CENTER FOR ECONOMIC BEHAVIOR & INEQUALITY

HANK models

Mini-Course: Heterogenous Agent Macro

Jeppe Druedahl

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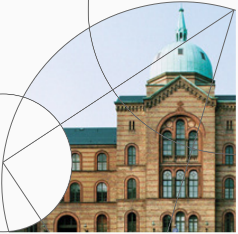

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Introduction

- **Now:** HANK - Heterogeneous Agent *New Keynesian* Model
  - Analytical insights (»opening the black box«)
    1. Zero-liquidity (Werning, 2015)
    2. Intertemporal Keynesian Cross (IKC) (Auclert et. al, 2024)
  - Sticky prices and sticky wages in practice (Kaplan, Moll, Violante, 2018)
  - Search-and-match labor market (Broer et. al., 2024)
  - Small open economy (Sundram 2025, Druedahl et. al. 2024, 2025)

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    (Kaplan, Moll, Violante, 2018)
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  - Small open economy (Sundram 2025, Druedahl et. al. 2024, 2025)
- **GEModelTools:**
  1. HANK-sticky-prices
  2. HANK-sticky-wages
  3. HANK-SAM
  4. I-HANK (mentioned, not covered)
  5. HANK-two-asset (not covered)

Plan

1. Introduction
2. Zero liquidity
3. Sticky prices
4. Sticky wages
5. IKC
6. HANK-SAM
7. I-HANK
8. Summary

Zero liquidity

Households

1. Preferences: $\sum_{t=0}^{\infty} \beta^t \mathbb{E}_0 \left[\frac{c_t^{1-\sigma}}{1-\sigma} \right]$
2. Idiosyncratic productivity, $s_t \sim \mathcal{S}$, which follows a Markov process
3. Risk-less bonds, b_{t-1}, with a real gross return of R_{t-1}
4. Income: $\gamma(s_t, Y_t)$ such that $Y_t = \int \gamma(s_t, Y_t) d\boldsymbol{D}_t$
5. Budget constraint, $c_t + b_t \leq \gamma(z_t, Y_t) + R_{t-1} b_{t-1}$
6. Borrowing constraint: $b_t \geq 0$
7. Optimal policy functions: $c_t^*(s_t, b_{t-1})$ and $b_t^*(s_t, b_{t-1})$.
8. Unconstrained, $b_t > 0$:

$$c_t^*(s_t, b_{t-1})^{-\sigma} = \beta R_t \mathbb{E}[c_t^*(s_{t+1}, b_t)^{-\sigma} \mid s_t]$$

9. Constrained, $b_t = 0$:

$$c_t^*(s_t, b_{t-1})^{-\sigma} \geq \beta R_t \mathbb{E}[c_t^*(s_{t+1}, b_t)^{-\sigma} \mid s_t]$$

- **Market clearing:**

  1. Goods:

  $$Y_t = C_t^{hh} = \int c_t^*(s_t, b_{t-1}) d\boldsymbol{D}_t$$

  2. Assets:

  $$B_t = B_t^{hh} \int b_t^*(s_t, b_{t-1}) d\boldsymbol{D}_t$$

- **Vanishing liquidity**, $B_t \to 0$ (equilibrium section rule):
  An infinitesimal increase in R_t in any given period makes at least one household willing to save more, i.e. buy more bonds.

  1. At least *one* household is on its Euler-equation
  2. Everybody consumes their own income each period (*autarky*)

Marginal saver

- **Equilibrium condition:** For a given $\{Y_t\}_{t\geq 0}$, the unique equilibrium price path is $\{R_t^*\}_{t\geq 0}$, where R_t^* is given by the Euler-equation of the *marginal saver*,

$$R_t^* \equiv R_t^*(\{Y_t\}_{t\geq 0}) = \min_{s_t \in \mathcal{S}} \tilde{R}_t(s_t)$$

where

$$\tilde{R}_t(s_t) \equiv \tilde{R}_t(s_t, \{Y_t\}_{t\geq 0}) = \beta^{-1} \frac{\gamma(s_t, Y_t)^{-\sigma}}{\mathbb{E}[\gamma(s_{t+1}, Y_{t+1})^{-\sigma} \mid s_t]}.$$

- **Intuition:**
  1. $R_t > R_t^*$: Some households would like to save.
  2. $R_t < R_t^*$: The Euler-equation would not bind for any household.
- **Marginal saver:** $s_t^* \equiv s_t^*(\{Y_t\}_{t\geq 0}) = \arg\min_{s_t \in \mathcal{S}} \tilde{R}_t(s_t)$,

Equilibrium

- **Equilibrium path:** $\{C_t, R_t\}_{t \geq 0}$ must satisfy an RA-like Euler-equation

$$\gamma(s_t^*, C_t)^{-\sigma} = \beta R_t \mathbb{E}_t^*[\gamma(s_{t+1}, C_{t+1})^{-\sigma}]$$

where $Y_t = C_t$ (market clearing) and $\mathbb{E}_t^*[\bullet] = \mathbb{E}[\bullet \mid s_t = s_t^*]$.

- **Amplification and propagation:**

$$\frac{d \log C_t}{d \log R_t}_{|d \log C_{t+1}=0} = \frac{-\sigma}{\varepsilon(s_t^*, C_t)}$$

$$\frac{d \log C_t}{d \log C_{t+1}}_{|d \log R_t=0} = \mathbb{E}_t^* \left[\frac{\gamma(s_{t+1}, C_{t+1})^{-\sigma}}{\mathbb{E}_t^* \left[(\gamma(s_{t+1}, C_{t+1}))^{-\sigma} \right]} \frac{\varepsilon(s_{t+1}, C_{t+1})}{\varepsilon(s_t^*, C_t)} \right]$$

where $\varepsilon(s_t, Y_t)$ is the elasticity of hh. income wrt. agg. income.

$$\varepsilon(s_t, Y_t) = \frac{\gamma_Y(s_t, Y_t) Y_t}{\gamma(s_t, Y_t)}$$

- **Neutrality of heterogeneity** if $\varepsilon(s_t, Y_t) = 1$

Example: Employed vs. unemployed

- **Employed:** $\overline{y}Y^{\gamma}, \gamma > 0$ (marginal saver)
- **Unemployed:** $\underline{y}Y^{\gamma}$, $\underline{y} < \overline{y}$
- **Unemployment risk**, $\lambda(Y)$: Satisfy
  $Y = (1 - \lambda(Y))\overline{y}Y^{\gamma} + \lambda(Y)\underline{y}Y^{\gamma}$
- **Marginal saver** is employed with Euler-equation

$$\left(\overline{y}Y^{\gamma}\right)^{-\sigma} = \beta R_t \left[(1 - \lambda(Y_{t+1}))\left(\overline{y}Y_{t+1}^{\gamma}\right)^{-\sigma} + \lambda(Y_{t+1})\left(\underline{y}Y_{t+1}^{\gamma}\right)^{-\sigma}\right] \Leftrightarrow$$

$$Y_t^{-\sigma} = \tilde{\beta}(Y_{t+1})R_t^{\frac{1}{\gamma}} Y_{t+1}^{-\sigma}$$

with $\tilde{\beta}(Y_{t+1}) = \left(\beta(1 - \lambda(Y_{t+1})) + \lambda(Y_{t+1})\left(\underline{y}/\overline{y}\right)^{-\sigma}\right)^{\frac{1}{\gamma}}$

- **Equivalence:** If $\gamma = 1$ with $\frac{\partial\lambda(Y)}{\partial Y} = 0$
- **Counter-cyclical income risk**, $\gamma < 1$: $\frac{\partial\lambda(Y)}{\partial Y} < 0$
  1. Amplification: $\frac{d\log C_t}{d\log R_t}_{|d\log C_{t+1}=0} \uparrow$ (from $R_t^{\frac{1}{\gamma}}$ with $\frac{1}{\gamma} > 1$)
  2. Propagation: $\frac{d\log C_t}{d\log C_{t+1}}_{|d\log R_t=0} \uparrow$ (because $\frac{\partial\tilde{\beta}(Y)}{\partial Y} < 0$)

Sticky prices

Overview

- **Households:**
  1. Differ by stochastic idiosyncratic productivity and savings
  2. Supply labor and choose consumption
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- **Central bank**: Set nominal interest rate

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$$\max_{y_{jt}\,\forall j} P_t Y_t - \int_0^1 p_{jt} y_{jt} dj \text{ s.t. } Y_t = \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj\right)^{\mu}$$

for given output price, P_t, and input prices, p_{jt}

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- **Demand curve** derived from FOC wrt. y_{jt}

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- **Note:** Zero profits (can be used to derive price index)

Derivation of demand curve

- **FOC wrt.** y_{jt}

$$
\begin{aligned}
0 &= P_t \mu \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu-1} \frac{1}{\mu} y_{jt}^{\frac{1}{\mu}-1} - p_{jt} \Leftrightarrow \\
\frac{p_{jt}}{P_t} &= \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu-1} y_{jt}^{\frac{1-\mu}{\mu}} \Leftrightarrow \\
\left(\frac{p_{jt}}{P_t} \right)^{\frac{\mu}{\mu-1}} &= \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu} y_{jt}^{-1} \Leftrightarrow \\
y_{jt} &= \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t
\end{aligned}
$$

Intermediary goods firms

- ***Dynamic* problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, n_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$

$$\text{s.t. } y_{jt} = \Gamma_t n_{jt}, \quad y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu - 1}} Y_t$$

$$\Omega(p_{jt}, p_{jt-1}) = \frac{\mu}{\mu - 1} \frac{1}{2\kappa} \left[\log \left(\frac{p_{jt}}{p_{jt-1}} \right) \right]^2$$

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- **NKPC** derived from FOC wrt. p_{jt} and envelope condition:

$$\log(1+\pi_t) = \kappa \left(\frac{w_t}{\Gamma_t} - \frac{1}{\mu} \right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}}, \;\; \pi_t \equiv P_t / P_{t-1} - 1$$

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- **Implied production:** $Y_t = y_{jt}$, $N_t = n_{jt}$ (from symmetry)
- **Implied dividends:** $d_t = Y_t - w_t N_t - \frac{\mu}{\mu-1} \frac{1}{2\kappa} \left[\log(1 + \pi_t) \right]^2 Y_t$

