



# Consumption-Saving

## Mini-Course: Heterogenous Agent Macro

---

Jeppe Druedahl  
2025



# Introduction

---

- **Generations of models:**
  1. **Permanent income hypothesis (PIH)** (Friedman, 1957)  
or life-cycle model (Modigliani and Brumberg, 1954)
  2. **Buffer-stock consumption model**  
Deaton (1991, 1992); Carroll (1992, 1997, 2019)
  3. **Multiple-asset buffer-stock consumption models**  
e.g. Kaplan and Violante (2014); Harmenberg and Öberg (2021)
- **Consumption-and-saving over the life-cycle dynamic**  
e.g. Gourinchas and Parker (2002); Druedahl and Martinello (2022)
- **Empirical MPCs and income risk**  
e.g. Fagereng et. al. (2021); Guvenen et. al. (2021)

**Book:** **The Economics of Consumption**, Jappelli and Pistaferri (2017)

1. Introduction
2. PIH
3. Buffer-stock
4. 3-periods
5. EGM
6. NEGM
7. Extra
8. Portfolio choice
9. Summary

**PIH**



$$v_0 = \max_{\{c_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t)$$

s.t.

$$a_t = (1 + r)a_{t-1} + wz_t - c_t$$

$$a_{T-1} \geq 0$$

- **Variables:**

Consumption:  $c_t$

Productivity:  $z_t$

End-of-period savings:  $a_t$  (*no debt at death*)

- **Parameters:**

Discount factor:  $\beta$

Wage:  $w$

Interest rate:  $r$  (define  $R \equiv 1 + r$  as interest factor)

# It is a *static* problem

$$v_0 = \max_{\{c_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t)$$

s.t.

$$a_t = (1 + r)a_{t-1} + wz_t - c_t$$

$$a_{T-1} \geq 0$$

- It is a *static* problem:

1. **Information:**  $z_t$  is known for all  $t$  at  $t = 0$
2. **Target:** Discounted utility,  $\sum_{t=0}^{T-1} \beta^t u(c_t)$
3. **Behavior:** Choose  $c_0, c_1, \dots, c_{T-1}$  *simultaneously*
4. **Solution:** Sequence of consumption *choices*  $c_0^*, c_1^*, \dots, c_{T-1}^*$

- **Substitution** implies *Intertemporal Budget Constraint* (IBC)

$$\begin{aligned}
 a_{T-1} &= Ra_{T-2} + wz_{T-1} - c_{T-1} \\
 &= R^2 a_{T-3} + R wz_{T-2} - Rc_{T-2} + wz_{T-1} - c_{T-1} \\
 &= R^T a_{-1} + \sum_{t=0}^{T-1} R^{T-1-t} (wz_t - c_t)
 \end{aligned}$$

- Use **terminal condition**  $a_{T-1} = 0$  (equality due utility max.)

$$R^{-(T-1)} a_{T-1} = 0 \Leftrightarrow s_0 + h_0 - \sum_{t=0}^{T-1} R^{-t} c_t = 0$$

where  $s_0 \equiv Ra_{-1}$  (after-interest assets)  
 and  $h_0 \equiv \sum_{t=0}^{T-1} R^{-t} wz_t$  (human capital)



$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t u(c_t) + \lambda \left[ \sum_{t=0}^{T-1} R^{-t} c_t - s_0 - h_0 \right]$$

- **First order conditions:**

$$\forall t : 0 = \beta^t u'(c_t) - \lambda(1+r)^{-t} \Leftrightarrow u'(c_t) = -\lambda(\beta R)^{-t}$$

- **Euler-equation** for  $k \in \{1, 2, \dots\}$ :

$$\frac{u'(c_t)}{u'(c_{t+k})} = \frac{-\lambda(\beta R)^{-t}}{-\lambda(\beta R)^{-(t+k)}} = (\beta R)^k$$

# Consumption choice

- **CRRA:**  $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$  imply Euler-equation

$$\frac{c_0^{-\sigma}}{c_t^{-\sigma}} = (\beta R)^t \Leftrightarrow c_t = (\beta R)^{\frac{t}{\sigma}} c_0$$

- Insert **Euler** into **IBC** to get consumption choice

$$\sum_{t=0}^{T-1} \left( (\beta R)^{1/\sigma} R^{-1} \right)^t c_0 = s_0 + h_0 \Leftrightarrow$$
$$c_0^* = \frac{1 - (\beta R)^{1/\sigma} R^{-1}}{1 - ((\beta R)^{1/\sigma} R^{-1})^T} (s_0 + h_0)$$

# Infinite horizon

- **Infinite horizon** for  $(\beta R)^{1/\sigma} R^{-1} < 1$ : Let  $T \rightarrow \infty$  to get

$$c_0^* = \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right) (s_0 + h_0)$$

$$\text{if } \forall z_t = 1 : c_0^* = \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right) \left(Ra_{-1} + \frac{R}{R-1}w\right)$$

- **Consume annuity value:**  $\beta R = 1, z_t = 1 \Rightarrow c_0^* = ra_{-1} + w$
- **Intertemporal elasticity of substitution** ( $\text{IES} = \frac{1}{\sigma}$ ):

$$\log c_{t+1} - \log c_t = \frac{1}{\sigma} \log \beta R$$

Constant consumption if:

1.  $\beta R = 1$
2.  $\sigma \rightarrow \infty$  (zero elasticity of substitution)

## Propensities to consume ( $\beta R \approx 1, z_t \approx 1$ )

$$c_0^* \approx \frac{r}{1+r} \left( (1+r)a_{-1} + \sum_{t=0}^{\infty} \frac{wz_t}{(1+r)^t} \right) \approx ra_{-1} + w$$

### Different types of shocks:

1. MPC of *windfall* income:  $\frac{\partial c_0}{\partial s_0} \approx \frac{r}{1+r}$
2. MPC of *future* income change:  $\frac{\partial c_0}{\partial wz_t} \approx \frac{r}{1+r} (1+r)^{-t}$
3. MPC of *permanent* income change:  $\frac{\partial c_0}{\partial w} \approx \frac{r}{1+r} \frac{1}{1-(1+r)^{-1}} = 1$

**Dynamic affects:** The same when  $\beta R = 1$ , for all  $k > 0$

$$\begin{aligned} \frac{\partial c_k}{\partial s_0} &= \frac{\partial c_0}{\partial s_0} \\ \frac{\partial c_k}{\partial wz_t} &= \frac{\partial c_0}{\partial wz_t} \\ \frac{\partial c_k}{\partial w} &= \frac{\partial c_0}{\partial w} \end{aligned}$$

# Savings (with $z_t = 1$ )

- **Constant savings** with  $\beta R = 1$

$$c_t = ra_{t-1} + w \Rightarrow a_t = Ra_{t-1} + w - c_t = a_{t-1}$$

- **Decreasing savings** with  $\beta R < 1$ :  $c_t \uparrow \Rightarrow a_t < a_{t-1}$
- **Increasing savings** with  $\beta R > 1$ :  $c_t \downarrow \Rightarrow a_t > a_{t-1}$

# Initial liquidity/borrowing constraint

- Implied period 0 **savings** are:

$$a_0 = Ra_{-1} + wz_0 - c_0$$

- Hard **borrowing constraint**:  $a_0 \geq -wb$
- **Maximum consumption**:  $\bar{c}_0 = Ra_{-1} + wz_0 + wb$
- **Optimal consumption**: Constrained or unconstrained.

$$c_0^* = \min \left\{ \bar{c}_0, \left( 1 - \frac{(\beta R)^{1/\sigma}}{R} \right) (s_0 + h_0) \right\}$$

- **Empirical realism**. Incl. high MPC of constrained.
- **Technical issue**: *Borrowing constraints further in the future complicates the analytical solution considerably.*

