin{aligned} \Pr[z_{t+1} = 1 \,|\, z_t = 1] &= \pi \\ \Pr[z_{t+1} = 1 - \Delta \,|\, z_t = 1] = \Pr[z_{t+1} = 1 + \Delta \,|\, z_t = 1] &= \frac{1-\pi}{2} \\ \Pr[z_{t+1} = z_t \,|\, z_t \in \{1-\Delta, 1+\Delta\}] &= 1 \end{aligned}$$

Bellman equation

$$
\begin{aligned}
v_t(z_t, a_{t-1}) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(z_{t+1}, a_t) \right] \\
&\text{s.t.} \\
y_t &= w z_t + \chi_t \\
m_t &= (1+r) a_{t-1} + y_t \\
a_t &= m_t - c_t \\
\Pr\left[z_{t+1} | z_t \right] &= \pi_t(z_t, z_{t+1}) \\
a_t &\geq \underline{a}
\end{aligned}
$$

where

$$v_3(z_3, a_2) = 0$$

Discretization

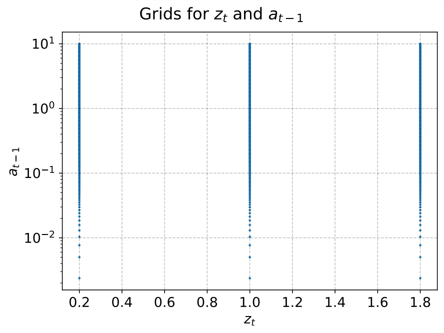
- **Discretization:** All state variables belong to discrete sets $\equiv$ *grids*,

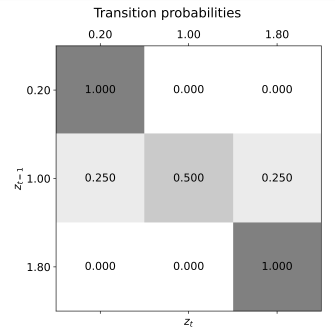
$$
\begin{aligned}
z_t \in \mathcal{G}_z &= \{z^0, z^1, \dots, z^{\#_z - 1}\} \\
a_t \in \mathcal{G}_a &= \{a^0, a^1, \dots, a^{\#_a - 1}\} \\
a^0 &= \underline{a}
\end{aligned}
$$

- **Expectation:** Numerical integration by

$$
\mathbb{E}_t \left[v_{t+1}(z_{t+1}, a_t) \right] = \sum_{z_{t+1} \in \{1-\Delta, 1, 1+\Delta\}} \pi_t(z_t, z_{t+1}) v_{t+1}(z_{t+1}, a_t)
$$

- **ConSav:** `grids.nonlinspace`, `grids.equilogspace`
- **ConSavNotebook:** `04. Tools/03. Grids.ipynb`





The size of risk is scaled by Δ

Baseline: $\Delta = 0.8$

Low risk: $\Delta = 0.4$

Linear interpolation

- **Linear interpolation** (function approximation):

  1. Assume v_{t+1} is known on $\mathcal{G}_z \times \mathcal{G}_a$ (tensor product)
  2. Evaluate $v_{t+1}(z^{i_z}, a)$ for arbitrary a by

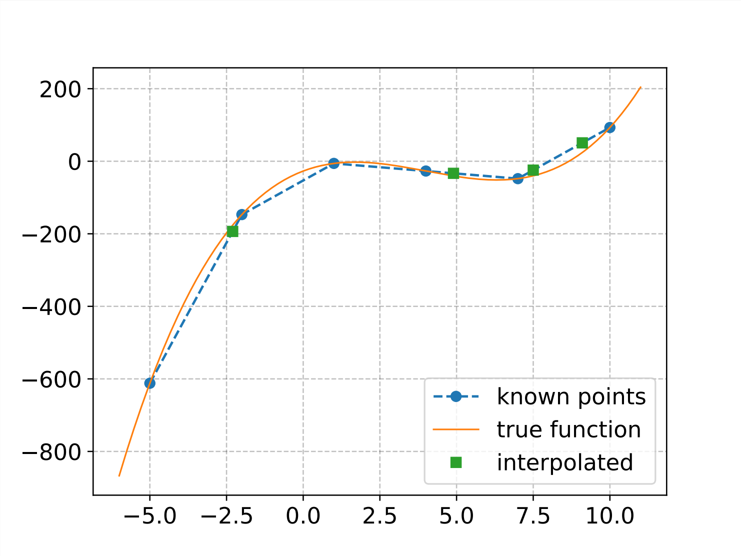
$$\breve{v}_{t+1}(z^{i_z}, a) = \text{baseline} + \text{slope} \times \text{distance}$$
$$= v_{t+1}(z^{i_z}, a^{\iota}) + \omega(a - a^{\iota})$$

where

$$\omega \equiv \frac{v_{t+1}(z^{i_z}, a^{\iota+1}) - v_{t+1}(z^{i_z}, a^{\iota})}{a^{\iota+1} - a^{\iota}}$$
$$\iota \equiv \text{largest } i_a \in \{0, 1, \dots, \#_a - 2\} \text{ such that } a^{i_a} \leq a$$

- **ConSav:** `linear_interp.interp1d`
- **ConSavNotebook:**
  `04. Tools/01. Linear interpolation.ipynb`

Linear interpolation



Value function iteration (VFI)

