CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

Consumption-Saving

Mini-Course: Heterogenous Agent Macro

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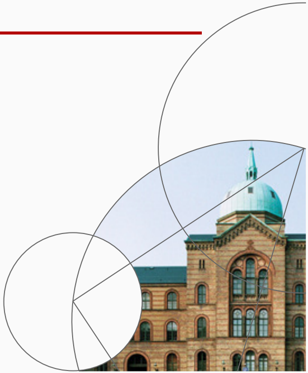


Introduction

Introduction

- **Generations of models:**
  1. **Permanent income hypothesis (PIH)** (Friedman, 1957) or life-cycle model (Modigliani and Brumburg, 1954)
  2. **Buffer-stock consumption model** Deaton (1991, 1992); Carroll (1992, 1997, 2019)
  3. **Multiple-asset buffer-stock consumption models** e.g. Kaplan and Violante (2014); Harmenberg and Öberg (2021)
- **Consumption-and-saving over the life-cycle dynamic** e.g. Gourinchas and Parker (2002); Druedahl and Martinello (2022)
- **Empirical MPCs and income risk** e.g. Fagereng et. al. (2021); Guvenen et. al. (2021)

Book: The Economics of Consumption, Jappelli and Pistaferri (2017)

Plan

1. Introduction
2. PIH
3. Buffer-stock
4. 3-periods
5. EGM
6. NEGM
7. Extra
8. Portfolio choice
9. Summary

PIH

Consumption-saving

$$v_0 = \max_{\{c_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t)$$

$$\text{s.t.}$$

$$a_t = (1+r)a_{t-1} + wz_t - c_t$$

$$a_{T-1} \geq 0$$

- **Variables:**
  Consumption: c_t
  Productivity: z_t
  End-of-period savings: a_t (*no debt at death*)
- **Parameters:**
  Discount factor: β
  Wage: w
  Interest rate: r (define $R \equiv 1 + r$ as interest factor)

It is a *static* problem

$$v_0 = \max_{\{c_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t)$$

$$\text{s.t.}$$

$$a_t = (1+r)a_{t-1} + wz_t - c_t$$

$$a_{T-1} \geq 0$$

- **It is a *static* problem:**
  1. **Information:** z_t is known for all t at $t=0$
  2. **Target:** Discounted utility, $\sum_{t=0}^{T-1} \beta^t u(c_t)$
  3. **Behavior:** Choose $c_0, c_1, \ldots, c_{T-1}$ *simultaneously*
  4. **Solution:** Sequence of consumption *choices* $c_0^*, c_1^*, \ldots, c_{T-1}^*$

IBC

- **Substitution** implies *Intertemporal Budget Constraint* (IBC)

$$
\begin{aligned}
a_{T-1} &= Ra_{T-2} + wz_{T-1} - c_{T-1} \\
&= R^2 a_{T-3} + Rwz_{T-2} - Rc_{T-2} + wz_{T-1} - c_{T-1} \\
&= R^T a_{-1} + \sum_{t=0}^{T-1} R^{T-1-t}(wz_t - c_t)
\end{aligned}
$$

- Use **terminal condition** $a_{T-1} = 0$ (equality due utility max.)

$$
R^{-(T-1)} a_{T-1} = 0 \quad \Leftrightarrow \quad s_0 + h_0 - \sum_{t=0}^{T-1} R^{-t} c_t = 0
$$

where $s_0 \equiv Ra_{-1}$ (after-interest assets)
and $h_0 \equiv \sum_{t=0}^{T-1} R^{-t} wz_t$ (human capital)

FOC and Euler-equation

$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t u(c_t) + \lambda \left[\sum_{t=0}^{T-1} R^{-t} c_t - s_0 - h_0 \right]$$

- **First order conditions**:

$$\forall t : 0 = \beta^t u'(c_t) - \lambda (1+r)^{-t} \Leftrightarrow u'(c_t) = -\lambda \left(\beta R\right)^{-t}$$

- **Euler-equation** for $k \in \{1, 2, \dots\}$:

$$\frac{u'(c_t)}{u'(c_{t+k})} = \frac{-\lambda \left(\beta R\right)^{-t}}{-\lambda \left(\beta R\right)^{-(t+k)}} = \left(\beta R\right)^k$$

Consumption choice

- **CRRA:** $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$ imply Euler-equation

$$\frac{c_0^{-\sigma}}{c_t^{-\sigma}} = (\beta R)^t \Leftrightarrow c_t = (\beta R)^{\frac{t}{\sigma}} c_0$$

- Insert **Euler** into **IBC** to get consumption choice

$$\begin{aligned}
\sum_{t=0}^{T-1} \left((\beta R)^{1/\sigma} R^{-1} \right)^t c_0 &= s_0 + h_0 \Leftrightarrow \\
c_0^* &= \frac{1 - (\beta R)^{1/\sigma} R^{-1}}{1 - \left((\beta R)^{1/\sigma} R^{-1} \right)^T} (s_0 + h_0)
\end{aligned}$$

- **Infinite horizon** for $(\beta R)^{1/\sigma} R^{-1} < 1$: Let $T \to \infty$ to get

$$c_0^* = \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right)(s_0 + h_0)$$

$$\text{if } \forall z_t = 1: \; c_0^* = \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right)\left(Ra_{-1} + \frac{R}{R-1}w\right)$$

- **Consume annuity value:** $\beta R = 1, z_t = 1 \Rightarrow c_0^* = ra_{-1} + w$
- **Intertemporal elasticity of substitution** (IES $= \frac{1}{\sigma}$)**:**

$$\log c_{t+1} - \log c_t = \frac{1}{\sigma} \log \beta R$$

Constant consumption if:

1. $\beta R = 1$
2. $\sigma \to \infty$ (zero elasticity of substitution)

Propensities to consume ($\beta R \approx 1, z_t \approx 1$)

$$c_0^* \approx \frac{r}{1+r}\left((1+r)a_{-1} + \sum_{t=0}^{\infty} \frac{wz_t}{(1+r)^t}\right) \approx ra_{-1} + w$$

Different types of shocks:

1. MPC of *windfall* income: $\frac{\partial c_0}{\partial s_0} \approx \frac{r}{1+r}$
2. MPC of *future* income change: $\frac{\partial c_0}{\partial wz_t} \approx \frac{r}{1+r}(1+r)^{-t}$
3. MPC of *permanent* income change: $\frac{\partial c_0}{\partial w} \approx \frac{r}{1+r}\frac{1}{1-(1+r)^{-1}} = 1$

Dynamic affects: The same when $\beta R = 1$, for all $k > 0$

$$\frac{\partial c_k}{\partial s_0} = \frac{\partial c_0}{\partial s_0}$$
$$\frac{\partial c_k}{\partial wz_t} = \frac{\partial c_0}{\partial wz_t}$$
$$\frac{\partial c_k}{\partial w} = \frac{\partial c_0}{\partial w}$$

Savings (with $z_t = 1$)

- **Constant savings** with $\beta R = 1$

$$c_t = ra_{t-1} + w \Rightarrow a_t = Ra_{t-1} + w - c_t = a_{t-1}$$

- **Decreasing savings** with $\beta R < 1$: $c_t \uparrow \Rightarrow a_t < a_{t-1}$
- **Increasing savings** with $\beta R > 1$: $c_t \downarrow \Rightarrow a_t > a_{t-1}$

Initial liquidity/borrowing constraint

- Implied period 0 **savings** are:

$$a_0 = Ra_{-1} + wz_0 - c_0$$

- Hard **borrowing constraint:** $a_0 \geq -wb$
- **Maximum consumption:** $\overline{c}_0 = Ra_{-1} + wz_0 + wb$
- **Optimal consumption:** Constrained or unconstrained.

$$c_0^* = \min\left\{\overline{c}_0, \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right)(s_0 + h_0)\right\}$$