# Empirical MPCs

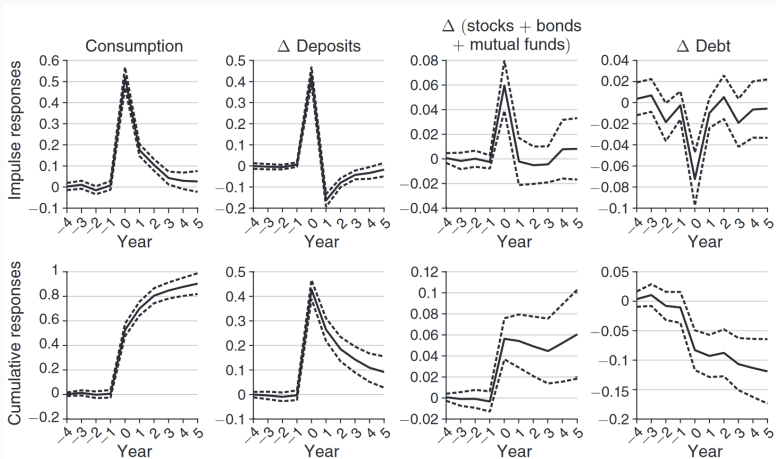


FIGURE 2. DYNAMIC HOUSEHOLD RESPONSES TO LOTTERY PRIZES

**Source:** Fagereng et. al. (2021)

# Buffer-stock

---



# Uncertainty and always borrowing constraint

$$v_0(z_0, a_{-1}) = \max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

s.t.

$$a_t = (1 + r)a_{t-1} + wz_t - c_t$$

$$z_{t+1} \sim \mathcal{Z}(z_t)$$

$$a_t \geq -wb$$

$$\lim_{t \rightarrow \infty} (1 + r)^{-t} a_t \geq 0 \quad [\text{No-Ponzi game}]$$

- **Stochastic income** from 1st order Markov-process,  $\mathcal{Z}$
- **A true dynamic problem:**
  1. **Information:**  $z_t$  is revealed period-by-period
  2. **Target:** Expected discounted utility,  $\mathbb{E}_0 \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right]$
  3. **Behavior:** Choose  $c_t$  *sequentially* as information is revealed
  4. **Solution:** Sequence of consumption *functions*,  $c_t^*(z_t, a_{t-1})$

- **Substitution** still implies:

$$R^{-(T-1)}a_{T-1} = 0 \Leftrightarrow s_0 + h_0 - \sum_{t=0}^{T-1} R^{-t}c_t = 0$$

- **What if  $T \rightarrow \infty$ ?** We must have  $\lim_{T \rightarrow \infty} R^{-(T-1)}a_{T-1} = 0$ 
  1.  $\lim_{T \rightarrow \infty} R^{-(T-1)}a_{T-1} > 0$ : Consumption can be increased
  2.  $\lim_{T \rightarrow \infty} R^{-(T-1)}a_{T-1} < 0$ : Violates No-Ponzi game condition
- For  $T \rightarrow \infty$  we have the **IBC**:

$$\sum_{t=0}^{\infty} R^{-t}c_t = Ra_{-1} + \sum_{t=0}^{\infty} R^{-t}wz_t$$

# Natural borrowing limit

- Denote **minimum possible productivity** by  $\underline{z}$
- **Consumption must be non-negative**  $\Rightarrow$   
*interest payments must be less than minimum income*

$$c_t \geq 0 \Rightarrow r(-a_t) \leq w\underline{z} \Leftrightarrow a_t \geq -\frac{w\underline{z}}{r}$$

If debt was larger it would in the worst case ( $\forall z_t = \underline{z}$ ) grow without bound even with zero consumption ( $\forall c_t = 0$ )

$$a_0 = -\frac{w\underline{z}}{r} - \Delta$$

$$a_1 = (1+r)a_0 + w\underline{z} = a_0 - (1+r)\Delta$$

$$a_2 = (1+r)a_1 + w\underline{z} = a_0 - (1+r)^2\Delta$$

$$\vdots$$

- **Natural borrowing constraint:**  $a_t \geq \underline{a} = -w \min \left\{ b, \frac{\underline{z}}{r} \right\}$

# Euler-equation from variation argument

- **Case I:** If  $u'(c_t) > \beta R \mathbb{E}_t [u'(c_{t+1})]$ :  
Increase  $c_t$  by marginal  $\Delta > 0$ , and lower  $c_{t+1}$  by  $R\Delta$ 
  1. **Feasible:** Yes, if  $a_t > \underline{a}$
  2. **Utility change:**  $u'(c_t) + \beta (-R) \mathbb{E}_t [u'(c_{t+1})] > 0$
- **Case II:** If  $u'(c_t) < \beta R \mathbb{E}_t [u'(c_{t+1})]$ :  
Lower  $c_t$  by marginal  $\Delta > 0$ , and increase  $c_{t+1}$  by  $R\Delta$ 
  1. **Feasible:** Yes (always)
  2. **Utility change:**  $u'(c_t) + \beta R \mathbb{E}_t [u'(c_{t+1})] > 0$
- **Conclusion:** By contradiction
  1. **Constrained:**  $a_t = \underline{a}$  and  $u'(c_t) \geq \beta R \mathbb{E}_t [u'(c_{t+1})]$ , or
  2. **Unconstrained:**  $a_t > \underline{a}$  and  $u'(c_t) = \beta R \mathbb{E}_t [u'(c_{t+1})]$
- **Sufficiency:** From concavity of value function  
FOC:  $c_t^{-\sigma} = \beta \mathbb{E}_t [v_a(z_{t+1}, a_t)]$   
Envelope:  $v_a(z_t, a_{t-1}) = (1 + r)c_t^{-\sigma}$

## Special case I: Quadratic utility

- **Quadratic utility:**  $u(c_t) = -\frac{1}{2}(\bar{c} - c)^2$  with  $\beta R = 1$  and »large«  $\bar{c}$
- **Euler-equation:** *Consumption = expected future consumption*

$$(\bar{c} - c_t) = \mathbb{E}_t [(\bar{c} - c_{t+k})] \Leftrightarrow c_t = \mathbb{E}_t [c_{t+k}]$$

- Use **IBC** in expectation to get **consumption function**:

$$\sum_{t=0}^{\infty} R^{-t} \mathbb{E}_0 [c_t] = Ra_{-1} + \sum_{t=0}^{\infty} R^{-t} w \mathbb{E}_0 [z_t] \Rightarrow$$
$$c^*(z_t, a_{t-1}) = c_0 = ra_{-1} + \frac{r}{R} \sum_{t=0}^T R^{-t} w \mathbb{E}_0 [z_t]$$

where we formally disregard the borrowing constraint

- **Certainty equivalence:** *Only expected income matter.*

## Special case II: CARA utility

- **CARA utility:**  $u(c_t) = -\frac{1}{\alpha} e^{-\alpha c}$
- **Productivity is absolute random walk:**

$$z_t = z_{t-1} + \psi_t$$

$$\psi_t \sim \mathcal{N}(0, \sigma_\psi^2)$$

- **Consumption function (see proof):**

$$c^*(a_{t-1}, z_t) = ra_{t-1} + wz_t - \frac{\log(\beta R)^{\frac{1}{\alpha}} + \alpha \frac{\sigma_\psi^2}{2}}{r^2}$$

where we formally disregard the borrowing constraint

- **Precautionary saving:**  $\sigma_\psi^2 \uparrow$  implies  $c_t^* \downarrow$  for given  $z_t$  and  $a_{t-1}$   
 $\Rightarrow$  *accumulation of buffer-stock*

# Dynamic solution: Bellman's Principle of Optimality

- **Origin:** Bellman, 1957, Chap. III.3.
- **Value function,  $v_t$ :** Defined *recursively* from

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1 + r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

with  $v_T(\bullet) = 0$ .

- **Policy function,  $c_t^*$ :** Is the same as

$$c_t^*(z_t, a_{t-1}) = \arg \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1 + r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1 + r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

1. **State variables:**  $z_t$  and  $a_{t-1}$
2. **Control (choice) variable:**  $c_t$
3. **Continuation value:**  $\beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$
4. **Parameters:**  $r$ ,  $w$ , and stuff in  $u(\bullet)$

**Note:** Straightforward to extend to more goods, more assets or other states, more complex uncertainty, bounded rationality etc.