Derivation of NKPC

- **FOC wrt.** p_{jt}**:**

$$0 = \left(1 - \frac{\mu}{\mu - 1}\right)\left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{P_t} + \frac{\mu}{\mu - 1}\frac{w_t}{\Gamma_t}\left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{p_{jt}}$$

$$- \frac{\mu}{\mu - 1}\frac{1}{\kappa}\frac{\log\left(\frac{p_{jt}}{p_{jt-1}}\right)}{p_{jt}} Y_t + \frac{J'_{t+1}(p_{jt})}{1 + r_{t+1}}$$

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- **FOC + Envelope + Symmetry +** $\pi_t = P_t/P_{t-1} - 1$

$$0 = \left(1 - \frac{\mu}{\mu-1}\right)\frac{Y_t}{P_t} + \frac{\mu}{\mu-1}\frac{w_t}{\Gamma_t}\frac{Y_t}{P_t}$$
$$+ \frac{\mu}{\mu-1}\frac{1}{\kappa}\log\left(1+\pi_t\right)\frac{Y_t}{P_t} + \frac{\frac{\mu}{\mu-1}\frac{1}{\kappa}\log\left(1+\pi_{t+1}\right)\frac{Y_{t+1}}{P_t}}{1+r_{t+1}}$$

Central NKPC intution

$$\log(1+\pi_t) = \kappa\left(\frac{w_t}{Z_t} - \frac{1}{\mu}\right) + \frac{Y_{t+1}}{Y_t}\frac{\log(1+\pi_{t+1})}{1+r_{t+1}}$$

1. **Zero-inflation steady state:**
   $\pi_t = 0 \rightarrow w_t = \frac{\Gamma_t}{\mu} \rightarrow$ wage is mark-downed relative to productivity

(Note: Sometimes a β^{firm} is used instead of $\frac{1}{1+r_{t+1}}$)

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   Less pass-through from marginal costs, $\frac{w_t}{Z_t}$, to inflation, π_t

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   Increase price today, $\pi_t \uparrow$
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4. **Dividends:** *Counter-cyclical* as wages increase more than prices

(Note: Sometimes a β^{firm} is used instead of $\frac{1}{1+r_{t+1}}$)

Households

- **Household problem**: Distribution, $\boldsymbol{D}_t$, over z_{it} and a_{it-1}

$$v_t(z_{it}, a_{it-1}) = \max_{c_{it},\ell_{it}} \frac{c_{it}^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_{it}^{1+\nu}}{1+\nu} + \beta \mathbb{E}_t \left[v_{t+1}(z_{it+1}, a_{it}) \right]$$

$$\text{s.t. } a_{it} = (1+r_t)a_{it-1} + (w_t \ell_{it} - \tau_t + d_t)z_{it} - c_{it} \geq \underline{a}$$

$$\log z_{it+1} = \rho_z \log z_{it} + \psi_{it+1} \ \ , \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \ \mathbb{E}[z_{it}] = 1$$

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- **Dividends:** Distributed proportional to productivity (ad hoc)

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$$v_t(z_{it}, a_{it-1}) = \max_{c_{it}, \ell_{it}} \frac{c_{it}^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_{it}^{1+\nu}}{1+\nu} + \beta \mathbb{E}_t \left[v_{t+1}(z_{it+1}, a_{it}) \right]$$

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$$\log z_{it+1} = \rho_z \log z_{it} + \psi_{it+1} \ , \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \ \mathbb{E}[z_{it}] = 1$$

- **Dividends:** Distributed proportional to productivity (ad hoc)
- **Taxes:** Collected proportional to productivity (ad hoc)

Households

- **Household problem**: Distribution, $\boldsymbol{D}_t$, over z_{it} and a_{it-1}

$$
\begin{aligned}
v_t(z_{it}, a_{it-1}) &= \max_{c_{it}, \ell_{it}} \frac{c_{it}^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_{it}^{1+\nu}}{1+\nu} + \beta \mathbb{E}_t \left[v_{t+1}(z_{it+1}, a_{it}) \right] \\
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\end{aligned}
$$

- **Dividends:** Distributed proportional to productivity (ad hoc)
- **Taxes:** Collected proportional to productivity (ad hoc)
- **Optimality conditions:**

$$
\begin{aligned}
\text{FOC wrt. } c_{it} : 0 &= c_{it}^{-\sigma} - \beta \mathbb{E}_t \left[v_{a,t+1}(z_{it+1}, a_{it}) \right] \\
\text{FOC wrt. } \ell_{it} : 0 &= w_t z_{it} \beta \mathbb{E}_t \left[v_{a,t+1}(z_{it+1}, a_{it}) \right] - \varphi \ell_{it}^{\nu} \\
\text{Envelope condition: } v_{a,t}(z_{it}, a_{it-1}) &= (1+r_t) c_{it}^{-\sigma}
\end{aligned}
$$

Households

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$$\text{FOC wrt. } \ell_{it} : 0 = w_t z_{it} \beta \mathbb{E}_t \left[v_{a,t+1}(z_{it+1}, a_{it}) \right] - \varphi \ell_{it}^{\nu}$$

$$\text{Envelope condition: } v_{a,t}(z_{it}, a_{it-1}) = (1+r_t) c_{it}^{-\sigma}$$

- **Effective labor-supply:** $n_{it} = z_{it} \ell_{it}$

EGM I

- **Beginning-of-period value function:**

$$\underline{v}_{a,t}(z_{it-1}, a_{it-1}) = \mathbb{E}_t \left[v_{a,t}(z_{it}, a_{it-1}) \right] = \mathbb{E}_t \left[(1 + r_t) c_{it}^{-\sigma} \right]$$

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- **Endogenous grid method:** Vary z_t and a_t to find

$$\begin{aligned}
c_{it} &= \left(\beta \underline{v}_{a,t+1}(z_{it}, a_{it})\right)^{-\frac{1}{\sigma}} \\
\ell_{it} &= \left(\frac{w_t z_{it}}{\varphi} c_{it}^{-\sigma}\right)^{\frac{1}{\nu}} \\
m_{it} &= c_{it} + a_{it} - (w_t \ell_{it} - \tau_t + d_t) z_{it}
\end{aligned}$$

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- **Consumption and labor supply:** Use linear interpolation to find

$$c^*(z_{it}, a_{it-1}) \text{ and } \ell^*(z_{it}, a_{it-1}) \text{ with } m_{it} = (1 + r_t) a_{it-1}$$

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$$c^*(z_{it}, a_{it-1}) \text{ and } \ell^*(z_{it}, a_{it-1}) \text{ with } m_{it} = (1 + r_t) a_{it-1}$$

- **Savings:** $a^*(z_{it}, a_{it-1}) = (1 + r_t) a_{it-1} - c^*_{it} + (w_t \ell^*_{it} - \tau_t + d_t) z_{it}$

EGM II

- **Problem:** $a_t^*(z_{it}, a_{it-1}) < \underline{a}$ violate borrowing constraint

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  Find ℓ_{it}^* (and c_{it}^* and n_{it}^*) with *Newton solver* assuming $a_{it}^* = \underline{a}$

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  Find ℓ_{it}^* (and c_{it}^* and n_{it}^*) with *Newton solver* assuming $a_{it}^* = \underline{a}$
  1. Stop if $f(\ell_{it}^*) = \ell_{it}^* - \left(\frac{w_t z_{it}}{\varphi}\right)^{\frac{1}{\nu}} (c_{it}^*)^{-\frac{\sigma}{\nu}} <$ tol. where

$$c_{it}^* = (1 + r_t)a_{it-1} + (w_t \ell_{it}^* - \tau_t + d_t)z_{it}$$
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  2. Set

  $$\ell_{it}^* = \ell_{it}^* - \frac{f(\ell_{it}^*)}{f'(\ell_{it}^*)} = \frac{f(\ell_{it}^*)}{1 - \left(\frac{w_t z_{it}}{\varphi}\right)^{\frac{1}{\nu}} \left(-\frac{\sigma}{\nu}\right) (c_{it}^*)^{-\frac{\sigma}{\nu}} w_t z_{it}}$$

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  1. Stop if $f(\ell_{it}^*) = \ell_{it}^* - \left(\frac{w_t z_{it}}{\varphi}\right)^{\frac{1}{\nu}} (c_{it}^*)^{-\frac{\sigma}{\nu}} <$ tol. where

$$
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c_{it}^* &= (1 + r_t)a_{it-1} + (w_t \ell_{it}^* - \tau_t + d_t)z_{it} \\
n_{it} &= \ell_{it} z_{it}
\end{aligned}
$$

  2. Set

$$
\ell_{it}^* = \ell_{it}^* - \frac{f(\ell_{it}^*)}{f'(\ell_{it}^*)} = \frac{f(\ell_{it}^*)}{1 - \left(\frac{w_t z_{it}}{\varphi}\right)^{\frac{1}{\nu}} \left(-\frac{\sigma}{\nu}\right) (c_{it}^*)^{-\frac{\sigma}{\nu}} w_t z_{it}}
$$

  3. Return to step 1

Government and central bank

- **Monetary policy:** Follow Taylor-rule:

$$i_t = i_t^* + \phi\pi_t + \phi^Y(Y_t - Y_{ss})$$

where i_t^* is a shock

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$$r_t = (1 + i_{t-1})/(1 + \pi_t) - 1$$

Government and central bank

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- **Fisher relationship:**

$$r_t = (1 + i_{t-1})/(1 + \pi_t) - 1$$

- **Government:** Choose τ_t to keep debt constant and finance exogenous public consumption

$$\tau_t = r_t B_{ss} + G_t$$

Market clearing

1. Assets: $B_{ss} = \int a_t^*(z_{it}, a_{it-1}) d\boldsymbol{D}_t$
2. Labor: $N_t = \int n_t^*(z_{it}, a_{it-1}) d\boldsymbol{D}_t$ (in effective units)
3. Goods: $Y_t = \int c_t^*(z_{it}, a_{it-1}) d\boldsymbol{D}_t + G_t + \frac{\mu}{\mu-1}\frac{1}{2\kappa}\left[\log\left(1+\pi_t\right)\right]^2 Y_t$