- **Empirical realism.** Incl. high MPC of constrained.
- **Technical issue:** *Borrowing constraints further in the future complicates the analytical solution considerably.*

Empirical MPCs

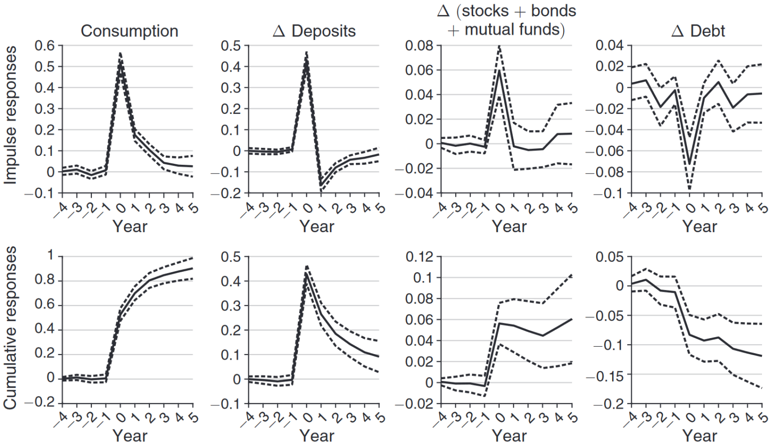


FIGURE 2. DYNAMIC HOUSEHOLD RESPONSES TO LOTTERY PRIZES

Source: Fagereng et. al. (2021)

Buffer-stock

Uncertainty and always borrowing constraint

$$v_0(z_0, a_{-1}) = \max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

$$\text{s.t.}$$

$$a_t = (1+r)a_{t-1} + wz_t - c_t$$

$$z_{t+1} \sim \mathcal{Z}(z_t)$$

$$a_t \geq -wb$$

$$\lim_{t\to\infty} (1+r)^{-t} a_t \geq 0 \quad \text{[No-Ponzi game]}$$

- **Stochastic income** from 1st order Markov-process, $\mathcal{Z}$
- ***A true dynamic problem:***
    1. **Information:** z_t is revealed period-by-period
    2. **Target:** *Expected* discounted utility, $\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$
    3. **Behavior:** Choose c_t *sequentially* as information is revealed
    4. **Solution:** Sequence of consumption *functions*, $c_t^*(z_t, a_{t-1})$

IBC

- **Substitution** still implies:

$$R^{-(T-1)}a_{T-1} = 0 \quad \Leftrightarrow \quad s_0 + h_0 - \sum_{t=0}^{T-1} R^{-t}c_t = 0$$

- **What if $T \to \infty$?** We must have $\lim_{T\to\infty} R^{-(T-1)}a_{T-1} = 0$
    1. $\lim_{T\to\infty} R^{-(T-1)}a_{T-1} > 0$: Consumption can be increased
    2. $\lim_{T\to\infty} R^{-(T-1)}a_{T-1} < 0$: Violates No-Ponzi game condition
- For $T \to \infty$ we have the **IBC**:

$$\sum_{t=0}^{\infty} R^{-t}c_t = Ra_{-1} + \sum_{t=0}^{\infty} R^{-t}wz_t$$

Natural borrowing limit

- Denote **minimum possible productivity** by $\underline{z}$
- **Consumption must be non-negative** $\Rightarrow$
  interest payments must be less than minimum income

$$c_t \geq 0 \Rightarrow r(-a_t) \leq w\underline{z} \Leftrightarrow a_t \geq -\frac{w\underline{z}}{r}$$

  If debt was larger it would in the worst case ($\forall z_t = \underline{z}$) grow without bound even with zero consumption ($\forall c_t = 0$)

$$\begin{aligned}
a_0 &= -\frac{w\underline{z}}{r} - \Delta \\
a_1 &= (1+r)a_0 + w\underline{z} = a_0 - (1+r)\Delta \\
a_2 &= (1+r)a_1 + w\underline{z} = a_0 - (1+r)^2\Delta \\
&\vdots
\end{aligned}$$

- **Natural borrowing constraint:** $a_t \geq \underline{a} = -w \min\left\{b, \frac{\underline{z}}{r}\right\}$

Euler-equation from variation argument

- **Case I:** If $u'(c_t) > \beta R \mathbb{E}_t [u'(c_{t+1})]$:
  Increase c_t by marginal $\Delta > 0$, and lower c_{t+1} by $R\Delta$
  1. **Feasible:** Yes, if $a_t > \underline{a}$
  2. **Utility change:** $u'(c_t) + \beta(-R)\mathbb{E}_t [u'(c_{t+1})] > 0$
- **Case II:** If $u'(c_t) < \beta R \mathbb{E}_t [u'(c_{t+1})]$:
  Lower c_t by marginal $\Delta > 0$, and increase c_{t+1} by $R\Delta$
  1. **Feasible:** Yes (always)
  2. **Utility change:** $u'(c_t) + \beta R \mathbb{E}_t [u'(c_{t+1})] > 0$
- **Conclusion:** By contradiction
  1. **Constrained:** $a_t = \underline{a}$ and $u'(c_t) \geq \beta R \mathbb{E}_t [u'(c_{t+1})]$, or
  2. **Unconstrained:** $a_t > \underline{a}$ and $u'(c_t) = \beta R \mathbb{E}_t [u'(c_{t+1})]$
- **Sufficiency:** From concavity of value function
  FOC: $c_t^{-\sigma} = \beta \mathbb{E}_t [v_a(z_{t+1}, a_t)]$
  Envelope: $v_a(z_t, a_{t-1}) = (1+r)c_t^{-\sigma}$

Special case I: Quadratic utility

- **Quadratic utility:** $u(c_t) = -\frac{1}{2}(\overline{c} - c)^2$ with $\beta R = 1$ and »large« $\overline{c}$
- **Euler-equation:** *Consumption = expected future consumption*

$$(\overline{c} - c_t) = \mathbb{E}_t\left[(\overline{c} - c_{t+k})\right] \Leftrightarrow c_t = \mathbb{E}_t\left[c_{t+k}\right]$$

- Use **IBC** in expectation to get **consumption function:**

$$\sum_{t=0}^{\infty} R^{-t}\mathbb{E}_0\left[c_t\right] = Ra_{-1} + \sum_{t=0}^{\infty} R^{-t} w \mathbb{E}_0\left[z_t\right] \Rightarrow$$

$$c^*(z_t, a_{t-1}) = c_0 = ra_{-1} + \frac{r}{R}\sum_{t=0}^{T} R^{-t} w \mathbb{E}_0\left[z_t\right]$$

where we formally disregard the borrowing constraint

- **Certainty equivalence:** *Only expected income matter.*

Special case II: CARA utility

- **CARA utility:** $u(c_t) = -\frac{1}{\alpha} e^{-\alpha c}$
- **Productivity is absolute random walk:**

$$z_t = z_{t-1} + \psi_t$$
$$\psi_t \sim \mathcal{N}(0, \sigma_\psi^2)$$

- **Consumption function** (see proof)**:**

$$c^*(a_{t-1}, z_t) = ra_{t-1} + wz_t - \frac{\log(\beta R)^{\frac{1}{\alpha}} + \alpha\frac{\sigma_\psi^2}{2}}{r^2}$$

where we formally disregard the borrowing constraint

- **Precautionary saving:** $\sigma_\psi^2 \uparrow$ implies $c_t^* \downarrow$ for given z_t and a_{t-1}
$\Rightarrow$ *accumulation of buffer-stock*

Dynamic solution: Bellman's Principle of Optimality

- **Origin:** Bellman, 1957, Chap. III.3.
- **Value function, v_t:** Defined *recursively* from

$$
\begin{aligned}
v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)] \\
\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq \underline{a}
\end{aligned}
$$

  with $v_T(\bullet) = 0$.

