



Consumption-Saving

Mini-Course: Heterogenous Agent Macro

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Introduction

- **Generations of models:**
 1. **Permanent income hypothesis (PIH)** (Friedman, 1957)
or life-cycle model (Modigliani and Brumberg, 1954)
 2. **Buffer-stock consumption model**
Deaton (1991, 1992); Carroll (1992, 1997, 2019)
 3. **Multiple-asset buffer-stock consumption models**
e.g. Kaplan and Violante (2014); Harmenberg and Öberg (2021)
- **Consumption-and-saving over the life-cycle dynamic**
e.g. Gourinchas and Parker (2002); Druedahl and Martinello (2022)
- **Empirical MPCs and income risk**
e.g. Fagereng et. al. (2021); Guvenen et. al. (2021)

Book: **The Economics of Consumption**, Jappelli and Pistaferri (2017)

1. Introduction
2. PIH
3. Buffer-stock
4. 3-periods
5. EGM
6. NEGM
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8. Summary

PIH



$$v_0 = \max_{\{c_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t)$$

s.t.

$$a_t = (1 + r)a_{t-1} + wz_t - c_t$$

$$a_{T-1} \geq 0$$

- **Variables:**

Consumption: c_t

Productivity: z_t

End-of-period savings: a_t (*no debt at death*)

- **Parameters:**

Discount factor: β

Wage: w

Interest rate: r (define $R \equiv 1 + r$ as interest factor)

It is a *static* problem

$$v_0 = \max_{\{c_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t)$$

s.t.

$$a_t = (1 + r)a_{t-1} + wz_t - c_t$$

$$a_{T-1} \geq 0$$

■ It is a *static* problem:

1. **Information:** z_t is known for all t at $t = 0$
2. **Target:** Discounted utility, $\sum_{t=0}^{T-1} \beta^t u(c_t)$
3. **Behavior:** Choose c_0, c_1, \dots, c_{T-1} *simultaneously*
4. **Solution:** Sequence of consumption *choices* $c_0^*, c_1^*, \dots, c_{T-1}^*$

- **Substitution** implies *Intertemporal Budget Constraint* (IBC)

$$\begin{aligned}
 a_{T-1} &= Ra_{T-2} + wz_{T-1} - c_{T-1} \\
 &= R^2 a_{T-3} + Rwz_{T-2} - Rc_{T-2} + wz_{T-1} - c_{T-1} \\
 &= R^T a_{-1} + \sum_{t=0}^{T-1} R^{T-1-t} (wz_t - c_t)
 \end{aligned}$$

- Use **terminal condition** $a_{T-1} = 0$ (equality due utility max.)

$$R^{-(T-1)} a_{T-1} = 0 \Leftrightarrow s_0 + h_0 - \sum_{t=0}^{T-1} R^{-t} c_t = 0$$

where $s_0 \equiv Ra_{-1}$ (after-interest assets)
 and $h_0 \equiv \sum_{t=0}^{T-1} R^{-t} wz_t$ (human capital)

$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t u(c_t) + \lambda \left[\sum_{t=0}^{T-1} R^{-t} c_t - s_0 - h_0 \right]$$

- **First order conditions:**

$$\forall t : 0 = \beta^t u'(c_t) - \lambda(1+r)^{-t} \Leftrightarrow u'(c_t) = -\lambda(\beta R)^{-t}$$

- **Euler-equation** for $k \in \{1, 2, \dots\}$:

$$\frac{u'(c_t)}{u'(c_{t+k})} = \frac{-\lambda(\beta R)^{-t}}{-\lambda(\beta R)^{-(t+k)}} = (\beta R)^k$$

Consumption choice

- **CRRA:** $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$ imply Euler-equation

$$\frac{c_0^{-\sigma}}{c_t^{-\sigma}} = (\beta R)^t \Leftrightarrow c_t = (\beta R)^{\frac{t}{\sigma}} c_0$$

- Insert **Euler** into **IBC** to get consumption choice

$$\sum_{t=0}^{T-1} \left((\beta R)^{1/\sigma} R^{-1} \right)^t c_0 = s_0 + h_0 \Leftrightarrow$$
$$c_0^* = \frac{1 - (\beta R)^{1/\sigma} R^{-1}}{1 - ((\beta R)^{1/\sigma} R^{-1})^T} (s_0 + h_0)$$

Infinite horizon

- **Infinite horizon** for $(\beta R)^{1/\sigma} R^{-1} < 1$: Let $T \rightarrow \infty$ to get

$$c_0^* = \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right) (s_0 + h_0)$$

- **Consume annuity value:**

$$\beta R = 1, z_t = 1 \Rightarrow c_0^* = ra_{-1} + w$$

- **Intertemporal elasticity of substitution** ($IES = \frac{1}{\sigma}$):

$$\log c_{t+1} - \log c_t = \frac{1}{\sigma} \log \beta R$$

Constant consumption if:

1. $\beta R = 1$
2. $\sigma \rightarrow \infty$ (zero elasticity of substitution)

Propensities to consume

$$c_0^* = \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right) (s_0 + h_0) \approx ra_{-1} + w$$

Different types of shocks:

1. MPC of *windfall* income: $\frac{\partial c_0}{\partial s_0} \approx \frac{r}{1+r}$
2. MPC of *future* income change: $\frac{\partial c_0}{\partial wz_t} \approx \frac{r}{1+r} (1+r)^{-t}$
3. MPC of *permanent* income change: $\frac{\partial c_0}{\partial w} \approx \frac{r}{1+r} \frac{1}{1-(1+r)^{-1}} = 1$

Dynamic affects: The same when $\beta R = 1$, for all $k > 0$

$$\begin{aligned}\frac{\partial c_k}{\partial s_0} &= \frac{\partial c_0}{\partial s_0} \\ \frac{\partial c_k}{\partial wz_t} &= \frac{\partial c_0}{\partial wz_t} \\ \frac{\partial c_k}{\partial w} &= \frac{\partial c_0}{\partial w}\end{aligned}$$

Savings (with $z_t = 1$)

- **Constant savings** with $\beta R = 1$

$$c_t = ra_{t-1} + w \Rightarrow a_t = Ra_{t-1} + w - c_t = a_{t-1}$$

- **Decreasing savings** with $\beta R < 1$: $c_t \uparrow \Rightarrow a_t < a_{t-1}$
- **Increasing savings** with $\beta R > 1$: $c_t \downarrow \Rightarrow a_t > a_{t-1}$

Initial liquidity/borrowing constraint

- Implied period 0 **savings** are:

$$a_0 = Ra_{-1} + wz_0 - c_0$$

- Hard **borrowing constraint**: $a_0 \geq -w \cdot b$
- **Maximum consumption**: $\bar{c}_0 = Ra_{-1} + wz_0 + wb$
- **Optimal consumption**: Constrained or unconstrained.

$$c_0^* = \min \left\{ \bar{c}_0, \left(1 - \frac{(\beta R)^{1/\sigma}}{R} \right) (s_0 + h_0) \right\}$$

- **Empirical realism**. Incl. high MPC of constrained.
- **Technical issue**: *Borrowing constraints further in the future complicates the analytical solution considerably.*

Empirical MPCs

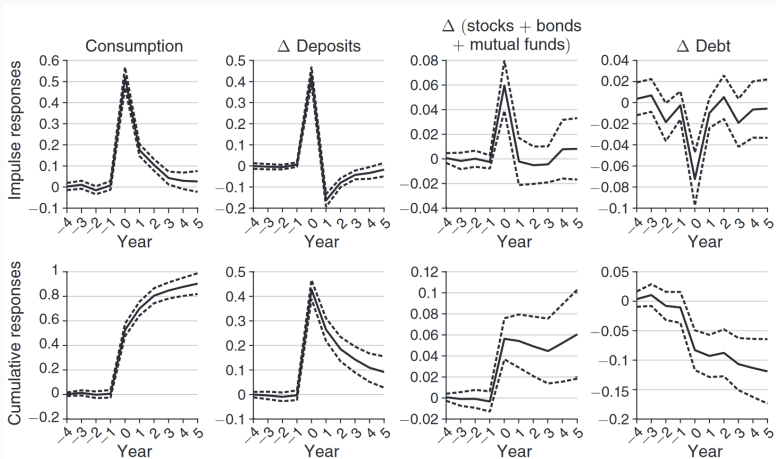


FIGURE 2. DYNAMIC HOUSEHOLD RESPONSES TO LOTTERY PRIZES

Source: Fagereng et. al. (2021)

Buffer-stock

Uncertainty and always borrowing constraint

$$v_0(z_0, a_{-1}) = \max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

s.t.

$$a_t = (1 + r)a_{t-1} + wz_t - c_t$$

$$z_{t+1} \sim \mathcal{Z}(z_t)$$

$$a_t \geq -wb$$

$$\lim_{t \rightarrow \infty} (1 + r)^{-t} a_t \geq 0 \quad [\text{No-Ponzi game}]$$

- **Stochastic income** from 1st order Markov-process, \mathcal{Z}
- **A true dynamic problem:**
 1. **Information:** z_t is revealed period-by-period
 2. **Target:** Expected discounted utility, $\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$
 3. **Behavior:** Choose c_t *sequentially* as information is revealed
 4. **Solution:** Sequence of consumption *functions*, $c_t^*(z_t, a_{t-1})$

- **Substitution** still implies:

$$R^{-(T-1)}a_{T-1} = 0 \Leftrightarrow s_0 + h_0 - \sum_{t=0}^{T-1} R^{-t}c_t = 0$$

- **What if $T \rightarrow \infty$?** We must have $\lim_{T \rightarrow \infty} R^{-(T-1)}a_{T-1} = 0$
 1. $\lim_{T \rightarrow \infty} R^{-(T-1)}a_{T-1} > 0$: Consumption can be increased
 2. $\lim_{T \rightarrow \infty} R^{-(T-1)}a_{T-1} < 0$: Violates No-Ponzi game condition
- For $T \rightarrow \infty$ we have the **IBC**:

$$\sum_{t=0}^{\infty} R^{-t}c_t = Ra_{-1} + \sum_{t=0}^{\infty} R^{-t}wz_t$$

Natural borrowing limit

- Denote **minimum possible productivity** by \underline{z}
- **Consumption must be non-negative** \Rightarrow
interest payments must be less than minimum income

$$c_t \geq 0 \Rightarrow r(-a_t) \leq w\underline{z} \Leftrightarrow a_t \geq -\frac{w\underline{z}}{r}$$