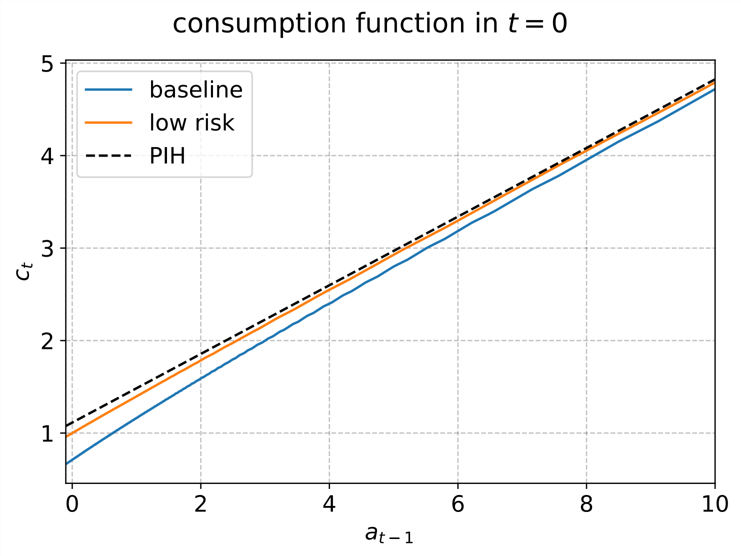
- **Maximize value-of-choice:**

$$v_t(z^{i_z}, a^{i_{a-}}) = \max_{c_t} v_t(z^{i_z}, a^{i_{a-}}|c_t)$$
$$\text{with } c_t \in [0, (1+r)a^{i_{a-}} + wz^{i_z} + \underline{a}]$$

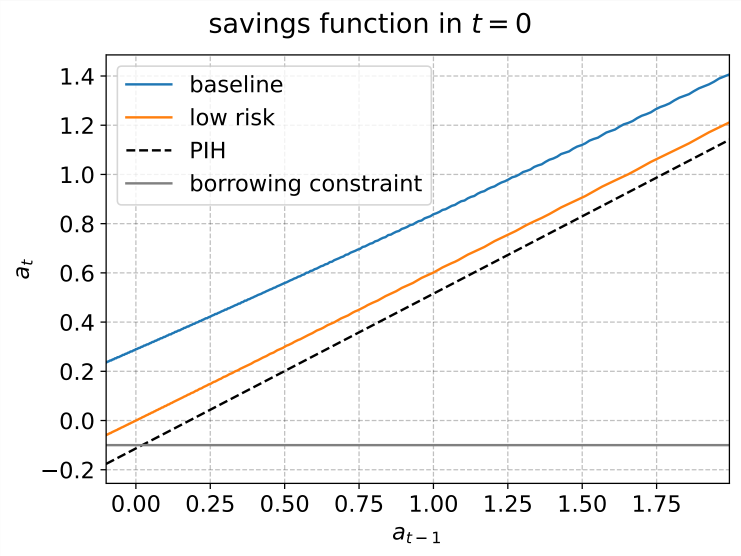
$$v_t(z^{i_z}, a^{i_{a-}}|c_t) = u(c_t) + \sum_{i_{z+1}=0}^{\#_z-1} \pi\left(i_z, i_{z+1}\right) \breve{v}_{t+1}(z^{i_z}, a_t)$$
$$\text{with } a_t = (1+r)a^{i_{a-}} + wz^{i_z} - c_t$$

- **Inner loop:** For each grid point in $\mathcal{G}_z \times \mathcal{G}_a$ find $c_t^*(z_t, a_{t-1})$ and therefore $v_t(z_t, a_{t-1})$ with a *numerical optimizer*
- **Outer loop:** Backwards from $t = T-1$ (note $\underline{v}_T = 0$, or known)
- **ConSav+QuantEcon:** Various optimizers in numba
- **ConSavNotebook:** `04. Tools/02. Optimization.ipynb`

