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A mathematical model for perishable products with price- and displayed-stock-dependent demand

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ABSTRACT

We introduce an economic order quantity model that incorporates product assortment, pricing and space-allocation decisions for a group of perishable products. The goal is to maximize the retailer's profit under shelf-space and backroom storage capacity constraints. We assume that the demand rate of a product is a function of the selling prices and the displayed stock levels of all the products in the assortment. We propose a Tabu Search based heuristic method to solve this complex problem.

1. Introduction

Walking into any grocery store, it may be striking to come across a salad-dressing aisle filled with dozens of varieties spread across two dozen brands, or to figure out what kind of Oreo brand cookie to choose among nearly 50 different versions of its classic cookie, or to stand in front of a Pantene hair-care aisle and decide on buying one of the 88 kinds of Pantene shampoo, conditioner and styling products. Apparently, nowadays, supermarkets are packing their shelves with an ever-expanding array of products in different brands, sizes, colors, flavors and prices. According to survey data from the Food Marketing Institute, nearly 47,000 distinct products filled a typical supermarket retailer's shelves by 2008, up more than 50% from 1996.[1] Likewise, *New Product News* reports that in the food category alone, 1677 new products were introduced in 2001 by the 20 largest food companies in the U.S., which was the highest figure since the mid-1990s. The fact is, companies are overwhelmingly following the 'more is better' route and increasing the number of alternatives they offer within their brands. In today's strategic landscape, a broader portfolio of products often can help a company capture more value as it increases not only the chances of appealing to a wider variety of customers but also the number of other products that can benefit from a hit product's popularity. The yellow flag here is that a broader product portfolio usually heightens three strategic challenges facing managers: management of scarce retail display areas/shelf-space, inventory management, and pricing. Recently, the number of new products offered to the supermarkets has grown at too fast a pace compared to the increase in the average size of a store and in the amount of shelf-space available. *USDA's Economic Research Service estimates the number of packaged food products available to American consumers to be about 320,000. However, a typical supermarket can accommodate only 50,000 products, including non-food items.*[2] This points to a constant battle between different products for the limited shelf space in stores - adding variety in a product category and allocating a larger shelf space to it can result in trimming variety and space for other products.

The proliferation of products competing for limited shelf spaces necessitates retailers to scrutinize their shelf space management strategies, at the heart of which lies the major concern for determining the appropriate amount of inventory displayed. For particular product categories, consumer demand increases when stores maintain higher inventory levels. Balakrishnan, Pangburn, and Stavrulaki (2004) suggest that tall stacks of an item can increase

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[1] http://www.gpnmag.com/something%E2%80%99s-happening-here-0, accessed September 19, 2012.

[2] http://www.ers.usda.gov/publications/foodreview/sep1997/sept97g.pdf, accessed June 16, 2011.

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sales for various reasons such as "increasing product visibility, kindling latent demand, signaling a popular product, or providing consumers an assurance of future availability." This demand stimulation factor particularly applies to impulse purchase of novelty items (e.g., fashion items, magazines, and consumer electronics products) that can be displayed effectively. The facings or display spaces let them be more visible, and the visibility in turn creates additional demand. This promotional role of inventory is often described by retail managers as the "stack them high, let them fly" phenomenon.

In a competitive market, prices of goods is another factor that affects the decision making process of a customer. Generally, a reduced price encourages a customer to buy more. Both the price and the shelf space allocated for an item are important factors that determine the demand of that item. Therefore, besides determining shelf spaces for the items, the retail managers should also determine the prices of the items. What is more interesting is that the demand of a particular product might also be affected by the prices and the spaces allocated for the other products on the shelves. That is, there might be cross dependencies between products on the shelves. Urban (1969) notes that interdependencies among the brands of a firm's product line should be considered when making marketing strategies. This implies that the retail managers should not only determine the prices and the display areas allocated to the stocked units, they should also determine which brands and/or products should be displayed together on the shelves to maximize their overall profits. The problem of determining the optimum portfolio of brands and/or products to get the best use of available shelves is called the product assortment problem.

The product assortment, pricing and shelf allocation decisions are particularly important when the items are perishable, because (i) it is more desirable to have higher demand rates for perishable items (in order to sell as much as possible in their short shelf lives) and (ii) due to their certain storage requirements (e.g., low temperature and humidity, bookkeeping to ensure on-time replenishments) the shelf spaces are limited for those items.

Deteriorating items are usually classified with respect to how their utilities change in time. Some items deteriorate continuously over time, where the deterioration rate is usually proportional to the amount of available inventory (see Goyal and Giri (2001) and Bakker, Riezebos, and Teunter (2012)). Some items have certain lifetimes after which they deteriorate completely. These items are said to deteriorate instantaneously (see Urban (2005) and Bakker et al. (2012)). For continuously deteriorating items, deterioration increases the depletion rate of the items in the stock. Since the demand is affected by the inventory level, deterioration affects instantaneous the demand rate of the item. On the other hand, for instantaneously deteriorating items (i.e., items with predetermined shelf lives), shelf life restrictions force the retailers to keep fewer number of items in stocks, although the retailer has a tendency to keep high levels of inventory to stimulate demand.

These observations motivate the focus of this paper. We establish a product assortment model in which items are perishable and the demand rates of the items depend on the displayed stock levels and the selling prices of the items in the assortment list. To the best of our knowledge, no study has yet integrated a perishable inventory model with assortment, pricing and shelf-space allocation decisions. In our model, we assume that items have predetermined shelf lives after which they must be discarded. We assume that items are of constant value to the consumer as long as they don't expire. Some examples are prescription drugs, pharmaceuticals (e.g., vitamins, cosmetics), chemicals (e.g., household cleaning products), batteries, photographic films, and frozen food. These products have a fixed shelf life, and they undergo decay but face no appreciable decrease in value during their usable lifetime. This also includes product categories that undergo unobservable change in storage so that they may become obsolete but offer non-decaying utility.

The remainder of the paper is organized as follows. Section 2 presents some related literature on inventory models that incorporate effects of displayed stock levels, prices and perishability. We define our product assortment problem, list our notation and assumptions in Section 3. We present a mathematical formulation of the problem in Section 4. In Section 5 we propose a metaheuristic solution approach to this problem. The numerical results and some managerial insights are reported in Section 6. Section 7 concludes the paper with a discussion of the main results and potential extensions of the model.