If debt was larger it would in the worst case ($\forall z_t = \underline{z}$) grow without bound even with zero consumption ($\forall c_t = 0$)

$$a_0 = -\frac{w\underline{z}}{r} - \Delta$$

$$a_1 = (1+r)a_0 + w\underline{z} = a_0 - (1+r)\Delta$$

$$a_2 = (1+r)a_1 + w\underline{z} = a_0 - (1+r)^2\Delta$$

$$\vdots$$

- **Natural borrowing constraint:** $a_t \geq \underline{a} = -w \min \left\{ b, \frac{\underline{z}}{r} \right\}$

Euler-equation from variation argument

- **Case I:** If $u'(c_t) > \beta R \mathbb{E}_t [u'(c_{t+1})]$:
Increase c_t by marginal $\Delta > 0$, and lower c_{t+1} by $R\Delta$
 1. **Feasible:** Yes, if $a_t > \underline{a}$
 2. **Utility change:** $u'(c_t) + \beta (-R) \mathbb{E}_t [u'(c_{t+1})] > 0$
- **Case II:** If $u'(c_t) < \beta R \mathbb{E}_t [u'(c_{t+1})]$:
Lower c_t by marginal $\Delta > 0$, and increase c_{t+1} by $R\Delta$
 1. **Feasible:** Yes (always)
 2. **Utility change:** $u'(c_t) + \beta R \mathbb{E}_t [u'(c_{t+1})] > 0$
- **Conclusion:** By contradiction
 1. **Constrained:** $a_t = \underline{a}$ and $u'(c_t) \geq \beta R \mathbb{E}_t [u'(c_{t+1})]$, or
 2. **Unconstrained:** $a_t > \underline{a}$ and $u'(c_t) = \beta R \mathbb{E}_t [u'(c_{t+1})]$
- **Sufficiency:** From concavity of value function
FOC: $c_t^{-\sigma} = \beta \mathbb{E}_t [v_a(z_{t+1}, a_t)]$
Envelope: $v_a(z_t, a_{t-1}) = (1 + r)c_t^{-\sigma}$

Special case I: Quadratic utility

- **Quadratic utility:** $u(c_t) = -\frac{1}{2}(\bar{c} - c)^2$ with $\beta R = 1$ and »large« \bar{c}
- **Euler-equation:** *Consumption = expected future consumption*

$$(\bar{c} - c_t) = \mathbb{E}_t [(\bar{c} - c_{t+k})] \Leftrightarrow c_t = \mathbb{E}_t [c_{t+k}]$$

- Use **IBC** in expectation to get **consumption function**:

$$\sum_{t=0}^{\infty} R^{-t} \mathbb{E}_0 [c_t] = Ra_{-1} + \sum_{t=0}^{\infty} R^{-t} w \mathbb{E}_0 [z_t] \Rightarrow$$
$$c^*(z_t, a_{t-1}) = c_0 = ra_{-1} + \frac{r}{R} \sum_{t=0}^T R^{-t} w \mathbb{E}_0 [z_t]$$

where we formally disregard the borrowing constraint

- **Certainty equivalence:** *Only expected income matter.*

Special case II: CARA utility

- **CARA utility:** $u(c_t) = -\frac{1}{\alpha} e^{-\alpha c}$
- **Productivity is absolute random walk:**

$$z_t = z_{t-1} + \psi_t$$

$$\psi_t \sim \mathcal{N}(0, \sigma_\psi^2)$$

- **Consumption function (see proof):**

$$c^*(a_{t-1}, z_t) = ra_{t-1} + wz_t - \frac{\log(\beta R)^{\frac{1}{\alpha}} + \alpha \frac{\sigma_\psi^2}{2}}{r^2}$$

where we formally disregard the borrowing constraint

- **Precautionary saving:** $\sigma_\psi^2 \uparrow$ implies $c_t^* \downarrow$ for given z_t and a_{t-1}
 \Rightarrow *accumulation of buffer-stock*

Dynamic solution: Bellman's Principle of Optimality

- **Origin:** Bellman, 1957, Chap. III.3.
- **Value function, v_t :** Defined *recursively* from

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1 + r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

with $v_T(\bullet) = 0$.

- **Policy function, c_t^* :** Is the same as

$$c_t^*(z_t, a_{t-1}) = \arg \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1 + r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1 + r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

1. **State variables:** z_t and a_{t-1}
2. **Control (choice) variable:** c_t
3. **Continuation value:** $\beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$
4. **Parameters:** r , w , and stuff in $u(\bullet)$

Note: Straightforward to extend to more goods, more assets or other states, more complex uncertainty, bounded rationality etc.

Infinite horizon: $T \rightarrow \infty$?