As an equation system

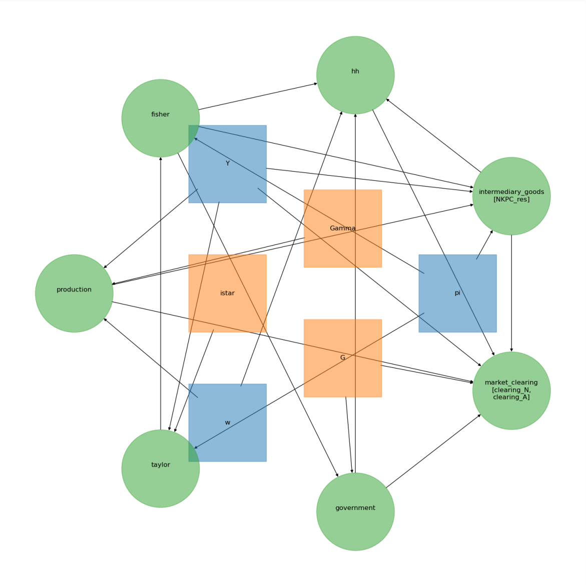
$$\boldsymbol{H}(\boldsymbol{\pi}, \boldsymbol{w}, \boldsymbol{Y}, \boldsymbol{i}^*, \boldsymbol{\Gamma}, \boldsymbol{G}, \underline{\boldsymbol{D}}_0) = \mathbf{0}$$

$$\begin{bmatrix} \log(1+\pi_t) - \left[\kappa\left(\frac{w_t}{Z_t} - \frac{1}{\mu}\right) + \frac{Y_{t+1}}{Y_t}\frac{\log(1+\pi_{t+1}}{1+r_{t+1}})\right] \\ N_t - \int n_t^*(z_{it}, a_{it-1}) d\boldsymbol{D}_t \\ B_{ss} - \int a_t^*(z_{it}, a_{it-1}) d\boldsymbol{D}_t \end{bmatrix} = \mathbf{0}$$

The rest of the model is given by

$$\boldsymbol{X} = \boldsymbol{M}(\boldsymbol{\pi}, \boldsymbol{w}, \boldsymbol{Y}, \boldsymbol{i}^*, \boldsymbol{\Gamma})$$

As a DAG


hh
fisher
Y
intermediary_goods
[NKPC_res]
Gamma
production
istar
pi
G
market_clearing
[clearing_N,
clearing_A]
w
taylor
government

Steady state

- **Chosen:** B_{ss}, G_{ss}, r_{ss}
- **Analytically:**
  1. **Normalization:** $Z_{ss} = N_{ss} = 1$
  2. **Zero-inflation:** $\pi_{ss} = 0 \Rightarrow i_{ss} = i^*_{ss} = (1 + r_{ss})(1 + \pi_{ss}) - 1$
  3. **Firms:** $Y_{ss} = Z_{ss} N_{ss}$, $w_{ss} = \frac{Z_{ss}}{\mu}$ and $d_{ss} = Y_{ss} - w_{ss} N_{ss}$
  4. **Government:** $\tau_{ss} = r_{ss} B_{ss} + G_{ss}$
  5. **Assets:** $A_{ss} = B_{ss}$
- **Numerically:** Choose β and φ to get market clearing

Transmission mechanism to monetary policy shock

1. **Monetary policy shock:** $i_t^* \downarrow \Rightarrow i_t = i_t^* + \phi\pi_t \downarrow$
2. **Real interest rate:** $r_t = \frac{1+i_{t-1}}{1+\pi_t} \downarrow$
3. **Taxes:** $\tau_t = r_t B_{ss} \downarrow$
4. **Household consumption**, $C_t^{hh} \uparrow$, due to $r_t \downarrow$ and $\tau_t \downarrow$
5. **Firms production,** $Y_t \uparrow$, and **labor demand**, $N_t \uparrow$
6. **Inflation,** $\pi_t \uparrow$, and **wage**, $w_t \uparrow$ and **dividends**, $d_t \downarrow$
7. **Household labor supply**, $N_t^{hh} \uparrow$, due to $w_t \uparrow$ and $d_t \downarrow$,
   but dampened $\tau_t \downarrow$
8. **Nominal rate**, $i_t \uparrow$ due to $\pi_t \uparrow$ implying $r_t \uparrow$
9. **Household consumption**, $C_t^{hh} \uparrow$, due to $w_t \uparrow$
   but dampened by $d_t \downarrow$ and $r_t \uparrow$

Transmission mechanism to monetary policy shock

- **Notebook:** GEModelToolsNotebooks/HANK-sticky-prices
- Look at:
  1. Stationary equilibrium (policies + distribution)
  2. Non-linear and linear transition path
  3. Effect of changing slope of Phillips curve (κ)
  4. Decomposition of consumption and labor supply

Representative agent (RANK)

- **Replace market clearing conditions with FOCs:**

$$C_t^{-\sigma} = \beta(1 + r_{t+1}) C_{t+1}^{-\sigma}$$
$$\varphi N_t^{\nu} = w_t C_t^{-\sigma}$$

- From resource constraint: $C_t = Y_t - G_t - \frac{\mu}{\mu-1} \frac{1}{2\kappa} \left[\log\left(1 + \pi_t\right)\right]^2 Y_t$
- Ensure same steady state: $\beta^{RA} = \frac{1}{1+r_{ss}}, \;\; \varphi^{RA} = \frac{w_{ss}\left(C_{ss}^{hh}\right)^{-\sigma}}{(N_{ss})^{\nu}}$
- **Intertemporal budget constraint:**

$$C_0 + \frac{C_1}{1 + r_1} + \ldots = (1 + r_0) A_{-1} + Y_0^{RA} + \frac{Y_1^{RA}}{1 + r_1} \ldots$$

where $Y_t^{RA} = w_t N_t + d_t - \tau_t$ is household income

Direct effect vs. indirect effects

- Euler-equation implies: $C_{t+1} = (\beta(1 + r_{t+1}))^{\frac{1}{\sigma}} C_t$
- **Direct effect:** Full effect of $\{r_t\}$

$$C_t = (\beta(1 + r_t))^{\frac{t}{\sigma}} C_0$$

$$C_0 = \frac{(1 + r_0)A_{ss}^{hh} + \sum_{t=0}^{\infty} q_t (w_{ss} N_{ss} - \tau_{ss} + d_{ss})}{\sum_{t=0}^{\infty} q_t \nu_t}$$

where

$$q_t = (1 + r_t)^{-1} q_{t-1}, \ q_0 = 1$$

$$\nu_t = (\beta(1 + r_t))^{\frac{1}{\sigma}} \nu_0, \ \nu_0 = 1$$

- **Indirect effect:** Residual (from $\{w_t N_t - \tau_t + d_t\}$)
- **Note:** We keep labor supply fixed here
- **Notebook:** GEModelToolsNotebooks/HANK-Sticky-prices

Kaplan, Moll and Violante (2018): *Indirect effects dominate in a large HANK model with both liquid and illiquid assets and detailed calibration*

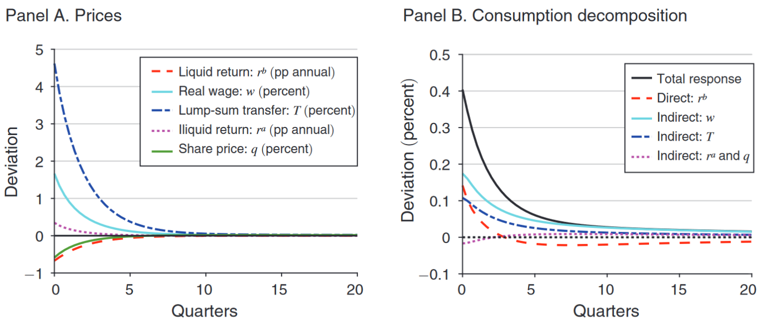


Figure 4. Direct and Indirect Effects of Monetary Policy in HANK

Sticky wages

Households

- **Household problem:**

$$
\begin{aligned}
v_t(z_t, a_{t-1}) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_t^{1+\nu}}{1+\nu} + \beta \mathbb{E}_t \left[v_{t+1}(z_{t+1}, a_t) \right] \\
\text{s.t. } a_t + c_t &= (1 + r_t^a) a_{t-1} + (1 - \tau_t) w_t \ell_t z_t + \chi_t \\
\log z_{t+1} &= \rho_z \log z_t + \psi_{t+1} \ \ , \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \ \mathbb{E}[z_t] = 1 \\
a_t &\geq 0
\end{aligned}
$$

- **Active decisions:** Consumption-saving, c_t (and a_t)
- **Union decision:** Labor supply, ℓ_t (same for everybody)
  (Alternative: Greenwood–Hercowitz–Huffman preferences)
- **Consumption function:** $C_t^{hh} = C^{hh}\left(\{r_s^a, \tau_s, w_s, \ell_s, \chi_s\}_{s \geq 0}\right)$

Firms

- **Production and profits:**

$$Y_t = \Gamma_t L_t$$
$$\Pi_t = P_t Y_t - W_t L_t$$

- **First order condition:**

$$\frac{\partial \Pi_t}{\partial L_t} = 0 \Leftrightarrow P_t \Gamma_t - W_t = 0 \Leftrightarrow w_t \equiv W_t / P_t = \Gamma_t$$

Zero profits: $\Pi_t = 0$

- **Wage and price inflation:**

$$\pi_t^w \equiv W_t / W_{t-1} - 1$$
$$\pi_t \equiv \frac{P_t}{P_{t-1}} - 1 = \frac{W_t / \Gamma_t}{W_{t-1} / \Gamma_{t-1}} - 1 = \frac{1 + \pi_t^w}{\Gamma_t / \Gamma_{t-1}} - 1$$

Union

- Everybody works the same:

$$\ell_t = L_t^{hh}$$

- Unspecified *wage adjustment costs* imply a **New Keynesian Wage (Phillips) Curve** (NKWPC or NKWC)

$$\pi_t^w = \kappa \left(\varphi \left(L_t^{hh} \right)^{\nu} - \frac{1}{\mu} \left(1 - \tau_t \right) w_t \left(C_t^{hh} \right)^{-\sigma} \right) + \beta \pi_{t+1}^w$$

(Can also be micro-founded)

Government

- **Spending:** G_t
- **Tax bill:** T_t

$$T_t = \int \tau_t w_t \ell_t z_t d\boldsymbol{D}_t = \tau_t \Gamma_t L_t = \tau_t Y_t$$

- If **one-period bonds:**

$$B_t = (1 + r_t^b) B_{t-1} + G_t + \chi_t - T_t$$

- If **long-term bonds:** Geometrically declining payment stream of $1, \delta, \delta^2, \dots$ for $\delta \in [0, 1]$. The bond price is q_t.

$$q_t (B_t - \delta B_{t-1}) = B_{t-1} + G_t + \chi_t - T_t$$

- Potential **tax-rule:**

$$\tau_t = \tau_{ss} + \omega q_{ss} \frac{B_{t-1} - B_{ss}}{Y_{ss}}$$

Central bank

- Standard **Taylor rule:**

$$1 + i_t = (1 + i_{t-1})^{\rho_i} \left((1 + r_{ss}) (1 + \pi_t)^{\phi_\pi} \right)^{1-\rho_i}$$