## Infinite horizon: $T \rightarrow \infty$ ?

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1 + r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

- **Contraction mapping result:** *If  $\beta$  is low enough (strong enough impatience) then the value and policy functions converge to  $v(z_t, a_{t-1})$  and  $c^*(z_t, a_{t-1})$  for large enough  $T$*
- **In practice:**
  1. Make arbitrary initial guess (e.g.  $v_{t+1} = 0$ )
  2. Solve backwards until value and policy functions does not change anymore (given some tolerance)

**3-periods**

---

### 3-period model

- **Expected discounted utility:**  $v(z_0, a_{-1}) = \mathbb{E}_0 \sum_{t=0}^2 \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$
- **Income = wage  $\times$  productivity + transfer:**

$$y_t = w z_t + \chi_t$$

- **Cash-on-hand, savings and borrowing constraint:**

$$m_t = (1 + r)a_{t-1} + y_t$$

$$a_t = m_t - c_t$$

$$a_t \geq \underline{a}$$

- **Stochastic transition:**  $\Pr[z_{t+1}|z_t] = \pi_t(z_t, z_{t+1})$  such that

$$\Pr[z_{t+1} = 1 | z_t = 1] = \pi$$

$$\Pr[z_{t+1} \in \{1 - \Delta, 1 + \Delta\} | z_t = 1] = \frac{1 - \pi}{2}$$

$$\Pr[z_{t+1} = z_t | z_t \in \{1 - \Delta, 1 + \Delta\}] = 1$$

# Bellman equation

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t [v_{t+1}(z_{t+1}, a_t)]$$

s.t.

$$y_t = wz_t + \chi_t$$

$$m_t = (1+r)a_{t-1} + y_t$$

$$a_t = m_t - c_t$$

$$\Pr[z_{t+1}|z_t] = \pi_t(z_t, z_{t+1})$$

$$a_t \geq \underline{a}$$

where

$$v_3(z_3, a_2) = 0$$

- **Discretization:** All state variables belong to discrete sets  $\equiv$  *grids*,

$$z_t \in \mathcal{G}_z = \{z^0, z^1, \dots, z^{\#z-1}\}$$

$$a_t \in \mathcal{G}_a = \{a^0, a^1, \dots, a^{\#a-1}\}$$

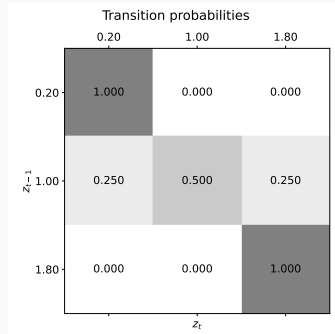
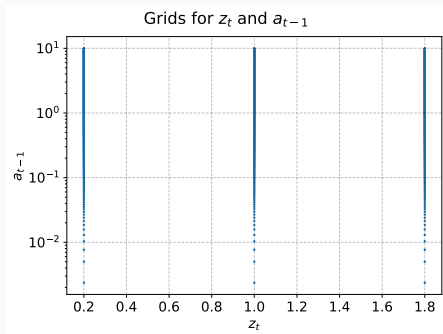
$$a^0 = \underline{a}$$

- **Expectation:** Numerical integration by

$$\mathbb{E}_t [v_{t+1}(z_{t+1}, a_t)] = \sum_{z_{t+1} \in \{1-\Delta, 1, 1+\Delta\}} \pi_t(z_t, z_{t+1}) v_{t+1}(z_{t+1}, a_t)$$

- **ConSav:** `grids.nonlinspace`, `grids.equilogspace`
- **ConSavNotebook:** 04. Tools/03. Grids.ipynb

# Grids and transition probabilities



# Linear interpolation

- **Linear interpolation** (function approximation):

1. Assume  $v_{t+1}$  is known on  $\mathcal{G}_z \times \mathcal{G}_a$  (tensor product)
2. Evaluate  $v_{t+1}(z^{i_z}, a)$  for arbitrary  $a$  by

$$\begin{aligned}\check{v}_{t+1}(z^{i_z}, a) &= \text{baseline} + \text{slope} \times \text{distance} \\ &= v_{t+1}(z^{i_z}, a^{\iota}) + \omega(a - a^{\iota})\end{aligned}$$

where

$$\omega \equiv \frac{v_{t+1}(z^{i_z}, a^{\iota+1}) - v_{t+1}(z^{i_z}, a^{\iota})}{a^{\iota+1} - a^{\iota}}$$

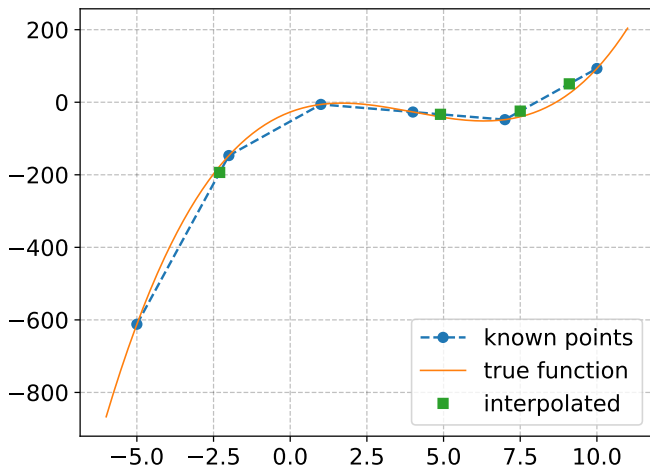
$$\iota \equiv \text{largest } i_a \in \{0, 1, \dots, \#_a - 2\} \text{ such that } a^{i_a} \leq a$$

- **ConSav:** `linear_interp.interp1d`

- **ConSavNotebook:**

04. Tools/01. Linear interpolation.ipynb

# Linear interpolation





# Value function iteration (VFI)

- **Maximize value-of-choice:**

$$v_t(z^{i_z}, a^{i_a}) = \max_{c_t} v_t(z^{i_z}, a^{i_a} | c_t)$$

$$\text{with } c_t \in [0, (1+r)a^{i_a} + wz^{i_z} + \underline{a}]$$

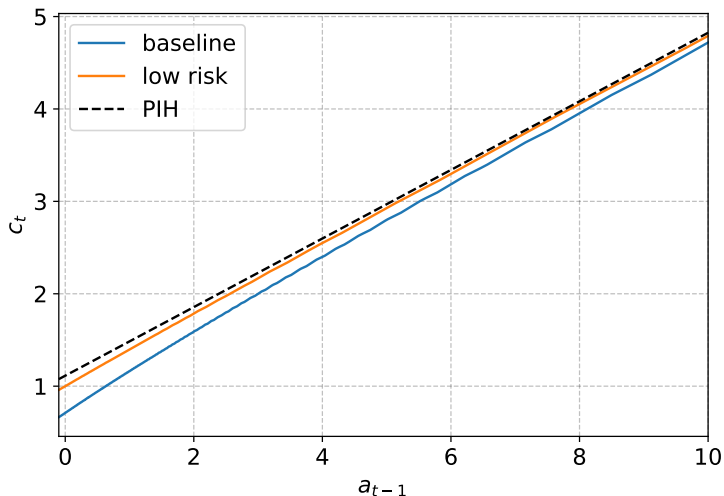
$$v_t(z^{i_z}, a^{i_a} | c_t) = u(c_t) + \sum_{i_{z+1}=0}^{\#_z-1} \pi(i_z, i_{z+1}) \check{v}_{t+1}(z^{i_z}, a)$$

$$\text{with } a_t = (1+r)a^{i_a} + wz^{i_z} - c_t$$

- **Inner loop:** For each grid point in  $\mathcal{G}_z \times \mathcal{G}_a$  find  $c_t^*(z_t, a_{t-1})$  and therefore  $v_t(z_t, a_{t-1})$  with a *numerical optimizer*
- **Outer loop:** Backwards from  $t = T - 1$  (note  $\underline{v}_T = 0$ , or known)
- **ConSav+QuantEcon:** Various optimizers in numba
- **ConSavNotebook:** 04. Tools/02. Optimization.ipynb