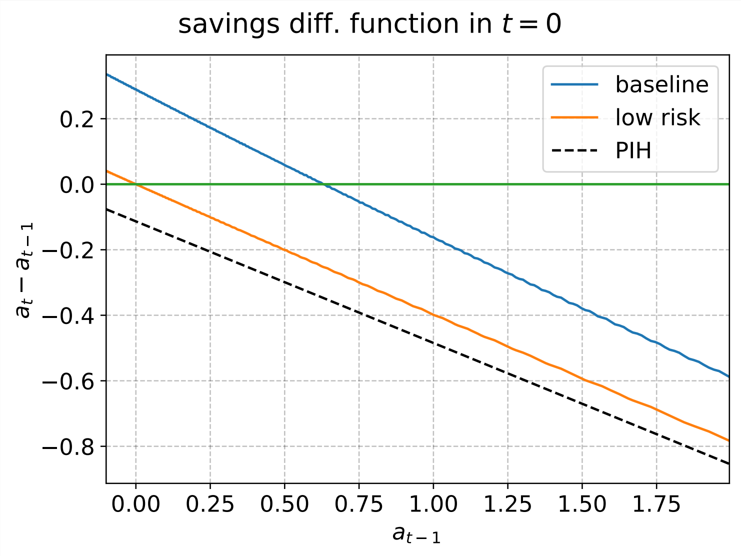
Consumption function



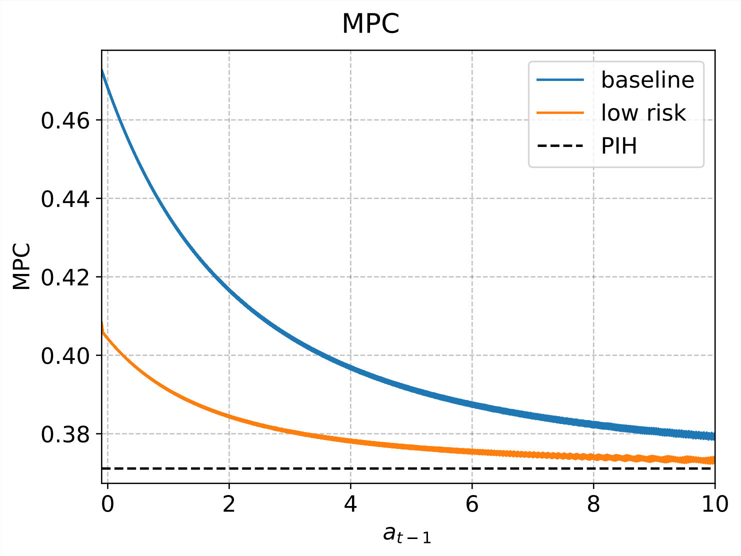
Savings function



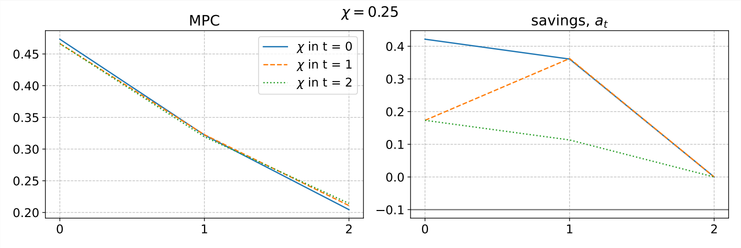
Change in savings function

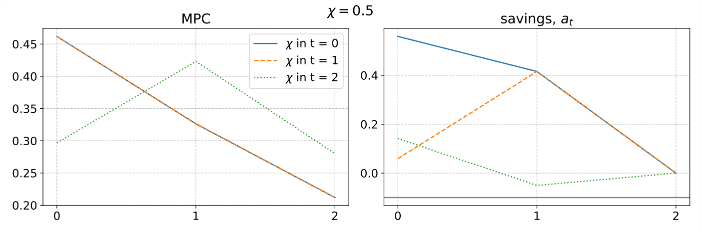


MPC



Intertemporal MPC





Note: No wealth effect as $r = 0$

Economic insights

- **Notebook:** 01. ConSavModel.ipynb
- **Consumption lower than under PIH and concave in assets**
  Intuition: *Precautionary saving motive is relatively larger for asset poor households because income risk is the same for everybody*
  Implications:
  1. Windfall gives safety and increases average consumption
     $\Rightarrow$ MPC decreasing in assets
  2. Attraction towards a buffer-stock target $a_t = a_{t-1}$ despite $\beta R < 1$
  3. Larger effective discounting of future income
     (extreme: no effect of future income changes if constrained before)

Numerical Monte Carlo simulation

- **Initial distribution:** Draw $z_{i,-1}$ and $a_{i,-1}$ for $i \in \{0, 1, \dots, N-1\}$
- **Simulation:** Forwards in time from $t = 0$ and in each time period
  1. Draw z_{it} given transition probabilities
  2. Use linear interpolation to evaluate

$$
\begin{aligned}
c_{it} &= \breve{c}_t^*(z_{it}, a_{it-1}) \\
a_{it} &= (1+r)a_{it-1} + wz_{it} - c_{it}
\end{aligned}
$$

- **Review:**
  - **Pro:** Simple to implement
  - **Con:** Computationally costly and introduces randomness
- **Infinite horizon:**
  1. Assume z_{it} has an ergodic distribution
  2. Ergodic distribution of a_{it} around buffer-stock target

Taking stock

- **Value Function Iteration (VFI)**
  1. Solve all consumption-saving models
  2. Accurate with dense enough grids
  3. Relatively simple code and easy to run in parallel
  4. Finding optimal choices is the computational bottleneck
     (especially with multi-starts in non-convex models)
- **Potential technical improvements:**
  1. More advanced interpolation methods (e.g. cubic)
     (typically slower per grid point and code more complex)
  2. Howard improvement steps (Rendahl, 2024), only in infinite horizon
  3. Adaptive sparse grids (Scheidegger, 2017)
  4. (Analytical or automatic differentiation)
  5. (Approximate value and policy function with global polynomials)
- **Now:** *Use more model information*

EGM

Time iteration

- **Replace numerical optimization with root-finding**
- **Time iteration:** For each a_{t-1} and z_t find c_t to solve the Euler-equation

$$c_t^{-\sigma} = \beta(1+r)\mathbb{E}_t[c_{t+1}^{-\sigma}]$$

  Note: *Necessary and sufficient* (for interior choices, else $a_t = \underline{a}$)
- **EGM:** No need for any numerical optimization or root-finding

Endogenous grid-point method (EGM)

1. Calculate **post-decision marginal value of cash:**

$$q(z^{i_z}, a^{i_a}) = \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z,i_{z+}} c_+^*(z^{i_{z+}}, a^{i_a})^{-\sigma}$$

2. **Invert Euler-equation:**

$$c(z^{i_z}, a^{i_a}) = (\beta(1+r)q(z^{i_z}, a^{i_a}))^{-\frac{1}{\sigma}}$$

3. **Endogenous cash-on-hand:**

$$m(z^{i_z}, a^{i_a}) = a^{i_a} + c(z^{i_z}, a^{i_a})$$

4. **Consumption function:** Calculate $m = (1+r)a^{i_{a-}} + wz^{i_z}$

   4.1 Binding constraint: If $m \leq m(z^{i_z}, a^0)$ then

   $$c^*(z^{i_z}, a^{i_{a-}}) = m + \underline{a}$$

   4.2 Interior choice: Else

   $$c^*(z^{i_z}, a^{i_{a-}}) = \text{interpolate } m(z^{i_z}, m) \to c(z^{i_z}, m)$$

NEGM

An illiquid asset

- **Illiuid asset:** Worth k in period T, else $(1-\gamma)k$ for $\gamma \in [0,1]$
  Note: *You can sell, but never buy*
- **Recursive problem:** For $\text{own}_{t-1} \in \{0,1\}$ and $y_t = 1$

$$
\begin{aligned}
v_t(\text{own}_{t-1}, a_{t-1}) &= \max_{c_t, \text{sell}_t \in \{0,1\}} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(\text{own}_t, a_t) \right] \\
\text{s.t. } \text{own}_t &= (1-\text{sell}_t)\text{own}_{t-1} \\
m_t &= a_{t-1} + \text{sell}_t \text{own}_{t-1}(1-\gamma)k \\
a_t &= m_t - c_t \\
a_t &\geq 0
\end{aligned}
$$

- **Terminal period:** $v_T(\text{own}_{T-1}, a_{T-1}) = \frac{(a_{T-1}+\text{own}_{T-1}k)^{1-\sigma}}{1-\sigma}$
- **Euler-equation:** Fix end-of-period ownership, own_t
  1. Necessary: From variational argument conditional on sell_{t+1}
  2. Non-sufficient: Savings low with $\text{sell}_{t+1} = 1$, or high with $\text{sell}_{t+1} = 0$