- **Policy function, c_t^*:** Is the same as

$$
\begin{aligned}
c_t^*(z_t, a_{t-1}) = \arg\max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)] \\
\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq \underline{a}
\end{aligned}
$$

Vocabulary

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

1. **State variables:** z_t and a_{t-1}
2. **Control** (choice) **variable:** c_t
3. **Continuation value:** $\beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$
4. **Parameters:** r, w, and stuff in $u(\bullet)$

Note: Straightforward to extend to more goods, more assets or other states, more complex uncertainty, bounded rationality etc.

Infinite horizon: $T \to \infty$?

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$

$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

- **Contraction mapping result:** *If β is low enough (strong enough impatience) then the value and policy functions converge to $v(z_t, a_{t-1})$ and $c^*(z_t, a_{t-1})$ for large enough T*
- **In practice:**
  1. Make arbitrary initial guess (e.g. $v_{t+1} = 0$)
  2. Solve backwards until value and policy functions does not change anymore (given some tolerance)

3-periods

3-period model

- **Expected discounted utility:** $v(z_0, a_{-1}) = \mathbb{E}_0 \sum_{t=0}^{2} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$
- **Income = wage × productivity + transfer:**

$$y_t = wz_t + \chi_t$$

- **Cash-on-hand, savings and borrowing constraint:**

$$\begin{aligned} m_t &= (1+r)a_{t-1} + y_t \\ a_t &= m_t - c_t \\ a_t &\geq \underline{a} \end{aligned}$$

- **Stochastic transition:** $\Pr[z_{t+1}|z_t] = \pi_t(z_t, z_{t+1})$ such that

$$\Pr[z_{t+1} = 1 \,|\, z_t = 1] = \pi$$
$$\Pr[z_{t+1} \in \{1-\Delta, 1-\Delta\} \,|\, z_t = 1] = \frac{1-\pi}{2}$$
$$\Pr[z_{t+1} = z_t \,|\, z_t \in \{1-\Delta, 1+\Delta\}] = 1$$

Bellman equation

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(z_{t+1}, a_t) \right]$$

$$\text{s.t.}$$

$$y_t = w z_t + \chi_t$$

$$m_t = (1+r) a_{t-1} + y_t$$

$$a_t = m_t - c_t$$

$$\Pr\left[z_{t+1} | z_t \right] = \pi_t(z_t, z_{t+1})$$

$$a_t \geq \underline{a}$$

where

$$v_3(z_3, a_2) = 0$$

Discretization

- **Discretization:** All state variables belong to discrete sets $\equiv$ *grids*,

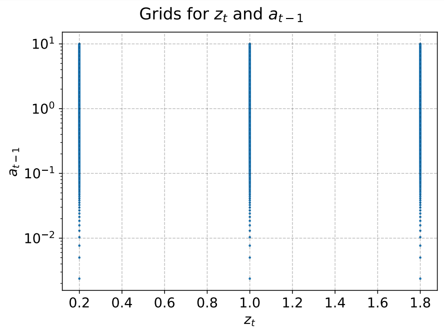
$$z_t \in \mathcal{G}_z = \{z^0, z^1, \ldots, z^{\#_z - 1}\}$$
$$a_t \in \mathcal{G}_a = \{a^0, a^1, \ldots, a^{\#_a - 1}\}$$
$$a^0 = \underline{a}$$

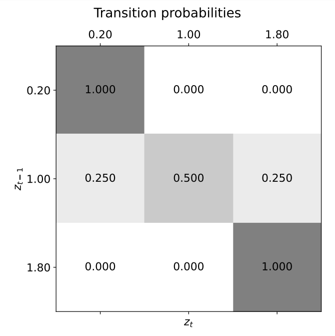
- **Expectation:** Numerical integration by

$$\mathbb{E}_t\left[v_{t+1}(z_{t+1}, a_t)\right] = \sum_{z_{t+1} \in \{1-\Delta, 1, 1+\Delta\}} \pi_t(z_t, z_{t+1}) v_{t+1}(z_{t+1}, a_t)$$

- **ConSav:** `grids.nonlinspace`, `grids.equilogspace`
- **ConSavNotebook:** `04. Tools/03. Grids.ipynb`

Grids and transition probabilities





Linear interpolation

- **Linear interpolation** (function approximation):
  1. Assume v_{t+1} is known on $\mathcal{G}_z \times \mathcal{G}_a$ (tensor product)
  2. Evaluate $v_{t+1}(z^{i_z}, a)$ for arbitrary a by

$$
\begin{aligned}
\breve{v}_{t+1}(z^{i_z}, a) &= \text{baseline} + \text{slope} \times \text{distance} \\
&= v_{t+1}(z^{i_z}, a^{\iota}) + \omega(a - a^{\iota})
\end{aligned}
$$

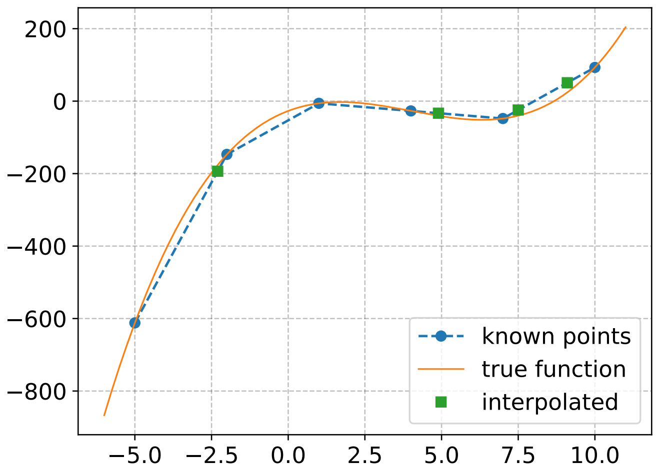
where

$$
\omega \equiv \frac{v_{t+1}(z^{i_z}, a^{\iota+1}) - v_{t+1}(z^{i_z}, a^{\iota})}{a^{\iota+1} - a^{\iota}}
$$

$$
\iota \equiv \text{largest } i_a \in \{0, 1, \ldots, \#_a - 2\} \text{ such that } a^{i_a} \leq a
$$

- **ConSav:** `linear_interp.interp1d`
- **ConSavNotebook:**
  `04. Tools/01. Linear interpolation.ipynb`

Linear interpolation



Value function iteration (VFI)

