CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

Transitional Dynamics

Mini-Course: Heterogenous Agent Macro

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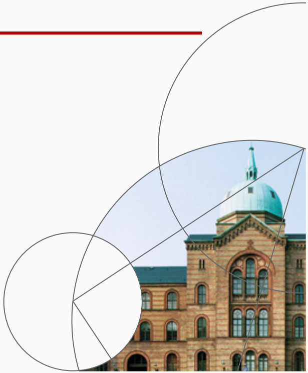
2025



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Introduction

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  1. Based on the GEModelTools package
  2. Example from GEModelToolsNotebooks/HANC

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- **Previously:** *Stationary equilibrium*
- **Now:** *Transitional dynamics*
- **Model:** Heterogeneous Agent Neo-Classical (HANC) model
- **Code:**
  1. Based on the GEModelTools package
  2. Example from GEModelToolsNotebooks/HANC
- **Literature:**
  1. Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«
  2. Documentation for GEModelTools
     (except stuff on *linearized solution* and *simulation*)

Plan

Ramsey

- **Simplified form:**

$$u'(C_t^{hh}) = \beta(1 + F_K(\Gamma_t, K_t, 1) - \delta)u'(C_{t+1}^{hh})$$
$$K_t = (1-\delta)K_{t-1} + F(\Gamma_t, K_{t-1}, 1) - C_t^{hh}$$

- **Production function:** $\Gamma_t K_{t-1}^{\alpha} L_t^{1-\alpha}$
- **Utility function:** $\frac{\left(C_t^{hh}\right)^{1-\sigma}}{1-\sigma}$
- **Steady state:**

$$K_{ss} = \left(\frac{\left(\frac{1}{\beta} - 1 + \delta\right)}{\Gamma_{ss}\alpha}\right)^{\frac{1}{\alpha-1}}$$
$$C_{ss}^{hh} = (1-\delta)K_{ss} + \Gamma_{ss}K_{ss}^{\alpha} - K_{ss}$$

As an equation system

$$
\begin{bmatrix}
r_t^K - \alpha \Gamma_t K_t^{\alpha-1} L_t^{1-\alpha} \\
w_t - (1-\alpha) \Gamma_t K_t^{\alpha} L_t^{-\alpha} \\
r_t - (r_t^K - \delta) \\
A_t - K_t \\
C_t^{hh,-\sigma} - \beta(1 + r_{t+1}) C_{t+1}^{hh,-\sigma} \\
L_t^{hh} - 1 \\
A_t^{hh} - \left((1 + r_t) A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}\right) \\
A_t - A_t^{hh} \\
L_t - L_t^{hh} \\
\forall t \in \{0, 1, \dots\}, \text{given } K_{-1}
\end{bmatrix} = \mathbf{0}
$$

Remember: Perfect foresight

Truncated, reduced vector form

$$\boldsymbol{H}\left(\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{\Gamma}, K_{-1}\right) = \begin{bmatrix} A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0, 1, \dots, T-1\} \end{bmatrix} = \mathbf{0}$$

where $\boldsymbol{X} = (X_0, X_1, \dots, X_{T-1})$, $A_{-1}^{hh} = K_{-1}$ and

$$\begin{aligned}
r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\
w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\
A_t &= K_t \\
r_t &= r_t^K - \delta \\
C_t^{hh} &= \left(\beta(1+r_{t+1})\right)^{-\sigma} C_{t+1}^{hh} \text{ (backwards, } C_T^{hh} = C_{ss}) \\
L_t^{hh} &= 1 \\
A_t^{hh} &= (1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh} \text{ (forwards, } A_{-1}^{hh} \text{ known)}
\end{aligned}$$

Truncation: $T < \infty$ fine when $\Gamma_t = \Gamma_{ss}$ for all $t > \underline{t}$ with $\underline{t} \ll T$

Further reduced

$$\boldsymbol{H}(\boldsymbol{K}, \Gamma, K_{-1}) = \left[\begin{array}{c} A_t - A_t^{hh} \\ \forall t \in \{0, 1, \ldots, T-1\} \end{array} \right] = \mathbf{0}$$

where $\boldsymbol{X} = (X_0, X_1, \ldots, X_{T-1})$, $A_{-1}^{hh} = K_{-1}$ and

$$\begin{aligned}
L_t = L_t^{hh} &= 1 \\
r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\
w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\
A_t &= K_t \\
r_t &= r_t^K - \delta \\
C_t^{hh} &= (\beta(1+r_{t+1}))^{-\sigma} C_{t+1}^{hh} \;\; \text{(backwards, } C_T^{hh} = C_{ss}\text{)} \\
A_t^{hh} &= (1+r_t) A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh} \;\; \text{(forwards, } A_{-1}^{hh} \text{ known)}
\end{aligned}$$

Solution method

1. **Set truncation** T
2. **Find Jacobian around steady state** $\boldsymbol{H}_{\boldsymbol{K}}$
   by *numerical differentiation*
3. **Solve** $\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{\Gamma}, K_{-1})$ in $\boldsymbol{K}$ for given $\boldsymbol{\Gamma}$ and K_{-1} with a quasi-Newton solver such as Broyden's method

- **Notebook:** *Ramsey.ipynb*

Intermezzo: Newton's method I

- **Simple example:** Solve 1 eq. with 1 unknown (x is a scalar):

$$f(x) = 0$$

- **How to find** x?
  First-order Taylor approximation around current guess x^i:

$$f(x) \approx f\left(x^i\right) + f'\left(x^i\right)\left(x - x^i\right)$$

- Set $f(x) = 0$ and solve for x to get:

$$x = x^i - \frac{f\left(x^i\right)}{f'\left(x^i\right)}$$

Intermezzo: Newton's method II

- **Newton's method:** Given initial guess x^0, update guess for x from i to $i+1$ as:

$$x^{i+1} = x^i - \frac{f\left(x^i\right)}{f'\left(x^i\right)}$$

  Continue until $\left|f\left(x^i\right)\right| < \epsilon$

- **Derivative:** $f'\left(x^i\right)$, use numerical approximation for small h

$$f'\left(x^i\right) \approx \frac{f\left(x^i + h\right) - f\left(x^i\right)}{h}$$

- **How well does it work?**
  1. If $f(x)$ is linear this solves $f(x) = 0$ in 1 iteration
  2. If $f(x)$ is non-linear we typically need more iterations, but works well if initial guess is within basis of attraction

Intermezzo: Multivariate Newton's method

- **Vector-valued, multivariate functions**
  $[f_1(x_{1,}x_2), f_2(x_{1,}x_2)]' = \boldsymbol{f}(\boldsymbol{x})$ with $\boldsymbol{x} = (x_{1,}x_2)'$:

$$\boldsymbol{x}^{i+1} = \boldsymbol{x}^i - \boldsymbol{J}\left(\boldsymbol{x}^i\right)^{-1} \boldsymbol{f}\left(\boldsymbol{x}^i\right)$$

  where $\boldsymbol{J}\left(\boldsymbol{x}^i\right)$ is the *Jacobian* of $f(\boldsymbol{x})$ w.r.t $\boldsymbol{x}^i$:

$$\boldsymbol{J}(\boldsymbol{x}_i) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1^i} & \frac{\partial f_1}{\partial x_2^i} \\ \frac{\partial f_2}{\partial x_1^i} & \frac{\partial f_2}{\partial x_2^i} \end{bmatrix}$$