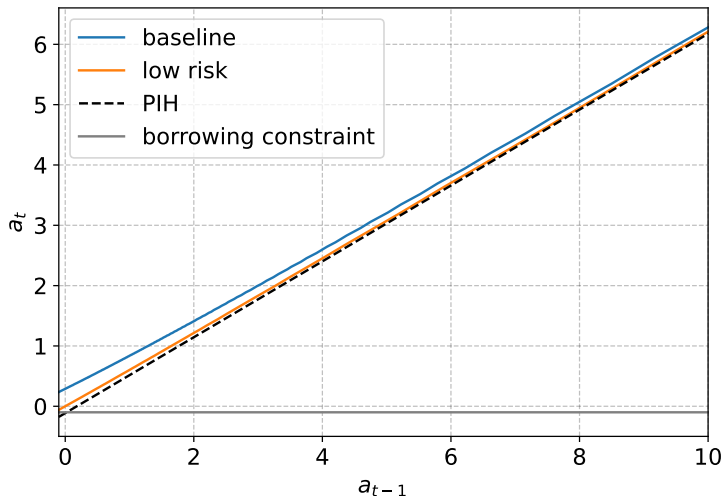
# Consumption function

consumption function in  $t = 0$



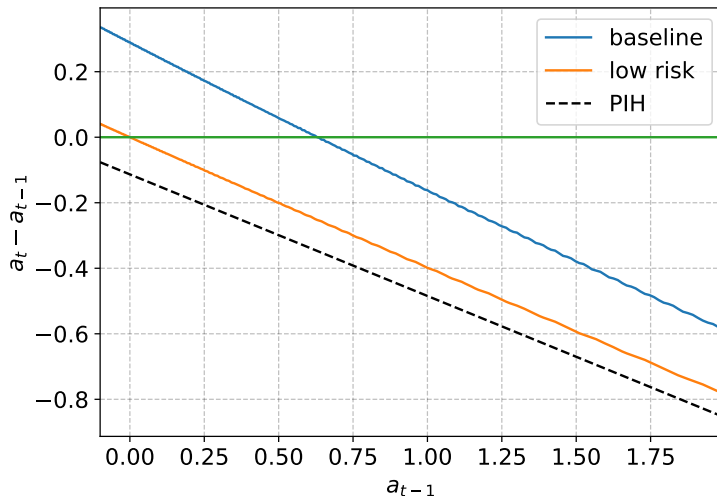
# Savings function

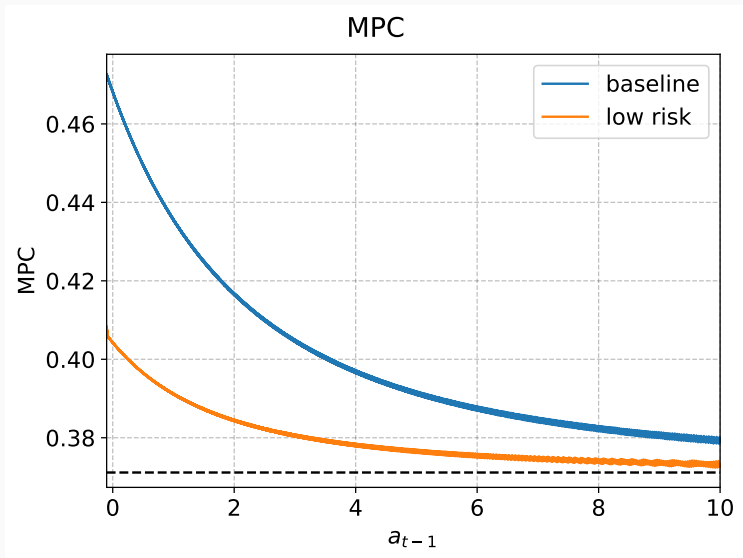
savings function in  $t = 0$



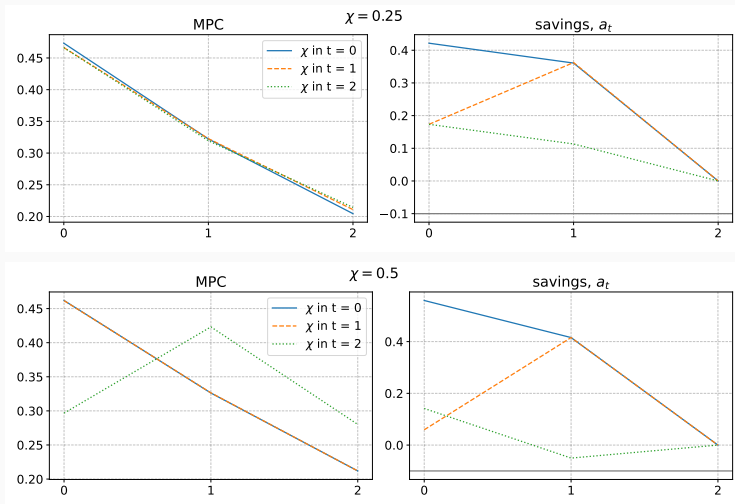
# Change in savings function

savings diff. function in  $t = 0$





# Intertemporal MPC



- **Notebook:** 01. ConSavModel.ipynb
- **Consumption lower than under PIH and concave in assets**

**Intuition:** *Precautionary saving motive is relatively larger for asset poor households because income risk is the same for everybody*

**Implications:**

1. Windfall gives safety and increases average consumption  
⇒ MPC decreasing in assets
2. Attraction towards a buffer-stock target  $a_t = a_{t-1}$  despite  $\beta R < 1$
3. Larger effective discounting of future income  
(extreme: no effect of future income changes if constrained before)

# Numerical Monte Carlo simulation

- **Initial distribution:** Draw  $z_{i,-1}$  and  $a_{i,-1}$  for  $i \in \{0, 1, \dots, N - 1\}$
- **Simulation:** Forwards in time from  $t = 0$  and in each time period
  1. Draw  $z_{it}$  given transition probabilities
  2. Use linear interpolation to evaluate

$$c_{it} = \check{c}_t^*(z_{it}, a_{it-1})$$

$$a_{it} = (1 + r)a_{it-1} + wz_{it} - c_{it}$$

- **Review:**
  - **Pro:** Simple to implement
  - **Con:** Computationally costly and introduces randomness
- **Infinite horizon:**
  1. Assume  $z_{it}$  has an ergodic distribution
  2. Ergodic distribution of  $a_{it}$  around buffer-stock target



**EGM**



# Time iteration

- **Replace numerical optimization with root-finding**
- **Time iteration:** For each  $a_{t-1}$  and  $z_t$  find  $c_t$  to solve the Euler-equation

$$c_t^{-\sigma} = \beta(1+r)\mathbb{E}_t[c_{t+1}^{-\sigma}]$$

Note: *Necessary and sufficient* (for interior choices, else  $a_t = \underline{a}$ )

# Endogenous grid-point method (EGM)

1. Calculate **post-decision marginal value of cash**:

$$q(z^{i_z}, a^{i_a}) = \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z, i_{z+}} c_+^*(z^{i_{z+}}, a^{i_a})^{-\sigma}$$

2. **Invert Euler-equation**:

$$c(z^{i_z}, a^{i_a}) = (\beta(1+r)q(z^{i_z}, a^{i_a}))^{-\frac{1}{\sigma}}$$

3. **Endogenous cash-on-hand**:

$$m(z^{i_z}, a^{i_a}) = a^{i_a} + c(z^{i_z}, a^{i_a})$$

4. **Consumption function**: Calculate  $m = (1+r)a^{i_{a-}} + wz^{i_z}$

- 4.1 Binding constraint: If  $m \leq m(z^{i_z}, a^0)$  then

$$c^*(z^{i_z}, a^{i_{a-}}) = m + \underline{a}$$

- 4.2 Interior choice: Else

$$c^*(z^{i_z}, a^{i_{a-}}) = \text{interpolate } m(z^{i_z}, m) \rightarrow c(z^{i_z}, m)$$