Timing and nesting

- **Beginning-of-period states:** own_{t-1} and m_t
- **Discrete choice:** If $\text{own}_{t-1} = 1$ choose $\text{sell}_t \in \{0, 1\}$

$$\underline{v}_t(m_t) = \max\left\{v_t(0, m_t^{\text{sell}}), v_t(1, m_t)\right\}$$
$$m_t^{\text{sell}} = m_t + (1-\gamma)k$$

- **Continuous choice:**

$$v_t(\text{own}_t, m_t) = \max_{c_t \in [0, m_t]} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta w_t(\text{own}_t, m_t - c_t)$$

- **Post-decision value function:**

$$w_t(\text{own}_t, a_t) = \mathbb{E}_t\left[\begin{cases} v_{t+1}(\text{own}_t, m_{t+1}) & \text{if } \text{own}_t = 0 \text{ or } t = T-1 \\ \underline{v}_{t+1}(m_{t+1}) & \text{if } \text{own}_t = 1 \text{ and } t < T-1 \end{cases}\right]$$
$$m_{t+1} = a_t + y_{t+1}$$

Nesting

- **Post-decision value function:**

$$w_t(\text{own}_t, a_t) = \mathbb{E}_t \left[\begin{cases} v_{t+1}(\text{own}_t, m_{t+1}) & \text{if own}_t = 0 \text{ or } t = T-1 \\ \overline{v}_{t+1}(m_{t+1}) & \text{if own}_t = 1 \text{ and } t < T-1 \end{cases} \right]$$

$$m_{t+1} = a_t + y_{t+1}$$

$$\text{where } v_T(\text{own}_{T-1}, m_T) = \frac{(m_T + \text{own}_{T-1} k)^{1-\sigma}}{1-\sigma}$$

- **Re-written Bellman:**

$$v_t(\text{own}_t, m_t) = \max_{c_t \in [0, m_t]} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta w_t(\text{own}_t, m_t - c_t)$$

- **Post-decision marginal value of cash:**

$$q_t(\text{own}_t, a_t) = \mathbb{E}_t \left[\begin{cases} (m_T + \text{own}_{T-1} k)^{-\sigma} & \text{if } t = T-1 \\ c^*_{t+1}(0, m^{\text{sell}}_{t+1})^{-\sigma} & \text{else if } \text{sell}_{t+1} = 1 \\ c^*_{t+1}(1, m_{t+1})^{-\sigma} & \text{else} \end{cases} \right]$$

$$m_{t+1} = a_t + y_{t+1}$$

$$m^{\text{sell}}_{t+1} = m_{t+1} + (1-\gamma)k$$

$$\text{sell}_{t+1} = \begin{cases} 1 & \text{if } v_{t+1}(0, m^{\text{sell}}_{t+1}) > v_{t+1}(1, m_{t+1}) \\ 0 & \text{else} \end{cases}$$

- **Euler-equation:**

$$u'(c_t) = \beta q_t \Leftrightarrow c_t^{-\sigma} = \beta q_t$$

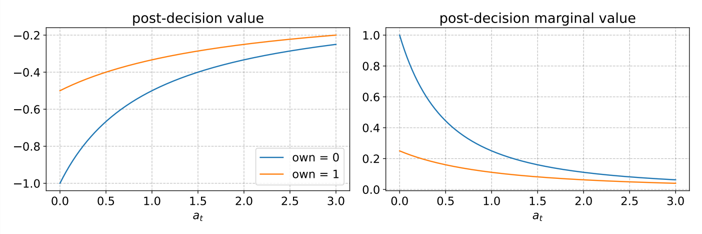
Solution method

For $t \in \{T-1, T-2, \ldots, 0\}$:

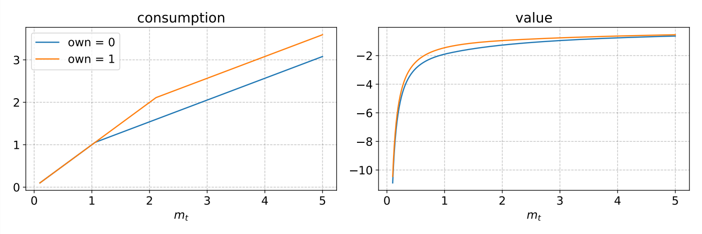
1. Calculate $q_t(\text{own}_t, a_t)$ and $w_t(\text{own}_t, a_t)$
2. Use inverted Euler-equation to get $c_t(\text{own}_t, a_t)$ and $m_t(\text{own}_t, a_t)$
3. Use *upper envelope* (see below) to get $c_t(\text{own}_t, m_t)$ on common grid for m_t
4. Calculate $v_t(\text{own}_t, m_t)$
5. Calculate $\underline{v}_t(m_t)$ (simple look-up)

Conditional consumption function in $t = T - 1$

Step 1. Use formulas for $v_T(\bullet)$ and $q_{T-1}(\bullet)$

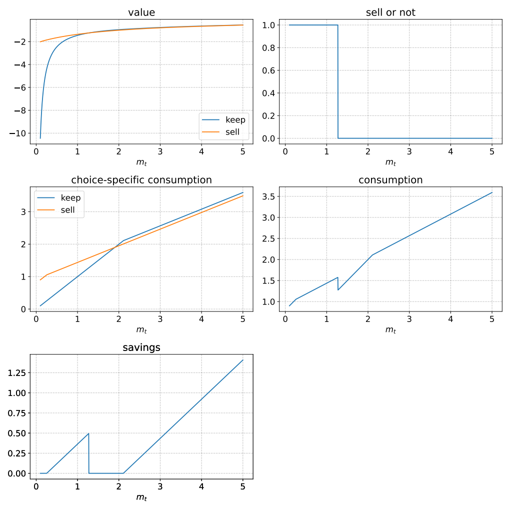


Step 2+3: Standard EGM (no upper envelope)



Unconditional owner behavior, $t = T - 1$

Step 4: Use $\underline{v}_t(m_t) = \max\left\{v_t(0, m_t^{\text{sell}}), v_t(1, m_t)\right\}$



Candidate points (given own_{it})