- Otherwise the same as before

Intermezzo: Broyden's method I

- **Newton's method** updates Jacobian $\boldsymbol{J}$ in **every iteration**
  - often the computational bottleneck
- **Broyden's method:**
  1. Calculate J at the initial guess
  2. Each iteration use the (linear) update

$$\boldsymbol{J}^{i-1} + \frac{(\boldsymbol{f}^{i} - \boldsymbol{f}^{i-1}) - \boldsymbol{J}^{i-1}(\boldsymbol{x}^{i} - \boldsymbol{x}^{i-1})}{\left|\boldsymbol{x}^{i} - \boldsymbol{x}^{i-1}\right|_2} \left(\boldsymbol{x}^{i} - \boldsymbol{x}^{i-1}\right)'$$

Intermezzo: Broyden's method II

1. Guess $\boldsymbol{x}^0$ and set $i = 0$
2. Calculate the Jacobian at the initial guess, $\boldsymbol{J_0}$
3. Calculate $\boldsymbol{f}^i = \boldsymbol{f}(\boldsymbol{x}^i)$.
4. Stop if $\left|\boldsymbol{f}^i\right|$ below tolerance ϵ
5. Calculate Jacobian by
$$\boldsymbol{J}^i = \begin{cases} \boldsymbol{J_0} & \text{if } i = 0 \\ \boldsymbol{J}^{i-1} + \frac{(\boldsymbol{f}^i - \boldsymbol{f}^{i-1}) - \boldsymbol{J}^{i-1}(\boldsymbol{x}^i - \boldsymbol{x}^{i-1})}{|\boldsymbol{x}^i - \boldsymbol{x}^{i-1}|_2} \left(\boldsymbol{x}^i - \boldsymbol{x}^{i-1}\right)' & \text{if } i > 0 \end{cases}$$
6. Update guess by $\boldsymbol{x}^{i+1} = \boldsymbol{x}^i - \left(\boldsymbol{J}^i\right)^{-1} \boldsymbol{f}^i$
7. Increment i and return to step 3

Jacobian

$$\boldsymbol{H_K} = \begin{bmatrix} \frac{\partial(A_0 - A_0^{hh})}{\partial K_0} & \frac{\partial(A_0 - A_0^{hh})}{\partial K_1} & \cdots \\ \frac{\partial(A_1 - A_1^{hh})}{\partial K_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

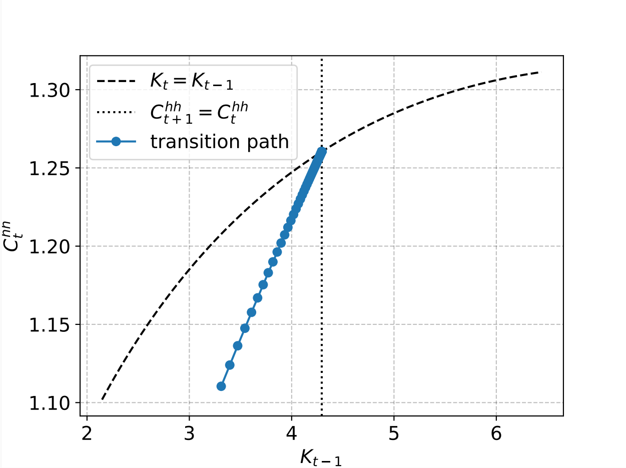
- **Column** s**:** Dynamic effect of change in capital in period s
- **Decomposition:**

$$\boldsymbol{H_K} = \boldsymbol{I} - \left(\mathcal{J}^{A^{hh},r} \mathcal{J}^{r,K} + \mathcal{J}^{A^{hh},r} \mathcal{J}^{w,K} \right)$$

1. Mechanic effect: $\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{K}} = \boldsymbol{I}$
2. Pricing through firms: $\mathcal{J}^{r,K}$ and $\mathcal{J}^{w,K}$
3. Consumption-saving through households: $\mathcal{J}^{A^{hh},r}$ and $\mathcal{J}^{A^{hh},w}$

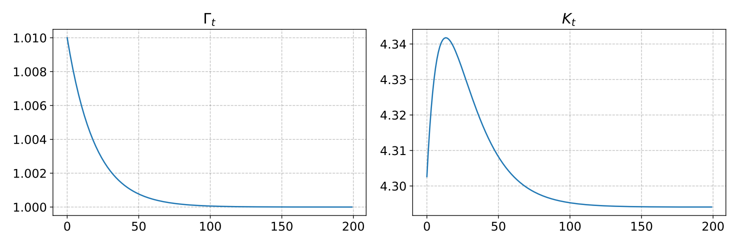
Example 1: Initially low capital

Initially away from steady state: $K_{-1} = 0.75K_{ss}$



Example 2: Technology shock

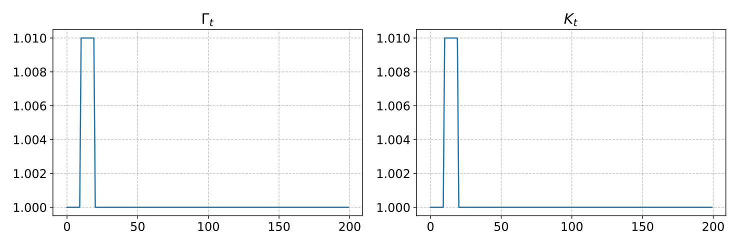
Technology shock: $\Gamma_t = 0.01\Gamma_{ss}0.95^t$ (exogenous, deterministic)



Example 3: Future technology shock

Technology shock: $\Gamma_t = \begin{cases} 1.01 \cdot \Gamma_{ss} & \text{if } t \in [10, 20) \\ \Gamma_{ss} & \text{else} \end{cases}$

(exogenous, deterministic)



Transition path

Heterogeneous households

- **Utility maximization** for household i:

$$v_0(\beta_i, \phi_i, z_{it}, a_{it-1}) = \max_{\{c_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta_i^t u(c_{it})$$

s.t.

$$\begin{aligned}
\ell_{it} &= z_{it} \\
a_{it} &= (1+r_t)a_{it-1} + w_t \phi_i \ell_{it} - c_{it} + \Pi_t \\
\log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1}, \;\; \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \;\; \mathbb{E}[z_{it}] = 1 \\
a_{it} &\geq 0
\end{aligned}$$

Distributions and aggregates

- **Policy functions:** Aggregate prices are hidden as inputs, i.e.