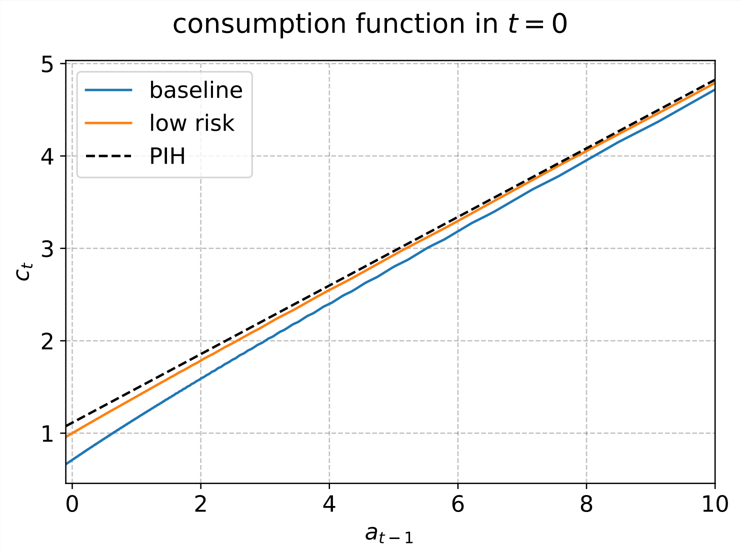
- **Maximize value-of-choice:**

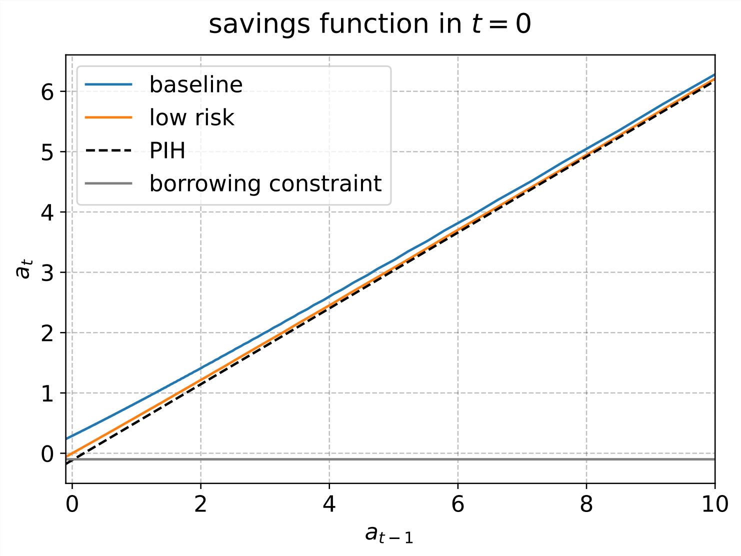
$$v_t(z^{i_z}, a^{i_{a-}}) = \max_{c_t} v_t(z^{i_z}, a^{i_{a-}} | c_t)$$
$$\text{with } c_t \in [0, (1+r)a^{i_{a-}} + wz^{i_z} + \underline{a}]$$

$$v_t(z^{i_z}, a^{i_{a-}} | c_t) = u(c_t) + \sum_{i_{z+1}=0}^{\#_z-1} \pi(i_z, i_{z+1}) \breve{v}_{t+1}(z^{i_z}, a)$$
$$\text{with } a_t = (1+r)a^{i_{a-}} + wz^{i_z} - c_t$$

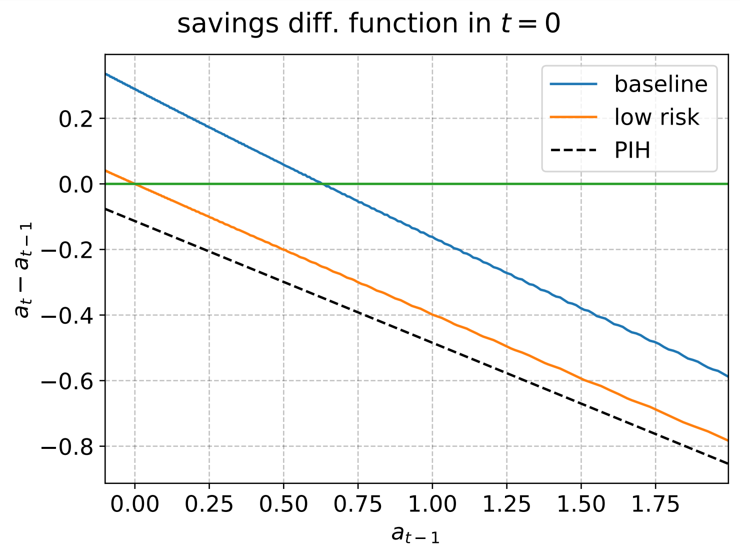
- **Inner loop:** For each grid point in $\mathcal{G}_z \times \mathcal{G}_a$ find $c_t^*(z_t, a_{t-1})$ and therefore $v_t(z_t, a_{t-1})$ with a *numerical optimizer*
- **Outer loop:** Backwards from $t = T - 1$ (note $\underline{v}_T = 0$, or known)
- **ConSav+QuantEcon:** Various optimizers in numba
- **ConSavNotebook:** `04. Tools/02. Optimization.ipynb`

Consumption function

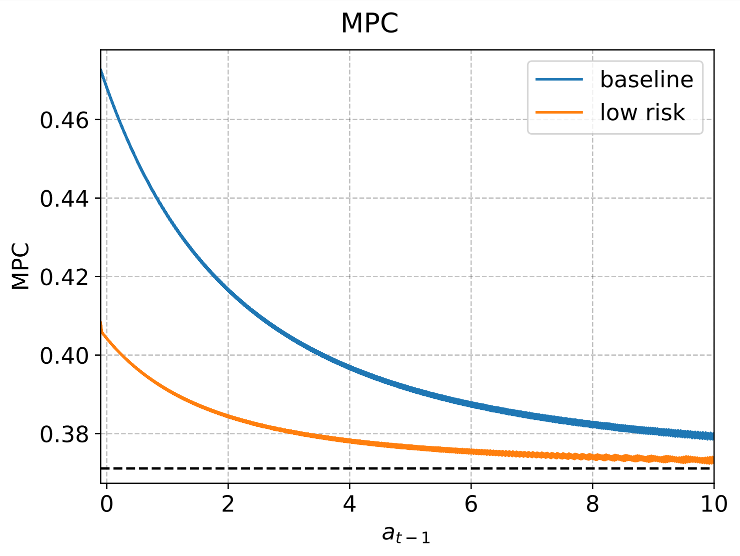



savings function in $t = 0$
baseline
low risk
PIH
borrowing constraint
a_t
a_{t-1}
0
1
2
3
4
5
6
0
2
4
6
8
10

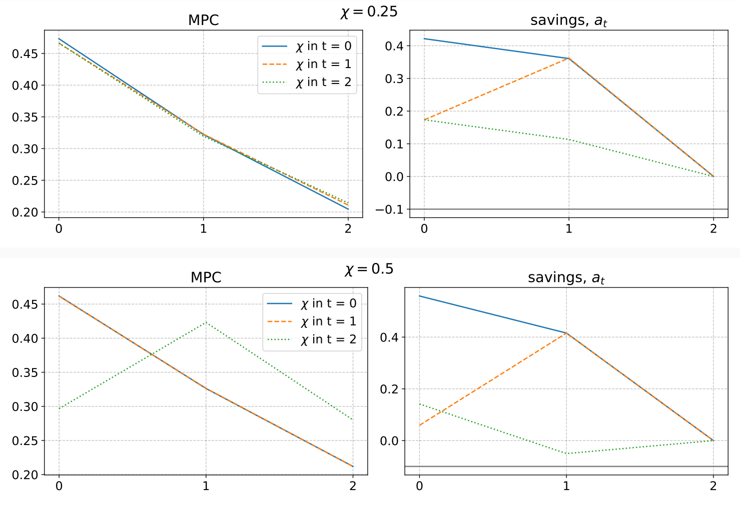
Change in savings function



MPC



Intertemporal MPC



Economic insights

- **Notebook:** 01. ConSavModel.ipynb
- **Consumption lower than under PIH and concave in assets**
  Intuition: *Precautionary saving motive is relatively larger for asset poor households because income risk is the same for everybody*
  Implications:
  1. Windfall gives safety and increases average consumption
     $\Rightarrow$ MPC decreasing in assets
  2. Attraction towards a buffer-stock target $a_t = a_{t-1}$ despite $\beta R < 1$
  3. Larger effective discounting of future income
     (extreme: no effect of future income changes if constrained before)

Numerical Monte Carlo simulation

- **Initial distribution:** Draw $z_{i,-1}$ and $a_{i,-1}$ for $i \in \{0, 1, \ldots, N-1\}$
- **Simulation:** Forwards in time from $t = 0$ and in each time period
  1. Draw z_{it} given transition probabilities
  2. Use linear interpolation to evaluate

$$\begin{aligned} c_{it} &= \breve{c}_t^*(z_{it}, a_{it-1}) \\ a_{it} &= (1+r)a_{it-1} + wz_{it} - c_{it} \end{aligned}$$

- **Review:**
  - **Pro:** Simple to implement
  - **Con:** Computationally costly and introduces randomness
- **Infinite horizon:**
  1. Assume z_{it} has an ergodic distribution
  2. Ergodic distribution of a_{it} around buffer-stock target