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1 + r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

- **Contraction mapping result:** *If β is low enough (strong enough impatience) then the value and policy functions converge to $v(z_t, a_{t-1})$ and $c^*(z_t, a_{t-1})$ for large enough T*
- **In practice:**
 1. Make arbitrary initial guess (e.g. $v_{t+1} = 0$)
 2. Solve backwards until value and policy functions does not change anymore (given some tolerance)

3-periods

3-period model

- **Expected discounted utility:** $v(z_0, a_{-1}) = \mathbb{E}_0 \sum_{t=0}^2 \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$
- **Income = wage \times productivity + transfer:**

$$y_t = wz_t + \chi_t$$

- **Cash-on-hand, savings and borrowing constraint:**

$$m_t = (1 + r)a_{t-1} + y_t$$

$$a_t = m_t - c_t$$

$$a_t \geq \underline{a}$$

- **Stochastic transition:** $\Pr[z_{t+1}|z_t] = \pi_t(z_t, z_{t+1})$ such that

$$\Pr[z_{t+1} | z_t = 1] = \pi_t$$

$$\Pr[z_{t+1} \in \{1 - \Delta, 1 + \Delta\} | z_t = 1] = \frac{1 - \pi_t}{2}$$

$$\Pr[z_{t+1} = z_t | z_t \in \{1 - \Delta, 1 + \Delta\}] = 1$$

Bellman equation

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t [v_{t+1}(z_{t+1}, a_t)]$$

s.t.

$$y_t = wz_t + \chi_t$$

$$m_t = (1+r)a_{t-1} + y_t$$

$$a_t = m_t - c_t$$

$$\Pr[z_{t+1}|z_t] = \pi_t(z_t, z_{t+1})$$

$$a_t \geq \underline{a}$$

where

$$v_3(z_3, a_2) = 0$$

- **Discretization:** All state variables belong to discrete sets \equiv *grids*,

$$z_t \in \mathcal{G}_z = \{z^0, z^1, \dots, z^{\#z-1}\}$$

$$a_t \in \mathcal{G}_a = \{a^0, a^1, \dots, a^{\#a-1}\}$$

$$a^0 = \underline{a}$$

- **Expectation:** Numerical integration by

$$\mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)] = \sum_{z_{t+1} \in \{1-\Delta, 1, 1+\Delta\}} \pi_t(z_t, z_{t+1}) v_{t+1}(z_{t+1}, a_t)$$

- **ConSav:** `grids.nonlinspace`, `grids.equilogspace`
- **ConSavNotebook:** 04. Tools/03. Grids.ipynb

Linear interpolation

- **Linear interpolation** (function approximation):

1. Assume v_{t+1} is known on $\mathcal{G}_z \times \mathcal{G}_a$ (tensor product)
2. Evaluate $v_{t+1}(z^{i_z}, a)$ for arbitrary a by

$$\begin{aligned}\check{v}_{t+1}(z^{i_z}, a) &= \text{baseline} + \text{slope} \times \text{distance} \\ &= v_{t+1}(z^{i_z}, a^{\iota}) + \omega(a - a^{\iota})\end{aligned}$$

where

$$\omega \equiv \frac{v_{t+1}(z^{i_z}, a^{\iota+1}) - v_{t+1}(z^{i_z}, a^{\iota})}{a^{\iota+1} - a^{\iota}}$$

$$\iota \equiv \text{largest } i_a \in \{0, 1, \dots, \#_a - 2\} \text{ such that } a^{i_a} \leq a$$

- **ConSav:** `linear_interp.interp1d`

- **ConSavNotebook:**

04. Tools/01. Linear interpolation.ipynb

Value function iteration (VFI)

- **Maximize value-of-choice:**

$$v_t(z^{i_z}, a^{i_a-}) = \max_{c_t} v_t(z^{i_z}, a^{i_a-} | c_t)$$

$$\text{with } c_t \in [0, (1+r)a^{i_a-} + wz^{i_z} + \underline{a}]$$

$$v_t(z^{i_z}, a^{i_a-} | c_t) = u(c_t) + \sum_{i_{z+1}=0}^{\#_z-1} \pi(i_z, i_{z+1}) v_{t+1}(z^{i_z}, a^{i_a})$$

$$\text{with } a_t = (1+r)a^{i_a-} + wz^{i_z} - c_t$$

- **Inner loop:** For each grid point in $\mathcal{G}_z \times \mathcal{G}_a$ find $c_t^*(z_t, a_{t-1})$ and therefore $v_t(z_t, a_{t-1})$ with a *numerical optimizer*
- **Outer loop:** Backwards from $t = T - 1$ (note $\underline{v}_T = 0$, or known)
- **ConSav+QuantEcon:** Various optimizers in numba
- **ConSavNotebook:** 04. Tools/02. Optimization.ipynb

- **Consumption lower than under PIH and concave in assets**

Intuition: *Precautionary saving motive is relatively larger for asset poor households because income risk is the same for everybody*

Implications:

1. Windfall gives safety and increases average consumption
⇒ MPC decreasing in assets
2. Attraction towards a buffer-stock target $a_t = a_{t-1}$ despite $\beta R < 1$
3. Larger effective discounting of future income
(extreme: no effect of future income changes if constrained before)

Numerical Monte Carlo simulation

- **Initial distribution:** Draw $z_{i,-1}$ and $a_{i,-1}$ for $i \in \{0, 1, \dots, N - 1\}$
- **Simulation:** Forwards in time from $t = 0$ and in each time period
 1. Draw z_{it} given transition probabilities
 2. Use linear interpolation to evaluate

$$c_{it} = \check{c}_t^*(z_{it}, a_{it-1})$$

$$a_{it} = (1 + r)a_{it-1} + wz_{it} - c_{it}$$

- **Review:**
 - **Pro:** Simple to implement
 - **Con:** Computationally costly and introduces randomness

EGM



Time iteration

- **Replace numerical optimization with root-finding**
- **Time iteration:** For each a_{t-1} and z_t find c_t to solve the Euler-equation

$$c_t^{-\sigma} = \beta(1+r)\mathbb{E}_t[c_{t+1}^{-\sigma}]$$

Note: *Necessary and sufficient* (for interior choices, else $a_t = \underline{a}$)