  Alternative: Real rate rule

$$1 + i_t = (1 + r_{ss})(1 + \pi_{t+1})$$

  Indeterminacy: Consider limit or assume future tightening

- **Fisher-equation:**

$$1 + r_t = \frac{1 + i_t}{1 + \pi_{t+1}}$$

Arbitrage

1. One-period *real* bond, $q_t = 1$:

$$t > 0: \; r_t^b = r_t^a = r_{t-1}$$
$$r_0^b = r_0^a = 1 + r_{ss}$$

2. or, one-period *nominal* bond, $q_t = 1$:

$$t > 0: \; r_t^b = r_t^a = r_{t-1}$$
$$t > 0: \; r_0^b = r_0^a = (1 + r_{ss})(1 + \pi_{ss}) / (1 + \pi_0)$$

3. or, long-term *(real)* bonds:

$$\frac{1 + \delta q_{t+1}}{q_t} = 1 + r_t$$
$$1 + r_t^b = 1 + r_t^a = \frac{1 + \delta q_t}{q_{t-1}} = \begin{cases} \frac{1+\delta q_0}{q_{ss}} & \text{if } t = 0 \\ 1 + r_{t-1} & \text{else} \end{cases}$$

Market clearing

1. Asset market: $q_t B_t = A_t^{hh}$
2. Labor market: $L_t = L_t^{hh}$
3. Goods market: $Y_t = C_t^{hh} + G_t$

Equation system

Taylor-rule and long-term government debt:

$$
\begin{bmatrix}
w_t - \Gamma_t \\
Y_t - \Gamma_t L_t \\
1 + \pi_t - \frac{1+\pi_t^w}{\Gamma_t / \Gamma_{t-1}} \\
1 + i_t - (1 + i_{t-1})^{\rho_i} \left((1 + r_{ss}) (1 + \pi_t)^{\phi_\pi} \right)^{1-\rho_i} \\
1 + r_t - \frac{1+i_t}{1+\pi_{t+1}} \\
\frac{1+\delta q_{t+1}}{q_t} - (1 + r_t) \\
1 + r_t^a - \frac{1+\delta q_t}{q_{t-1}} \\
\tau_t - \left[\tau_{ss} + \omega q_{ss} \frac{B_{t-1} - B_{ss}}{Y_{ss}} \right] \\
q_t (B_t - \delta B_{t-1}) - [B_{t-1} + G_t + \chi_t - \tau_t Y_t] \\
q_t B_t - A_t^{hh} \\
\pi_t^w - \left[\kappa \left(\varphi \left(L_t^{hh} \right)^{\nu} - \frac{1}{\mu} (1 - \tau_t) w_t \left(C_t^{hh} \right)^{-\sigma} \right) + \beta \pi_{t+1}^w \right]
\end{bmatrix} = \mathbf{0}
$$

Reduced equation system with ordered blocks

$$\boldsymbol{H}(\boldsymbol{\pi^w}, \boldsymbol{L}, \boldsymbol{G}, \chi, \boldsymbol{\Gamma}) = \left[\begin{array}{c} q_t B_t - A_t^{hh} \\ \pi_t^w - \left[\kappa \left(\varphi \left(L_t^{hh}\right)^{\nu} - \frac{1}{\mu}\left(1-\tau_t\right) w_t \left(C_t^{hh}\right)^{\text{-}\sigma}\right) + \beta \pi_{t+1}^W\right] \end{array} \right] = \mathbf{0}$$

$$\begin{aligned}
\text{Production: } w_t &= \Gamma_t \\
Y_t &= \Gamma_t L_t \\
\pi_t &= \frac{1+\pi_t^w}{\Gamma_t / \Gamma_{t-1}} - 1 \\
\text{Central bank: } i_t &= \left(1+i_{t-1}\right)^{\rho_i} \left(\left(1+r_{ss}\right)\left(1+\pi_t\right)^{\phi_\pi}\right)^{1-\rho_i} - 1 \text{ (forwards)} \\
r_t &= \frac{1+i_t}{1+\pi_{t+1}} - 1 \\
\text{Mutual fund: } q_t &= \frac{1+\delta q_{t+1}}{1+r_t} \text{ (backwards)} \\
r_t^a &= \frac{1+\delta q_t}{q_{t-1}} - 1
\end{aligned}$$

$$\text{Government: } \left[\begin{array}{c} \tau_t \\ B_t \end{array} \right] = \left[\begin{array}{c} \tau_{ss} + \omega q_{ss} \frac{B_{t-1} - B_{ss}}{Y_{ss}} \\ \frac{(1+\delta q_t) B_{t-1} + G_t + \chi_t - \tau_t Y_t}{q_t} \end{array} \right] \text{ (forwards)}$$

DAG


central_bank
mutual_fund
pi_w
NKWC
[NKWC_res]
G
government
chi
Gamma
production
L
market_clearing
[clearing_A]
hh

Fiscal multiplier

- **Notebook:** GEModelToolsNotebooks/HANK-sticky-wages
  1. Stationary equilibrium is similar, but no labor supply
  2. Realistic MPC by varying $q_{ss}B_{ss}$ target
  3. Fiscal multiplier increasing in MPC
     3.1 Impact: $\frac{\partial Y_0}{\partial G_0}$
     3.2 Cumulative: $\frac{\sum_{t=0}^{\infty}(1+r_{ss})^{-t}dY_t}{\sum_{t=0}^{\infty}(1+r_{ss})^{-t}dG_t}$
  4. Fiscal multiplier depends on fiscal and monetary rules
  5. The transfer multiplier is smaller

Extension: Endogenous Idiosyncratic risk

- **Baseline:** Idiosyncratic risk
- **Extension:** Endogenous idiosyncratic risk
- **Empirical evidence:**
  1. Unequal exposure (Guvenen et al., 2017)
  2. Cyclical income risk (Storesletten et al., 2004, Guvenen et al., 2014)
- **Specification 1** from Auclert et al. (2024)

$$\ell_{it} = L_t \frac{z_t^{\upsilon \log L_t}}{\mathbb{E}\left[z_t^{\upsilon \log L_t}\right]}$$

- **Specification 2** from Acharya et al. (2023)

$$\sigma_{\psi,t} = \sigma_\psi + \upsilon \log L_t$$

and use a scaling factor to ensure $\mathbb{E}\left[z_{it}\right] = 1$ always

Extension: Larger fiscal multipliers

- **Counter-cyclical income risk and inequality with $\upsilon < \mathbf{0}$**
- **Effect:** Substantial boost to consumption
  1. Impact fiscal multiplier increase substantially
  2. Cumulative fiscal multiplier also increase
- **Tight calibration is hard...**
- **Notebook:** GEModelToolsNotebooks/HANK-sticky-wages

IKC

Simpler consumption function

- **Assumptions:**
  1. One-period real bond
  2. No lump-sum transfers, $\chi_t = 0$
  3. Real rate rule: $r_t = r_{ss}$
  4. Fiscal policy in terms of dG_t and dT_t satisfying IBC

$$\sum_{t=0}^{\infty} (1 + r_{ss})^{-t} (dG_t - dT_t) = 0$$

- **Tax-bill:** $T_t = \tau_t w_t \int \ell_t z_t d\boldsymbol{D}_t = \tau_t \Gamma_t L_t = \tau_t Y_t$
- **Household income:** $(1 - \tau_t) w_t \ell_t z_t = \underbrace{(Y_t - T_t)}_{\equiv Z_t} z_t = Z_t z_t$
- **Consumption function:** Simplifies to

$$C_t^{hh} = C^{hh}(\{Y_s - T_s\}_{s \geq 0}) \Rightarrow \boldsymbol{C}^{hh} = C^{hh}(\boldsymbol{Y} - \boldsymbol{T}) = C^{hh}(\boldsymbol{Z})$$

Side-note: Two-equation version in $\boldsymbol{Y}$ and $\boldsymbol{r}$

$$
\begin{aligned}
\boldsymbol{Y} &= \boldsymbol{G} + C^{hh}(\boldsymbol{r}, \boldsymbol{Y} - \boldsymbol{T}) \\
\boldsymbol{r} &= \mathcal{R}(\boldsymbol{Y}; \boldsymbol{G}, \boldsymbol{T})
\end{aligned}
$$

- **First equation:** Goods market clearing
- **Second equation:**
  1. Government: $\boldsymbol{T}, \boldsymbol{Y} \to \boldsymbol{\tau}$
  2. Resource constraint: $\boldsymbol{G}, \boldsymbol{Y} \to \boldsymbol{C}$
  3. Firm behavior I: $\boldsymbol{\Gamma}, \boldsymbol{Y} \to \boldsymbol{L}, \boldsymbol{w}$
  4. NKWC: $\boldsymbol{L}, \boldsymbol{C}, \boldsymbol{w}, \boldsymbol{\tau} \to \boldsymbol{\pi}^{\boldsymbol{w}}$
  5. Firm behavior II: $\boldsymbol{\pi}^{\boldsymbol{w}}, \boldsymbol{\Gamma} \to \boldsymbol{\pi}$
  6. Central bank: $\boldsymbol{\pi} \to \boldsymbol{i}$
  7. Fisher: $\boldsymbol{i}, \boldsymbol{\pi} \to \boldsymbol{r}$
- **Heterogeneity does not enter** $\mathcal{R}(\boldsymbol{Y}; \boldsymbol{G}, \boldsymbol{T})$
- **Real rate rule:** *Inflation is a side-show*

Intertemporal Keynesian Cross

$$\boldsymbol{Y} = \boldsymbol{G} + C^{hh}(\boldsymbol{Y} - \boldsymbol{T})$$

- **Total differentiation:**

$$dY_t = dG_t + \sum_{s=0}^{\infty} \frac{\partial C_t^{hh}}{\partial Z_s} dZ_s = dG_t + \sum_{s=0}^{\infty} \frac{\partial C_t^{hh}}{\partial Z_s} (dY_s - dT_s)$$

- **Intertemporal Keynesian Cross** in vector form

$$d\boldsymbol{Y} = d\boldsymbol{G} + \boldsymbol{M}(d\boldsymbol{Y} - d\boldsymbol{T}) \Leftrightarrow$$
$$(\boldsymbol{I} - \boldsymbol{M})d\boldsymbol{Y} = d\boldsymbol{G} - \boldsymbol{M}d\boldsymbol{T}$$

where $M_{t,s} = \frac{\partial C_t^{hh}}{\partial Z_s}$ encodes the entire *complexity*