EGM

Time iteration

- **Replace numerical optimization with root-finding**
- **Time iteration:** For each a_{t-1} and z_t find c_t to solve the Euler-equation

$$c_t^{-\sigma} = \beta(1+r)\mathbb{E}_t[c_{t+1}^{-\sigma}]$$

Note: *Necessary and sufficient* (for interior choices, else $a_t = \underline{a}$)

Endogenous grid-point method (EGM)

1. Calculate **post-decision marginal value of cash:**

$$q(z^{i_z}, a^{i_a}) = \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z,i_{z+}} c_+^*(z^{i_{z+}}, a^{i_a})^{-\sigma}$$

2. **Invert Euler-equation:**

$$c(z^{i_z}, a^{i_a}) = (\beta(1+r)q(z^{i_z}, a^{i_a}))^{-\frac{1}{\sigma}}$$

3. **Endogenous cash-on-hand:**

$$m(z^{i_z}, a^{i_a}) = a^{i_a} + c(z^{i_z}, a^{i_a})$$

4. **Consumption function:** Calculate $m = (1+r)a^{i_{a-}} + wz^{i_z}$

   4.1 Binding constraint: If $m \leq m(z^{i_z}, a^0)$ then

   $$c^*(z^{i_z}, a^{i_{a-}}) = m + \underline{a}$$

   4.2 Interior choice: Else

   $$c^*(z^{i_z}, a^{i_{a-}}) = \text{interpolate } m(z^{i_z}, m) \to c(z^{i_z}, m)$$

NEGM

An illiquid asset

- **Illiuid asset:** Worth k in period T, else $(1-\gamma)k$ for $\gamma \in [0,1]$
- **Terminal period:** $v_T(\text{own}_{T-1}, a_{T-1}, y_T) = \frac{(a_{T-1}+y_T+\text{own}_{T-1}k)^{1-\sigma}}{1-\sigma}$
- **Recursive problem:** For $\text{own}_{t-1} \in \{0,1\}$

$$
\begin{aligned}
v_t(\text{own}_{t-1}, a_{t-1}, y_t) = \max_{c_t,\text{sell}_t \in \{0,1\}} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(\text{own}_t, a_t, y_{t+1}) \right] \\
\text{s.t. } \text{own}_t &= (1-\text{sell}_t)\text{own}_{t-1} \\
m_t &= a_{t-1} + y_t + \text{sell}_t \text{own}_{t-1}(1-\gamma)k \\
a_t &= m_t - c_t \\
y_{t+1} &= 1 \\
a_t &\geq 0
\end{aligned}
$$

- **Euler-equation:** Still *necessary*, but no longer *sufficient*
  1. Necessary: From variation argument conditional on own/sell choice
  2. Not sufficient due to *non-convexity*
     (more savings can trigger sell with fall in consumption)

Nesting

- **Sell or not?**

$$\overline{v}_t(m_t) = \max\left\{v_t(0, m_t^{\text{sell}}), v_t(1, m_t)\right\}$$
$$m_t^{\text{sell}} = m_t + (1-\gamma)k$$

- **Post-decision value function:**

$$w_t(\text{own}_t, a_t) = \mathbb{E}_t\left[\begin{cases} v_{t+1}(\text{own}_t, m_{t+1}) & \text{if } \text{own}_t = 0 \text{ or } t = T-1 \\ \overline{v}_{t+1}(m_{t+1}) & \text{if } \text{own}_t = 1 \text{ and } t < T-1 \end{cases}\right]$$
$$m_{t+1} = a_t + y_{t+1}$$
$$\text{where } v_T(\text{own}_{T-1}, m_T) = \frac{(m_T + \text{own}_{T-1}k)^{1-\sigma}}{1-\sigma}$$

- **Re-written Bellman:**

$$v_t(\text{own}_t, m_t) = \max_{c_t \in [0, m_t]} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta w_t(\text{own}_t, m_t - c_t)$$

Post-decision marginal value of cash

- **Post-decision marginal value of cash:**

$$q_t(\text{own}_t, a_t) = \mathbb{E}_t \left[\begin{cases} (m_T + \text{own}_{T-1} k)^{-\sigma} & \text{if } t = T-1 \\ c^*_{t+1}(0, m^{\text{sell}}_{t+1})^{-\sigma} & \text{else if sell}_{t+1} = 1 \\ c^*_{t+1}(1, m_{t+1})^{-\sigma} & \text{else} \end{cases} \right]$$

$$m_{t+1} = a_t + y_{t+1}$$

$$m^{\text{sell}}_{t+1} = m_{t+1} + (1-\gamma)k$$

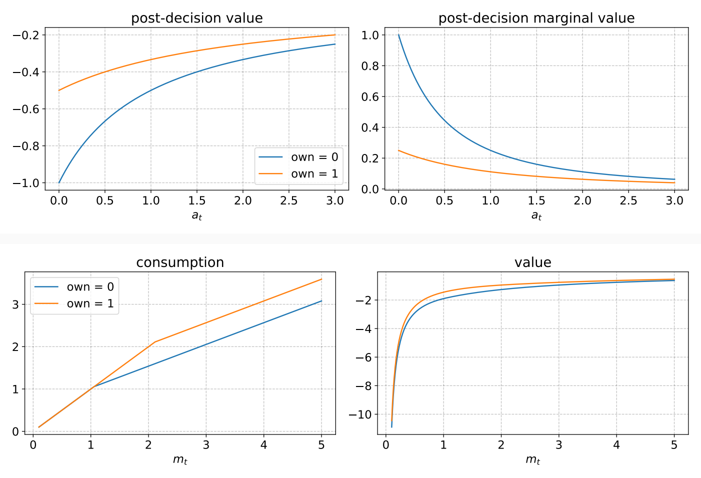
$$\text{sell}_{t+1} = \begin{cases} 1 & \text{if } v_{t+1}(0, m^{\text{sell}}_{t+1}) > v_{t+1}(1, m_{t+1}) \\ 0 & \text{else} \end{cases}$$

- **Euler-equation:**

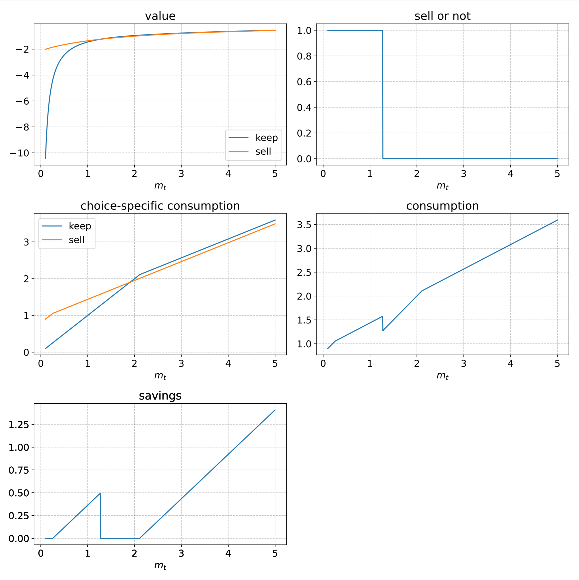
$$u'(c_t) = \beta q_t \Leftrightarrow c_t^{-\sigma} = \beta q_t$$

Conditional consumption function in $t = T - 2$

Solution method: EGM



Unconditional owner behavior, $t = T - 2$



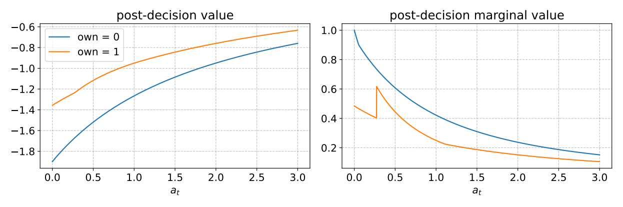
Upper envelope (given own_t)

Step 1: Generate candidate points, $\forall i_a$

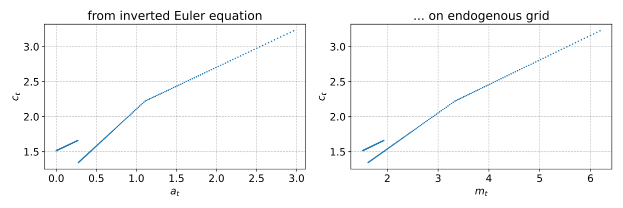
$$
\begin{aligned}
w^{i_a} &= w_t(\text{own}_t, a^{i_a}) \\
q^{i_a} &= q_t(\text{own}_t, a^{i_a}) \\
c^{i_a} &= u'^{-1}\left(\beta q^{i_a}\right) = \left(\beta q^{i_a}\right)^{-\frac{1}{\sigma}} \\
m^{i_a} &= a^{i_a} + c^{i_a} \\
v^{i_a} &= u\left(c^{i_a}\right) + w^{i_a}
\end{aligned}
$$

Problems with EGM

Post-decision values in $t = T - 2$:



Problems for owners:



Reason: Non-sufficient Euler-equation!