Endogenous grid-point method (EGM)

1. Calculate **post-decision marginal value of cash**:

$$q(z^{i_z}, a^{i_a}) = \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z, i_{z+}} c_+^*(z^{i_{z+}}, a^{i_a})^{-\sigma}$$

2. **Invert Euler-equation**:

$$c(z^{i_z}, a^{i_a}) = (\beta(1+r)q(z^{i_z}, a^{i_a}))^{-\frac{1}{\sigma}}$$

3. **Endogenous cash-on-hand**:

$$m(z^{i_z}, a^{i_a}) = a^{i_a} + c(z^{i_z}, a^{i_a})$$

4. **Consumption function**: Calculate $m = (1+r)a^{i_{a-}} + wz^{i_z}$

If $m \leq m(z^{i_z}, a^0)$ constraint binds: $c^*(z^{i_z}, a^{i_{a-}}) = m + \underline{a}$

Else: $c^*(z^{i_z}, a^{i_{a-}}) = \text{interpolate } m(z^{i_z}, \cdot) \text{ to } c(z^{i_z}, \cdot) \text{ at } m$

NEGM



An illiquid asset

- **Illiquid asset:** Worth k in period T , else $(1 - \gamma)k$ for $\gamma \in [0, 1]$
- **Terminal period:** $v_T(\text{own}_{T-1}, a_{T-1}, y_T) = \frac{(a_{T-1} + y_T + \text{own}_{T-1}k)^{1-\sigma}}{1-\sigma}$
- **Recursive problem:** For $\text{own}_{t-1} \in \{0, 1\}$

$$v_t(\text{own}_{t-1}, a_{t-1}, y_t) = \max_{c_t, \text{sell}_t \in \{0, 1\}} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t [v_{t+1}(\text{own}_t, a_t, y_{t+1})]$$

$$\text{s.t. } \text{own}_t = (1 - \text{sell}_t)\text{own}_{t-1}$$

$$m_t = a_{t-1} + y_t + \text{sell}_t \text{own}_{t-1}(1 - \gamma)k$$

$$a_t = m_t - c_t$$

$$y_{t+1} = 1$$

$$a_t \geq 0$$

- **Euler-equation:** Still *necessary*, but no longer *sufficient*
 1. Necessary: From variation argument conditional on own_t
 2. Not sufficient due to *non-convexity*
(more savings can trigger sell with fall in consumption)

- **Sell or not?**

$$\bar{v}_t(m_t) = \max \{v_t(0, m_t^{\text{sell}}), v_t(1, m_t)\}$$
$$m_t^{\text{sell}} = m_t + (1 - \gamma)k$$

- **Post-decision value function:**

$$w_t(\text{own}_t, a_t) = \mathbb{E}_t \left[\begin{cases} v_{t+1}(\text{own}_t, m_{t+1}) & \text{if } \text{own}_t = 0 \text{ or } t = T - 1 \\ \bar{v}_{t+1}(m_{t+1}) & \text{if } \text{own}_t = 1 \text{ and } t < T - 1 \end{cases} \right]$$

$$m_{t+1} = a_t + y_{t+1}$$

$$\text{where } v_T(\text{own}_{T-1}, m_T) = \frac{(m_T + \text{own}_{T-1}k)^{1-\sigma}}{1-\sigma}$$

- **Re-written Bellman:**

$$v_t(\text{own}_t, m_t) = \max_{c_t \in [0, m_t]} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta w_t(\text{own}_t, m_t - c_t)$$

- Post-decision marginal value of cash:

$$q_t(\text{own}_t, a_t) = \mathbb{E}_t \left[\begin{cases} (m_T + \text{own}_{T-1}k)^{-\sigma} & \text{if } t = T - 1 \\ c_{t+1}^*(0, m_{t+1}^{\text{sell}})^{-\sigma} & \text{else if } \text{sell}_{t+1} = 1 \\ c_{t+1}^*(1, m_{t+1})^{-\sigma} & \text{else} \end{cases} \right]$$

$$m_{t+1} = a_t + y_{t+1}$$

$$m_{t+1}^{\text{sell}} = m_{t+1} + (1 - \gamma)k$$

$$\text{sell}_{t+1} = \begin{cases} 1 & \text{if } v_{t+1}(0, m_{t+1}^{\text{sell}}) > v_{t+1}(1, m_{t+1}) \\ 0 & \text{else} \end{cases}$$

- Euler-equation Bellman:

$$u'(c_t) = \beta q_t \Leftrightarrow c_t^{-\sigma} = \beta q_t$$

Upper envelope (given own_t)

Step 1: Generate candidate points, $\forall i_a$

$$w^{i_a} = w_t(\text{own}_t, a^{i_a})$$

$$q^{i_a} = q_t(\text{own}_t, a^{i_a})$$

$$c^{i_a} = u'^{-1}(\beta q^{i_a}) = (\beta q^{i_a})^{-\frac{1}{\sigma}}$$

$$m^{i_a} = a^{i_a} + c^{i_a}$$

$$v^{i_a} = u(c^{i_a}) + w^{i_a}$$

Upper envelope (given own_t)