  $x_t^*(\beta_i, \phi_i, z_{it}, a_{it-1}) = x^*(\beta_i, \phi_i, z_{it}, a_{it-1}, \{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t})$ for $x \in \{a, \ell, c\}$

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  1. Beginning-of-period: $\underline{\boldsymbol{D}}_t$ over β_i, ϕ_i, z_{it-1} and a_{it-1}

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  3. Savings transition: $\underline{\boldsymbol{D}}_{t+1} = \Lambda_t' \boldsymbol{D}_t$ where again

     $$\Lambda_t = \Lambda\left(\{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t}\right)$$

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  $$\Lambda_t = \Lambda\left(\{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t}\right)$$

- **Aggregate consumption and savings:**

$$\begin{aligned} X_t^{hh} &= \int x_t^*(\beta_i, \phi_i, z_{it}, a_{it-1}) d\boldsymbol{D}_t \text{ for } x \in \{a, \ell, c\} \\ &= X^{hh}\left(\{r_\tau, w_\tau, \Pi_\tau\}_{\tau=0}^{t}, \underline{\boldsymbol{D}}_0\right) \\ &= \boldsymbol{x}_t^{*\prime} \boldsymbol{D}_t \end{aligned}$$

Equation system

The model can be written as an **equation system**

$$\begin{bmatrix} r_t^K - F_K(K_{t-1}, L_t) \\ w_t - F_L(K_{t-1}, L_t) \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ \boldsymbol{D}_t - \Pi_z \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} - \Lambda_t \boldsymbol{D}_t \\ A_t - \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\ L_t - \boldsymbol{\ell}_t^{*\prime} \boldsymbol{D}_t \\ \forall t \in \{0, 1, \dots\}, \text{given } \underline{\boldsymbol{D}}_0 \end{bmatrix} = \mathbf{0}$$

where $\{\Gamma_t\}_{t \geq 0}$ is a given technology path and $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$

Transition path - close to verbal definition

For a given $\underline{\boldsymbol{D}}_0$ and a path $\{\Gamma_t\}$

1. Quantities $\{K_t\}$ and $\{L_t\}$,
2. prices $\{r_t\}$ and $\{w_t\}$,
3. the distributions $\{\boldsymbol{D}_t\}$ over β_i, z_t and a_{t-1}
4. and the policy functions $\{a_t^*\}$, $\{\ell_t^*\}$ and $\{c_t^*\}$

are such that in all periods

1. Firms maximize profits (prices)
2. Household maximize expected utility (policy functions)
3. $\boldsymbol{D}_t$ is implied by simulating the household problem forwards from $\underline{\boldsymbol{D}}_0$
4. Mutual fund balance sheet is satisfied
5. The capital market clears
6. The labor market clears
7. The goods market clears

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What are we finding

- **Underlying assumption:** No aggregate uncertainty
- **»Shock«, Γ:** A fully unexpected non-recurrent event $\equiv$ *MIT shock*
- **Transition path, $\boldsymbol{K}$:** Non-linear perfect foresight response to
    1. Initial distribution, $\underline{\boldsymbol{D}}_0 \neq \boldsymbol{D}_{ss}$, *or to*
    2. Shock, $\Gamma_t \neq \Gamma_{ss}$ for some t (i.e. impulse-response)

Truncated, reduced vector form

$$\boldsymbol{H}\left(\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{\Gamma}, \underline{\boldsymbol{D}}_0\right) = \begin{bmatrix} A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0, 1, \ldots, T-1\} \end{bmatrix} = \mathbf{0}$$

where $\boldsymbol{X} = (X_0, X_1, \ldots, X_{T-1})$, $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$ and

$$\begin{aligned} r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\ w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\ A_t &= K_t \\ \boldsymbol{D}_t &= \Pi_z' \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} &= \Lambda_t' \boldsymbol{D}_t \\ A_t^{hh} &= \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\ L_t^{hh} &= \boldsymbol{\ell}_t^{*\prime} \boldsymbol{D}_t \\ &\forall t \in \{0, 1, \ldots, T-1\} \end{aligned}$$

Truncation: $T < \infty$ fine when $\Gamma_t = \Gamma_{ss}$ for all $t > \underline{t}$ with $\underline{t} \ll T$

Further reduction

$$\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{\Gamma}, \underline{\boldsymbol{D}}_0) = \begin{bmatrix} A_t - A_t^{hh} \\ \forall t \in \{0, 1, \ldots, T-1\} \end{bmatrix} = \mathbf{0}$$

where $\boldsymbol{X} = (X_0, X_1, \ldots, X_{T-1})$, $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$ and

$$\begin{aligned} L_t &= 1 \\ A_t &= K_t \\ r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\ w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\ \boldsymbol{D}_t &= \Pi_z' \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} &= \Lambda_t' \boldsymbol{D}_t \\ A_t^{hh} &= \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\ & \forall t \in \{0, 1, \ldots, T-1\} \end{aligned}$$

Truncation: $T < \infty$ fine when $\Gamma_t = \Gamma_{ss}$ for all $t > \underline{t}$ with $\underline{t} \ll T$

Use Broyden's method?

1. Guess $\boldsymbol{K}^0$ and set $i = 0$
2. Calculate the steady state Jacobian $\boldsymbol{H}_{\boldsymbol{K},ss} = \boldsymbol{H}_{\boldsymbol{K}}(\boldsymbol{K}_{ss}, \boldsymbol{\Gamma}_{ss}, K_{ss})$
3. Calculate $\boldsymbol{H}^i = \boldsymbol{H}(\boldsymbol{\Gamma}, \boldsymbol{K}^i, K_{-1})$
4. Stop if $\left|\boldsymbol{H}^i\right|_\infty$ below tolerance
5. Update Jacobian by
$$\boldsymbol{H}_{\boldsymbol{K}}^i = \begin{cases} \boldsymbol{H}_{\boldsymbol{K},ss} & \text{if } i = 0 \\ \boldsymbol{H}_{\boldsymbol{K}}^{i-1} + \frac{(\boldsymbol{H}^i - \boldsymbol{H}^{i-1}) - \boldsymbol{H}_{\boldsymbol{K}}^{i-1}(\boldsymbol{K}^i - \boldsymbol{K}^{i-1})}{\left|\boldsymbol{K}^i - \boldsymbol{K}^{i-1}\right|_2} \left(\boldsymbol{K}^i - \boldsymbol{K}^{i-1}\right)' & \text{if } i > 0 \end{cases}$$
6. Update guess by $\boldsymbol{K}^{i+1} = \boldsymbol{K}^i - \left(\boldsymbol{H}_{\boldsymbol{K}}^i\right)^{-1} \boldsymbol{H}^i$
7. Increment i and return to step 3

Note: *We find the fully non-linear solution*

Much more stable than relaxation (esp. with many variables)

Bottleneck: How do we find the Jacobian?

1. **Naive approach:** For each $s \in \{0, 1, ..., T-1\}$ do
   1.1 Set $K_t = K_{ss} + \mathbf{1}\{t = s\} \cdot \Delta, \qquad \Delta = 10^{-4}$
   1.2 Find $\boldsymbol{r}$ and $\boldsymbol{w}$
   1.3 Solve household problem backwards along transition path
   1.4 Simulate households forward along transition path
   1.5 Calculate $\frac{\partial H_t}{\partial K_s} = \frac{K_t - A_t^{hh}}{\Delta}$ for all t

   Bottleneck: We need T^2 solution steps and simulation steps!

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   Bottleneck: We need T^2 solution steps and simulation steps!
2. **Fake news algorithm:** From household Jacobian to full Jacobian

$$\boldsymbol{H_K} = \boldsymbol{I} - \left(\mathcal{J}^{A^{hh},r} \mathcal{J}^{r,K} + \mathcal{J}^{A^{hh},w} \mathcal{J}^{w,K} \right)$$

   $\mathcal{J}^{r,K}, \mathcal{J}^{w,K}$: Fast from the onset - *only involve aggregates*
   $\mathcal{J}^{A^{hh},r}, \mathcal{J}^{A^{hh},w}$: Only requires T solution steps and simulation steps!