Upper envelope (given own_t)

Step 2: Apply upper-envelope, $\forall i_m$, $c^*(m^{i_m}) = c^{i_m,j^*}$

$$j^* = \arg\max_{j \in \{0,1,\dots\#_a-2\}} u\left(c^{i_m,j}\right) + w^{i_m,j}$$

s.t.

$$\text{potential segment: } m^{i_m} \in \begin{cases} \left[m^j, m^{j+1}\right] & \text{if } j < \#_a - 2 \\ \left[m^j, \infty\right] & \text{if } j = \#_a - 2 \end{cases}$$

$$\text{interpolation + constraint } c^{i_m,j} = \min\left\{c^j + \frac{c^{j+1} - c^j}{m^{j+1} - m^j}\left(m^{i_m} - m^j\right), m^{i_m}\right\}$$

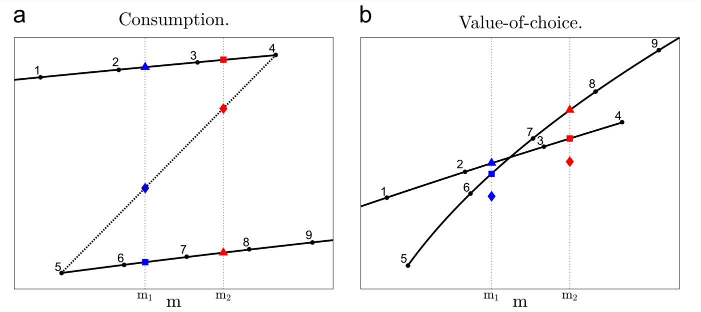
$$\text{continuation value: } w^{i_m,j} = \text{interp } \left\{a^{i_a}\right\} \to \left\{w^{i_a}\right\} \text{ at } a^{i_m,j}$$

$$a^{i_m,j} = m^{i_m} - c^{i_m,j}$$

ConSav: `upperenvelope`

ConSavNotebook: `04. Tools/06. Upper envelope.ipynb`

Illustration

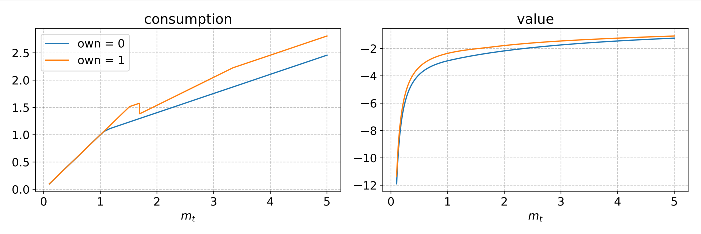


1. **Numbering:** Different levels of end-of-period assets, a^{i_a}
2. **Problem:** Find the consumption function at m_1 and m_2
3. **Largest value-of-choice:** Denoted by the *triangles*

Source: Druedahl and Jørgensen (2017), *G^2EGM*

Druedahl (2021), *NEGM*

Conditional consumption function in $t = T - 1$



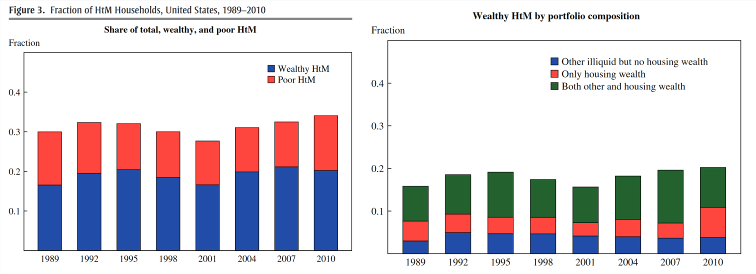
Economic insights

Notebook: 02. Illiquid.ipynb

1. **Simultaneous high total wealth and high MPC**
   1.1 Poor hands-to-mouth households
   1.2 Wealthy hands-to-mouth households
2. **The MPC is strongly size-dependent**
3. **Precautionary savings:**
   3.1 Frequent shocks: Liquid assets important
   3.2 Infrequent shocks: Illiquid assets enough

   (see Larkin (2024)

Empirical evidence for hands-to-mouth households



Poor HtM: Low liquid net worth, low total net worth

Wealthy HtM: Low liquid net worth, high total net worth

Source: Kaplan et. al. (2014)

Extra: Adding smoothing

- **Taste shocks:** Following Iskhakov et. al., 2017)

$$\overline{v}_t(m_t) = \max\left\{v_t(0, m_t^{\text{sell}}) + \sigma_\varepsilon \varepsilon(0), v_t(1, m_t) + \sigma_\varepsilon \varepsilon(1)\right\}$$
$$\varepsilon(x) \sim \text{Extreme value}$$

- **Logit-formula:**

$$\overline{v}_t(m_t) = \sigma_\varepsilon \log\left(\exp\frac{v_t(0, m_t^{\text{sell}})}{\sigma_\varepsilon} + \exp\frac{v_t(1, m_t)}{\sigma_\varepsilon}\right)$$

in choice probabilities:

$$P_t^{\text{sell}}(1, m_t) = \frac{\exp\frac{v_t(0, m_t^{\text{sell}})}{\sigma_\varepsilon}}{\exp\frac{v_t(0, m_t^{\text{sell}})}{\sigma_\varepsilon} + \exp\frac{v_t(1, m_t)}{\sigma_\varepsilon}}$$
$$\overline{v}_t(m_t) = P_t^{\text{sell}} v_t(0, m_t^{\text{sell}}) + \left(1 - P_t^{\text{sell}}\right) v_t(1, m_t)$$

Extra

1. Permanent transitory income process

- **Persistent-transitory income process:**

$$z_t = \tilde{z}_t \xi_t, \;\; \log \xi_t \sim \mathcal{N}(\mu_\xi, \sigma_\xi)$$
$$\log \tilde{z}_{t+1} = \rho_z \log \tilde{z}_t + \psi_{t+1}, \;\; \psi_{t+1} \sim \mathcal{N}(\mu_\psi, \sigma_\psi)$$

  1. Transitory shock: ξ_t
  2. Persistent shock: ψ_t
  3. Normalization using μ_ψ and μ_ξ: $\mathbb{E}[z_t] = \mathbb{E}[\tilde{z}_t] = 1$

- **ConSav:** `qudarature.log_normal_gauss_hermite`
- **ConSavNotebook:** `04. Tools/04. Quadrature.ipynb`