Step 2: Apply upper-envelope, $\forall i_m, c^*(m^{i_m}) = c^{i_m, j^*}$

$$j^* = \arg \max_{j \in \{0, 1, \dots, \#_a - 2\}} u(c^{i_m, j}) + w^{i_m, j}$$

s.t.

$$\text{potential segment: } m^{i_m} \in \begin{cases} [m^j, m^{j+1}] & \text{if } j < \#_a - 2 \\ [m^j, \infty] & \text{if } j = \#_a - 2 \end{cases}$$

$$\text{interpolation + constraint } c^{i_m, j} = \min \left\{ c^j + \frac{c^{j+1} - c^j}{m^{j+1} - m^j} (m^{i_m} - m^j), m^{i_m} \right\}$$

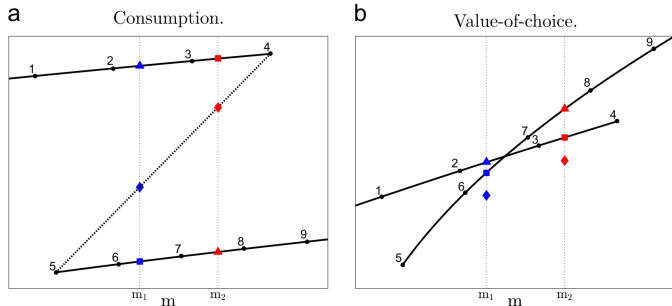
$$\text{continuation value: } w^{i_m, j} = \text{interp } \{a^{i_a}\} \rightarrow \{w^{i_a}\} \text{ at } a^{i_m, j}$$

$$a^{i_m, j} = m^{i_m} - c^{i_m, j}$$

ConSav: upperenvelope

ConSavNotebook: 04. Tools/06. Upper envelope.ipynb

Illustration



1. **Numbering:** Different levels of end-of-period assets, a^i_a
2. **Problem:** Find the consumption function at m_1 and m_2
3. **Largest value-of-choice:** Denoted by the *triangles*

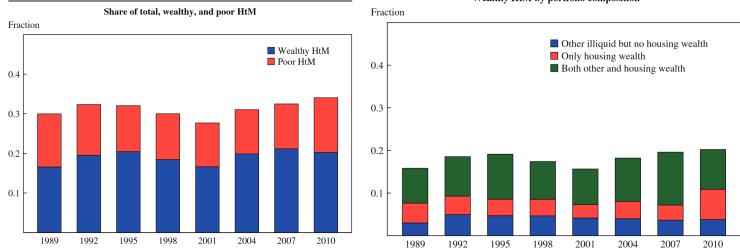
Sources: Druedahl and Jørgensen (2017), G^2EGM

Drueahl (2021), $NEGM$

1. **Simultaneous high total wealth and high MPC**
 - 1.1 Poor hands-to-mouth households
 - 1.2 Wealthy hands-to-mouth households
2. **The MPC is strongly size-dependent**
3. **Precautionary savings:**
 - 3.1 Frequent shocks: Liquid assets important
 - 3.2 Infrequent shocks: Illiquid assets enough

Empirical evidence for hands-to-mouth households

Figure 3. Fraction of HtM Households, United States, 1989–2010



Poor HtM: Low liquid net worth, low total net worth

Wealthy HtM: Low liquid net worth, high total net worth

Source: Kaplan et. al. (2014)

Extra: Adding smoothing

- **Taste shocks:** Following Iskhakov et. al., 2017)

$$\bar{v}_t(m_t) = \max \{ v_t(0, m_t^{\text{sell}}) + \sigma_\varepsilon \varepsilon(0), v_t(1, m_t) + \sigma_\varepsilon \varepsilon(1) \}$$
$$\vartheta(x) \sim \text{Extreme value}$$

- **Logit-formula:**

$$\bar{v}_t(m_t) = \sigma_\varepsilon \log \left(\exp \frac{v_t(0, m_t^{\text{sell}})}{\sigma_\varepsilon} + \exp \frac{v_t(1, m_t)}{\sigma_\varepsilon} \right)$$

in choice probabilities:

$$P_t^{\text{sell}}(1, m_t) = \frac{\exp \frac{v_t(0, m_t^{\text{sell}})}{\sigma_\varepsilon}}{\exp \frac{v_t(0, m_t^{\text{sell}})}{\sigma_\varepsilon} + \exp \frac{v_t(1, m_t)}{\sigma_\varepsilon}}$$
$$\bar{v}_t(m_t) = P_t^{\text{sell}} v_t(0, m_t^{\text{sell}}) + (1 - P_t^{\text{sell}}) v_t(1, m_t)$$

Extra

1. Permanent transitory income process

- **Persistent-transitory income process:**

$$z_t = \tilde{z}_t \xi_t, \quad \log \xi_t \sim \mathcal{N}(\mu_\xi, \sigma_\xi)$$
$$\log \tilde{z}_{t+1} = \rho_z \log \tilde{z}_t + \psi_{t+1}, \quad \psi_{t+1} \sim \mathcal{N}(\mu_\psi, \sigma_\psi)$$

1. Transitory shock: ξ_t
2. Persistent shock: ψ_t
3. Normalization using μ_ψ and μ_ξ : $\mathbb{E}[z_t] = \mathbb{E}[\tilde{z}_t] = 1$

- **ConSav:** `qudarature.log_normal_gauss_hermite`
- **ConSavNotebook:** 04. Tools/04. Quadrature.ipynb