   $\Rightarrow$ *details discussed later*

Full block structure

- **Shocks** are $\boldsymbol{Z} = \boldsymbol{\Gamma}$ and **unknowns** are $\boldsymbol{U} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{L} \end{bmatrix}'$
- **Ordered blocks:**
  1. Production firm: $\boldsymbol{\Gamma}, \boldsymbol{K}, \boldsymbol{L}, K_{-1} \to \boldsymbol{r^K}, \boldsymbol{w}$
  2. Mutual fund: $\boldsymbol{K}, \boldsymbol{r^K} \to \boldsymbol{A}, \boldsymbol{r}$
  3. Households: $\boldsymbol{r}, \boldsymbol{w}, \underline{\boldsymbol{D}}_0 \to \boldsymbol{A^{hh}}, \boldsymbol{L^{hh}}$
  4. Market clearing: $\boldsymbol{A}, \boldsymbol{L}, \boldsymbol{A^{hh}}, \boldsymbol{L^{hh}} \to \boldsymbol{A} - \boldsymbol{A^{hh}}, \boldsymbol{L} - \boldsymbol{L^{hh}}$
- **Jacobian:**

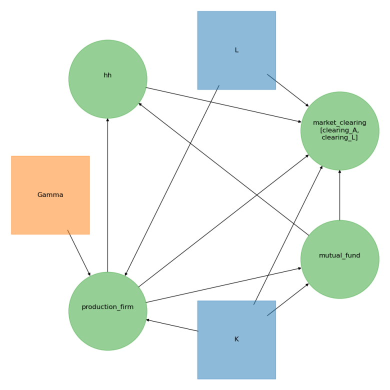
$$\boldsymbol{H_U} = \begin{bmatrix} \boldsymbol{H_K} & \boldsymbol{H_L} \end{bmatrix}$$

$$\boldsymbol{H_K} = \begin{bmatrix} \mathcal{J}^{A,K} - \left(\mathcal{J}^{A^{hh},r} \mathcal{J}^{r,r^K} \mathcal{J}^{r^K,K} + \mathcal{J}^{A^{hh},w} \mathcal{J}^{w,K} \right) \\ \boldsymbol{0} \end{bmatrix}$$

$$\boldsymbol{H_L} = \begin{bmatrix} \mathcal{J}^{A^{hh},r} \mathcal{J}^{r,r^K} \mathcal{J}^{r^K,L} + \mathcal{J}^{A^{hh},w} \mathcal{J}^{w,L} \\ \boldsymbol{I} \end{bmatrix}$$

DAG: Directed Acyclical Growth

- **Orange square:** Shocks (exogenous)
- **Purple square:** Unknowns (endogenous)
- **Green circles:** Blocks (with variables and targets inside)



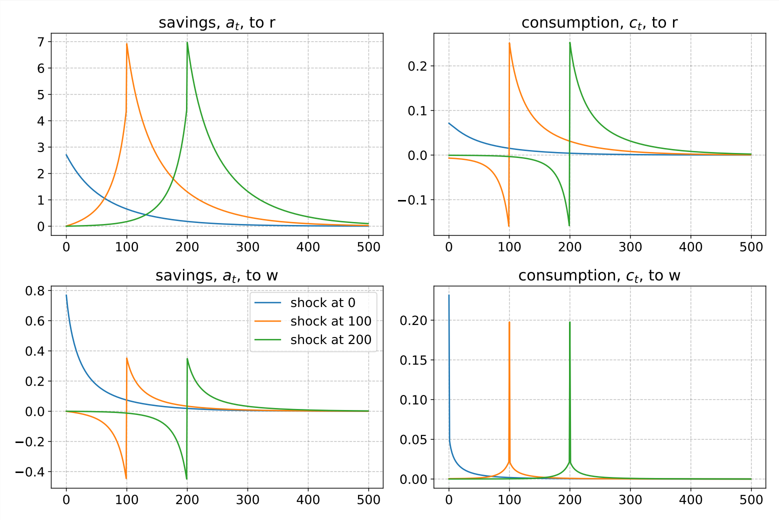
Interpreting the household Jacobians

- **Jacobian of consumption wrt. wage:** *What happens to consumption in period t when the wage (and thus income) increases in period s?*

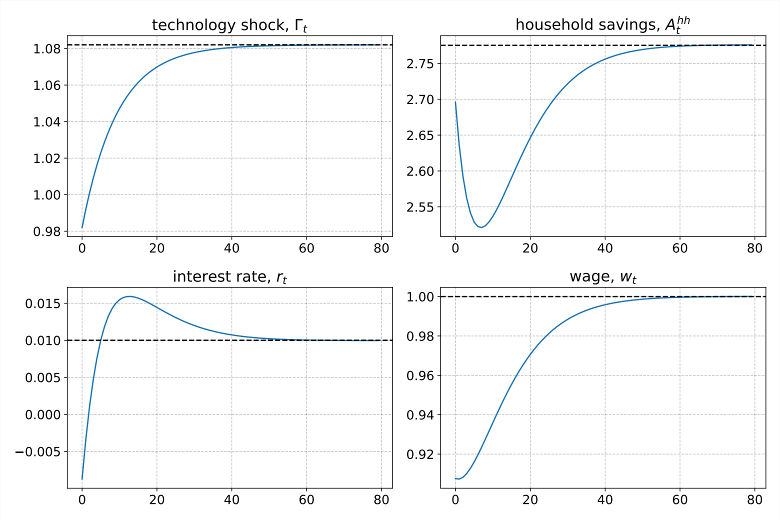
$$\mathcal{J}^{C^{hh},w} = \begin{bmatrix} \frac{\partial C_0^{hh}}{\partial w_0} & \frac{\partial C_0^{hh}}{\partial w_1} & \cdots \\ \frac{\partial C_1^{hh}}{\partial w_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