2. Literature review

A considerable amount of research has been done on inventory control policies assuming that demand is affected by the amount of available inventory. This research area has been extended in several ways by incorporating pricing, perishability and assortment decisions.

Empirical evidence on the dependence of demand on inventory of certain products has been provided by Larson and DeMarais (1990), who refer to inventory that stimulates demand as "psychic" stock, and more recently by Koschat (2008). In particular, Koschat (2008) presents empirical evidence that demand can indeed vary with inventory by investigating a major US magazine publisher, and quantifies the magnitude of the dependence of demand of a brand on its own inventory as well as the inventory of a substitutable magazine brand. In addition to these empirical papers, a number of theoretical models have been developed to determine the appropriate inventory policy with the impact of inventory on sales. Gupta and Vrat (1986) introduced the notion of endogenous, inventory-dependent, deterministic demand as a function of the initial stock level. Later on, Baker and Urban (1988) developed an inventory model where the demand rate depends not on the initial, but on the instantaneous inventory level throughout sales. In their model, they assumed that the zero-inventory-order (ZIO) property holds, that is, new items are not ordered unless the items in stocks are consumed completely. Urban (1992) improved this model by relaxing the ZIO property. Recently, Alfares (2007) has proposed an inventory model with stock-level dependent demand rate and variable holding cost, and Soni and Shah (2008) has formulated optimal ordering policies when demand is partially constant and partially dependent on the stock. More recent examples on inventory models with inventory dependent-demand include, Pando, García-Laguna, and San-José (2012), which assumes nonlinear holding costs; and Alfares (2015), which assumes finite production rates.

Some researchers focused their attention on the inventory models that coordinate pricing decisions with stock-dependent demand. Urban and Baker (1997) generalized the EOQ model by formulating the demand as a multivariate function of price, time and inventory level. Datta and Paul (2001) and You and Hsieh (2007) analyzed multi-period models with stock-dependent and price-sensitive demand rate.

There is also a huge stream of research that incorporates perishability into EOQ models with inventory dependent demand. Earlier examples include Padmanabhan and Vrat (1990),Giri, Pal, and Chaudhuri (1996), Hwang and Hahn (2000), Zhou and Yang (2003), and Dye and Ouyang (2005). Teng and Chang (2005) established an economic production quantity model with finite production rate, and Chang, Chen, Tsai, and Wu (2010) proposed an EOQ model, for continuously deteriorating items, where the demand rate depends on the selling price and the displayed stock level of the item. Wu, Ouyang, and Yang (2006) considered a replenishment policy for a continuously deteriorating item item with backlogging and stock dependent demand. Recently, Zhang, Wang, Lu, and Tang (2015) and Lu, Zhang, and Tang (2016) developed inventory models that incorporates a dynamic pricing strategy to the EOQ model for a continuously deteriorating item with stock dependent demand. Chintapalli (2015) considered the inventory control problem of perishable goods where the demand is price sensitive and customers are free to choose between new and old units.

All the models cited above consider items in isolation. However, the demand for an item in the store may depend on the inventory level and the prices of the other items in the store as well. Considering this, Corstjens and Doyle (1981) proposed a model which allocates shelf spaces to a given selection of products to maximize total profit of a retailer. In this model, the demand of each product is a function of the shelf facings allocated to all the items in the store. Martínez-de-Albéniz and Roels (2011) used the shelf space model of Corstjens and Doyle (1981) and considered a retailer that optimizes its shelf space allocation among a given selection of products. The manufacturers of the products set their prices so as to obtain larger shelf spaces. On the other hand, Urban (1969) developed a model to identify which products should be included in a firm's product line. In the model, the demand rate of an item is a function of the shelf facings allocated to each product in the assortment list. To identify the most profitable assortment list, Urban (1969) proposed to use a trial and error search technique. Urban (1998) improved the work in Urban (1969) by modeling the demand as a function of the instantaneous inventory level and the shelf spaces allocated to other items in the assortment. Later, Hariga, Al-Ahmari, and Mohamed (2007) proposed an optimization model to determine the product assortment for a retailer. Hariga et al. (2007) extended the demand model of Urban (1998) such that demand of an item depends also on the location of the items in the assortment list as well. Irion, Lu, Al-Khayyal, and Tsao (2012) worked on a product assortment problem where the demand of an item is a function of the shelf faces allocated for the items in the assortment. They proposed a piecewise linearization technique to approximate the nonlinear space allocation model. Katsifou, Seifert, and Tancrez (2014) modeled a "customer attraction" model where the retailer carries a combined product assortment of standard and fashionable short lived products to attract heterogeneous classes of customers. Most recently, Hübner, Kuhn, and Kühn (2016) considered a newsvendor based decision model that jointly optimized the assortment and the associated order quantities, and Goyal, Levi, and Segev (2016) considered a single-period joint assortment and inventory planning problem under dynamic substitution with stochastic demands.

In this paper, we consider the assortment problem of a retailer who has to choose among a set of perishable items. We assume that items have deterministic shelf lives. We extend the demand model of Urban (1998), by incorporating the effect of prices of the items in the assortment. We propose a Tabu Search based heuristic, where we utilize several other heuristic approaches in each step of the search process.

3. Model definition

We consider a single retail store with a shelf space capacity of S^{capa} and a backroom space capacity of B^{capa}. The store must decide on an assortment out of n items. The items in consideration are represented by the set $N = \{1, 2, \dots, n\}$, whereas the items in the assortment are represented by the set N^+, where $N^+ \subseteq N$. Besides assortment, the store must decide on prices, p_i, and the shelf spaces, S_i, allocated for each item $i \in N^+$. Both the price and the shelf space allocated for the product affect its consumption rate. Moreover, there are cross dependencies between products such that the demand rate of an item depends also on the prices and shelf spaces allocated for the other items in the assortment.

We assume that item i has a predetermined shelf life, T_i, after which it is salvaged. We assume that items do not lose their utilities until the end of their shelf lives. We assume that replenishments to the retailer are independent for each item (no joint replenishments) and they are sent instantaneously to the backroom storage area with zero lead time. As soon as the order is received, S_i units are immediately transferred to the display area. Thereafter, the display space allocated for item i is continuously refilled from the backroom area to maintain it fully stocked. During this time period, item i is displayed at a constant level S_i on the shelf. When the backroom stock is depleted for the item, on-display stock level starts decreasing since replenishment from backroom storage can no longer be accomplished. The replenishment cycle ends either when all the items in the stocks are consumed or when the lifetime for the item is reached. In the latter case, remaining items are not good for consumption anymore, and they are sold at a salvage value of r_i per unit.