Illustration

Writing out the IKC:

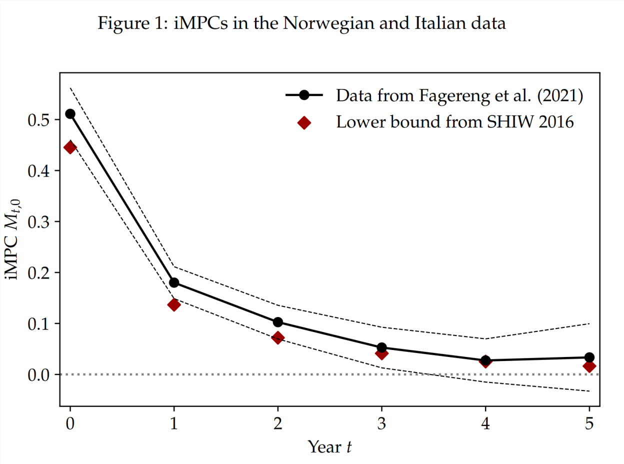
$$\begin{bmatrix} dY_0 \\ dY_1 \\ dY_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} dG_0 \\ dG_1 \\ dG_2 \\ \vdots \end{bmatrix} + \begin{bmatrix} \frac{\partial C_0^{hh}}{\partial Z_0} & \frac{\partial C_0^{hh}}{\partial Z_1} & \frac{\partial C_0^{hh}}{\partial Z_2} & \cdots \\ \frac{\partial C_1^{hh}}{\partial Z_0} & \frac{\partial C_1^{hh}}{\partial Z_1} & \frac{\partial C_1^{hh}}{\partial Z_2} & \cdots \\ \frac{\partial C_2^{hh}}{\partial Z_0} & \frac{\partial C_2^{hh}}{\partial Z_1} & \frac{\partial C_2^{hh}}{\partial Z_2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \left(\begin{bmatrix} dY_0 \\ dY_1 \\ dY_2 \\ \vdots \end{bmatrix} - \begin{bmatrix} dT_0 \\ dT_1 \\ dT_2 \\ \vdots \end{bmatrix} \right)$$

iMPC matrix

- $\boldsymbol{M}$ is the Jacobian of aggregate C w.r.t (post-tax) labor income
  - Column s: Response of C at different dates to unit change in Z at date s (IRF)
  - Row s: Change in C at date s to change in income Z at different dates

$$\boldsymbol{M} = \begin{bmatrix} \frac{\partial C_0^{hh}}{\partial Z_0} & \frac{\partial C_0^{hh}}{\partial Z_1} & \cdots \\ \frac{\partial C_1^{hh}}{\partial Z_0} & \frac{\partial C_1^{hh}}{\partial Z_1} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

iMPCs in the data



Other columns: Druedahl et al. (2023) show in micro-data that consumption responds today to news about future income.

Perspective: Static Keynesian Cross

- **Old Keynesians:** Consumption only depends on current income

$$Y_t = G_t + C^{hh}(Y_t - T_t)$$

- **Total differentiate:**

$$\begin{aligned} dY_t &= dG_t + \frac{\partial C_t^{hh}}{\partial Z_t}(dY_t - dT_t) \\ &= dG_t + \text{mpc} \cdot (dY_t - dT_t) \end{aligned}$$

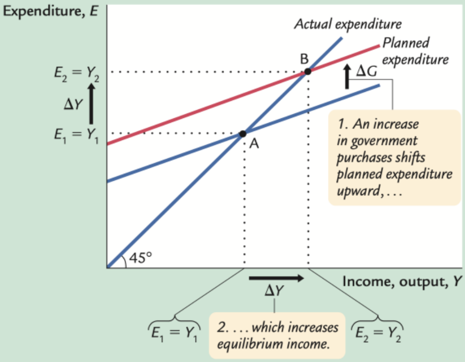
- **Solution**

$$dY_t = \frac{1}{1-\text{mpc}}\left(dG_t - \text{mpc} \cdot dT_t\right)$$

from multiplier-process $1 + \text{mpc} + \text{mpc}^2 \cdots = \frac{1}{1-\text{mpc}}$

Static Keynesian Cross

figure 10-5



An Increase in Government Purchases in the Keynesian Cross An increase in government purchases of ΔG raises planned expenditure by that amount for any given level of income. The equilibrium moves from point A to point B, and income rises from Y_1 to Y_2. Note that the increase in income ΔY exceeds the increase in government purchases ΔG. Thus, fiscal policy has a multiplied effect on income.

NPV-vector

- **NPV-vector:** $\boldsymbol{q} \equiv [1, (1+r_{ss})^{-1}, (1+r_{ss})^{-2}, \dots]'$
- **Government:** IBC holds

$$\sum_{t=0}^{\infty} (1+r_{ss})^{-t} (dG_t - dT_t) = 0 \Leftrightarrow \boldsymbol{q}'(d\boldsymbol{G} - d\boldsymbol{T}) = 0$$

- **Households:** IBC holds

$$\sum_{t=0}^{\infty} (1+r_{ss})^{-t} C_t^{hh} = (1+r_{ss})A_{-1} + \sum_{t=0}^{\infty} (1+r_{ss})^{-t} Z_t \Rightarrow$$

$$\sum_{t=0}^{\infty} (1+r_{ss})^{-t} M_{t,s} = \frac{1}{(1+r_{ss})^s} \Rightarrow \boldsymbol{q}'\boldsymbol{M} = \boldsymbol{q}' \Leftrightarrow \boldsymbol{q}'(\boldsymbol{I} - \boldsymbol{M}) = \boldsymbol{0}$$

- **Problem:** $(\boldsymbol{I} - \boldsymbol{M})^{-1}$ cannot exist because this leads to a contradiction

$$\boldsymbol{q}'(\boldsymbol{I} - \boldsymbol{M})(\boldsymbol{I} - \boldsymbol{M})^{-1} = \boldsymbol{0}(\boldsymbol{I} - \boldsymbol{M})^{-1} \Leftrightarrow \boldsymbol{q}' = \boldsymbol{0}$$

Form of unique solution

- **Result:** Define $\boldsymbol{K} \equiv -\sum_{t=1}^{\infty}(1+r_{ss})\boldsymbol{F}^t$ then if unique bounded solution exists to

$$(\boldsymbol{I}-\boldsymbol{M})d\boldsymbol{Y} = (d\boldsymbol{G}-\boldsymbol{M}d\boldsymbol{T})$$

  it is given by

$$d\boldsymbol{Y} = \mathcal{M}(d\boldsymbol{G}-\boldsymbol{M}d\boldsymbol{T})$$

  where

$$\mathcal{M} = \left(\boldsymbol{K}(\boldsymbol{I}-\boldsymbol{M})\right)^{-1}\boldsymbol{K}$$

- **Indeterminancy:** Still work-in-progress (Auclert et. al., 2023)

Response of consumption

$$\begin{aligned}
d\boldsymbol{Y} &= d\boldsymbol{G} + \boldsymbol{M}(d\boldsymbol{Y} - d\boldsymbol{T}) \Leftrightarrow \\
d\boldsymbol{Y} - d\boldsymbol{G} &= \boldsymbol{M}(d\boldsymbol{G} - d\boldsymbol{T}) + \boldsymbol{M}(d\boldsymbol{Y} - d\boldsymbol{G}) \Leftrightarrow \\
(I - \boldsymbol{M})\,(d\boldsymbol{Y} - d\boldsymbol{G}) &= \boldsymbol{M}(d\boldsymbol{G} - d\boldsymbol{T}) \Leftrightarrow \\
d\boldsymbol{Y} - d\boldsymbol{G} &= \mathcal{M}\boldsymbol{M}(d\boldsymbol{G} - d\boldsymbol{T}) \Leftrightarrow \\
d\boldsymbol{C} &= \mathcal{M}\boldsymbol{M}(d\boldsymbol{G} - d\boldsymbol{T})
\end{aligned}$$

Fiscal multipliers

$$d\boldsymbol{Y} = d\boldsymbol{G} + \underbrace{\mathcal{M}\boldsymbol{M}(d\boldsymbol{G} - d\boldsymbol{T})}_{d\boldsymbol{C}}$$

- **Balanced budget multiplier:**

$$d\boldsymbol{G} = d\boldsymbol{T} \Rightarrow d\boldsymbol{Y} = d\boldsymbol{G}, d\boldsymbol{C} = 0$$

  Note: Central that income and taxes affect household income proportionally in exactly the same way = no redistribution

- **Deficit multiplier:** $d\boldsymbol{G} \neq d\boldsymbol{T}$
  1. Larger effect of $d\boldsymbol{G}$ than $d\boldsymbol{T}$
  2. *Numerical results needed*

Comparison with RA model

- **From lecture 1:** $\beta(1+r_{ss}) = 1$ implies

$$C_t = (1-\beta)\sum_{s=0}^{\infty}\beta^s Y_{t+s}^{hh} + r_{ss}a_{-1}$$

- The **iMPC-matrix** becomes

$$\boldsymbol{M}^{RA} = \begin{bmatrix} (1-\beta) & (1-\beta)\beta & (1-\beta)\beta^2 & \cdots \\ (1-\beta) & (1-\beta)\beta & (1-\beta)\beta^2 & \cdots \\ (1-\beta) & (1-\beta)\beta & (1-\beta)\beta^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = (1-\beta)\mathbf{1}\boldsymbol{q}'$$

- **Consumption response** is zero

$$\begin{aligned} d\boldsymbol{C}^{\boldsymbol{RA}} &= \mathcal{M}\boldsymbol{M}^{\boldsymbol{RA}}(d\boldsymbol{G} - d\boldsymbol{T}) \\ &= \mathcal{M}(1-\beta)\mathbf{1}\boldsymbol{q}'(d\boldsymbol{G} - d\boldsymbol{T}) \\ &= \mathbf{0} \Leftrightarrow d\boldsymbol{Y} = d\boldsymbol{G} \end{aligned}$$