- **Columns:** The full dynamic response to a shock in period s

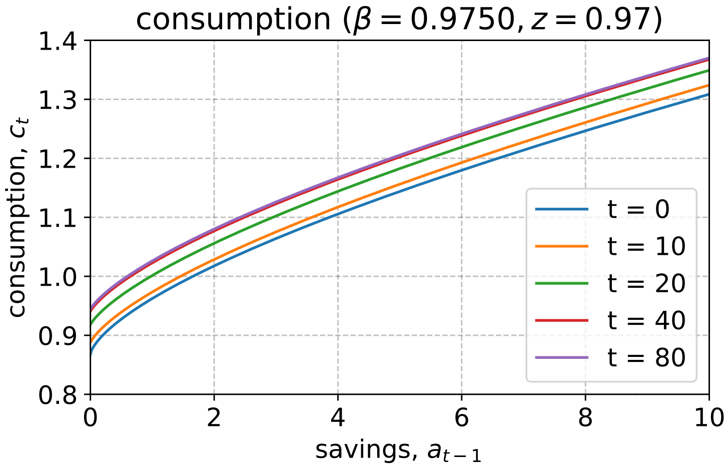
Household Jacobians



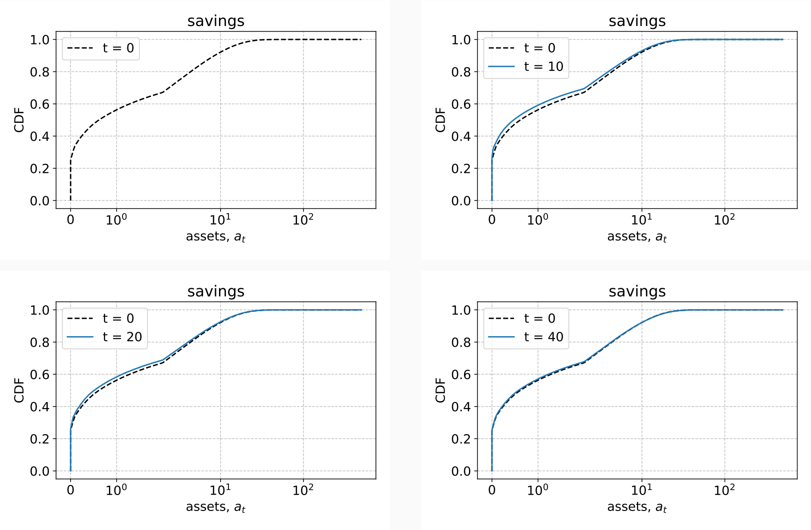
Transition path to technology shock



Consumption functions along transition path



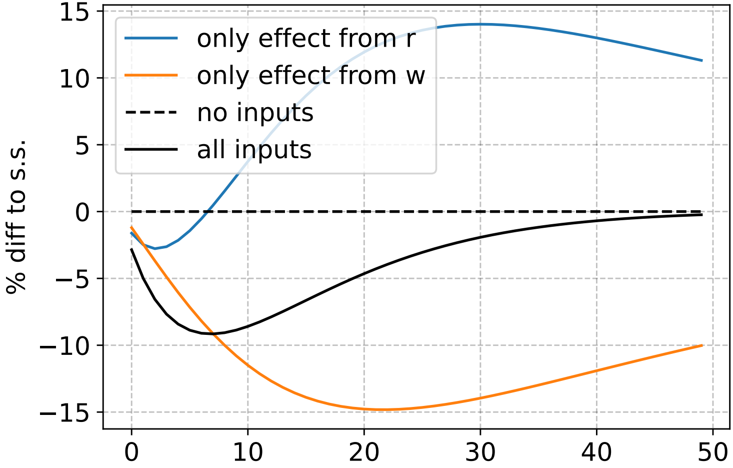
Distributions along transition path



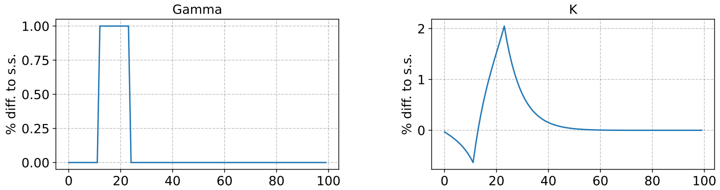
Decomposition of GE response

- **GE transition path:** $\boldsymbol{r}^*$ and $\boldsymbol{w}^*$
- **PE response of each:**
  1. Set $(\boldsymbol{r}, \boldsymbol{w}) \in \{(\boldsymbol{r}^*, \boldsymbol{w}_{ss}), (\boldsymbol{r}_{ss}, \boldsymbol{w}^*)\}$
  2. Solve household problem backwards along transition path
  3. Simulate households forward along transition path
  4. Calculate outcomes of interest
- **Additionally:** We can vary the initial distribution, $\underline{\boldsymbol{D}}_0$, to find the response of sub-groups

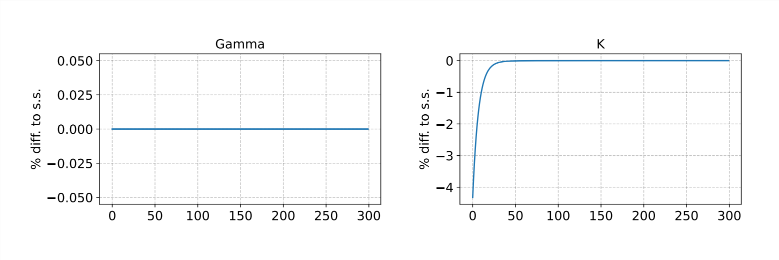
Decomposition of savings



More shocks: Future technology shock



More shocks: 5% less capital



Distribution: *Proportional reduction of savings for everybody*

DAGs

General model class I

1. Time is discrete (index t).
2. There is a continuum of households (index i, when needed).
3. There is *perfect foresight* wrt. all aggregate variables, $\boldsymbol{X}$, indexed by $\mathcal{N}$, $\boldsymbol{X} = \{\boldsymbol{X}_t\}_{t=0}^{\infty} = \{\boldsymbol{X}^j\}_{j\in\mathcal{N}} = \{X_t^j\}_{t=0,j\in\mathcal{N}}^{\infty}$, where $\mathcal{N} = \mathcal{Z} \cup \mathcal{U} \cup \mathcal{O}$, and $\mathcal{Z}$ are *exogenous shocks,* $\mathcal{U}$ are *unknowns*, $\mathcal{O}$ are outputs, and $\mathcal{H} \in \mathcal{O}$ are *targets.*
4. The model structure is described in terms of a set of *blocks* indexed by $\mathcal{B}$, where each block has inputs, $\mathcal{I}_b \subset \mathcal{N}$, and outputs, $\mathcal{O}_b \subset \mathcal{O}$, and there exists functions $h^o(\{\boldsymbol{X}^i\}_{i\in\mathcal{I}_b})$ for all $o \in \mathcal{O}_b$.
5. The blocks are *ordered* such that (i) each output is *unique* to a block, (ii) the first block only have shocks and unknowns as inputs, and (iii) later blocks only additionally take outputs of previous blocks as inputs. This implies the blocks can be structured as a *directed acyclical graph* (DAG).

General model class II

6. The number of targets are equal to the number of unknowns, and an *equilibrium* implies $\boldsymbol{X}^o = 0$ for all $o \in \mathcal{H}$. Equivalently, the model can be summarized by an *target equation system* from the unknowns and shocks to the targets,

$$\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{Z}) = \boldsymbol{0},$$

and an *auxiliary model equation* to infer all variables

$$\boldsymbol{X} = \boldsymbol{M}(\boldsymbol{U}, \boldsymbol{Z}).$$

A *steady state* satisfy

$$\boldsymbol{H}(\boldsymbol{U}_{ss}, \boldsymbol{Z}_{ss}) = 0 \text{ and } \boldsymbol{X}_{ss} = \boldsymbol{M}(\boldsymbol{U}_{ss}, \boldsymbol{Z}_{ss})$$

General model class III

7. The *discretized household block* can be written recursively as

$$
\begin{aligned}
\boldsymbol{v}_t &= v(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh}) \\
\underline{\boldsymbol{v}}_t &= \Pi(\boldsymbol{X}_t^{hh})\boldsymbol{v}_t \\
\boldsymbol{D}_t &= \Pi(\boldsymbol{X}_t^{hh})'\underline{\boldsymbol{D}}_t \\
\underline{\boldsymbol{D}}_{t+1} &= \Lambda(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh})'\boldsymbol{D}_t \\
\boldsymbol{a}_t^* &= \boldsymbol{a}^*(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh}) \\
\boldsymbol{Y}_t^{hh} &= \boldsymbol{y}(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh})'\boldsymbol{D}_t \\
& \underline{\boldsymbol{D}}_0 \text{ is given,} \\
& \boldsymbol{X}_t^{hh} = \{\boldsymbol{X}_t^i\}_{i\in\mathcal{I}_{hh}}, \boldsymbol{Y}_t^{hh} = \{\boldsymbol{X}_t^o\}_{o\in\mathcal{O}_{hh}},
\end{aligned}
$$

   where $\boldsymbol{Y}_t$ is aggregated outputs with $\boldsymbol{y}(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh})$ as individual level measures (savings, consumption labor supply etc.).