We assume that no product uses the area dedicated to another product even if the available inventory is inadequate to fill it completely. Shortage situations are not considered because, with the demand-rate pattern under consideration, there is no demand when the inventory level reaches zero. Furthermore, the retailer cannot markdown the price of any item to stimulate sales within a replenishment cycle. Unit procurement, holding, and shelf-space costs for each item are known and constant over time. The objective is to maximize the net profit comprised of the gross revenue with salvage value less the procurement cost, inventory investment cost, and backroom and display area maintenance costs.

4. Mathematical model

In this section we build a mathematical model of the problem. First, we present the details on the assumed demand structure. Then, we show how we compute the cost and revenue terms. Finally, we present the full mathematical formulation of the problem.

4.1. Demand function

Inspired with the existing literature (see Urban (1998) and Hariga et al. (2007)), we assume that the consumption rate of an individual product i at time t, $d_i(t)$, is defined as follows:

$$d_i(t) = \alpha_i \left[s_i(t)^{\beta_i} \prod_{\substack{j \neq i \\ j \in N^+}} s_j(t)^{\delta_{ij}} \right] \left[p_i^{\gamma_i} \prod_{\substack{j \neq i \\ j \in N^+}} p_j^{\mu_{ij}} \right], \tag{1}$$

where for $i = 1, \dots, n$,

- α_i: scale parameter of product i. This represents the average demand rate of the item without the price and stock availability effect. For demand to be positive, we assume $\alpha_i > 0$.
- β_i: main space elasticity of product i. It represents the positive effect of displayed inventory. We assume $\beta_i > 0$.
- δ_{ij}: cross-space elasticity between products i and j. It represents the change in quantity demanded for product i that occurs in response to a change in the number of units displayed of product j. The cross-space elasticity of demand can be negative, zero, or positive in cases where two items are complements, independent or substitutes.
- γ_i: main price elasticity of product i. It represents the change in quantity demanded for item i in response to a change in price of item i. We assume $\gamma_i < 0$.

μ_{ij}: cross-price elasticity between products i and j. It represents the change in quantity demanded for product i in response to a change in price of product j. The cross-price elasticity of demand can be negative, zero, or positive in cases where two items are complements, independent or substitutes.

$s_i(t)$: instantaneous on-display inventory level of item i at time t.

Defining $d_i(t)$ as a function of instantaneous on-display stock levels of products included in the assortment undoubtedly increases the complexity of the model. In order to simplify the evaluation of $d_i(t)$, Urban (1998) approximates demand rate as a function of the allocated shelf-space, not the instantaneous on-display inventory level, of all other products in the assortment. In this regard, replacing $s_j(t)$ in Eq. (1) with S_j, demand rate of an item i is simplified as a function of its on-display inventory level and the shelf-spaces dedicated to other products in the assortment. Therefore, throughout the rest of the paper, we are going to assume that $d_i(t)$ is defined as follows:

$$d_i(t) = \alpha_i \left[s_i(t)^{\beta_i} \prod_{\substack{j \neq i \\ j \in N^+}} S_j^{\delta_{ij}} \right] \left[p_i^{\gamma_i} \prod_{\substack{j \neq i \\ j \in N^+}} p_j^{\mu_{ij}} \right]. \tag{2}$$

Note that with this demand structure, the demand rate for product i at time zero is given by

$$d_i(0) = \alpha_i \left[S_i^{\beta_i} \prod_{\substack{j \neq i \\ j \in N^+}} S_j^{\delta_{ij}} \right] \left[p_i^{\gamma_i} \prod_{\substack{j \neq i \\ j \in N^+}} p_j^{\mu_{ij}} \right]. \tag{3}$$

Then, it is straightforward to see that the following relation holds:

$$d_i(t) = d_i(0) \left[\frac{s_i(t)}{S_i} \right]^{\beta_i}. \tag{4}$$

4.2. Cost and revenue terms

Since the cost parameters are stationary over time and the planning horizon is infinite, for any choice of p_j and S_j for $j \in N^+$, there exits a cost minimizing inventory plan that is composed of identical replenishment cycles. Since the replenishment cycles are identical, to determine the long run cost and revenues per unit time, it is sufficient to calculate those terms for a single cycle.

Let Q_i represent the order quantity, and CT_i represent the total duration of a single replenishment cycle of a generic item i. We set t to 0 at the beginning of the replenishment cycle so that each replenishment cycle is defined on the interval $[0, CT_i]$. Then, we divide the whole cycle into two subintervals: $[0, \tau_i^{BS}]$ and $[\tau_i^{BS}, CT_i]$, where τ_i^{BS} refers to the time duration required to deplete the backroom storage of item i. We also let $\tau_i^{DA} = CT_i - \tau_i^{BS}$ be the remaining time until the end of the replenishment cycle after the backroom storage is depleted. Clearly, we have $0 \leqslant \tau_i^{BS} \leqslant CT_i = \tau_i^{BS} + \tau_i^{DA} \leqslant T_i$.

Consider the interval $[0, \tau_i^{BS}]$. Since $s_i(t) = S_i$ for $t \in [0, \tau_i^{BS}]$, due to (4), we have that

$$d_i(t) = d_i(0), \text{ for } t \in [0, \tau_i^{BS}].$$

That is, inventory is consumed at a constant rate of $d_i(0)$ in $[0, \tau_i^{BS}]$. Moreover, observe that in an optimal solution, no items deteriorate in the backroom storage. Otherwise we can always order less and this does not change anything but reduce costs in the replenishment cycle. Therefore, Q_i cannot exceed $S_i + T_i d_i(0)$, and, given τ_i^{BS}, we have

$$Q_i = S_i + \tau_i^{BS} d_i(0).$$

Letting $I_i(t)$ be the instantaneous total inventory level of item i (i.e., the total inventory of item i both in the display area and in the backroom storage) at time t, we have

$$I_i(t) = Q_i - d_i(0)t = S_i + d_i(0)\left(\tau_i^{BS} - t\right) \text{ for } 0 \leqslant t \leqslant \tau_i^{BS}.$$

Now, consider the interval $[\tau_i^{BS}, CT_i]$, where (4) implies that

$$d\frac{s_i(t)}{dt} = -d_i(t) = -d_i(0)\left(\frac{s_i(t)}{S_i}\right)^{\beta_i}. \tag{5}$$