1. Transition probabilities

- **Discretization of ξ_t:** Derive $\mathcal{G}_\xi$ and $\pi_{i_{\xi-},i_\xi}$ given σ_ξ using Gauss-Hermite quadrature

$$x \sim \mathcal{N}(\mu,\sigma^2): \;\; \mathbb{E}[h(x)] \approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^{n} \omega_i h(\sqrt{2}\sigma x_i + \mu)$$

  where nodes, x_i, and weights, ω_i, have analytical expressions
- **Discretization of $\tilde{z}_t$:** Derive $\mathcal{G}_{\tilde{z}}$ and $\pi_{i_{\tilde{z}-},i_{\tilde{z}}}$ given $\rho_z < 1$ and σ_ψ (using a method such as Tauchen (1986) or Rouwenhorst (1995))
  If $\rho_z = 1$: Also use quadrature here.
- **Combined:** Derive $\mathcal{G}_z = \mathcal{G}_{\tilde{z}} \times \mathcal{G}_\xi$ (tensor product) and use independence of $\tilde{z}_t$ and ξ_t to get transition probabilities π_{i_{z-},i_z} (kronecker product)
- **ConSav:** `markov.log_rouwenhorst`, `markov.log_tauchen`
- **ConSavNotebook:** `04. Tools/05. Markov.ipynb`

1. Cash-on-hand formulation

Naive formulation:

$$v_t(\tilde{z}_t, \xi_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(\tilde{z}_{t+1}, \xi_{t+1}, a_t) \right]$$

s.t.

$$\begin{aligned}
z_t &= \tilde{z}_t \xi_t \\
y_t &= w z_t \\
m_t &= (1+r) a_{t-1} + y_t \\
a_t &= m_t - c_t \\
\tilde{z}_{t+1} &= \tilde{z}_t^{\rho_z} \psi_{t+1} \\
a_t &\geq -w b \tilde{z}_t
\end{aligned}$$

Cash-on-hand formulation (1 less state variable)

$$v_t(\tilde{z}_t, m_t) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(\tilde{z}_{t+1}, a_t) \right]$$

s.t.

$$\begin{aligned}
a_t &= m_t - c_t \\
\tilde{z}_{t+1} &= \tilde{z}_t^{\rho_z} \psi_{t+1} \\
m_{t+1} &= (1+r) a_{t+1} + w \tilde{z}_{t+1} \xi_{t+1} \\
a_t &\geq -wb\tilde{z}_t
\end{aligned}$$

1. Normalization if $\rho_z = 1$

- **Assumption:** $\rho_z = 1 \Leftrightarrow \tilde{z}_{t+1} = \tilde{z}_t \psi_{t+1}$
- **Define normalized variables:** $\boldsymbol{x}_t = x_t / \tilde{z}_t$ and $\boldsymbol{v}_t(\boldsymbol{m}_t) = \frac{v_t(\tilde{z}_t, m_t)}{\tilde{z}_t^{1-\rho}}$
- **Normalized Bellman equation:**

$$
\begin{aligned}
\boldsymbol{v}_t(\boldsymbol{m}_t) &= \max_{\boldsymbol{c}_t} \frac{\boldsymbol{c}_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[\psi_{t+1}^{1-\rho} \boldsymbol{v}_{t+1}(\boldsymbol{m}_{t+1}) \right] \\
\text{s.t.} \quad \boldsymbol{a}_t &= \boldsymbol{m}_t - \boldsymbol{c}_t \\
\boldsymbol{m}_{t+1} &= \frac{1+r}{\psi_{t+1}} \boldsymbol{a}_t + w \xi_{t+1} \\
\boldsymbol{a}_t &\geq -wb
\end{aligned}
$$

- **Normalized Euler-equation:**

$$
c_t^{-\sigma} = \beta(1+r)\mathbb{E}_t \left[c_{t+1}^{-\sigma} \right] \Leftrightarrow \boldsymbol{c}_t^{-\sigma} = \beta(1+r)\mathbb{E}_t \left[\left(\psi_{t+1} \boldsymbol{c}_{t+1} \right)^{-\sigma} \right]
$$

- **Simulation speed-up:** Harmenberg (2021)

2. Life-cycle (I)

- **Basically:**
    1. Born, working, retied, die
    2. Age-varying parameters (esp. income)
- **Add-ons:**
    1. Labor supply, human capital, occupation
    2. Portfolio choice and entrepreneurship
    3. Family formation
    4. Health, mortality

    etc.
- **Good starting example:** »Life-Cycle Consumption and Children: Evidence from a Structural Estimation«, Jørgensen (2017)

2. Life-cycle (II)

Paper: Gourinchas and Parker (2021)
Life-cycle consumption-saving model with retirement

- Young households:
  Save for precautionary reasons (buffer)
- Older households:
  Save for retirement (life-cycle)

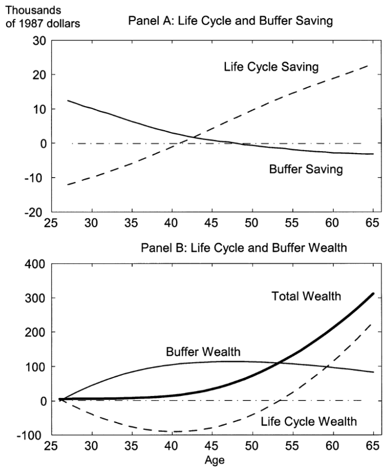
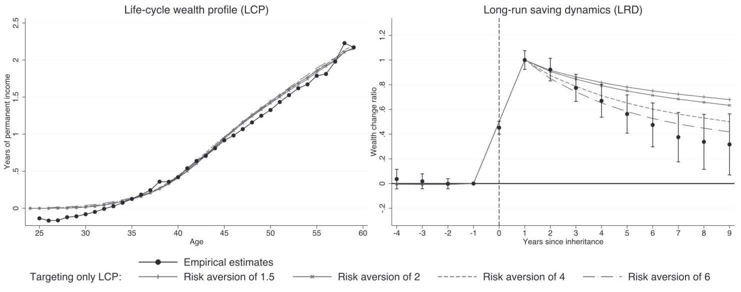


FIGURE 7.—The role of risk in saving and wealth accumulation.

2. Life-cycle (III)

- **Natural experiment:** Wealth depletion after sudden inheritance
- **Results:**
  1. Life-cycle profile of wealth fitted for many levels of risk-aversion (by varying the discount factor)
  2. Fast wealth depletation requires high risk-aversion (or high perceived risk)

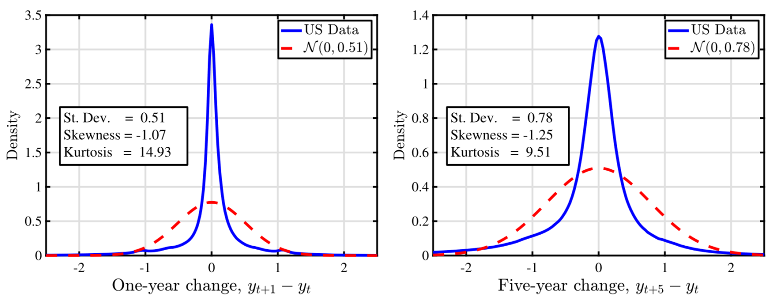
FIGURE 3.—REPLICATING LONG-RUN SAVING DYNAMICS



Source: Druedahl and Martinello (2022)

3. More realistic income risk (I)

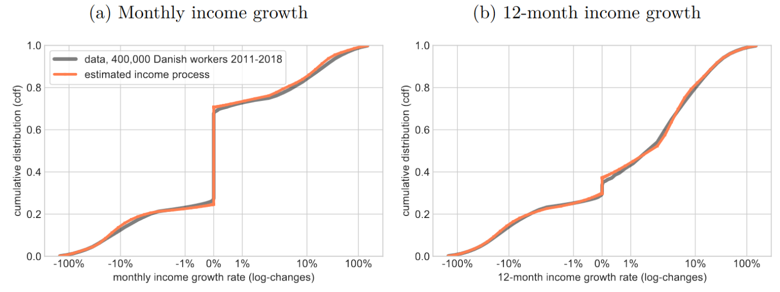
Annual earnings-changes are far from log-normal:



Source: Guvenen et. al. (2021)

3. More realistic income risk (II)

Many with zero-growth month-month:



Source: Druedahl et. al. (2021)