8. Given the sequence of shocks, $\boldsymbol{Z}$, there exists a *truncation period*, T, such all variables return to steady state beforehand.

Fake News Algorithm

Fake news algorithm

- **Household block:**

$$\boldsymbol{Y}^{hh} = hh(\boldsymbol{X}^{hh})$$

- **Goal:** Fast computation of

$$\mathcal{J}^{hh} = \frac{dhh(\boldsymbol{X}_{ss}^{hh})}{d\boldsymbol{X}^{hh}}$$

- **Naive approach:** Requires T^2 solution and simulation steps
- **Next slides:** *Sketch of much faster approach* (with $\Pi_t = \Pi_{ss}$ for notational simplicity)

Forward looking behavior

- **Notation:** $\bullet_t^{s,i}$ when there in period s is a shock to variable i
- **Time to shock:** Sufficient statistic for value and policy functions

$$\underline{\boldsymbol{v}}_t^{s,i} = \begin{cases} \underline{\boldsymbol{v}}_{ss} & \text{for } t > s \\ \underline{\boldsymbol{v}}_{T-1-(s-t)}^{T-1,i} & \text{for } t \leq s \end{cases} \text{ and } \boldsymbol{v}_t^{s,i} = \begin{cases} \boldsymbol{v}_{ss} & \text{for } t > s \\ \boldsymbol{v}_{T-1-(s-t)}^{T-1,i} & \text{for } t \leq s \end{cases}$$

$$\boldsymbol{y}_t^{s,i} = \begin{cases} \boldsymbol{y}_{ss} & t > s \\ \boldsymbol{y}_{T-1-(s-t)}^{T-1,i} & t \leq s \end{cases} \text{ and } \Lambda_t^{s,i} = \begin{cases} \Lambda_{ss} & t > s \\ \Lambda_{T-1-(s-t)}^{T-1,i} & t \leq s \end{cases}$$

- **Computation:** Only a single backward iteration required!
- **Note:** This is *not* an approximation

The first steps forward

- **Effect on output variable o in period 0:**

$$\mathcal{Y}_{0,s}^{o,i} \equiv \frac{dY_0^{o,s,i}}{dx} = \frac{\left(d\boldsymbol{y}_0^{o,s,i}\right)'}{dx}\Pi_{ss}'\underline{\boldsymbol{D}}_{ss}$$

- **Effect on beginning-of-period distribution in period 1:**

$$\underline{\mathcal{D}}_{1,s}^{i} \equiv \frac{d\underline{\boldsymbol{D}}_1^{s,i}}{dx} = \frac{\left(d\Lambda_0^{s,i}\right)'}{dx}\Pi_{ss}'\underline{\boldsymbol{D}}_{ss}$$

- **Expectation vector:** $\mathcal{E}_t^o \equiv \left(\Pi_{ss}\Lambda_{ss}\right)^t \Pi_{ss}\boldsymbol{y}_{ss}^o,$
- **Computational cost:**
  1. The cost of computing $\mathcal{Y}_{0,s}^{o,i}$ and $\underline{\mathcal{D}}_{1,s}^{i}$ for $s \in \{0, 1, \ldots, T-1\}$ are similar to a full forward simulation for T periods.
  2. The cost of computing $\mathcal{E}_s^o$ is negligible in comparison and can be done recursively, $\mathcal{E}_t^o = \Pi_{ss}\Lambda_{ss}\mathcal{E}_{t-1}^o$ with $\mathcal{E}_0^o = \Pi_{ss}\boldsymbol{y}_{ss}^o$.

Main result

- **Result:** Tedious algebra imply the Jacobian can be constructed from the known objects as

$$\mathcal{F}_{t,s}^{,i,o} \equiv \begin{cases} \mathcal{Y}_{0,s}^{o,i} & t = 0 \\ \left(\mathcal{E}_{t-1}^{o}\right)' \underline{\mathcal{D}}_{1,s}^{i} & t \geq 1 \end{cases}$$

$$\mathcal{J}_{t,s}^{hh,i,o} = \sum_{k=0}^{\min\{t,s\}} \mathcal{F}_{t-k,s-k}^{i,o}$$

- **Intuition:** ???
- **Mathematically:** Use the chain-rule over and over again
- **Note:** Use linearity and that we start from steady state

Chain-rule unfolding $t = 0$

$$\mathcal{J}^{hh,i,o}_{0,s} = \mathcal{F}^{i,o}_{0,s} = \mathcal{Y}^{o,i}_{0,s} = \underbrace{\frac{dY_0^{o,s,i}}{dx}}_{\text{change in policy}}$$

Chain-rule unfolding $t = 1$

$$\mathcal{J}^{hh,i,o}_{1,0} = \mathcal{F}^{i,o}_{1,0} = (\mathcal{E}^{o}_{0})'\,\underline{\mathcal{D}}^{i}_{1,0} = \underbrace{(\boldsymbol{y}^{o}_{ss})'\,\Pi'_{ss}\frac{d\underline{\boldsymbol{D}}^{0,i}_{1}}{dx}}_{\text{change in distribution}}$$

$$s \geq 1:\ \mathcal{J}^{hh,i,o}_{1,s} = \mathcal{F}^{i,o}_{1,s} + \mathcal{F}^{i,o}_{0,s-1} = \underbrace{(\boldsymbol{y}^{o}_{ss})'\,\Pi'_{ss}\frac{d\underline{\boldsymbol{D}}^{s,i}_{1}}{dx}}_{\text{change in distribution}} + \underbrace{\frac{dY^{o,s-1,i}_{0}}{dx}}_{\text{change in policy}}$$

Chain-rule unfolding $t = 2$

$$\mathcal{J}_{2,0}^{hh,i,o} = \mathcal{F}_{2,0}^{i,o} = \underbrace{(\boldsymbol{y}_{ss}^{o})' \, \Pi_{ss}' \Lambda_{ss}' \Pi_{ss}' \frac{d\underline{\boldsymbol{D}}_1^{0,i}}{dx}}_{\text{change in distribution}}$$

$$\mathcal{J}_{2,1}^{hh,i,o} = \mathcal{F}_{2,1}^{i,o} + \mathcal{F}_{1,0}^{i,o} = \underbrace{(\boldsymbol{y}_{ss}^{o})' \, \Pi_{ss}' \Lambda_{ss}' \Pi_{ss}' \frac{d\underline{\boldsymbol{D}}_1^{1,i}}{dx} + (\boldsymbol{y}_{ss}^{o})' \, \Pi_{ss}' \frac{d\underline{\boldsymbol{D}}_1^{0,i}}{dx}}_{\text{change in distribution}}$$

$$\begin{aligned} s \geq 2: \ \mathcal{J}_{2,s}^{hh,i,o} &= \mathcal{F}_{2,s}^{i,o} + \mathcal{F}_{1,s-1}^{i,o} + \mathcal{F}_{0,s-2}^{i,o} \\ &= \underbrace{(\boldsymbol{y}_{ss}^{o})' \, \Pi_{ss}' \Lambda_{ss}' \Pi_{ss}' \frac{d\underline{\boldsymbol{D}}_1^{s,i}}{dx} + (\boldsymbol{y}_{ss}^{o})' \, \Pi_{ss}' \frac{d\underline{\boldsymbol{D}}_1^{s-1,i}}{dx}}_{\text{change in distribution}} + \underbrace{\frac{dY_0^{o,s-2,i}}{dx}}_{\text{change in policy}} \end{aligned}$$