Closer look at step 3: The generated candidate points from inverting the Euler-equation is

$$
\begin{aligned}
w^{i_a} &= w_t(\text{own}_t, a^{i_a}) \\
q^{i_a} &= q_t(\text{own}_t, a^{i_a}) \\
c^{i_a} &= u'^{-1}\left(\beta q^{i_a}\right) = \left(\beta q^{i_a}\right)^{-\frac{1}{\sigma}} \\
m^{i_a} &= a^{i_a} + c^{i_a}
\end{aligned}
$$

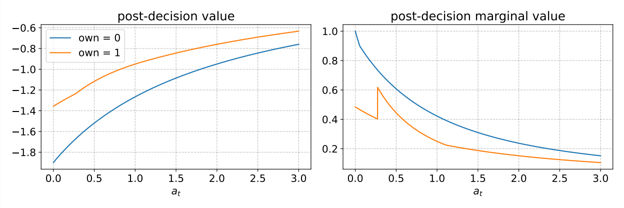
Problem: Small increase in a_t when $sell_{t+1}$ goes from 1 to 0 $\Rightarrow$

1. Downward jump c_{t+1} and upward jump in q_t $\Rightarrow$
2. Downward jump in c_t and m_t

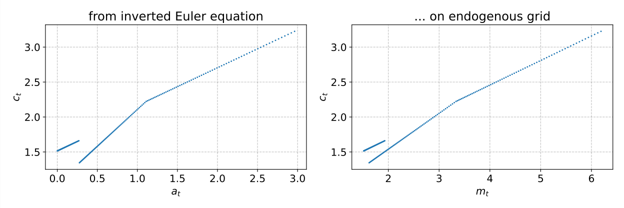
$\Rightarrow$ the m_t's will be overlapping - which one to choose?

Problems with EGM

Step 1 in $t = T-2$:



Step 2 in $t = T-2$ for owners:



Reason: Non-sufficient Euler-equation!

Upper envelope (given own_t)

Upper-envelope: $\forall i_m,\ c^*(m^{i_m}) = c^{i_m,j^*}$

$$j^* = \arg\max_{j\in\{0,1,\dots\#_a-2\}} u\left(c^{i_m,j}\right) + w^{i_m,j}$$

s.t.

$$\text{potential segment: } m^{i_m} \in \begin{cases} \left[m^j, m^{j+1}\right] & \text{if } j < \#_a - 2 \\ \left[m^j, \infty\right] & \text{if } j = \#_a - 2 \end{cases}$$

$$\text{interpolation + constraint } c^{i_m,j} = \min\left\{c^j + \frac{c^{j+1}-c^j}{m^{j+1}-m^j}\left(m^{i_m}-m^j\right), m^{i_m}\right\}$$

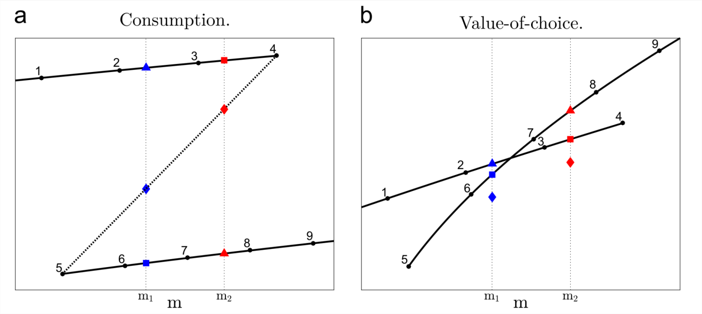
$$\text{continuation value: } w^{i_m,j} = \text{interp } \left\{a^{i_a}\right\} \to \left\{w^{i_a}\right\} \text{ at } a^{i_m,j}$$

$$a^{i_m,j} = m^{i_m} - c^{i_m,j}$$

ConSav: `upperenvelope`

ConSavNotebook: `04. Tools/06. Upper envelope.ipynb`

Illustration



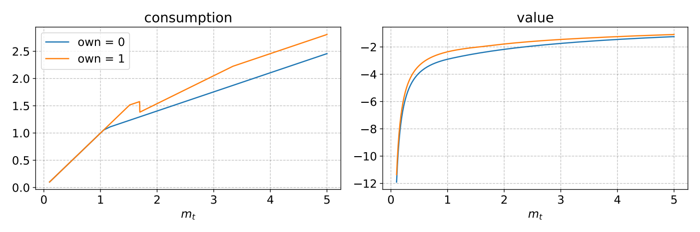
1. **Numbering:** Different levels of end-of-period assets, a^{i_a}
2. **Problem:** Find the consumption function at m_1 and m_2
3. **Largest value-of-choice:** Denoted by the *triangles*

Source: Druedahl and Jørgensen (2017), *G^2EGM*

Druedahl (2021), *NEGM*

Conditional consumption function in $t = T - 2$

Step 3: After applying upper envelope



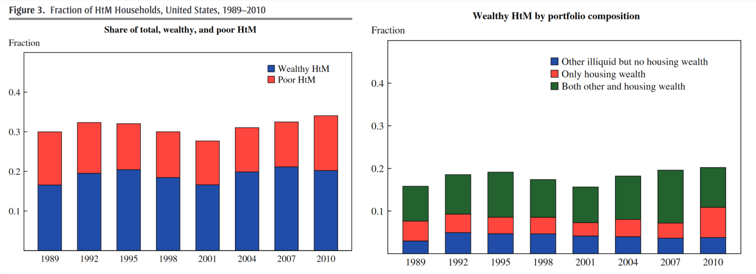
Economic insights

Notebook: 02. Illiquid.ipynb

1. **Simultaneous high total wealth and high MPC**
   1.1 Poor hands-to-mouth households
   1.2 Wealthy hands-to-mouth households
2. **The MPC is strongly size-dependent**
3. **Precautionary savings:**
   3.1 Frequent shocks: Liquid assets important
   3.2 Infrequent shocks: Illiquid assets enough

   (see Larkin (2024))

Empirical evidence for hands-to-mouth households



Poor HtM: Low liquid net worth, low total net worth

Wealthy HtM: Low liquid net worth, high total net worth

Source: Kaplan et. al. (2014)