With the condition $s_i(t) = S_i$ for $t = \tau_i^{BS}$, the differential Eq. (5) implies

$$s_i(t) = \left[S_i^{1-\beta_i} - \frac{d_i(0)(1-\beta_i)}{S_i^{\beta_i}} (t - \tau_i^{BS}) \right]^{\frac{1}{1-\beta_i}}, \text{ for } \tau_i^{BS} \leqslant t \leqslant CT_i. \tag{6}$$

Then, since $I_i(t) = s_i(t)$ for $t \in [\tau_i^{BS}, CT_i]$, the instantaneous total inventory level within the whole replenishment cycle of item i is defined as follows:

$$I_i(t) = \begin{cases} S_i + d_i(0)\left(\tau_i^{BS} - t\right), & \text{for } 0 \leqslant t \leqslant \tau_i^{BS} \\ \left[S_i^{1-\beta_i} - \frac{d_i(0)(1-\beta_i)}{S_i^{\beta_i}} (t - \tau_i^{BS}) \right]^{\frac{1}{1-\beta_i}}, & \text{for } \tau_i^{BS} \leqslant t \leqslant CT_i. \end{cases}$$

There are two cases to consider. In one, the displayed inventory is consumed before the items deteriorate (as illustrated in Fig. 1a), and in the other one items deteriorate before the displayed inventory is depleted (as illustrated in Fig. 1b). In the former case, solving for $s_i(t) = 0$ in (6), we obtain $\tau_i^{DA} = \frac{S_i}{d_i(0)(1-\beta_i)}$. In the latter case, we have $CT_i = T_i$, and hence $\tau_i^{DA} = T_i - \tau_i^{BS}$. Therefore, we have the following equality:

$$\tau_i^{DA} = \min\left\{ T_i - \tau_i^{BS}, \frac{S_i}{d_i(0)(1-\beta_i)} \right\} \text{ for } i = 1, \ldots, n. \tag{7}$$

In the following subsections, we are going to define cost and revenue terms in a replenishment cycle of item i. Our aim is to define these terms solely as functions of S_i, p_i, $d_i(0)$ (which itself is a function of S_j and p_j for $j \in N+$), τ_i^{BS}, and τ_i^{DA}.

4.2.1. Cost terms

The retailer has three major cost components: (i) inventory investment costs, (ii) backroom storage and display area maintenance costs, and (iii) procurement costs.

(*a*) *Inventory investment costs*

We assume that the retailer incurs an investment cost of h_i per item held in the inventories per unit time. Then, the inventory investment cost per replenishment cycle of item i ($ICPRC_i$) is calculated as follows:

$$\begin{aligned} ICPRC_i &= \int_0^{CT_i} h_i I_i(t)dt = \int_0^{\tau_i^{BS}} h_i \left[S_i + d_i(0)\left(\tau_i^{BS} - t\right) \right] dt \\ &+ \int_{\tau_i^{BS}}^{\tau_i^{BS}+\tau_i^{DA}} h_i \left[S_i^{1-\beta_i} - \frac{d_i(0)(1-\beta_i)}{S_i^{\beta_i}} (t - \tau_i^{BS}) \right]^{\frac{1}{1-\beta_i}} dt \\ &= h_i \frac{2S_i + \tau_i^{BS} d_i(0)}{2} \tau_i^{BS} \\ &+ hi \left[\frac{S_i^2}{d_i(0)(2-\beta_i)} - \frac{S_i^{\beta_i}}{d_i(0)(2-\beta_i)} \left[S_i^{1-\beta_i} - \frac{d_i(0)}{S_i^{\beta_i}} (1-\beta_i) \tau_i^{DA} \right]^{\frac{2-\beta_i}{1-\beta_i}} \right]. \end{aligned}$$

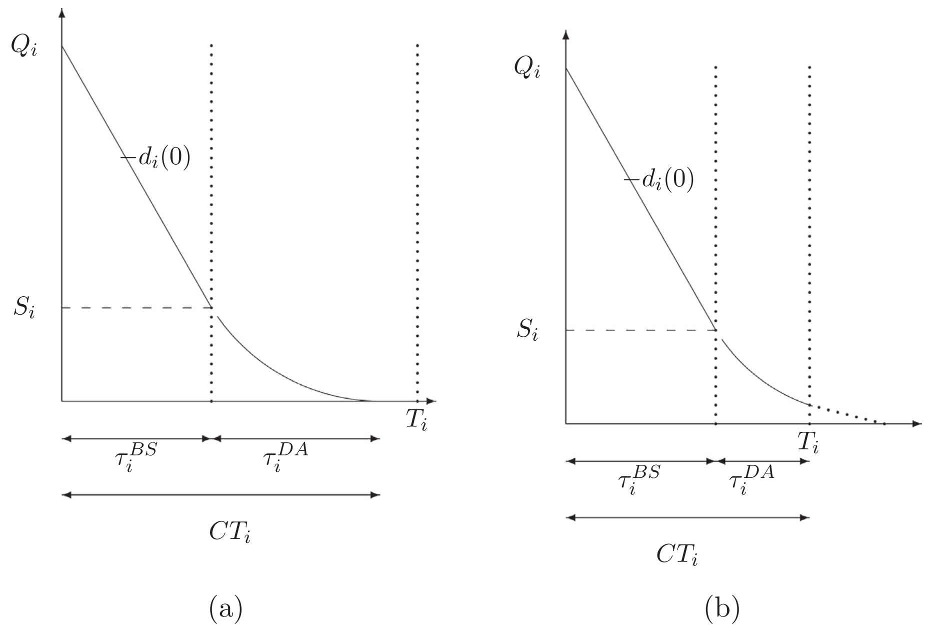

Fig. 1. Variation of total inventory level of item i over time.

Clearly, $ICPRC_i$ is a function of $S_i,\ d_i(0),\ \tau_i^{BS}$ and τ_i^{DA}. We are going to explicitly indicate this dependency by referring to this cost term as $ICPRC_i(S_i, d_i(0), \tau_i^{BS}, \tau_i^{DA})$.