Details on matrix formulation

$$(1-\beta)\mathbf{1}\boldsymbol{q}' = \begin{bmatrix} (1-\beta) & (1-\beta) & (1-\beta) & \cdots \\ (1-\beta) & (1-\beta) & (1-\beta) & \cdots \\ (1-\beta) & (1-\beta) & (1-\beta) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & (1+r_{ss})^{-1} & (1+r_{ss})^{-2} & \cdots \end{bmatrix}$$

$$= \begin{bmatrix} (1-\beta) & (1-\beta) & (1-\beta) & \cdots \\ (1-\beta) & (1-\beta) & (1-\beta) & \cdots \\ (1-\beta) & (1-\beta) & (1-\beta) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & \beta & \beta^2 & \cdots \end{bmatrix}$$

$$= \begin{bmatrix} (1-\beta) & (1-\beta)\beta & (1-\beta)\beta^2 & \cdots \\ (1-\beta) & (1-\beta)\beta & (1-\beta)\beta^2 & \cdots \\ (1-\beta) & (1-\beta)\beta & (1-\beta)\beta^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Comparison with TA model

- **Hand-to-Mouth (HtM) households:** λ share have $C_t = Y_t^{hh}$

$$\boldsymbol{M}^{TA} = (1-\lambda)\boldsymbol{M}^{RA} + \lambda \boldsymbol{I}$$

- **Intertemporal Keynesian Cross** becomes

$$(\boldsymbol{I} - \boldsymbol{M}^{TA})d\boldsymbol{Y} = d\boldsymbol{G} - \boldsymbol{M}^{TA}d\boldsymbol{T}$$
$$(\boldsymbol{I} - \boldsymbol{M}^{RA})d\boldsymbol{Y} = \underbrace{\frac{1}{1-\lambda}\left[d\boldsymbol{G} - \lambda d\boldsymbol{T}\right]}_{d\tilde{\boldsymbol{G}}_t} - \boldsymbol{M}^{RA}d\boldsymbol{T}$$

- **Same solution-form as RA:** $d\boldsymbol{Y} = d\tilde{\boldsymbol{G}}_t$

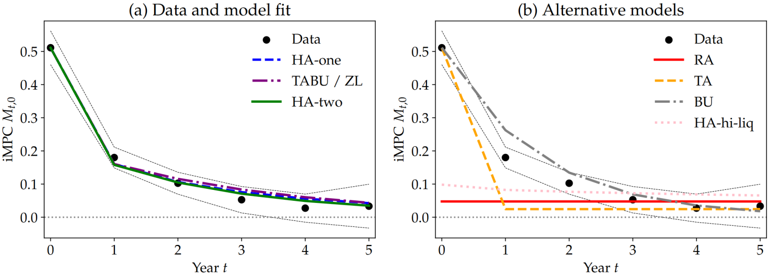
$$d\boldsymbol{Y} = d\tilde{\boldsymbol{G}}_t = d\boldsymbol{G}_t + \frac{\lambda}{1-\lambda}\left[d\boldsymbol{G} - d\boldsymbol{T}\right]$$

Cumulative multiplier still one

$$\frac{\boldsymbol{q}' d\boldsymbol{Y}}{\boldsymbol{q}' d\boldsymbol{G}} = \frac{\boldsymbol{q}' d\boldsymbol{G}_t + \frac{\lambda}{1-\lambda}\boldsymbol{q}' \left[d\boldsymbol{G} - d\boldsymbol{T}\right]}{\boldsymbol{q}' d\boldsymbol{G}}$$
$$= \frac{\boldsymbol{q}' d\boldsymbol{G}_t}{\boldsymbol{q}' d\boldsymbol{G}_t}$$
$$= 1$$

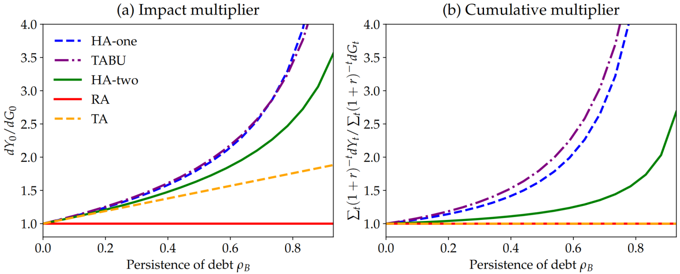
iMPCs in models

Figure 2: iMPCs in the Norwegian data and several models



Multipliers and debt-financing

Figure 5: Multipliers according to the IKC



Note. These figures assume a persistence of government spending equal to $\rho_G = 0.76$, and vary ρ_B in $dB_t = \rho_B(dB_{t-1} + dG_t)$. See section 7.1 for details on calibration choices.

Generalized IKC

- **Budget constraint** can be written with initial capital gain

$$a_t + c_t = (Y_t - T_t)z_t + \chi_t + \begin{cases} (1 + r_{t-1})a_{t-1} & \text{if } t > 0 \\ (1 + r_{ss} + \text{cap}_0)a_{t-1} & \text{if } t = 0 \end{cases}$$

1. Real bond: $\text{cap}_0 = 0$
2. Nominal bond:

$$\text{cap}_0 = \frac{(1 + r_{ss})(1 + \pi_{ss})}{1 + \pi_0} - (1 + r_{ss})$$

3. Long-term bond:

$$\text{cap}_0 = \frac{1 + \delta q_0}{q_{ss}} - (1 + r_{ss})$$

Generalized IKC

- **Consumption-function** $\boldsymbol{C}^{hh} = C^{hh}(\boldsymbol{r}, \boldsymbol{Y} - \boldsymbol{T}, \chi, \mathbf{cap}_0)$ implies

$$d\boldsymbol{C}^{hh} = \boldsymbol{M}^r d\boldsymbol{r} + \boldsymbol{M}(d\boldsymbol{Y} - d\boldsymbol{T}) + \boldsymbol{M}^{\chi} d\boldsymbol{\chi} + \boldsymbol{m}^{\text{cap}} \text{cap}_0$$

  where

$$\boldsymbol{M}^r_{t,s} = \left[\frac{\partial C^{hh}_t}{\partial r_s}\right], \boldsymbol{M}^{\chi}_{t,s} = \left[\frac{\partial C^{hh}_t}{\partial \chi_s}\right], \boldsymbol{m}^{\text{cap}}_t = \left[\frac{\partial C^{hh}_t}{\partial \text{cap}_0}\right]$$

- Why are $\boldsymbol{M}^{\chi}$ and $\boldsymbol{M}$ different?

HANK-SAM

Overview

- Intermediate **producers:**
  1. Hire and fire in search-and-matching labor market
  2. Sell homogeneous good at price p_t^X.
- Wholesale **price-setters:**
  1. Set prices in monopolistic competition subject to adjustment costs
  2. Pay out dividends
- Final producers: Aggregate to final good
- **Government:**
  1. Pay transfers and unemployment insurance
  2. Collect taxes and issues debt
- **Central bank:** Sets nominal interest rate
- **Households:** Consume and save

Equilibrium dynamics

1. **Incomplete markets:** Unemployment risk $\rightarrow$ demand

   Complete markets / representative agent:
   Only total income matters

2. **Sticky prices:** Demand $\rightarrow$ profitability

3. **Frictional labor market:** Profitability $\rightarrow$ unemployment risk

Household problem

$$v_t(\beta_i, u_{it}, a_{it-1}) = \max_{c_{it}, a_{it}} \frac{c_{it}^{1-\sigma}}{1-\sigma} + \beta_i \mathbb{E}_t \left[v_{t+1} \left(\beta_i, u_{it+1}, a_{it} \right) \right]$$

$$\text{s.t. } a_{it} + c_{it} = (1+r_t)a_{it-1} + (1-\tau_t) y_t(u_{it}) + \text{div}_t + \text{transfer}_t$$

$$a_{it} \geq 0$$

1. **Dividends and government transfers:** div_t and transfer_t
2. **Real wage**: w_{ss}
3. **Income tax:** τ_t
4. **Separation rate** for employed: δ_{ss}
5. **Job-finding rate** for unemployed: $\lambda_t^{u,s} s(u_{it-1})$
   (where $s(u_{it-1})$ is exogenous search effectiveness)
6. US-style duration-dependent **UI system:**
   a) High replacement rate $\overline{\phi}$, first $\overline{u}$ months
   b) Low replacement rate $\underline{\phi}$, after $\overline{u}$ months

Income process

- **Income** is

$$y_{it}(u_{it}) = w_{ss} \cdot \begin{cases} 1 & \text{if } u_{it} = 0 \\ \overline{\phi}\mathrm{UI}_{it} + (1 - \mathrm{UI}_{it})\underline{\phi} & \text{else} \end{cases}$$

where the share of the month with UI is

$$\mathrm{UI}_{it} = \begin{cases} 0 & \text{if } u_{it} = 0 \\ 1 & \text{else if } u_{it} < \overline{u} \\ 0 & \text{else if } u_{it} > \overline{u} + 1 \\ \overline{u} - (u_{it} - 1) & \text{else} \end{cases}$$

- **Note:** *Hereby $\overline{u}$ becomes a continuous variable.*

Transition probabilities

- **Beginning-of-period value function:**

$$\underline{v}_t\left(\beta_i, u_{it-1}, a_{it-1}\right) = \mathbb{E}\left[v_t(\beta_i, u_{it}, a_{it-1}) \,\middle|\, u_{it-1}, a_{it-1}\right]$$

- **Grid:** $u_{it} \in \{0, 1, \ldots, \#_u - 1\}$
- **Employed** with $u_{it-1} = 0$: $u_{it} = \begin{cases} 0 & \text{with prob. } 1 - \delta_{ss} \\ 1 & \text{with prob. } \delta_{ss} \end{cases}$
- **Unemployed** with $u_{it-1} = 1$:

$$u_{it} = \begin{cases} 0 & \text{with prob. } \lambda_t^{u,s} s(u_{it-1}) \\ u_{it-1} + 1 & \text{with prob. } 1 - \lambda_t^{u,s} s(u_{it-1}) \end{cases}$$

  Trick: $u_{it} = \min\left\{u_{it-1} + 1, \#_u - 1\right\}$

- **All unemployed search:** $s(u_{it-1}) = \begin{cases} 0 & \text{if } u_{it-1} = 0 \\ 1 & \text{else} \end{cases}$

Aggregation

- **Distributions:**
    1. Beginning-of-period: $\underline{\boldsymbol{D}}_t$ over β_i, u_{it-1} and a_{it-1}
    2. At decision: $\boldsymbol{D}_t$ over β_i, u_{it} and a_{it-1}
- **Stochastic (time-varying) transition matrix:** $\Pi_{t,z} = \Pi_z(\lambda_t^u)$
- **Deterministic savings policy matrix:** Λ_t'
- **Transition steps:**

$$\boldsymbol{D}_t = \Pi_{t,z}' \underline{\boldsymbol{D}}_t$$

$$\underline{\boldsymbol{D}}_{t+1} = \Lambda_t' \boldsymbol{D}_t$$

- **Searchers:** $S_t = \int s(\beta_i, u_{it-1}, a_{it-1}) d\underline{\boldsymbol{D}}_t$
- **Savings:** $A_t^{hh} = \int a_t^*(\beta_i, u_{it}, a_{it-1}) d\boldsymbol{D}_t$
- **Consumption:** $C_t^{hh} = \int c_t^*(\beta_i, u_{it}, a_{it-1}) d\boldsymbol{D}_t$