Extra: Adding smoothing

- **Taste shocks:** Following Iskhakov et. al., 2017)

$$\overline{v}_t(m_t) = \max\left\{v_t(0, m_t^{\text{sell}}) + \sigma_\varepsilon \varepsilon(0), v_t(1, m_t) + \sigma_\varepsilon \varepsilon(1)\right\}$$
$$\varepsilon(x) \sim \text{Extreme value}$$

- **Logit-formula:**

$$\overline{v}_t(m_t) = \sigma_\varepsilon \log\left(\exp \frac{v_t(0, m_t^{\text{sell}})}{\sigma_\varepsilon} + \exp \frac{v_t(1, m_t)}{\sigma_\varepsilon}\right)$$

in choice probabilities:

$$P_t^{\text{sell}}(1, m_t) = \frac{\exp \frac{v_t(0, m_t^{\text{sell}})}{\sigma_\varepsilon}}{\exp \frac{v_t(0, m_t^{\text{sell}})}{\sigma_\varepsilon} + \exp \frac{v_t(1, m_t)}{\sigma_\varepsilon}}$$
$$\overline{v}_t(m_t) = P_t^{\text{sell}} v_t(0, m_t^{\text{sell}}) + \left(1 - P_t^{\text{sell}}\right) v_t(1, m_t)$$

Extra

- **Persistent-transitory income process:**

$$z_t = \tilde{z}_t \xi_t, \;\; \log \xi_t \sim \mathcal{N}(\mu_\xi, \sigma_\xi)$$
$$\log \tilde{z}_{t+1} = \rho_z \log \tilde{z}_t + \psi_{t+1}, \;\; \psi_{t+1} \sim \mathcal{N}(\mu_\psi, \sigma_\psi)$$

  1. Transitory shock: ξ_t
  2. Persistent shock: ψ_t
  3. Normalization using μ_ψ and μ_ξ: $\mathbb{E}[z_t] = \mathbb{E}[\tilde{z}_t] = 1$

- **ConSav:** `qudarature.log_normal_gauss_hermite`
- **ConSavNotebook:** `04. Tools/04. Quadrature.ipynb`

1. Transition probabilities

- **Discretization of ξ_t:** Derive $\mathcal{G}_\xi$ and $\pi_{i_{\xi-},i_\xi}$ given σ_ξ using Gauss-Hermite quadrature

$$x \sim \mathcal{N}(\mu, \sigma^2): \;\; \mathbb{E}[h(x)] \approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^{n} \omega_i h(\sqrt{2}\sigma x_i + \mu)$$

  where nodes, x_i, and weights, ω_i, have analytical expressions

- **Discretization of $\tilde{z}_t$:** Derive $\mathcal{G}_{\tilde{z}}$ and $\pi_{i_{\tilde{z}-},i_{\tilde{z}}}$ given $\rho_z < 1$ and σ_ψ (using a method such as Tauchen (1986) or Rouwenhorst (1995))
  If $\rho_z = 1$: Also use quadrature here.
- **Combined:** Derive $\mathcal{G}_z = \mathcal{G}_{\tilde{z}} \times \mathcal{G}_\xi$ (tensor product) and use independence of $\tilde{z}_t$ and ξ_t to get transition probabilities π_{i_{z-},i_z} (kronecker product)
- **ConSav:** `markov.log_rouwenhorst`, `markov.log_tauchen`
- **ConSavNotebook:** `04. Tools/05. Markov.ipynb`

1. Cash-on-hand formulation

Naive formulation:

$$v_t(\tilde{z}_t, \xi_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(\tilde{z}_{t+1}, \xi_{t+1}, a_t) \right]$$

s.t.

$$\begin{aligned}
z_t &= \tilde{z}_t \xi_t \\
y_t &= w z_t \\
m_t &= (1+r) a_{t-1} + y_t \\
a_t &= m_t - c_t \\
\tilde{z}_{t+1} &= \tilde{z}_t^{\rho_z} \psi_{t+1} \\
a_t &\geq -w b \tilde{z}_t
\end{aligned}$$

Cash-on-hand formulation (1 less state variable)

$$v_t(\tilde{z}_t, m_t) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(\tilde{z}_{t+1}, a_t) \right]$$

s.t.

$$\begin{aligned}
a_t &= m_t - c_t \\
\tilde{z}_{t+1} &= \tilde{z}_t^{\rho_z} \psi_{t+1} \\
m_{t+1} &= (1+r)a_{t+1} + w\tilde{z}_{t+1}\xi_{t+1} \\
a_t &\geq -wb\tilde{z}_t
\end{aligned}$$

1. Normalization if $\rho_z = 1$

- **Assumption:** $\rho_z = 1 \Leftrightarrow \tilde{z}_{t+1} = \tilde{z}_t \psi_{t+1}$
- **Define normalized variables:** $\boldsymbol{x}_t = x_t / \tilde{z}_t$ and $\boldsymbol{v}_t(\boldsymbol{m}_t) = \frac{v_t(\tilde{z}_t, m_t)}{\tilde{z}_t^{1-\rho}}$
- **Normalized Bellman equation:**

$$
\begin{aligned}
\boldsymbol{v}_t(\boldsymbol{m}_t) &= \max_{\boldsymbol{c}_t} \frac{\boldsymbol{c}_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[\psi_{t+1}^{1-\rho} \boldsymbol{v}_{t+1}(\boldsymbol{m}_{t+1}) \right] \\
\text{s.t.} \quad \boldsymbol{a}_t &= \boldsymbol{m}_t - \boldsymbol{c}_t \\
\boldsymbol{m}_{t+1} &= \frac{1+r}{\psi_{t+1}} \boldsymbol{a}_t + w \xi_{t+1} \\
\boldsymbol{a}_t &\geq -wb
\end{aligned}
$$

- **Normalized Euler-equation:**

$$
c_t^{-\sigma} = \beta(1+r)\mathbb{E}_t \left[c_{t+1}^{-\sigma} \right] \Leftrightarrow \boldsymbol{c}_t^{-\sigma} = \beta(1+r)\mathbb{E}_t \left[\left(\psi_{t+1} \boldsymbol{c}_{t+1} \right)^{-\sigma} \right]
$$

- **Simulation speed-up:** Harmenberg (2021)