(b) *Backroom storage and display area maintenance costs*

We assume that backroom storage and display area costs are charged per reserved capacity per unit time. In particular, we assume that the cost of reserving space for one unit of item i in the backroom area and the display area per unit time are c_i^{BS} and c_i^{DA}, respectively. Hence, backroom storage costs and display area costs per replenishment cycle of item i ($BCPRC_i$ and $DCPRC_i$, respectively) are defined as follows:

$$BCPRC_i = c_i^{BS} CT_i [Q_i - S_i] = c_i^{BS}\left[\tau_i^{BS} + \tau_i^{DA}\right] d_i(0) \tau_i^{BS}.$$

$$DCPRC_i = c_i^{DA} CT_i S_i = c_i^{DA}\left[\tau_i^{BS} + \tau_i^{DA}\right] S_i.$$

To indicate the dependency of these cost terms on $S_i, d_i(0), \tau_i^{BS}$ and τ_i^{DA}, we are going to refer to them as $BCPRC_i\left(d_i(0), \tau_i^{BS}, \tau_i^{DA}\right)$ and $DCPRC_i\left(S_i, \tau_i^{BS}, \tau_i^{DA}\right)$, respectively.

(c) *Procurement costs*

We assume that the retailer incurs a fixed setup cost of K_i associated with every order arrival and c_i for every unit purchased. Therefore, total purchase cost for item i per replenishment cycle ($PCPRC_i$) is as follows:

$$PCPRC_i = K_i + c_i Q_i = K_i + c_i\left[S_i + \tau_i^{BS} d_i(0)\right].$$

We are going to refer to this cost term by $PCPRC_i\left(d_i(0), \tau_i^{BS}\right)$.

4.2.2. Revenue terms

We can compute the revenue per replenishment cycle for product i ($RPRC_i$) as follows:

$$\begin{aligned} RPRC_i &= p_i[Q_i - S_i] + p_i[S_i - s_i(CT_i)] + r_i s_i(CT_i) \\ &= \tau_i^{BS} d_i(0) p_i + p_i S_i + (r_i \\ &\quad - p_i)\left[S_i - \left(S_i^{1-\beta_i} - \frac{d_i(0)}{S^{\beta_i}}(1-\beta_i)\tau_i^{DA}\right)^{\frac{1}{1-\beta_i}}\right] \end{aligned}$$

The first term in the first line corresponds to the revenue obtained in the interval $[0, \tau_i^{BS}]$; the second term corresponds to the revenue obtained in the interval $[\tau_i^{BS}, CT_i]$; and the third term corresponds to the revenue obtained through salvage, where $s_i(CT_i)$ is equal to the quantity deteriorated at the end of the cycle. Note that $s_i(CT_i) = 0$ if $CT_i \leqslant T_i$. We are going to refer to the total revenue term by $RPRC_i\left(S_i, p_i, d_i(0), \tau_i^{BS}, \tau_i^{DA}\right)$ to stress that it is a function of $S_i,\ p_i,\ d_i(0),\ \tau_i^{BS},\ \tau_i^{DA}$.

4.2.3. Net profit

Net profit per replenishment cycle of item i ($NPPRC_i$) is the difference between the revenues and the costs in that replenishment cycle. We define it as a function of $S_i,\ p_i,\ d_i(0),\ \tau_i^{BS}$ and τ_i^{DA} as follows:

$$\begin{aligned} NPPRC_i\left(S_i, p_i, d_i(0), \tau_i^{BS}, \tau_i^{DA}\right) &= RPRC_i\left(S_i, p_i, d_i(0), \tau_i^{BS}, \tau_i^{DA}\right) \\ &\quad - ICPRC_i\left(S_i, d_i(0), \tau_i^{BS}, \tau_i^{DA}\right) \\ &\quad - BCPRC_i\left(d_i(0), \tau_i^{BS}, \tau_i^{DA}\right) \\ &\quad - DCPRC_i\left(S_i, \tau_i^{BS}, \tau_i^{DA}\right) \\ &\quad - PCPRC_i\left(d_i(0), \tau_i^{BS}\right). \end{aligned}$$

4.3. Formulation

In addition to the variables introduced earlier, let y_i be the binary decision variable to indicate whether or not an item i is in the assortment. Moreover, we introduce auxiliary decision variable NP_i, which, we will show, is equal to the net profit per unit time of item i in an optimal solution. Furthermore, we let Rev_i^{max} represent the upper bound for the total revenues that can be obtained by the sales of item i per unit time, which can as well be an upper bound for NP_i. Rev_i^{max} is simply equal to the product of p_i^{max}, the upper bound for the price of item i, and $d_i(0)^{max}$, the upper bound for the demand rate of item i. Since p_i^{max} is exogenous to the problem, to calculate Rev_i^{max}, it is sufficient to determine $d_i(0)^{max}$, which can be calculated as follows:

$$d_i(0)^{max} = \alpha_i \left[S_i^{\beta_i} \prod_{\substack{j \neq i \\ \delta_{ij} \geqslant 0}} \left(S_j^{max}\right)^{\delta_{ij}}\right]\left[p_i^{\gamma_i} \prod_{\substack{j \neq i \\ \mu_{ij} \geqslant 0}} \left(p_j^{max}\right)^{\mu_{ij}}\right].$$

Then, a mixed integer nonlinear programming (MINLP) formulation of our problem (which we denote by (P) in short) can be written as follows.

$$\text{Maximize} \sum_{i=1}^{n} NP_i \qquad \text{(P)}$$

subject to

$$NP_i \leqslant \frac{NPPRC_i(S_i, p_i, d_i(0), \tau_i^{BS}, \tau_i^{DA})}{\tau_i^{BS} + \tau_i^{DA}} \quad \text{for } i = 1, \ldots, n, \tag{8}$$

$$\log(d_i(0)) = \log(\alpha_i) + \beta_i \log(S_i) + \sum_{j \neq i} \log(S_j)\delta_{ij} y_j + \gamma_i \log(p_i)$$

$$+ \sum_{j \neq i} \mu_{ij} \log(p_j) y_j \quad \text{for } i = 1, \ldots, n, \tag{9}$$

$$NP_i \leqslant Rev_i^{max} y_i \tag{10}$$

$$\sum_{i=1}^{n} d_i(0)\tau_i^{BS} \leqslant B^{cap} \tag{11}$$

$$\sum_{i=1}^{n} S_i y_i \leqslant S^{cap} \tag{12}$$

$$\tau_i^{BS} + \tau_i^{DA} \leqslant T_i \quad \text{for } i = 1, \ldots, n, \tag{13}$$

$$\left[\tau_i^{DA} - \frac{S_i}{d_i(0)(1-\beta_i)}\right]\left[T_i - \tau_i^{BS} - \tau_i^{DA}\right] = 0 \quad \text{for } i = 1, \ldots, n, \tag{14}$$