**NEGM**



# An illiquid asset

- **Illiquid asset:** Worth  $k$  in period  $T$ , else  $(1 - \gamma)k$  for  $\gamma \in [0, 1]$

Note: *You can sell, but never buy*

- **Recursive problem:** For  $\text{own}_{t-1} \in \{0, 1\}$  and  $y_t = 1$

$$v_t(\text{own}_{t-1}, a_{t-1}) = \max_{c_t, \text{sell}_t \in \{0, 1\}} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t[v_{t+1}(\text{own}_t, a_t)]$$

$$\text{s.t. } \text{own}_t = (1 - \text{sell}_t)\text{own}_{t-1}$$

$$m_t = a_{t-1} + y_t + \text{sell}_t \text{own}_{t-1}(1 - \gamma)k$$

$$a_t = m_t - c_t$$

$$a_t \geq 0$$

- **Terminal period:**  $v_T(\text{own}_{T-1}, a_{T-1}) = \frac{(a_{T-1} + y_T + \text{own}_{T-1}k)^{1-\sigma}}{1-\sigma}$
- **Euler-equation:** Fix end-of-period ownership,  $\text{own}_t$ 
  1. Necessary: From variational argument conditional on  $\text{sell}_{t+1}$
  2. Non-sufficient: Savings low with  $\text{sell}_{t+1} = 1$ , or high with  $\text{sell}_{t+1} = 0$

# Timing and nesting

- **Beginning-of-period states:**  $\text{own}_{t-1}$  and  $m_t$
- **Discrete choice:** If  $\text{own}_{t-1} = 1$  choose  $\text{sell}_t \in \{0, 1\}$

$$\underline{v}_t(m_t) = \max \{v_t(0, m_t^{\text{sell}}), v_t(1, m_t)\}$$
$$m_t^{\text{sell}} = m_t + (1 - \gamma)k$$

- **Continuous choice:**

$$v_t(\text{own}_t, m_t) = \max_{c_t \in [0, m_t]} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta w_t(\text{own}_t, m_t - c_t)$$

- **Post-decision value function:**

$$w_t(\text{own}_t, a_t) = \mathbb{E}_t \left[ \begin{cases} v_{t+1}(\text{own}_t, m_{t+1}) & \text{if } \text{own}_t = 0 \text{ or } t = T - 1 \\ \underline{v}_{t+1}(m_{t+1}) & \text{if } \text{own}_t = 1 \text{ and } t < T - 1 \end{cases} \right]$$

$$m_{t+1} = a_t + y_{t+1}$$

- **Post-decision value function:**

$$w_t(\text{own}_t, a_t) = \mathbb{E}_t \left[ \begin{cases} v_{t+1}(\text{own}_t, m_{t+1}) & \text{if } \text{own}_t = 0 \text{ or } t = T - 1 \\ \bar{v}_{t+1}(m_{t+1}) & \text{if } \text{own}_t = 1 \text{ and } t < T - 1 \end{cases} \right]$$

$$m_{t+1} = a_t + y_{t+1}$$

$$\text{where } v_T(\text{own}_{T-1}, m_T) = \frac{(m_T + \text{own}_{T-1}k)^{1-\sigma}}{1-\sigma}$$

- **Re-written Bellman:**

$$v_t(\text{own}_t, m_t) = \max_{c_t \in [0, m_t]} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta w_t(\text{own}_t, m_t - c_t)$$

# Post-decision marginal value of cash

- Post-decision marginal value of cash:

$$q_t(\text{own}_t, a_t) = \mathbb{E}_t \left[ \begin{cases} (m_T + \text{own}_{T-1}k)^{-\sigma} & \text{if } t = T - 1 \\ c_{t+1}^*(0, m_{t+1}^{\text{sell}})^{-\sigma} & \text{else if } \text{sell}_{t+1} = 1 \\ c_{t+1}^*(1, m_{t+1})^{-\sigma} & \text{else} \end{cases} \right]$$

$$m_{t+1} = a_t + y_{t+1}$$

$$m_{t+1}^{\text{sell}} = m_{t+1} + (1 - \gamma)k$$

$$\text{sell}_{t+1} = \begin{cases} 1 & \text{if } v_{t+1}(0, m_{t+1}^{\text{sell}}) > v_{t+1}(1, m_{t+1}) \\ 0 & \text{else} \end{cases}$$

- Euler-equation:

$$u'(c_t) = \beta q_t \Leftrightarrow c_t^{-\sigma} = \beta q_t$$



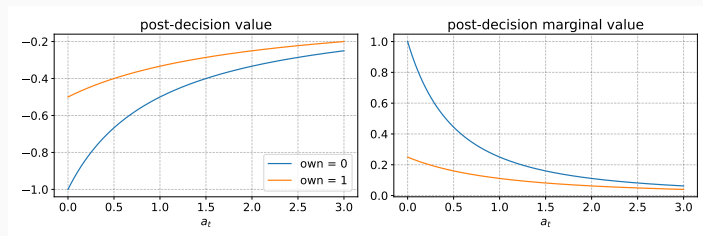
# Solution method

For  $t \in \{T-1, T-2, \dots, 0\}$ :

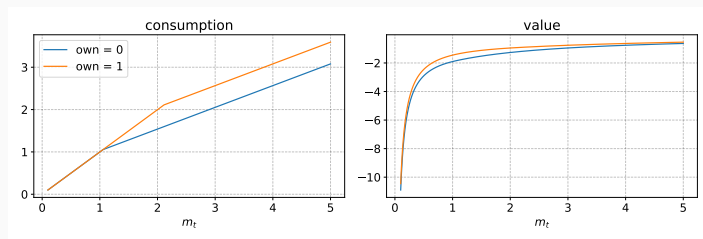
1. Calculate  $q_t(\text{own}_t, a_t)$  and  $w_t(\text{own}_t, a_t)$
2. Use inverted Euler-equation to get  $c_t(\text{own}_t, a_t)$  and  $m_t(\text{own}_t, a_t)$
3. Use *upper envelope* (see below) to get  $c_t(\text{own}_t, m_t)$   
on common grid for  $m_t$
4. Calculate  $v_t(\text{own}_t, m_t)$  and then  $\underline{v}_t(m_t)$

# Conditional consumption function in $t = T - 1$

**Step 1.** Use formulas for  $v_T(\bullet)$  and  $q_{T-1}(\bullet)$

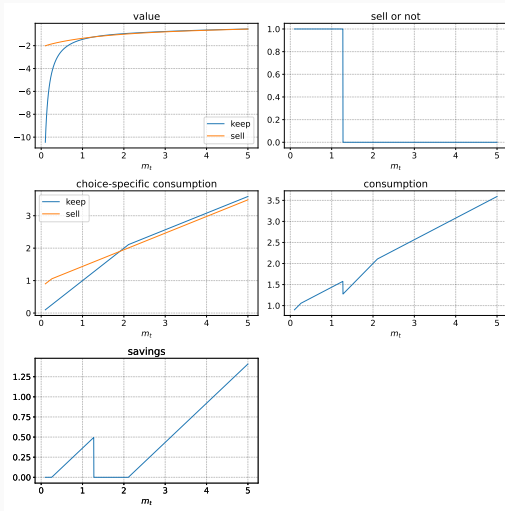


**Step 2+3:** Standard EGM (no upper envelope)



# Unconditional owner behavior, $t = T - 1$

**Step 4:** Use  $\underline{v}_t(m_t) = \max \{v_t(0, m_t^{\text{sell}}), v_t(1, m_t)\}$



## Candidate points (given $\text{own}_{it}$ )

**Closer look at step 3:** The generated candidate points from inverting the Euler-equation is

$$w^{i_a} = w_t(\text{own}_t, a^{i_a})$$

$$q^{i_a} = q_t(\text{own}_t, a^{i_a})$$

$$c^{i_a} = u'^{-1}(\beta q^{i_a}) = (\beta q^{i_a})^{-\frac{1}{\sigma}}$$

$$m^{i_a} = a^{i_a} + c^{i_a}$$

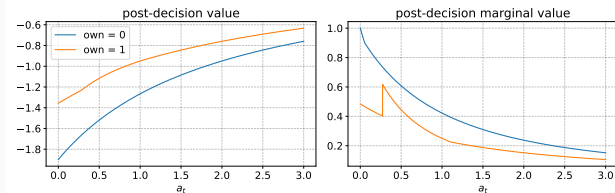
**Problem:** Small increase in  $a_t$  when  $\text{sell}_{t+1}$  goes from 1 to 0