2. Life-cycle (I)

- **Basically:**
    1. Born, working, retied, die
    2. Age-varying parameters (esp. income)
- **Add-ons:**
    1. Labor supply, human capital, occupation
    2. Portfolio choice and entrepreneurship
    3. Family formation
    4. Health, mortality

       etc.
- **Good starting example:** »Life-Cycle Consumption and Children: Evidence from a Structural Estimation«, Jørgensen (2017)

2. Life-cycle (II)

Paper: Gourinchas and Parker (2021)
Life-cycle consumption-saving model with retirement

- Young households:
  Save for precautionary reasons (buffer)
- Older households:
  Save for retirement (life-cycle)

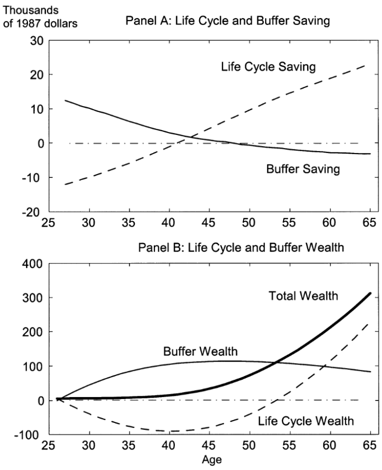
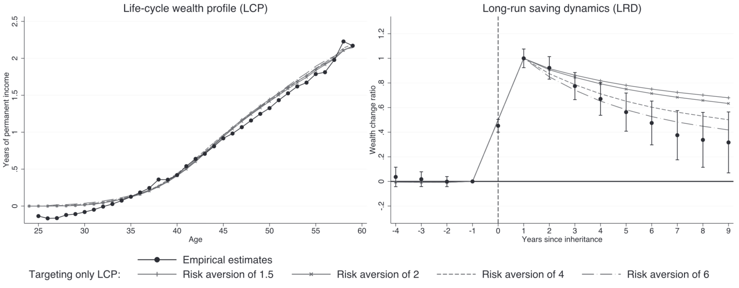


FIGURE 7.—The role of risk in saving and wealth accumulation.

- **Natural experiment:** Wealth depletion after sudden inheritance
- **Results:**
    1. Life-cycle profile of wealth fitted for many levels of risk-aversion (by varying the discount factor)
    2. Fast wealth depletation requires high risk-aversion (or high perceived risk)

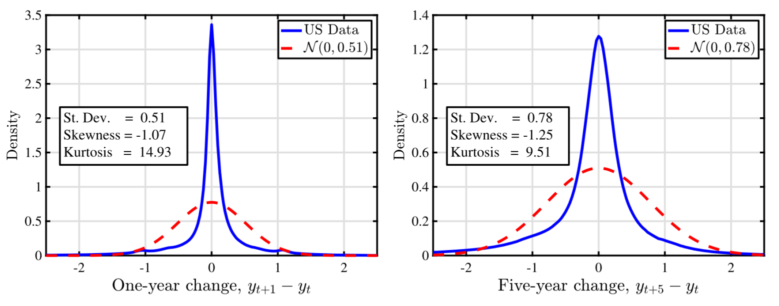
FIGURE 3.—REPLICATING LONG-RUN SAVING DYNAMICS



Source: Druedahl and Martinello (2022)

3. More realistic income risk (I)

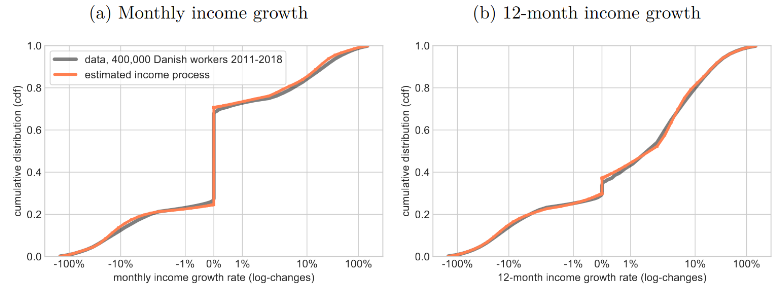
Annual earnings-changes are far from log-normal:



Source: Guvenen et. al. (2021)

3. More realistic income risk (II)

Many with zero-growth month-month:



Source: Druedahl et. al. (2021)

4. Epstein-Zin

$$
\begin{aligned}
v_t\left(z_t, m_t\right) \quad &= \quad \max_{c_t}\left[(1-\beta)\cdot c_t^{1-\sigma}+\beta\cdot w_{t+1}^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \\
\text{s.t.} \quad & w_{t+1}\equiv\mathbb{E}_t\left[v_{t+1}\left(z_{t+1}, m_{t+1}\right)^{1-\rho}\right]^{\frac{1}{1-\rho}} \\
& m_{t+1}=(1+r)(m_t-c_t)+y_{t+1}
\end{aligned}
$$

- **Preferences:**
  1. Patience: β
  2. Intertemporal substitution: σ
  3. Risk-aversion: ρ
- **Euler-equation:** $c_t^{-\sigma}=\beta R\cdot\mathbb{E}_t\left[c_{t+1}^{-\sigma}\cdot\left(\frac{w_{t+1}}{v_{t+1}}\right)^{\rho-\sigma}\right]$
  1. FOC: $0=v_t^{\sigma}\cdot\left[(1-\beta)\cdot c_t^{-\sigma}-\beta R\cdot w_{t+1}^{\rho-\sigma}\cdot\mathbb{E}_t\left[v_{t+1}^{-\rho}\cdot\frac{\partial v_{t+1}}{\partial m_{t+1}}\right]\right]$
  2. Envelope condition: $\frac{\partial v_t(z_t,m_t)}{\partial m_t}=v_t^{\sigma}\cdot(1-\beta)\cdot c_t^{-\sigma}$

5. Deep learning

- **Curse of dimensionality:**
  1. Many states
  2. Many choices
  3. Many shocks
- **Deep (reinforcement) learning:**
  1. Approximate value and policy functions with *neural networks*
  2. Approximate on simulation sample rather than on grid
  3. Automatic differentiation (backpropagation) and GPUs for speed
- **Examples:** Maliar and Maliar (2021) and Azinovic and Scheidegger (2022)
- **Working paper:** Druedahl and Røpke (2025)
  Python package: EconDLSolvers