$$p_i^{min} \leqslant p_i \leqslant p_i^{max}, S_i^{min} \leqslant S_i \leqslant S_i^{max} \quad \text{for } i = 1, \ldots, n, \tag{15}$$

$$0 \leqslant \tau_i^{BS} \leqslant T_i \quad \text{for } i = 1, \ldots, n, \tag{16}$$

$$y_i \in \{0, 1\}. \tag{17}$$

All the variables and parameters used in this formulation are summarized in Table 6 in Appendix A. Eq. (10) guarantees that if the item is not in the assortment, NP_i should be less than or equal to 0, otherwise, it can be as high as its upper bound. For any choice of decision variables S_i, p_i, $d_i(0)$, τ_i^{BS} and τ_i^{DA}, the right hand side of the constraint (8) is equal to the net profit of item i per unit time. Since the objective function maximizes the sum of NP_i values, together with (10), the inequality (8) guarantees that NP_i is equal to the net profit per unit time of item i in an optimal solution. Eq. (9) is a restatement of (3) where we take the logarithm of both sides. It ties $d_i(0)$ to S_j and p_j values for all items $j \in N^+$. Constraint (14) implements (7). It determines whether or not there will be salvage. If the cycle time is less than the shelf life of the item, i.e., $CT_i = \tau_i^{BS} + \tau_i^{DA} < T_i$, then, we should have $\tau_i^{DA} = \frac{S_i}{d_i(0)(1-\beta_i)}$, which implies that there is no salvage and that the displayed inventory drops down to zero solely through demand (as in Fig. 1a). Otherwise, $\tau_i^{DA} = T_i - \tau_i^{BS}$, and some items might deteriorate, and salvage revenue might be obtained (see function $RPRC_i(S_i, p_i, d_i(0), \tau_i^{BS}, \tau_i^{DA})$ and Fig. 1b). Constraints (11) and (12) ensure that backroom and display area capacities are not exceeded. Constraint (15) guarantees that display areas and prices set for the items are within the predefined bounds. Constraint (16) forces τ_i^{BS} to be nonnegative and less than T_i. Finally, constraint (17) forces y_i to be binary.

5. Solution approach

Constraint (9) ties binary variables y to variables S and p, which are input to the function on the right hand side of constraint (8), which is neither a convex nor a concave function of input variables. This makes it difficult for standard MINLP solvers to set correct bounds in their branch-and-bound trees. As a result, they often stop at a suboptimal solution or it may take several hours to find a feasible solution even for relatively small sized problems. This fact justifies heuristic approaches. Therefore, in this section we describe a heuristic approach to solve problem (P). As a first step, we introduce an approximate problem ($\tilde{\text{P}}$) where we approximate the inventory holding cost and revenue functions of problem (P), assuming that the inventory changes linearly over $[\tau_i^{BS}, CT_i]$.

5.1. Approximate problem ($\tilde{\text{P}}$)

Let $\tilde{I}_i(t)$ be the approximate instantaneous total inventory level, and $\tilde{s}_i(t)$ be the approximate displayed inventory level under the assumption that there is a constant demand rate of $d_i(0)(1 - \beta_i)$ on $[\tau_i^{BS}, CT_i]$ in a replenishment cycle of item i. Then, we have

$$\tilde{s}_i(t) = \begin{cases} S_i, & \text{for } 0 \leqslant t \leqslant \tau_i^{BS} \\ S_i - d_i(0)(1-\beta_i)t, & \text{for } \tau_i^{BS} \leqslant t \leqslant CT_i. \end{cases}$$

$$\tilde{I}_i(t) = \begin{cases} S_i + d_i(0)(\tau_i^{BS} - t), & \text{for } 0 \leqslant t \leqslant \tau_i^{BS} \\ S_i - d_i(0)(1-\beta_i)t, & \text{for } \tau_i^{BS} \leqslant t \leqslant CT_i. \end{cases}$$

Clearly, $\tilde{I}_i(t)$ and $\tilde{s}_i(t)$ overestimates $I_i(t)$ and $s_i(t)$ for $t \in [\tau_i^{BS}, CT_i]$, respectively. As a result, inventory holding costs are overestimated in ($\tilde{\text{P}}$), and revenues (obtained either through sales or through salvage) are underestimated if there is salvage. This is illustrated in the graphs of the approximated inventory curve shown in Fig. 2a and b, for the cases where there is salvage and no salvage.

Let $IC\tilde{P}RC_i$ and $R\tilde{P}RC_i$ be the approximated inventory cost and revenues per replenishment cycle of item i in ($\tilde{\text{P}}$), respectively. With this approximation, we have

$$\begin{aligned} IC\tilde{P}RC_i &= \int_0^{CT_i} h_i \tilde{I}_i(t)dt \\ &= \int_0^{\tau_i^{BS}} h_i\left[S_i + d_i(0)\left[\tau_i^{BS} - t\right]\right]dt \\ &\quad + \int_{\tau_i^{BS}}^{\tau_i^{BS}+\tau_i^{DA}} h_i[S_i - d_i(0)(1-\beta_i)t]dt \\ &= h_i \frac{2S_i + \tau_i^{BS} d_i(0)}{2}\tau_i^{BS} + h_i \frac{2S_i - D_i(1-\beta_i)\tau_i^{DA}}{2}\tau_i^{DA}, \end{aligned}$$

and,

$$\begin{aligned} R\tilde{P}RC_i &= p_i[Q_i - S_i] + p_i[S_i - \tilde{s}_i(CT_i)] + r_i\tilde{s}_i(CT_i) \\ &= \tau_i^{BS} d_i(0)p_i + p_i S_i + (r_i - p_i)\left[S_i - d_i(0)(1-\beta_i)\tau_i^{DA}\right]. \end{aligned}$$