1. Transition probabilities

- **Discretization of ξ_t :** Derive \mathcal{G}_ξ and $\pi_{i_{\xi-}, i_\xi}$ given σ_ξ using Gauss-Hermite quadrature

$$x \sim \mathcal{N}(\mu, \sigma^2) : \mathbb{E}[h(x)] \approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^n \omega_i h(\sqrt{2}\sigma x_i + \mu)$$

where nodes, x_i , and weights, ω_i , have analytical expressions

- **Discretization of \tilde{z}_t :** Derive $\mathcal{G}_{\tilde{z}}$ and $\pi_{i_{\tilde{z}-}, i_{\tilde{z}}}$ given $\rho_z < 1$ and σ_ψ (using a method such as Tauchen (1986) or Rouwenhorst (1995))
If $\rho_z = 1$: Also use quadrature here.
- **Combined:** Derive $\mathcal{G}_z = \mathcal{G}_{\tilde{z}} \times \mathcal{G}_\xi$ (tensor product) and use independence of \tilde{z}_t and ξ_t to get transition probabilities π_{i_{z-}, i_z} (kronecker product)
- **ConSav:** `markov.log_rouwenhorst`, `markov.log_tauschen`
- **ConSavNotebook:** 04. Tools/05. Markov.ipynb

1. Cash-on-hand formulation

Naive formulation:

$$v_t(\tilde{z}_t, \xi_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t [v_{t+1}(\tilde{z}_{t+1}, \xi_{t+1}, a_t)]$$

s.t.

$$z_t = \tilde{z}_t \xi_t$$

$$y_t = w z_t$$

$$m_t = (1+r)a_{t-1} + y_t$$

$$a_t = m_t - c_t$$

$$\tilde{z}_{t+1} = \tilde{z}_t^{\rho_z} \psi_{t+1}$$

$$a_t \geq -wb\tilde{z}_t$$

1. Cash-on-hand formulation

Cash-on-hand formulation (1 less state variable)

$$v_t(\tilde{z}_t, m_t) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t [v_{t+1}(\tilde{z}_{t+1}, a_t)]$$

s.t.

$$a_t = m_t - c_t$$

$$\tilde{z}_{t+1} = \tilde{z}_t^{\rho_z} \psi_{t+1}$$

$$m_{t+1} = (1+r)a_{t+1} + w\tilde{z}_{t+1}\xi_{t+1}$$

$$a_t \geq -wb\tilde{z}_t$$

1. Normalization if $\rho_z = 1$

- **Assumption:** $\rho_z = 1 \Leftrightarrow \tilde{z}_{t+1} = \tilde{z}_t \psi_{t+1}$
- **Define normalized variables:** $\mathbf{x}_t = x_t / \tilde{z}_t$ and $\mathbf{v}_t(\mathbf{m}_t) = \frac{v_t(\tilde{z}_t, m_t)}{\tilde{z}_t^{1-\rho}}$
- **Normalized Bellman equation:**

$$\mathbf{v}_t(\mathbf{m}_t) = \max_{\mathbf{c}_t} \frac{\mathbf{c}_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[\psi_{t+1}^{1-\rho} \mathbf{v}_{t+1}(\mathbf{m}_{t+1}) \right]$$

s.t.

$$\mathbf{a}_t = \mathbf{m}_t - \mathbf{c}_t$$

$$\mathbf{m}_{t+1} = \frac{1+r}{\psi_{t+1}} \mathbf{a}_t + w \xi_{t+1}$$

$$\mathbf{a}_t \geq -wb$$

- **Normalized Euler-equation:**

$$c_t^{-\sigma} = \beta(1+r) \mathbb{E}_t [c_{t+1}^{-\sigma}] \Leftrightarrow \mathbf{c}_t^{-\sigma} = \beta(1+r) \mathbb{E}_t \left[(\psi_{t+1} \mathbf{c}_{t+1})^{-\sigma} \right]$$

2. Life-cycle (I)

- **Basically:**

1. Born, working, retired, die
2. Age-varying parameters (esp. income)

- **Add-ons:**

1. Labor supply, human capital, occupation
 2. Portfolio choice and entrepreneurship
 3. Family formation
 4. Health, mortality
- etc.

- **Good starting example:** »Life-Cycle Consumption and Children: Evidence from a Structural Estimation«, Jørgensen (2017)

2. Life-cycle (II)

Paper: Gourinchas and Parker (2021)
Life-cycle consumption-saving model with retirement

- Young households:
Save for precautionary reasons (buffer)
- Older households:
Save for retirement (life-cycle)