1. Downward jump  $c_{t+1}$
2. Upward jump in  $q_t$
3. Downward jump in  $c_t$
4. Downward jump in  $m_t$

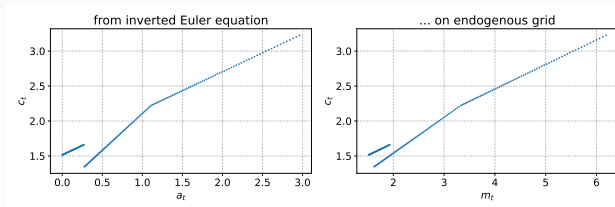
$\Rightarrow$  the  $m_t$ 's will be overlapping - which one to choose?

# Problems with EGM

## Step 1 in $t = T - 2$ :



## Step 2 in $t = T - 2$ for owners:



**Reason:** Non-sufficient Euler-equation!

# Upper envelope (given $\text{own}_t$ )

**Upper-envelope:**  $\forall i_m, c^*(m^{i_m}) = c^{i_m, j^*}$

$$j^* = \arg \max_{j \in \{0, 1, \dots, \#_a - 2\}} u(c^{i_m, j}) + w^{i_m, j}$$

s.t.

$$\text{potential segment: } m^{i_m} \in \begin{cases} [m^j, m^{j+1}] & \text{if } j < \#_a - 2 \\ [m^j, \infty] & \text{if } j = \#_a - 2 \end{cases}$$

$$\text{interpolation + constraint } c^{i_m, j} = \min \left\{ c^j + \frac{c^{j+1} - c^j}{m^{j+1} - m^j} (m^{i_m} - m^j), m^{i_m} \right\}$$

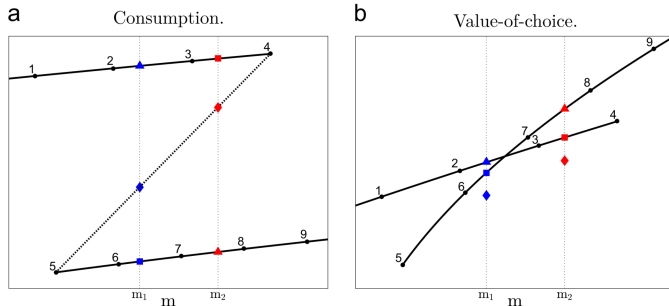
$$\text{continuation value: } w^{i_m, j} = \text{interp } \{a^{i_a}\} \rightarrow \{w^{i_a}\} \text{ at } a^{i_m, j}$$

$$a^{i_m, j} = m^{i_m} - c^{i_m, j}$$

**ConSav:** upperenvelope

**ConSavNotebook:** 04. Tools/06. Upper envelope.ipynb

# Illustration

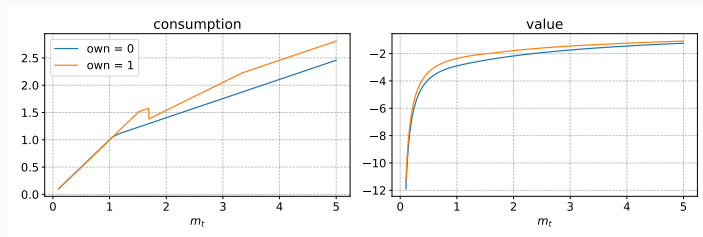


1. **Numbering:** Different levels of end-of-period assets,  $a^{i_a}$
2. **Problem:** Find the consumption function at  $m_1$  and  $m_2$
3. **Largest value-of-choice:** Denoted by the *triangles*

**Source:** Druedahl and Jørgensen (2017),  $G^2EGM$   
Drueahl (2021),  $NEGM$

# Conditional consumption function in $t = T - 1$

## Step 3: After applying upper envelope





**Notebook:** 02. Illiquid.ipynb

## 1. **Simultaneous high total wealth and high MPC**

- 1.1 Poor hands-to-mouth households
- 1.2 Wealthy hands-to-mouth households

## 2. **The MPC is strongly size-dependent**

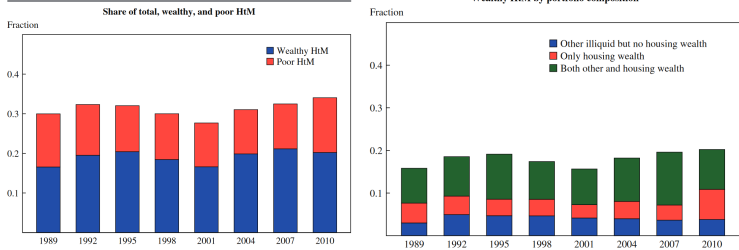
## 3. **Precautionary savings:**

- 3.1 Frequent shocks: Liquid assets important
- 3.2 Infrequent shocks: Illiquid assets enough

(see Larkin (2024))

# Empirical evidence for hands-to-mouth households

Figure 3. Fraction of HtM Households, United States, 1989–2010



Poor HtM: Low liquid net worth, low total net worth

Wealthy HtM: Low liquid net worth, high total net worth

Source: Kaplan et. al. (2014)

## Extra: Adding smoothing

- **Taste shocks:** Following Iskhakov et. al., 2017)

$$\bar{v}_t(m_t) = \max \{ v_t(0, m_t^{\text{sell}}) + \sigma_\varepsilon \varepsilon(0), v_t(1, m_t) + \sigma_\varepsilon \varepsilon(1) \}$$

$\varepsilon(x) \sim \text{Extreme value}$

- **Logit-formula:**

$$\bar{v}_t(m_t) = \sigma_\varepsilon \log \left( \exp \frac{v_t(0, m_t^{\text{sell}})}{\sigma_\varepsilon} + \exp \frac{v_t(1, m_t)}{\sigma_\varepsilon} \right)$$

in choice probabilities:

$$P_t^{\text{sell}}(1, m_t) = \frac{\exp \frac{v_t(0, m_t^{\text{sell}})}{\sigma_\varepsilon}}{\exp \frac{v_t(0, m_t^{\text{sell}})}{\sigma_\varepsilon} + \exp \frac{v_t(1, m_t)}{\sigma_\varepsilon}}$$
$$\bar{v}_t(m_t) = P_t^{\text{sell}} v_t(0, m_t^{\text{sell}}) + (1 - P_t^{\text{sell}}) v_t(1, m_t)$$

**Extra**

---

# 1. Permanent transitory income process

- **Persistent-transitory income process:**

$$z_t = \tilde{z}_t \xi_t, \quad \log \xi_t \sim \mathcal{N}(\mu_\xi, \sigma_\xi)$$

$$\log \tilde{z}_{t+1} = \rho_z \log \tilde{z}_t + \psi_{t+1}, \quad \psi_{t+1} \sim \mathcal{N}(\mu_\psi, \sigma_\psi)$$

1. Transitory shock:  $\xi_t$
2. Persistent shock:  $\psi_t$
3. Normalization using  $\mu_\psi$  and  $\mu_\xi$ :  $\mathbb{E}[z_t] = \mathbb{E}[\tilde{z}_t] = 1$

- **ConSav:** `qudarature.log_normal_gauss_hermite`
- **ConSavNotebook:** 04. Tools/04. Quadrature.ipynb