Portfolio choice

Portfolio choice model

- **Risk-free asset:** a_t with return r_f
- **Risky asset:** b_t with return $r_f + \nu_t$
- **Recursive formulation:**

$$v_t(z_t, \nu_t, a_{t-1}, b_{t-1}) = \max_{a_t, b_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1} \left(z_{t+1}, a_t, b_t \right) \right]$$

$$\begin{aligned}
&\text{s.t.} \\
m_t &= (1 + r_f) a_{t-1} + (1 + r_f + \nu_t) b_{t-1} + w z_t \\
c_t &= m_t - a_t - b_t \\
z_{t+1} &\sim F_z(z_t) \\
\nu_{t+1} &\sim F_\nu \\
a_t, b_t, c_t &\geq 0
\end{aligned}$$

Optimality conditions

- **Envelope conditions:**

$$\frac{\partial v_t}{\partial a_{t-1}} = (1+r_f)c_t^{-\sigma}, \quad \frac{\partial v_t}{\partial b_{t-1}} = (1+r_f+\nu_t)c_t^{-\sigma}$$

- **FOCs**

$$-c_t^{-\sigma} + \beta \mathbb{E}_t \left[\frac{\partial v_{t+1}}{\partial a_t} \right] = 0$$
$$-c_t^{-\sigma} + \beta \mathbb{E}_t \left[\frac{\partial v_{t+1}}{\partial b_t} \right] = 0$$

- **Combined:**

$$c_t^{-\sigma} = \beta(1+r_f)\mathbb{E}_t \left[c_{t+1}^{-\sigma} \right]$$
$$0 = \mathbb{E}_t \left[\nu_{v+1} c_{t+1}^{-\sigma} \right]$$

Reformulation with fewer states

- **Consumption-decision value function:**

$$v_t(z_t, m_t) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta w_t\left(z_t, a_t\right)$$

$$\text{s.t.}$$

$$a_t = m_t - c_t$$

$$a_t \geq 0$$

- **Portfolio-decision value function:**

$$w_t\left(z_t, a_t\right) = \max_{\alpha_t} \beta \mathbb{E}_t\left[v_{t+1}\left(z_{t+1}, m_{t+1}\right)\right]$$

$$\text{s.t.}$$

$$m_{t+1} = R_{t+1} a_t + z_{t+1}$$

$$R_{t+1} = 1 + r_f + \nu_{t+1}\alpha_t$$

Solution method

1. Solve for $\alpha_t^* \left(z_t, a_t\right)$ by root-finding on

$$
\begin{aligned}
0 &= \mathbb{E}_t \left[\nu_{v+1} c_{t+1}^{-\sigma}\right] \\
&\text{s.t.} \\
c_{t+1} &= c_{t+1}^* \left(z_{t+1}, m_{t+1}\right) \\
m_{t+1} &= R_{t+1} a_t + z_{t+1} \\
R_{t+1} &= 1 + r_f + \nu_{t+1} \alpha_t^* \left(z_t, a_t\right)
\end{aligned}
$$

2. Compute

$$
q_t \left(z_t, a_t\right) = \mathbb{E}_t \left[R_{t+1} c_{t+1}^{-\sigma}\right]
$$

3. Find $c_t^* \left(m_t, z_t\right)$ using EGM

$$
\begin{aligned}
c_t \left(a_t, z_t\right) &= \left(\beta q_t \left(z_t, a_t\right)\right)^{-\frac{1}{\sigma}} \\
m_t \left(a_t, z_t\right) &= c_t + a_t
\end{aligned}
$$

Extension with participation costs κ

$$v_t(z_t, m_t, \iota_{t-1}) = \max_{c_t \in [0, m_t]} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \underline{w}_t \left(z_t, m_t - c_t, \iota_{t-1}\right)$$

$$\underline{w}_t \left(z_t, a_t, \iota_{t-1}\right) = \max_{\iota_t} w_t \left(z_t, a_t, \iota_t\right) - \kappa \mathbf{1} \left\{\iota_t = 1 \land \iota_{t-1} = 0\right\}$$

s.t.

$$\iota_t \in \{1\} \text{ if } \iota_{t-1} = 1 \text{ else } \{0, 1\}$$

$$w_t \left(z_t, a_t, \iota_t\right) = \max_{\alpha_t} \beta \mathbb{E}_t \left[v_{t+1} \left(z_{t+1}, m_{t+1}, \iota_t\right)\right]$$

s.t.

$$m_{t+1} = R_{t+1} a_t + z_{t+1}$$

$$R_{t+1} = 1 + r_f + \nu_{t+1} \alpha_t$$

$$\alpha_t \in [0, 1] \text{ if } \iota_t = 1 \text{ else } \{0\}$$

Solution method with participation costs κ

- **Participation is an absorbing state**
    1. If $\iota_{t-1} = 1$ then $\iota_t = 1$
    2. The same solution method as before can be used
- **Before participation,** $\iota_t = 0$**:** The post-decision marginal value of cash no longer needs to be monotone

$$\underline{w}_t(z_t, a_t, 0) = \mathbb{E}_t\left[w_t(z_t, a_t, \iota_t) - \kappa \mathbf{1}\{\iota_t = 1\}\right]$$

$$q_t(z_t, a_t, \iota_t) = \mathbb{E}_t\left[R_{t+1} c_{t+1}^{-\sigma}\right]$$

$$\iota_t = \begin{cases} 1 & \text{if } w_t(z_t, a_t, 1) - \kappa > w_t(z_t, a_t, 0) \\ 0 & \text{else} \end{cases}$$

Same solution as before: *Apply an upper envelope*

Summary

Summary and what's next

- **This lecture:**
  1. Consumption-saving models
  2. Basic numerical dynamic programming
  3. EGM and NEGM
- **Next:** *Stationary equilibrium*
- **You should:**
  1. Study the code
  2. Glance at Aiyagari (1994),
     »Uninsured Idiosyncratic Risk and Aggregate Saving«