Bottlenecks

Bottlenecks

- **Small models:** Finding the stationary equilibrium
  - **Trick:** Howard improvements *(Modified policy function iteration)*
  - **Idea:** Multiple steps as once when finding the value function
    See e.g. Eslami and Phelan (2023) and Rendahl (2024)
- **Bigger models:** With many unknowns and targets both computing the Jacobian and solving the equation system can be costly
  $\Rightarrow$ *SSJ toolbox from Auclert et. al. (2021) has some methods for speeding this up not available in GEModelTools*
  1. Separate computation of non-household Jacobians
  2. Structured sparsity of non-household Jacobians

  GEModelTools: Just numerical differentiation of $\boldsymbol{H}$ using the household Jacobians computed with the fake news algorithm
- **Complex models** (e.g. no EGM)**:** Solving backwards is much more costly than simulating forwards $\Rightarrow$ *the second part of the fake news algorithm is not important*

IRFs and simulation

Reminder of model class

- **Unknowns:** $\boldsymbol{U}$
- **Shock:** $\boldsymbol{Z}$
- **Additional variables:** $\boldsymbol{X}$
- **Target equation system:**

$$\boldsymbol{H(U,Z) = 0}$$

- **Auxiliary model equations:**

$$\boldsymbol{X = M(U,Z)}$$

Linearized IRFs

- **New:** Just consider the *first order solution*

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    1. Solve for Impulse Response Functions (IRFs) for unknowns

$$\boldsymbol{H}(\boldsymbol{U},\boldsymbol{Z})=0\Rightarrow\boldsymbol{H}_{\boldsymbol{U}}d\boldsymbol{U}+\boldsymbol{H}_{\boldsymbol{Z}}d\boldsymbol{Z}=0\Leftrightarrow d\boldsymbol{U}=\underbrace{-\boldsymbol{H}_{\boldsymbol{U}}^{-1}\boldsymbol{H}_{\boldsymbol{Z}}}_{\equiv\boldsymbol{G}_{\boldsymbol{U}}}d\boldsymbol{Z}$$

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2. Derive all other IRFs for

$$\begin{aligned}\boldsymbol{X} = \boldsymbol{M}(\boldsymbol{U},\boldsymbol{Z}) \Rightarrow d\boldsymbol{X} &= M_{\boldsymbol{U}} d\boldsymbol{U} + M_{\boldsymbol{Z}} d\boldsymbol{Z} \\ &= \underbrace{(-M_{\boldsymbol{U}}\boldsymbol{H_U^{-1}}\boldsymbol{H_Z} + M_{\boldsymbol{Z}})}_{\equiv \boldsymbol{G}} d\boldsymbol{Z}\end{aligned}$$

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  1. Imprecise for *large* shocks

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- **Computation:** Same for $\boldsymbol{Z}$ as for $\boldsymbol{U}$
- **Limitations:**
  1. Imprecise for *large* shocks
  2. Imprecise in models with *aggregate non-linearities* (direct in aggregate equations or through micro-behavior)

Aggregate risk (dynamic equilibrium)

- **Aggregate stochastic variables:** $\boldsymbol{Z}$ follow some known process with innovations ϵ. *State space form*: RHS is what is known today

$$\begin{bmatrix} \underline{\boldsymbol{D}}_{t+1} \\ \boldsymbol{X}_t \\ \boldsymbol{Z}_t \end{bmatrix} = \mathcal{M}\left(\begin{bmatrix} \underline{\boldsymbol{D}}_t \\ \boldsymbol{X}_{t-1} \\ \boldsymbol{Z}_{t-1} \end{bmatrix}, \boldsymbol{\epsilon}_t \right)$$

$\neq$ perfect foresight wrt. future agg. variables as in *sequence-space*

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- **Observation:** Linearization of aggregate variables imply *certainty equivalence* with respect to these

$$\begin{bmatrix} \underline{\boldsymbol{D}}_{t+1} \\ \boldsymbol{X}_t \\ \boldsymbol{Z}_t \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} \underline{\boldsymbol{D}}_t \\ \boldsymbol{X}_{t-1} \\ \boldsymbol{Z}_{t-1} \end{bmatrix} + \boldsymbol{B}\boldsymbol{\epsilon}_t$$

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- **Insight:** *The IRF from an MIT shock is <u>equivalent</u> to the IRF in a model with aggregate risk, which is linearized in the aggregate variables* (Boppart et. al., 2018)

Comparisons

- **State-space approach with linearization**: Ahn et al. (2018); Bayer and Luetticke (2020); Bhandari et al. (2023); Bilal (2023)
  Con:
  1. Harder to implement in my view
  2. Valuable to be able to interpret Jacobians

  Pro**:**
  1. More similar to standard approaches for RBC and NK models
  2. Easier path to 2nd and higher order approximations
- **Global solution:** The distribution of households is a state variable for each household $\Rightarrow$ *explosion in complexity*
  1. Original: Krusell and Smith (1997, 1998); Algan et al. (2014);
  2. Deep learning: Fernández-Villaverde et al. (2021); Maliar et al. (2021); Han et al. (2021); Kase et al. (2022); Azinovic et al. (2022)

Example: Global HANC (Krusell-Smith)

- **Recursive formulation of household problem:**

$$v(\boldsymbol{D}_t, \Gamma_t, z_{it}, a_{it-1}) = \max_{a_{it}, c_{it}} u(c_{it}) + \beta \mathbb{E}_t \left[v(\boldsymbol{D}_{t+1}, \Gamma_{t+1}, z_{it+1}, a_{it}) \right]$$

s.t.

$$\begin{aligned}
K_{t-1} &= \int a_{it-1} d\boldsymbol{D}_t \\
r_t &= \alpha \Gamma_t K_{t-1}^{\alpha-1} - \delta \\
w_t &= (1-\alpha) \Gamma_t K_{t-1}^{\alpha} \\
a_{it} + c_{it} &= (1+r_t) a_{it-1} + w_t z_{it} \\
\log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1}, \;\; \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \;\; \mathbb{E}[z_{it}] = 1 \\
a_{it} &\geq 0,
\end{aligned}$$

- **Problem:** How to discretize $\boldsymbol{D}_t$?
  Note: $\boldsymbol{D}_t$ needed directly for K_{t-1} and indirectly for $K_t, K_{t+1} \ldots$

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.