EGM

- **Beginning-of-period value function:**

$$\underline{v}_{a,t}(\beta_i, u_{it-1}, a_{it-1}) = \mathbb{E}_t\left[v_{a,t}(\beta_i, u_{it}, a_{it-1})\right] = \mathbb{E}_t\left[(1+r_t)c_{it}^{-\sigma}\right]$$

- **Endogenous grid method:** Vary u_{it} and a_{it} to find

$$\begin{aligned} c_{it} &= \left(\beta \underline{v}_{a,t+1}(\beta_i, u_{it}, a_{it})\right)^{-\frac{1}{\sigma}} \\ m_{it} &= c_{it} + a_{it} \end{aligned}$$

- **Consumption and labor supply:** Use linear interpolation to find

$$c_t^*(\beta_i, u_{it}, a_{it-1}) \text{ with } m_{it} = (1+r_t)a_{it-1}$$

- **Savings:** $a^*(u_{it}, a_{it-1}) = (1+r_t)a_{it-1} - c_t^*(\beta_i, u_{it}, a_{it-1})$

Producers: Hiring and firing

- **Job value:**

$$V_t^j = p_t^X Z_t - w_{ss} + \beta^{\text{firm}} \mathbb{E}_t \left[(1 - \delta_{ss}) V_{t+1}^j \right]$$

- **Vacancy value:**

$$V_t^v = -\kappa + \lambda_t^v V_t^j + (1 - \lambda_t^v)(1 - \delta_{ss}) \beta^{\text{firm}} \mathbb{E}_t \left[V_{t+1}^v \right]$$

- **Free entry** implies

$$V_t^v = 0$$

- **Labor market tightness** is given by

$$\theta_t = \frac{\text{vacancies}_t}{\text{searchers}_t} = \frac{v_t}{S_t}$$

- **Cobb-Douglas matching function**

$$\text{matches}_t = AS_t^{\alpha} v_t^{1-\alpha}, \;\; \alpha \in (0,1)$$

implies the job-filling and job-finding rates:

$$\lambda_t^{v} = \frac{\text{matches}_t}{v_t} = A\theta_t^{-\alpha}$$
$$\lambda_t^{u,s} = \frac{\text{matches}_t}{S_t} = A\theta_t^{1-\alpha}$$

- **Law of motion for unemployment:**

$$u_t = u_{t-1} + \delta_t(1 - u_{t-1}) - \lambda_t^{u,s} S_t$$

Price setters

- **Intermediate goods price:** p_t^X
- Dixit-Stiglitz **demand curve** $\Rightarrow$ **Phillips curve** relating marginal cost, $MC_t = p_t^x$, and **final goods price inflation**, $\Pi_t = P_t / P_{t-1}$,

$$1 - \epsilon + \epsilon p_t^x = \phi \pi_t (1 + \pi_t) - \phi \beta^{\text{firm}} \mathbb{E}_t \left[\pi_{t+1} (1 + \pi_{t+1}) \frac{Y_{t+1}}{Y_t} \right]$$

  with output $Y_t = Z_t (1 - u_t)$
- **Flexible price limit:** $\phi \to 0$
- **Dividends:**

$$\text{div}_t = Y_t - w_t (1 - u_t)$$

Central bank

- **Taylor rule:**

$$1 + i_t \quad = \quad (1 + i_{ss}) \left(\frac{1 + \pi_t}{1 + \pi_{ss}} \right)^{\delta_\pi}$$

Government

- **Unemployment insurance:** $\Phi_t = w_{ss}\left(\overline{\phi}\mathrm{UI}_t^{hh} + \underline{\phi}\left(u_t - \mathrm{UI}_t^{hh}\right)\right)$
- **Total expenses:** $X_t = \Phi_t + G_t + \mathrm{transfer}_t$
- **Total taxes:** $\mathrm{taxes}_t = \tau_t\left(\Phi_t + w_{ss}(1-u_t)\right)$
- **Government budget** is

$$q_t B_t = (1 + q_t\delta_q)B_{t-1} + X_t - \mathrm{taxes}_t$$

  Long-term debt: Real payment stream is $1, \delta, \delta^2, \ldots$.
  The real bond price is q_t.

- **Tax rule:**

$$\tilde{\tau}_t = \frac{(1 + q_t\delta_q)\,B_{t-1} + X_t - q_{ss}B_{ss}}{\Phi_t + w_{ss}(1-u_t)}$$

$$\tau_t = \omega\tilde{\tau}_t + (1-\omega)\tau_{ss}$$

- **Transfers:** $\mathrm{transfer}_t = -\mathrm{div}_{ss}$

Financial markets: No arbitrage

1. **Pricing of government debt:**

$$\frac{1+\delta_q q_{t+1}}{q_t} = \frac{1+i_t}{1+\pi_{t+1}} = 1 + r_{t+1}$$

2. **Ex post real return:**

$$1 + r_t = \begin{cases} \frac{(1+\delta_q q_0)B_{-1}}{A^{hh}_{-1}} & \text{if } t = 0 \\ \frac{1+i_{t-1}}{1+\pi_t} & \text{else} \end{cases}$$

Market clearing

1. Asset market: $A_t^{hh} = q_t B_t$
2. Goods market: $Y_t = C_t^{hh} + G_t$

Tip: *You should be able to verify Walras' law.*

Market clearing

1. **Shocks:** G_t
2. **Unknowns:** p_t^X, V_t^j, v_t, u_t, S_t, π_t, $\mathrm{UI}_t^{\text{guess}}$
3. **Targets:**
   3.1 Error in Job Value
   3.2 Error in Vacancy Value
   3.3 Error in Law-of-Motion for u_t
   3.4 Error in Philips Curve
   3.5 Error in Asset Market Clearing
   3.6 $u_t = U_t^{hh} = \int 1\{u_{it} > 0\} d\boldsymbol{D}_t$
   3.7 $\mathrm{UI}_t^{\text{guess}} = \mathrm{UI}_t^{hh} = \int \mathrm{UI}_{it} d\boldsymbol{D}_t$

Steady State

1. **Zero inflation:** $\pi_t = 0$
2. **SAM:** Choose A and κ to ensure $\delta_{ss} = 0.02$ and $\lambda_{ss}^{u,s} = 0.30$
3. **HANK:** Enforce *asset market clearing*
   3.1 Set r_{ss}
   3.2 Calculate implied A_{ss}^{hh}
   3.3 Adjust G_{ss} so $q_{ss} B_{ss} = A_{ss}^{hh}$

Calibration

1. **Real interest rate:** $1 + r_t = 1.02^{\frac{1}{12}}$
2. **Households:** $\sigma = 2.0$
   30%: $\beta_i = \beta^{\text{HtM}} = 0$
   60%: $\beta_i = \beta^{\text{BS}} = 0.94^{\frac{1}{12}}$
   10%: $\beta_i = \beta^{\text{PIH}} = 0.975^{\frac{1}{12}}$
3. **Matching and bargaining:** $\alpha = 0.60$, $\theta = 0.60$, $w_{ss} = 0.90$
4. **Producers:** $\beta^{\text{firm}} = 0.975^{\frac{1}{12}}$
5. **Price-setters:** $\epsilon = 6$ and $\phi = 600$
6. **Monetary policy:** $\phi = 1.5$
7. **Government:**
   Tax: $\tau = 0.30$
   Debt: $\delta_q = 1 - \frac{1}{36}$ and $\omega = 0.05$
   UI: $\overline{\phi} = 0.70$, $\underline{\phi} = 0.40$, and $\overline{u} = 6$

Steady state analysis

In **steady state:**

1. Look at the consumption functions
2. Look at the distribution of savings
3. Look at how consumption evolves in unemployment

Policy analysis

Shock: Consider a 1% shock to government consumption

$$G_t - G_{ss} = 0.80^t \cdot 0.01 \cdot G_{ss}$$

Look at **impulse responses** for**:**

1. Output
2. Unemployment (risk)
3. Tax rate

What drives the consumption response?

1. Interest rate
2. Tax rate
3. Job-finding rate
4. Dividends

Is the effect from the job-finding rate larger than an equivalent change in income causes by wages? Why?

HANK-SAM

Stimulus Effects of Common Fiscal Policies

Stimulus Effects of Common Fiscal Policies

- **Many types of fiscal policy:**
  1. Government consumption, G_t
  2. Universal transfer, $T_t = \text{transfer}_t$
  3. Higher unemployment benefits, $\overline{\phi}_t$
  4. Longer unemployment benefit duration, $\overline{u}_t$
  5. Hiring subsidies, hs_t
  6. Retention subsidies, rs_t
- **Extended model:**
  1. Endogenous separations + sluggish entry
  2. Dividends distributed equally
  3. Decreasing search intensity/efficiency while unemployed
  4. Risk of no unemployment benefits
  5. More detailed calibration
- **Previous paper:** Broer et. al. (2024) in zero liquidity

Model summary

- **Notation:** $\boldsymbol{x} = [X_0 - X_{ss}, X_1 - X_{ss}, \dots]$
- **Household policies:**

$$\boldsymbol{h} = \left[\boldsymbol{g}, \boldsymbol{t}, \overline{\boldsymbol{\phi}}, \overline{\boldsymbol{u}}\right]'$$

- **Firm policies:**

$$\boldsymbol{f} = [\boldsymbol{hs}, \boldsymbol{rs}]'$$

- **Income process:**

$$\boldsymbol{inc} = [\boldsymbol{\delta}, \boldsymbol{\lambda^u}, \boldsymbol{div}]'$$

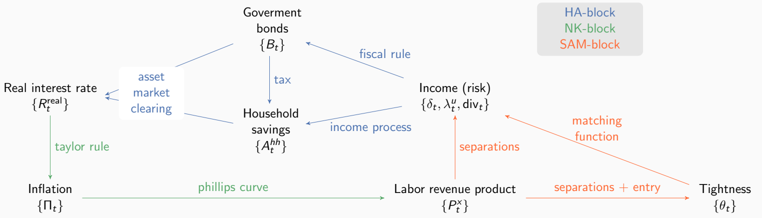
- **Model summary:**

$$\boldsymbol{r^{real}} = M_{HA}\boldsymbol{inc} + M_{h,r}\boldsymbol{h} + M_{f,r}\boldsymbol{f}, \tag{1}$$

$$\boldsymbol{p^x} = M_{NK}\boldsymbol{r^{real}}, \tag{2}$$

$$\boldsymbol{inc} = M_{SAM}\boldsymbol{p^x} + M_{s,inc}\boldsymbol{f}. \tag{3}$$