Then, formulation ($\tilde{\text{P}}$) can be obtained from formulation (P) by simply replacing constraints (8) and (14) by (18) and (19):

$$NP_i \leqslant \frac{NP\tilde{P}RC_i(S_i, p_i, d_i(0), \tau_i^{BS}, \tau_i^{DA})}{\tau_i^{BS} + \tau_i^{DA}} \quad \text{for } i = 1, \ldots, n, \tag{18}$$

$$\left[S_i - d_i(0)(1-\beta_i)\tau_i^{DA}\right]\left[T_i - \tau_i^{BS} - \tau_i^{DA}\right] = 0 \text{ for } i = 1, \ldots, n, \tag{19}$$

where

$$\begin{aligned} NP\tilde{P}RC_i(S_i, p_i, d_i(0), \tau_i^{BS}, \tau_i^{DA}) &= R\tilde{P}RC_i(S_i, p_i, d_i(0), \tau_i^{BS}, \tau_i^{DA}) \\ &\quad - IC\tilde{P}RC_i(S_i, d_i(0), \tau_i^{BS}, \tau_i^{DA}) \\ &\quad - BCPRC_i(d_i(0), \tau_i^{BS}, \tau_i^{DA}) \\ &\quad - DCPRC_i(S_i, \tau_i^{BS}, \tau_i^{DA}) \\ &\quad - PCPRC_i(d_i(0), \tau_i^{BS}). \end{aligned}$$

5.2. Overall framework of the solution approach

Note that in (P) (and ($\tilde{\text{P}}$)), once the values to the variables y_i, S_i, p_i and τ_i^{BS} are determined, the values of $d_i(0)$ and τ_i^{DA} for

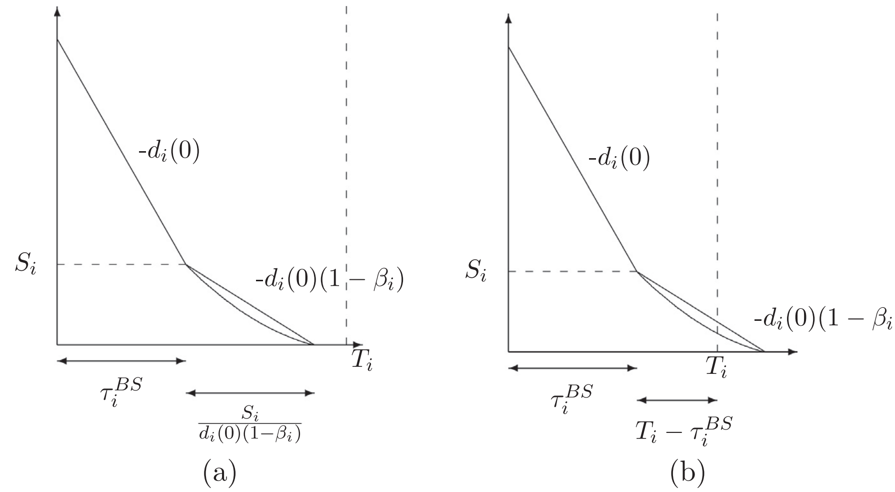

Fig. 2. Approximation of inventory curve on the interval $[\tau_i^{BS}, CT_i]$.

$i = 1,\dots,n$, are automatically determined using equality constraints (9) and (14). As a result, we call the former variables (y_i, S_i, p_i and τ_i^{BS}) the *free variables* and latter variables ($d_i(0)$ and τ_i^{DA}) the *dependent variables*. We can uniquely identify any solution x to (P) (and ($\tilde{\text{P}}$)) by the vector of free variables (y, S, p, τ^{BS}), where y is a $1 \times n$ vector for the assortment list; S is a $1 \times n$ vector for the display areas; p is a $1 \times n$ for the prices; and τ^{BS} is a $1 \times n$ vector for the backroom storage inventory depletion times of items.

Now, consider the two capacity constraints in (P) (and ($\tilde{\text{P}}$)): display area constraints (12) and backroom storage capacity constraints (11). If vectors y and S satisfy the display capacity constraints in ($\tilde{\text{P}}$), they also satisfy that constraint in (P). Likewise, if vectors y and τ^{BS} satisfy the backroom storage capacity constraint in ($\tilde{\text{P}}$), they also satisfy that constraint in (P). Hence, a vector (y, S, p, τ^{BS}) that is feasible to ($\tilde{\text{P}}$) is also feasible to (P). Therefore, we can solve problem ($\tilde{\text{P}}$) and determine the values for y, S, p and τ^{BS} vectors for a feasible solution to (P). Our solution approach exploits this observation and can be summarized as follows:

1. Solve the approximate problem ($\tilde{\text{P}}$).
2. Use the values of the variables y, S, p and τ^{BS} in the solution to ($\tilde{\text{P}}$) to determine the exact values of inventory holding cost and revenue functions in (P).

Although problem ($\tilde{\text{P}}$) is not easy to solve either, the linear approximation promotes implementation of some heuristic approaches. In the following subsection, we describe our tabu search based heuristic approach to solve problem ($\tilde{\text{P}}$).

5.3. Tabu search algorithm to solve ($\tilde{\text{P}}$)

As previously stated, Branch-and-Bound algorithms of standard MINLP solvers stop at local optimal solutions of ($\tilde{\text{P}}$) mostly because they cannot determine correct bounds. To overcome this problem, we propose using a Tabu Search algorithm, which does not (necessarily) require bound information. Tabu Search (see Glover (1989) and Glover (1990)) is an iterative metaheuristic search method that takes a potential solution, evaluates its immediate neighbors at each iteration, and moves to the best neighbor. To avoid being stuck in a suboptimal region, the algorithm allows moving to worse solutions, and using a Tabu list, it prohibits moving back to the previously visited solutions.

Our algorithm works as follows. First, we find a feasible solution to start the algorithm. We call this solution the *current solution*. We then start the first iteration of our tabu search procedure, where we evaluate every neighbor of this current solution. We say that a solution x with assortment list vector y is a neighbor of solution x' with assortment list vector y', if y and y' have the same values in all indexes except for exactly one. For instance, for a five item problem instance, $y = (1,0,0,1,1)$ and $y' = (1,0,1,1,1)$ are the assortment list vectors of two neighbor solutions. After we evaluate each neighbor, we move to the best neighbor (which might have a worse objective value than the objective value of the current solution) and define it the new current solution. We repeat the procedure in the next iteration (with the new current solution) and the tabu search procedure continues in this manner. In the next section, (Section 5.3.1) we describe the details on how we determine the starting feasible solution for our algorithm.