4. Epstein-Zin

$$
\begin{aligned}
v_t(z_t, m_t) \quad &= \quad \max_{c_t} \left[(1-\beta) \cdot c_t^{1-\sigma} + \beta \cdot w_{t+1}^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \\
\text{s.t.} \quad & w_{t+1} \equiv \mathbb{E}_t \left[v_{t+1}(z_{t+1}, m_{t+1})^{1-\rho} \right]^{\frac{1}{1-\rho}} \\
& m_{t+1} = (1+r)(m_t - c_t) + y_{t+1}
\end{aligned}
$$

- **Preferences:**
    1. Patience: β
    2. Intertemporal substitution: σ
    3. Risk-aversion: ρ
- **Euler-equation:** $v_t = \left[\beta R \cdot \mathbb{E}_t \left[c_{t+1}^{-\sigma} \cdot \left(\frac{w_{t+1}}{v_{t+1}} \right)^{\rho-\sigma} \right] \right]^{-\frac{1}{\sigma}}$
    1. FOC: $0 = v_t^{\sigma} \cdot \left[(1-\beta) \cdot c_t^{-\sigma} - \beta R \cdot w_{t+1}^{\rho-\sigma} \cdot \mathbb{E}_t \left[v_{t+1}^{-\rho} \cdot \frac{\partial v_{t+1}}{\partial m_{t+1}} \right] \right]$
    2. Envelope condition: $\frac{\partial v_t(z_t, m_t)}{\partial m_t} = v_t^{\sigma} \cdot (1-\beta) \cdot c_t^{-\sigma}$

5. Deep learning

- **Curse of dimensionality:**
  1. Many states
  2. Many choices
  3. Many shocks
- **Deep (reinforcement) learning:**
  1. Approximate value and policy functions with *neural networks*
  2. Approximate on simulation sample rather than on grid
  3. Automatic differentiation (backpropagation) and GPUs for speed
- **Examples:** Maliar and Maliar (2021) and Azinovic and Scheidegger (2022)
- **Working paper:** Druedahl and Røpke (2025)

Portfolio choice

Portfolio choice model

- **Risk-free asset:** a_t with return r_f
- **Risky asset:** b_t with return $r_f + \nu_t$
- **Recursive formulation:**

$$v_t(a_{t-1}, b_{t-1}, \nu_t, z_t) = \max_{a_t, b_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}\left(a_t, b_t, z_{t+1}\right)\right]$$

$$\begin{aligned}
&\text{s.t.} \\
m_t &= (1+r_f)a_{t-1} + (1+r_f+\nu_t)b_{t-1} + wz_t \\
c_t &= m_t - a_t - b_t \\
z_{t+1} &\sim F_z(z_t) \\
\nu_{t+1} &\sim F_\nu \\
a_t, b_t, c_t &\geq 0
\end{aligned}$$

Optimality conditions

- **Envelope conditions:**

$$\frac{\partial v_t}{\partial a_{t-1}} = (1+r_f)c_t^{-\sigma}, \;\; \frac{\partial v_t}{\partial a_{t-1}} = (1+r_f+\nu_t)c_t^{-\sigma}$$

- **FOCs**

$$-c_t^{-\sigma} + \beta \mathbb{E}_t\left[\frac{\partial v_{t+1}}{\partial a_t}\right] = 0$$
$$-c_t^{-\sigma} + \beta \mathbb{E}_t\left[\frac{\partial v_{t+1}}{\partial b_t}\right] = 0$$

- **Combined:**

$$c_t^{-\sigma} = \beta(1+r_f)\mathbb{E}_t\left[c_{t+1}^{-\sigma}\right]$$
$$0 = \mathbb{E}_t\left[\nu_{v+1} c_{t+1}^{-\sigma}\right]$$

Reformulation with fewer states

- **Consumption-decision value function:**

$$v_t(m_t, z_t) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta w_t\left(a_t, z_t\right)$$

$$\text{s.t.}$$

$$a_t = m_t - c_t$$

$$a_t \geq 0$$

- **Portfolio-decision value function:**

$$w_t\left(a_t, z_t\right) = \max_{\alpha_t} \beta \mathbb{E}_t\left[v_{t+1}\left(m_{t+1}, z_{t+1}\right)\right]$$

$$\text{s.t.}$$

$$m_{t+1} = R_{t+1} a_t + z_{t+1}$$

$$R_{t+1} = 1 + r_f + \nu_{t+1}\alpha_t$$

Solution method

1. Solve for $\alpha_t^*(a_t, z_t)$ by root-finding on

$$
\begin{aligned}
0 &= \mathbb{E}_t\left[\nu_{v+1} c_{t+1}^{-\sigma}\right] \\
&\text{s.t.} \\
c_{t+1} &= c_{t+1}^*(m_{t+1}, z_{t+1}) \\
m_{t+1} &= R_{t+1} a_t + z_{t+1} \\
R_{t+1} &= 1 + r_f + \nu_{t+1}\alpha_t^*(a_t, z_t)
\end{aligned}
$$

2. Compute

$$
q_t(a_t, z_t) = \mathbb{E}_t\left[R_{t+1} c_{t+1}^{-\sigma}\right]
$$

3. Find $c_t^*(m_t, z_t)$ using EGM

$$
\begin{aligned}
c_t(a_t, z_t) &= (\beta q_t(a_t, z_t))^{-\frac{1}{\sigma}} \\
m_t(a_t, z_t) &= c_t + a_t
\end{aligned}
$$

Extension with participation costs κ

$$v_t(m_t, z_t, \iota_{t-1}) = \max_{c_t \in [0, m_t]} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \underline{w}_t \left(m_t - c_t, z_t, \iota_{t-1}\right)$$

$$\underline{w}_t \left(a_t, z_t, \iota_{t-1}\right) = \max_{\iota_t} w_t \left(a_t, z_t, \iota_t\right) - \kappa \mathbf{1} \left\{\iota_t = 1 \wedge \iota_{t-1} = 0\right\}$$

s.t.

$$\iota_t \in \{1\} \text{ if } \iota_{t-1} = 1 \text{ else } \{0, 1\}$$

$$w_t \left(a_t, z_t, \iota_t\right) = \max_{\alpha_t} \beta \mathbb{E}_t \left[v_{t+1} \left(m_{t+1}, z_{t+1}, \iota_t\right)\right]$$

s.t.

$$m_{t+1} = R_{t+1} a_t + z_{t+1}$$

$$R_{t+1} = 1 + r_f + \nu_{t+1} \alpha_t$$

$$\alpha_t \in [0, 1] \text{ if } \iota_t = 1 \text{ else } \{0\}$$

Solution method with participation costs κ

- **Participation is an absorbing state**
  1. If $\iota_{t-1} = 1$ then $\iota_t = 1$
  2. The same solution method as before can be used
- **Before participation, $\iota_t = 0$:** The post-decision marginal value of cash no longer needs to be monotone

$$\underline{w}_t(a_t, z_t, 0) = \mathbb{E}_t\left[w_t(a_t, z_t, \iota_t) - \kappa \mathbf{1}\{\iota_t = 1\}\right]$$

$$q_t(a_t, z_t, \iota_t) = \mathbb{E}_t\left[R_{t+1} c_{t+1}^{-\sigma}\right]$$

$$\iota_t = \begin{cases} 1 & \text{if } w_t(a_t, z_t, 1) - \kappa > w_t(a_t, z_t, 0) \\ 0 & \text{else} \end{cases}$$

Same solution as before: *Apply an upper envelope*

Summary

Summary and what's next

- **This lecture:**
  1. Introduction to course
  2. Consumption-saving models
  3. Basic numerical dynamic programming
  4. EGM and NEGM
- **Next:** *Stationary equilibrium*
- **You should:**
  1. Study the code
  2. Glance at Aiyagari (1994),
     »Uninsured Idiosyncratic Risk and Aggregate Saving«