Directed Cycle Graph



Directed Cycle Process

Let $||\cdot||$ denote the operator norm. If $||M_{SAM}M_{NK}M_{HA}|| < 1$, there is a unique solution to the system (1)-(3) given by

$$\boldsymbol{inc} = \underbrace{\mathcal{G}}_{\text{GE}} \times \left(\underbrace{M_{\text{SAM}} M_{\text{NK}} \underbrace{M_{h,r}\boldsymbol{h}}_{\text{direct}}}_{\text{first round, household transfer policy}} + \underbrace{M_{SAM} M_{NK} \underbrace{M_{f,r}\boldsymbol{f}}_{\text{direct}} + \underbrace{M_{f,\text{inc}}\boldsymbol{f}}_{\text{direct}}}_{\text{first round, firm transfer policy}} \right),$$

where $\mathcal{G}$ is defined by

$$\mathcal{G} = (I - M_{SAM} M_{NK} M_{HA})^{-1}.$$

Fiscal multipliers

- **Fiscal multiplier:**

$$\mathcal{M} = \text{cumulative fiscal multiplier} = \frac{\mathbf{1}'\boldsymbol{y}}{\mathbf{1}'\boldsymbol{taxes}}.$$

$$\boldsymbol{taxes} = M_{\text{inc,taxes}}\boldsymbol{inc} + M_{h,taxes}\boldsymbol{h}$$

- **Household policies 0 and 1:** If same direct PE real interest rate

$$M_{h,r}\boldsymbol{h}^0 = M_{h,r}\boldsymbol{h}^1$$

then output and income are the same $\boldsymbol{y}^0 = \boldsymbol{y}^1$ and $\boldsymbol{inc}^0 = \boldsymbol{inc}^1$. Differences in taxes are due to direct fiscal costs

$$\mathbf{1}'\boldsymbol{taxes}^0 - \mathbf{1}'\boldsymbol{taxes}^1 = \mathbf{1}'M_{h,\text{taxes}}\boldsymbol{h}^0 - \mathbf{1}'M_{h,\text{taxes}}\boldsymbol{h}^1,$$

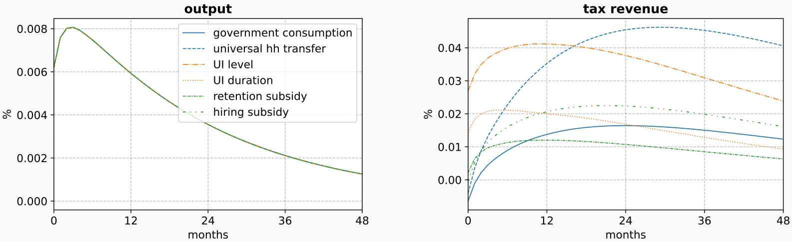
Fiscal multipliers are ordered by direct fiscal costs:

$$\mathcal{M}_{h^0} \gtreqless \mathcal{M}_{h^1} \iff \mathbf{1}'M_{h,\text{taxes}}\boldsymbol{h}^0 \lesseqgtr \mathbf{1}'M_{h,\text{taxes}}\boldsymbol{h}^1.$$

- **Firm policies:** Same result, but only with representative agent

Policy experiment

- **Experiment:** Same output path for different policies.



Different fiscal multipliers

	G [level]	Household transfers: Transfer	Household transfers: Level	Household transfers: Duration	Firm transfers: Retention	Firm transfers: Hiring
1. Relative fiscal multiplier	1.0 [0.99]	0.28	0.44	1.03	1.64	0.72
2. Relative tax response	1.00	3.64	2.29	0.97	0.61	1.39
3. PE relative tax response	1.47	4.11	2.77	1.45	0.57	1.56
4. GE relative tax response	-0.47	-0.47	-0.47	-0.47	0.04	-0.17

- **Relative fiscal multiplier:** $\frac{\mathcal{M}_{h^j}}{\mathcal{M}_{h^G}}$
- **Relative tax respones:** $\frac{\mathbf{1}'\boldsymbol{taxes}^j}{\mathbf{1}'\boldsymbol{taxes}^G}$
  Decomposition for household transfers:

$$\boldsymbol{taxes}^j = M_{\text{inc},\text{taxes}}\boldsymbol{inc}^j + M_{h,taxes}\boldsymbol{h}^j$$

$$\boldsymbol{taxes}^{j,\text{PE}} = M_{h,\text{taxes}}\boldsymbol{h}^j$$

$$\boldsymbol{taxes}^{j,\text{GE}} = M_{\text{inc},\text{taxes}}\boldsymbol{inc}^j$$

Determinants of fiscal multipliers

	G [level]	Household transfers: Transfer	Household transfers: Level	Household transfers: Duration	Firm transfers: Retention	Firm transfers: Hiring
1. Baseline	1.0 [0.99]	0.28	0.44	1.03	1.64	0.72
2. Less sticky prices ($\phi = 178$)	1.0 [0.61]	0.30	0.47	1.03	3.43	1.15
3. More reactive mp ($\delta_\pi = 2$)	1.0 [0.64]	0.30	0.47	1.03	3.33	1.13
4. Representative agent	1.0 [0.54]	0.00	0.00	0.00	1.92	0.57
5. Fewer HtM (17.4%)	1.0 [0.80]	0.19	0.41	1.11	1.80	0.69
6. More tax financing ($\omega = 0.10$)	1.0 [0.84]	0.19	0.40	1.10	1.70	0.67
7. Exo. separations ($\psi = 0$)	1.0 [0.13]	0.35	0.52	1.02	1.39	3.38
8. Free entry ($\xi = \infty$)	1.0 [0.54]	0.31	0.47	1.03	1.50	1.21
9. Wage rule ($\eta_e = 0.50$)	1.0 [0.73]	0.29	0.46	1.03	1.55	0.74
10. 95% of div. to PIH	1.0 [0.82]	0.28	0.43	0.99	0.72	0.16

I-HANK

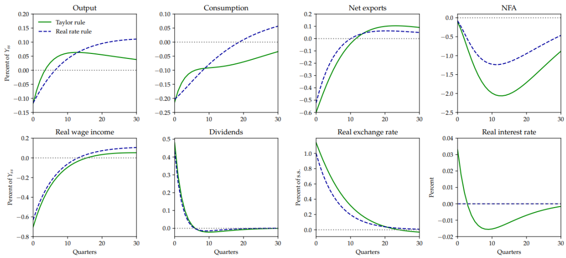
Models

- **So far:** Closed economy
- **Reality:** Many countries are small open economies
- **Baseline New Keynesian model:** Gali and Monacelli

Contractionary depreciations

- Auclert et. al. (2024): **Exchange Rates and Monetary Policy with Heterogeneous Agents: Sizing up the Real Income Channel**
  1. Depreciation causes a fall in real income
  2. Much stronger with high MPC households
  3. Depreciation can be contractionary

Figure 8: Contractionary depreciations



Note: impulse response in the quantitative model to the shock to ι_t^* from figure 2. The model with Taylor rule is our quantitative model; the one with real rate rule is our quantitative model without the Taylor rule.

Fiscal multipliers in small open economies

- Sundram (2024), **Fiscal Policy in Small Open Economies: The International Intertemporal Keynesian Cross**
    1. The cumulative fiscal multiplier is 1 with a real rate rule
    2. Consumption boom now, but bust later on as $NFA_t \rightarrow NFA_{ss}$
    3. The fiscal multiplier jumps in the limit of $\alpha \rightarrow 0$

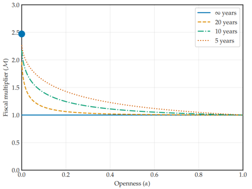


Figure 3: The Cumulative Fiscal Multiplier at Different Horizons

Note: The figure shows the cumulative fiscal multiplier from eq. (17) for different values of α and different horizons, T. The fiscal multipliers are computed using the model with an HA household side using the calibration from Appendix A.1.5.

- Druedahl et. al. (2025), **Fiscal Multipliers in Small Open Economies with Heterogeneous Households**
    1. A number of RA vs HA equivalence results can be proven
    2. Difference in fiscal multipliers smaller than in closed economies

Foreign demand shocks

- Druedahl et. al. (2024), **The Transmission of Foreign Demand Shocks.** Response to foreign demand shock:
  1. RA: Consumption falls
  2. HA: Consumption increase

$\Rightarrow$ foreign demand shocks important for international co-movement

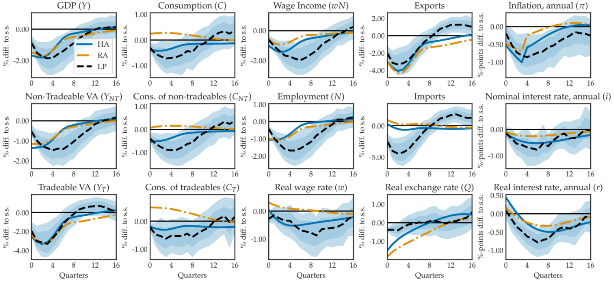


Figure 4: Response of the domestic economy to a foreign demand shock

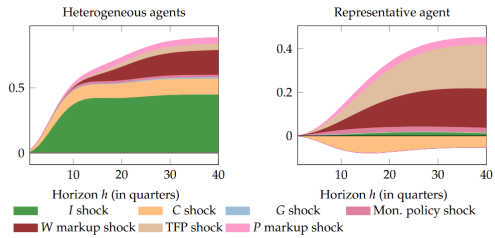
Note: The figure shows the response of domestic variables to a foreign demand shock in the HANK and the RANK model, alongside our empirical LP-based results from Section 2.

Summary

Summary

- **This lecture:** HANK models
  1. Some aggregate neutrality results - still distributional concerns
  2. Size of mechanisms are different - cash-flow effects important
  3. High MPC and precautionary saving become of central importance
- **Business cycles:** Corr. of C and I from shock to I (not vice versa)

Figure 13: Decomposition of forecast error covariance between consumption and investment



Source: Auclert et. al. (2020), Micro Jumps, Macro Humps: Monetary Policy and Business Cycles in an Estimated HANK Model