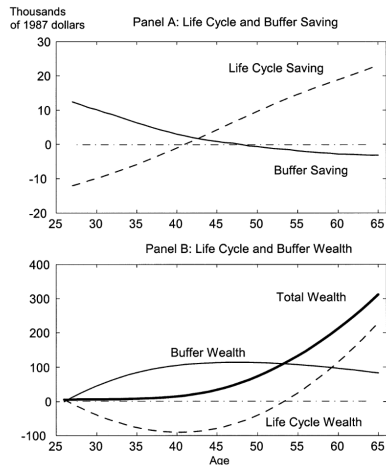
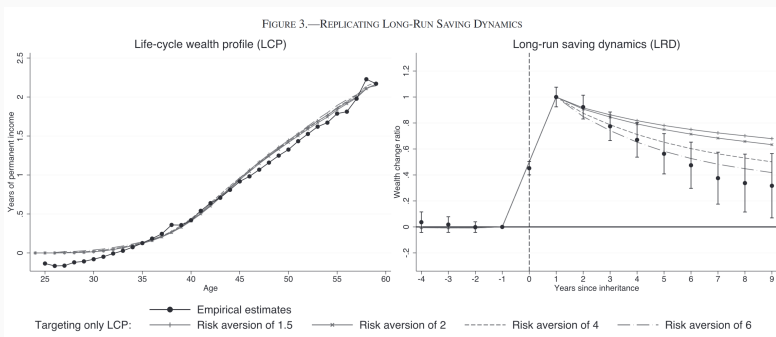


FIGURE 7.—The role of risk in saving and wealth accumulation.

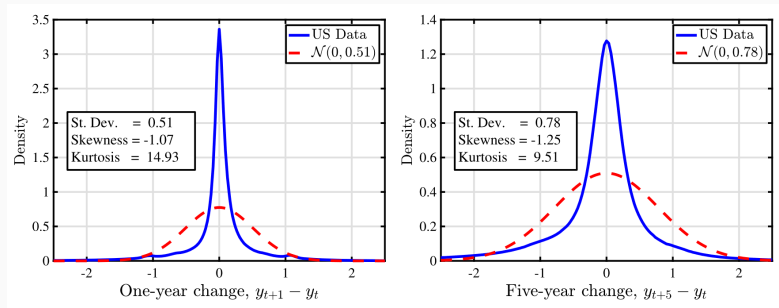
2. Life-cycle (III)

- **Natural experiment:** Wealth depletion after sudden inheritance
- **Results:**
 1. Life-cycle profile of wealth fitted for many levels of risk-aversion (by varying the discount factor)
 2. Fast wealth depletion requires high risk-aversion (or high perceived risk)



3. More realistic income risk (I)

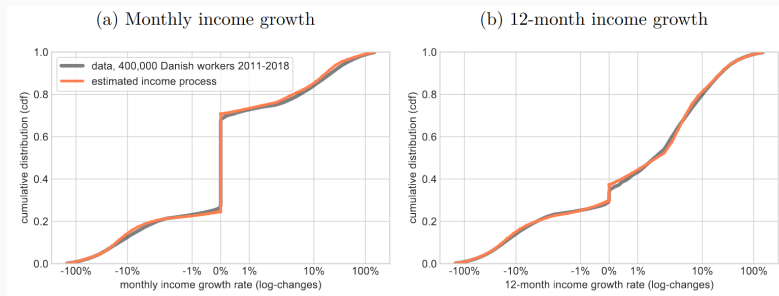
Annual earnings-changes are far from log-normal:



Source: Guvenen et. al. (2021)

3. More realistic income risk (II)

Many with zero-growth month-month:



Source: Druedahl et. al. (2021)

4. Epstein-Zin

$$\begin{aligned}v_t(z_t, m_t) &= \max_{c_t} \left[(1 - \beta) \cdot c_t^{1-\sigma} + \beta \cdot w_{t+1}^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \\ \text{s.t.} \quad w_{t+1} &\equiv \mathbb{E}_t \left[v_{t+1}(z_{t+1}, m_{t+1})^{1-\rho} \right]^{\frac{1}{1-\rho}} \\ m_{t+1} &= (1 + r)(m_t - c_t) + y_{t+1}\end{aligned}$$

- **Preferences:**

1. Patience: β
2. Intertemporal substitution: σ
3. Risk-aversion: ρ

- **Euler-equation:**
$$v_t = \left[\beta R \cdot \mathbb{E}_t \left[c_{t+1}^{-\sigma} \cdot \left(\frac{w_{t+1}}{v_{t+1}} \right)^{\rho-\sigma} \right] \right]^{-\frac{1}{\sigma}}$$

1. FOC: $0 = v_t^\sigma \cdot \left[(1 - \beta) \cdot c_t^{-\sigma} - \beta R \cdot w_{t+1}^{\rho-\sigma} \cdot \mathbb{E}_t \left[v_{t+1}^{-\rho} \cdot \frac{\partial v_{t+1}}{\partial m_{t+1}} \right] \right]$
2. Envelope condition: $\frac{\partial v_t(z_t, m_t)}{\partial m_t} = v_t^\sigma \cdot (1 - \beta) \cdot c_t^{-\sigma}$

5. Deep learning

- **Curse of dimensionality:**
 1. Many states
 2. Many choices
 3. Many shocks
- **Deep (reinforcement) learning:**
 1. Approximate value and policy functions with *neural networks*
 2. Approximate on simulation sample rather than on grid
 3. Various trick to get stability
(replay buffers, mini-batches, target networks)
 4. Automatic differentiation (backpropagation) and GPUs for speed
- **Examples:** Maliar and Maliar (2021) and Azinovic and Scheidegger (2022)
- **Work-in-progress:** Druedahl and Røpke (2024)

Summary

Summary and what's next

- **This lecture:**

1. Introduction to course
2. Consumption-saving models
3. Basic numerical dynamic programming

- **Next:** *Stationary equilibrium*

- **You should:**

1. Study today's code
2. Glance at Aiyagari (1994),
»Uninsured Idiosyncratic Risk and Aggregate Saving«