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- **Linearized simulation** (with truncation)**:**
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  2. Calculate the time series of shocks as $d\tilde{\boldsymbol{Z}}_t = \sum_{s=0}^{T-1} d\boldsymbol{Z}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$
     Note: $d\boldsymbol{Z}_s \tilde{\boldsymbol{\epsilon}}_{t-s} =$ effect of shock s periods ago today

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     Note: $d\boldsymbol{Z}_s \tilde{\boldsymbol{\epsilon}}_{t-s} =$ effect of shock s periods ago today
  3. Calculate the time series of other aggregate variables as

$$d\tilde{\boldsymbol{X}}_t = \sum_{s=0}^{T-1} d\boldsymbol{X}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$$

     where $d\boldsymbol{X}_s$ is the IRF to a *unit-shock* after s periods

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$$d\tilde{\boldsymbol{X}}_t = \sum_{s=0}^{T-1} d\boldsymbol{X}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$$

     where $d\boldsymbol{X}_s$ is the IRF to a *unit-shock* after s periods
- **Intuition:** Sum of first order effects from all previous shocks

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.
- **Linearized simulation** (with truncation)**:**
  1. Draw time series of innovations, $\tilde{\boldsymbol{\epsilon}}_t$
  2. Calculate the time series of shocks as $d\tilde{\boldsymbol{Z}}_t = \sum_{s=0}^{T-1} d\boldsymbol{Z}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$
     Note: $d\boldsymbol{Z}_s \tilde{\boldsymbol{\epsilon}}_{t-s} =$ effect of shock s periods ago today
  3. Calculate the time series of other aggregate variables as

     $$d\tilde{\boldsymbol{X}}_t = \sum_{s=0}^{T-1} d\boldsymbol{X}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$$

     where $d\boldsymbol{X}_s$ is the IRF to a *unit-shock* after s periods
- **Intuition:** Sum of first order effects from all previous shocks
- **Equivalence:** Same result if we linearize all aggregated equations and write the model in MA(∞) form

Generalized linearized simulation [advanced]

- **Generality:** Add auxiliary variables (incl. distributional moments) to calculate additional IRFs and simulations

Generalized linearized simulation [advanced]

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- **Full distribution:**

Generalized linearized simulation [advanced]

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- **Full distribution:**
  1. The IRF for grid point i_g in a policy function can be calculated as

$$da^*_{i_g,s} = \sum_{s'=s}^{T-1} \sum_{X^{hh} \in \boldsymbol{X}^{hh}} \frac{\partial a^*_{i_g}}{\partial X^{hh}_{s'-s}} dX^{hh}_{s'}.$$

where $\partial \boldsymbol{a}^*_{i_g} / \partial X^{hh}_k$ is the derivative to a k-period ahead shock to input X^{hh} (calculated in fake news algorithm)

Generalized linearized simulation [advanced]

- **Generality:** Add auxiliary variables (incl. distributional moments) to calculate additional IRFs and simulations
- **Full distribution:**
  1. The IRF for grid point i_g in a policy function can be calculated as

$$da_{i_g,s}^{*} = \sum_{s'=s}^{T-1} \sum_{X^{hh} \in \boldsymbol{X}^{hh}} \frac{\partial a_{i_g}^{*}}{\partial X_{s'-s}^{hh}} dX_{s'}^{hh}.$$

     where $\partial \boldsymbol{a}_{i_g}^{*} / \partial X_{k}^{hh}$ is the derivative to a k-period ahead shock to input X^{hh} (calculated in fake news algorithm)

  2. The policy function can there be simulated as

$$\boldsymbol{a}_{i_g,t}^{*} = \sum_{s=0}^{T-1} da_{i_g,s}^{*} \tilde{\boldsymbol{\epsilon}}_{t-s}$$

Generalized linearized simulation [advanced]

- **Generality:** Add auxiliary variables (incl. distributional moments) to calculate additional IRFs and simulations
- **Full distribution:**
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     where $\partial \boldsymbol{a}^*_{i_g} / \partial X^{hh}_k$ is the derivative to a k-period ahead shock to input X^{hh} (calculated in fake news algorithm)
  2. The policy function can there be simulated as

$$\boldsymbol{a}^*_{i_g,t} = \sum_{s=0}^{T-1} da^*_{i_g,s} \tilde{\boldsymbol{\epsilon}}_{t-s}$$

  3. Distribution can then be simulated forwards

Calculating moments - variance

- **Identical and independent distributed innovations:**

$$\mathbb{E}\left[\epsilon_t^i \epsilon_{t'}^j\right] = \begin{cases} \sigma_i^2 & \text{if } t = t' \text{ and } i = j \\ 0 & \text{el} \end{cases}$$

Calculating moments - variance

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$$\mathbb{E}\left[\epsilon_t^i \epsilon_{t'}^j\right] = \begin{cases} \sigma_i^2 & \text{if } t = t' \text{ and } i = j \\ 0 & \text{el} \end{cases}$$

- **Calculating moments such as var**(dC_t) from the IRFs:

$$\begin{aligned} \text{var}(dC_t) &= \mathbb{E}\left[\left(\sum_{i \in \mathcal{Z}} \sum_{s=0}^{T-1} dC_s \epsilon_{t-s}^i\right)^2\right] \\ &= \sum_{i \in \mathcal{Z}} \sum_{s=0}^{T-1} \mathbb{E}\left[\epsilon_{t-s}^i \epsilon_{t-s}^i\right] \left(dC_s^i\right)^2 \\ &= \sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1} \left(dC_s^i\right)^2 \end{aligned}$$

where dC_s^i is the IRF to a unit-shock to i after s periods and σ_i is the standard deviation of shock i

Calculating moments - covariance

- **Covariances:**

$$\text{cov}(dC_t, dY_{t+k}) = \sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1-k} dC_s^i dY_{s+k}^i$$

Calculating moments - covariance

- **Covariances:**

$$\text{cov}(dC_t, dY_{t+k}) = \sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1-k} dC_s^i dY_{s+k}^i$$

- **Covariance decomposition:**

$$\frac{\text{contribution from one shock}}{\text{contributions from all shocks}} = \frac{\sigma_j^2 \sum_{s=0}^{T-1-k} dC_s^j dY_{s+k}^j}{\sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1-k} dC_s^i dY_{s+k}^i}$$

Estimation

- **The simplest approaches:**
  1. Impulse Response Function (IRF) matching
  2. Minimum distance / simulated method of methods (SMM)

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- **Also possible:** *Bayesian likelihood estimation* (see SSJ)

Estimation

- **The simplest approaches:**
  1. Impulse Response Function (IRF) matching
  2. Minimum distance / simulated method of methods (SMM)
- **Also possible:** *Bayesian likelihood estimation* (see SSJ)
- **Speed:** For a new set of parameters?
  1. Only shock processes change $\Rightarrow$ *same Jacobians* ($\boldsymbol{G_U}$, $\boldsymbol{G}$)
  2. Only need to re-compute Jacobian of aggregate variables? (only single block?)
  3. Also need to re-compute Jacobian of household problem?
  4. Also need to find stationary equilibrium again?

Summary

Summary and what's next

- **This lecture:**
  1. The concept of a transition path
  2. Decomposition of GE responses
  3. DAGs
  4. Equivalence of sequence space and linearization
  5. Simulation
  6. Details of the GEModelTools package
- **You should:**
  1. Study the code
  2. Glance at Kaplan, Moll and Violante (2018)