# 1. Transition probabilities

- **Discretization of  $\xi_t$ :** Derive  $\mathcal{G}_\xi$  and  $\pi_{i_{\xi-}, i_\xi}$  given  $\sigma_\xi$  using Gauss-Hermite quadrature

$$x \sim \mathcal{N}(\mu, \sigma^2) : \mathbb{E}[h(x)] \approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^n \omega_i h(\sqrt{2}\sigma x_i + \mu)$$

where nodes,  $x_i$ , and weights,  $\omega_i$ , have analytical expressions

- **Discretization of  $\tilde{z}_t$ :** Derive  $\mathcal{G}_{\tilde{z}}$  and  $\pi_{i_{\tilde{z}-}, i_{\tilde{z}}}$  given  $\rho_z < 1$  and  $\sigma_\psi$  (using a method such as Tauchen (1986) or Rouwenhorst (1995))  
If  $\rho_z = 1$ : Also use quadrature here.
- **Combined:** Derive  $\mathcal{G}_z = \mathcal{G}_{\tilde{z}} \times \mathcal{G}_\xi$  (tensor product) and use independence of  $\tilde{z}_t$  and  $\xi_t$  to get transition probabilities  $\pi_{i_{z-}, i_z}$  (kronecker product)
- **ConSav:** `markov.log_rouwenhorst`, `markov.log_tauschen`
- **ConSavNotebook:** 04. Tools/05. Markov.ipynb

# 1. Cash-on-hand formulation

Naive formulation:

$$v_t(\tilde{z}_t, \xi_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t [v_{t+1}(\tilde{z}_{t+1}, \xi_{t+1}, a_t)]$$

s.t.

$$z_t = \tilde{z}_t \xi_t$$

$$y_t = wz_t$$

$$m_t = (1+r)a_{t-1} + y_t$$

$$a_t = m_t - c_t$$

$$\tilde{z}_{t+1} = \tilde{z}_t^{\rho_z} \psi_{t+1}$$

$$a_t \geq -wb\tilde{z}_t$$

# 1. Cash-on-hand formulation

**Cash-on-hand formulation** (1 less state variable)

$$v_t(\tilde{z}_t, m_t) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t [v_{t+1}(\tilde{z}_{t+1}, a_t)]$$

s.t.

$$a_t = m_t - c_t$$

$$\tilde{z}_{t+1} = \tilde{z}_t^{\rho_z} \psi_{t+1}$$

$$m_{t+1} = (1+r)a_{t+1} + w\tilde{z}_{t+1}\xi_{t+1}$$

$$a_t \geq -wb\tilde{z}_t$$



# 1. Normalization if $\rho_z = 1$

- **Assumption:**  $\rho_z = 1 \Leftrightarrow \tilde{z}_{t+1} = \tilde{z}_t \psi_{t+1}$
- **Define normalized variables:**  $\mathbf{x}_t = x_t / \tilde{z}_t$  and  $\mathbf{v}_t(\mathbf{m}_t) = \frac{v_t(\tilde{z}_t, m_t)}{\tilde{z}_t^{1-\rho}}$
- **Normalized Bellman equation:**

$$\mathbf{v}_t(\mathbf{m}_t) = \max_{\mathbf{c}_t} \frac{\mathbf{c}_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[ \psi_{t+1}^{1-\rho} \mathbf{v}_{t+1}(\mathbf{m}_{t+1}) \right]$$

$$\text{s.t. } \mathbf{a}_t = \mathbf{m}_t - \mathbf{c}_t$$

$$\mathbf{m}_{t+1} = \frac{1+r}{\psi_{t+1}} \mathbf{a}_t + w \xi_{t+1}$$

$$\mathbf{a}_t \geq -wb$$

- **Normalized Euler-equation:**

$$c_t^{-\sigma} = \beta(1+r) \mathbb{E}_t [c_{t+1}^{-\sigma}] \Leftrightarrow \mathbf{c}_t^{-\sigma} = \beta(1+r) \mathbb{E}_t \left[ (\psi_{t+1} \mathbf{c}_{t+1})^{-\sigma} \right]$$

- **Simulation speed-up:** Harmenberg (2021)

## 2. Life-cycle (I)

- **Basically:**

1. Born, working, retired, die
2. Age-varying parameters (esp. income)

- **Add-ons:**

1. Labor supply, human capital, occupation
  2. Portfolio choice and entrepreneurship
  3. Family formation
  4. Health, mortality
- etc.

- **Good starting example:** »Life-Cycle Consumption and Children: Evidence from a Structural Estimation«, Jørgensen (2017)

## 2. Life-cycle (II)

**Paper:** Gourinchas and Parker (2021)

*Life-cycle consumption-saving model with retirement*

- Young households:  
Save for precautionary reasons (buffer)
- Older households:  
Save for retirement (life-cycle)

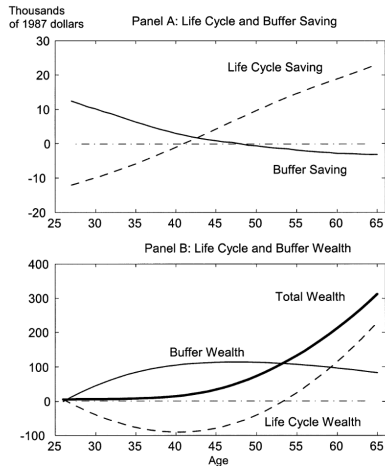
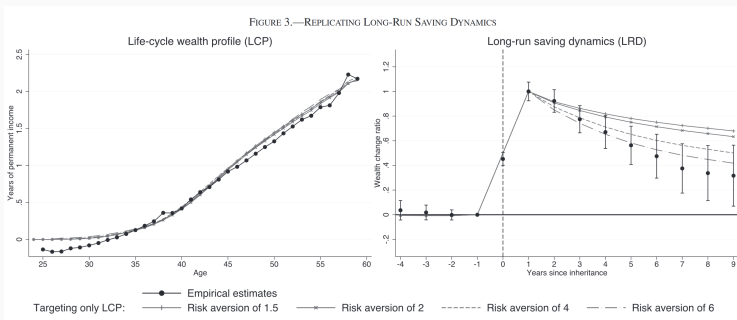


FIGURE 7.—The role of risk in saving and wealth accumulation.

## 2. Life-cycle (III)

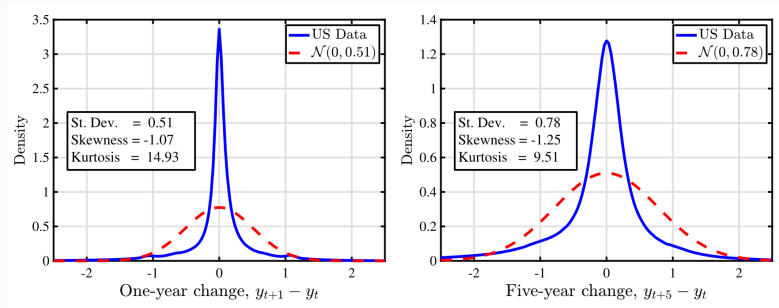
- **Natural experiment:** Wealth depletion after sudden inheritance
- **Results:**
  1. Life-cycle profile of wealth fitted for many levels of risk-aversion (by varying the discount factor)
  2. Fast wealth depletion requires high risk-aversion (or high perceived risk)



Source: Druedahl and Martinello (2022)

### 3. More realistic income risk (I)

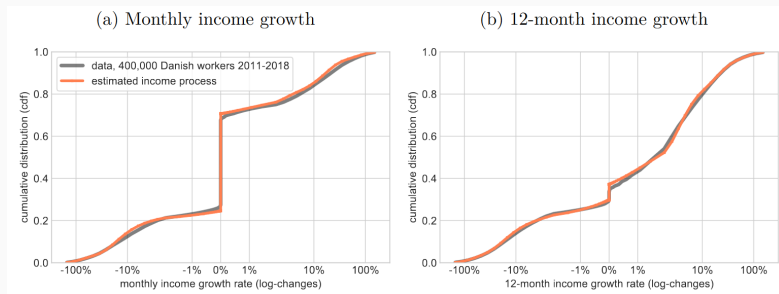
Annual earnings-changes are far from log-normal:



Source: Guvenen et. al. (2021)

### 3. More realistic income risk (II)

Many with zero-growth month-month:



Source: Druedahl et. al. (2021)

## 4. Epstein-Zin

$$\begin{aligned}v_t(z_t, m_t) &= \max_{c_t} \left[ (1 - \beta) \cdot c_t^{1-\sigma} + \beta \cdot w_{t+1}^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \\ \text{s.t.} \quad w_{t+1} &\equiv \mathbb{E}_t \left[ v_{t+1}(z_{t+1}, m_{t+1})^{1-\rho} \right]^{\frac{1}{1-\rho}} \\ m_{t+1} &= (1 + r)(m_t - c_t) + y_{t+1}\end{aligned}$$

- **Preferences:**

1. Patience:  $\beta$
2. Intertemporal substitution:  $\sigma$
3. Risk-aversion:  $\rho$

- **Euler-equation:**  $v_t = \left[ \beta R \cdot \mathbb{E}_t \left[ c_{t+1}^{-\sigma} \cdot \left( \frac{w_{t+1}}{v_{t+1}} \right)^{\rho-\sigma} \right] \right]^{-\frac{1}{\sigma}}$

1. FOC:  $0 = v_t^\sigma \cdot \left[ (1 - \beta) \cdot c_t^{-\sigma} - \beta R \cdot w_{t+1}^{\rho-\sigma} \cdot \mathbb{E}_t \left[ v_{t+1}^{-\rho} \cdot \frac{\partial v_{t+1}}{\partial m_{t+1}} \right] \right]$
2. Envelope condition:  $\frac{\partial v_t(z_t, m_t)}{\partial m_t} = v_t^\sigma \cdot (1 - \beta) \cdot c_t^{-\sigma}$

## 5. Deep learning

- **Curse of dimensionality:**
  1. Many states
  2. Many choices
  3. Many shocks
- **Deep (reinforcement) learning:**
  1. Approximate value and policy functions with *neural networks*
  2. Approximate on simulation sample rather than on grid
  3. Automatic differentiation (backpropagation) and GPUs for speed
- **Examples:** Maliar and Maliar (2021) and Azinovic and Scheidegger (2022)
- **Working paper:** Druedahl and Røpke (2025)



# Portfolio choice

---

# Portfolio choice model

- **Risk-free asset:**  $a_t$  with return  $r_f$
- **Risky asset:**  $b_t$  with return  $r_f + \nu_t$
- **Recursive formulation:**

$$v_t(a_{t-1}, b_{t-1}, \nu_t, z_t) = \max_{a_t, b_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t [v_{t+1}(a_t, b_t, z_{t+1})]$$

s.t.

$$m_t = (1 + r_f)a_{t-1} + (1 + r_f + \nu_t)b_{t-1} + wz_t$$

$$c_t = m_t - a_t - b_t$$

$$z_{t+1} \sim F_z(z_t)$$

$$\nu_{t+1} \sim F_\nu$$

$$a_t, b_t, c_t \geq 0$$

# Optimality conditions

- **Envelope conditions:**

$$\frac{\partial v_t}{\partial a_{t-1}} = (1 + r_f)c_t^{-\sigma}, \quad \frac{\partial v_t}{\partial a_{t-1}} = (1 + r_f + \nu_t)c_t^{-\sigma}$$

- **FOCs**

$$-c_t^{-\sigma} + \beta \mathbb{E}_t \left[ \frac{\partial v_{t+1}}{\partial a_t} \right] = 0$$

$$-c_t^{-\sigma} + \beta \mathbb{E}_t \left[ \frac{\partial v_{t+1}}{\partial b_t} \right] = 0$$

- **Combined:**

$$c_t^{-\sigma} = \beta(1 + r_f)\mathbb{E}_t [c_{t+1}^{-\sigma}]$$

$$0 = \mathbb{E}_t [\nu_{t+1}c_{t+1}^{-\sigma}]$$

- **Consumption-decision value function:**

$$v_t(m_t, z_t) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta w_t(a_t, z_t)$$

s.t.

$$a_t = m_t - c_t$$

$$a_t \geq 0$$

- **Portfolio-decision value function:**

$$w_t(a_t, z_t) = \max_{\alpha_t} \beta \mathbb{E}_t[v_{t+1}(m_{t+1}, z_{t+1})]$$

s.t.

$$m_{t+1} = R_{t+1}a_t + z_{t+1}$$

$$R_{t+1} = 1 + r_f + \nu_{t+1}\alpha_t$$

1. Solve for  $\alpha_t^*(a_t, z_t)$  by root-finding on

$$0 = \mathbb{E}_t [\nu_{t+1} c_{t+1}^{-\sigma}]$$

s.t.

$$c_{t+1} = c_{t+1}^*(m_{t+1}, z_{t+1})$$

$$m_{t+1} = R_{t+1} a_t + z_{t+1}$$

$$R_{t+1} = 1 + r_f + \nu_{t+1} \alpha_t^*(a_t, z_t)$$

2. Compute

$$q_t(a_t, z_t) = \mathbb{E}_t [R_{t+1} c_{t+1}^{-\sigma}]$$

3. Find  $c_t^*(m_t, z_t)$  using EGM

$$c_t(a_t, z_t) = (\beta q_t(a_t, z_t))^{-\frac{1}{\sigma}}$$

$$m_t(a_t, z_t) = c_t + a_t$$

## Extension with participation costs $\kappa$

$$v_t(m_t, z_t, \iota_{t-1}) = \max_{c_t \in [0, m_t]} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \underline{w}_t(m_t - c_t, z_t, \iota_{t-1})$$

$$\underline{w}_t(a_t, z_t, \iota_{t-1}) = \max_{\iota_t} w_t(a_t, z_t, \iota_t) - \kappa \mathbf{1}\{\iota_t = 1 \wedge \iota_{t-1} = 0\}$$

s.t.

$$\iota_t \in \{1\} \text{ if } \iota_{t-1} = 1 \text{ else } \{0, 1\}$$

$$w_t(a_t, z_t, \iota_t) = \max_{\alpha_t} \beta \mathbb{E}_t[v_{t+1}(m_{t+1}, z_{t+1}, \iota_t)]$$

s.t.

$$m_{t+1} = R_{t+1}a_t + z_{t+1}$$

$$R_{t+1} = 1 + r_f + \nu_{t+1}\alpha_t$$

$$\alpha_t \in [0, 1] \text{ if } \iota_t = 1 \text{ else } \{0\}$$

# Solution method with participation costs $\kappa$

- **Participation is an absorbing state**

1. If  $\iota_{t-1} = 1$  then  $\iota_t = 1$
2. The same solution method as before can be used

- **Before participation,  $\iota_t = 0$ :** The post-decision marginal value of cash no longer needs to be monotone

$$\underline{w}_t(a_t, z_t, 0) = \mathbb{E}_t [w_t(a_t, z_t, \iota_t) - \kappa \mathbf{1}\{\iota_t = 1\}]$$

$$q_t(a_t, z_t, \iota_t) = \mathbb{E}_t [R_{t+1} c_{t+1}^{-\sigma}]$$

$$\iota_t = \begin{cases} 1 & \text{if } w_t(a_t, z_t, 1) - \kappa > w_t(a_t, z_t, 0) \\ 0 & \text{else} \end{cases}$$

Same solution as before: *Apply an upper envelope*

# Summary

---



# Summary and what's next

- **This lecture:**

1. Introduction to course
2. Consumption-saving models
3. Basic numerical dynamic programming
4. EGM and NEGM

- **Next:** *Stationary equilibrium*

- **You should:**

1. Study the code
2. Glance at Aiyagari (1994),  
»Uninsured Idiosyncratic Risk and Aggregate Saving«