5.3.1. Initialization

There are two capacity constraints that every feasible solution should obey in (P) (and ($\tilde{\text{P}}$)): constraint (12) and constraint (11). Due to constraint (12), for an assortment list vector y to be feasible, we should have

$$\sum_i S_i^{min} y_i \leqslant S^{cap}. \tag{20}$$

As long as inequality (20) is satisfied for a vector y, a feasible solution to (P) (and ($\tilde{\text{P}}$)) is guaranteed to exist: we can always set $S_i = S_i^{min}$, and for the constraint (11) to hold, we can set $\tau_i^{BS} = 0$ for $i = 1,\dots,n$. Moreover, as long as the values for p_i are within the bounds, the values for the remaining variables do not pose any problems regarding feasibility. This suggests that we can find a good feasible starting solution quickly. To do so, we follow a two phase method. In the first phase, we solve the following problem, which we denote by (R):

$$\text{Maximize} \sum_{i=1}^{n} e^{p_i' + d_i(0)'} y_i$$

subject to (R)

$$\sum_{i=1}^{n} e^{S_i'} y_i \leqslant S^{cap} \tag{21}$$

$$d_i(0)' = \alpha_i' + \beta_i S_i' + \sum_{j \neq i} \delta_{ij} y_j S_j' + \gamma_i p_i' + \sum_{j \neq i} \mu_{ij} y_j p_j'$$

$$(p_i^{min})' \leqslant p_i' \leqslant (p_i^{max})'$$

$$(S_i^{min})' \leqslant S_i' \leqslant (S_i^{max})'$$

$$y_i \in \{0, 1\}.$$

where for any variable or parameter $a,\ a' = \log(a)$.

It is easy to see that the objective function of (R) maximizes $\sum_{i=1}^{N} p_i d_i(0) y_i$ (i.e., total revenues from sales) and the constraint (21) is equivalent to the backroom capacity constraint (12). In other words, problem (R) determines an assortment vector y together with S and p values to maximize total revenues from sales with no regard to deterioration. The $y,\ S$ and p vectors are feasible to problem ($\tilde{\mathrm{P}}$) because constraint (12) is satisfied and constraint (11) can be easily satisfied by setting $\tau_i^{BS} = 0$ for $i = 1, \ldots, n$. However, instead of setting τ^{BS} vector to 0, we move to the second phase where we solve problem ($\tilde{\mathrm{P}}$) for fixed values of y S and p to determine best possible τ^{BS} values. Throughout the paper, we are going to denote this problem by ($\tilde{\mathrm{P}}_{|y,S,p}$).

Observe that for any given set of y variables, objective function and the feasible region of problem (R) are convex. Hence, standard MINLP solvers can solve problem (R) quite effectively. Problem ($\tilde{\mathrm{P}}_{|y,S,p}$) can also be solved effectively as well. Therefore, using this two phase method, we can quickly find a good starting solution for our Tabu Search algorithm.

5.3.2. Search process and parameters

Once a starting solution is found, we begin the first iteration of our tabu search procedure, where we evaluate the neighbors of the current solution. To evaluate a neighbor, we need to solve problem ($\tilde{\mathrm{P}}$) for a given assortment vector y, which we denote by ($\tilde{\mathrm{P}}_{|y}$). Even though the assortment list is given, ($\tilde{\mathrm{P}}_{|y}$) is still difficult to solve by an optimization software. Moreover, for a problem instance with n products, there are n neighbors of any current solution. Therefore, to speed up this process, we don't solve ($\tilde{\mathrm{P}}_{|y}$) exactly. Instead, as we did in the initialization step, we follow a two phase heuristic approach. In the first phase, we aim to find good S and p values (not necessarily optimal for ($\tilde{\mathrm{P}}_{|y}$)) for the items in the assortment list. In this phase, we solve ($\mathrm{R}_{|y}$), i.e., problem (R) with fixed assortment vector y, and determine vectors S and p. In the second phase, we solve ($\tilde{\mathrm{P}}_{|(y,S,p)}$) to determine the vector (τ^{BS}) that represents the backroom depletion times for that particular neighbor. We do this for every neighbor of the current solution. The neighbor with the highest objective function value becomes the current solution in the next iteration.

The size of the tabu list, which determines the number of iterations that an item should stay in the tabu list is an important factor that affects the performance of the tabu search algorithm. In our experiments we decided to set the tabu list size to $\lfloor \frac{n}{3} \rfloor$ and iteration limit to $\lceil 1.5 \times n \rceil$. For example, for a 10 period problem instance, the tabu list size is 3 and iteration limit is 15. The experiments show that this tabu list size is enough for the algorithm to avoid getting stuck in suboptimal regions and that the number of iterations give sufficient time to evaluate high potential solutions in a reasonable amount of time.

6. Computational results

In this section we assess the performance of our heuristic approach in terms of solution quality. Our aim is to prove that our algorithm can search the feasible region more effectively and hence can find much better solutions (i.e., solutions that yield more profit) than commercial MINLP solvers.

6.1. Performance of our heuristic

In order to test the performance of our heuristic approach, we created random instances of problem (P). These instances were created in such a manner that, each one had certain properties that might affect the performance of any solution approach. Since a natural property that might affect the performance is the number of items, we generate groups of 10, 20 and 30 item instances. There are 108 instances for each group (i.e., a total of 324 instances). Likewise, since it is also natural to expect that the performance might be affected by the tightness of capacities, within each group, we generate subgroups of instances with low, medium and high display area and backroom storage capacities. In addition to these natural factors, our aim is to observe how the variance of cross-price and cross-space elasticities among items affects the performance of solution approaches. As the variance of cross elasticities increase, the assortment problem becomes more important to solve. In case of high variance, demand rates can be significantly improved with the right choice of items in the assortment, which might lead to significant improvements in revenues. So, if an algorithm performs poorly in these instances (compared to other instances with low variance), it can imply that the algorithm cannot solve the assortment problem well. Therefore, within each subgroup, we let half of the instances have low variance, and let the other half have high variance among cross-price and cross-space elasticities. Overall, we created 54 groups ($3 \times 3 \times 3 \times 2$) of problem instances, where each group contained 6 problem instances with exactly the same characteristics regarding the number of items, tightness of display and backroom capacities and the variance of cross-price and cross-space elasticities. See Appendix B for details on data generation.

We solved each instance using our heuristic approach. That is, we solved the approximate reformulation ($\tilde{\mathrm{P}}$) using our tabu search based algorithm and determined y $S,\ p$ and τ^{BS} vectors. Then, we used the values of these variables in the solution to (P) and computed exact revenues and inventory holding costs. This whole process took approximately 1.5, 4 and 10 min for 10, 20 and 30 item instances respectively.

To make a fair comparison with a standard MINLP solver, we didn't let the solver work on the more complicated formulation (P). Instead, we followed the same procedure as we did for our heuristic approach. That is, we solved formulation ($\tilde{\mathrm{P}}$) using the MINLP solver to determine y S, p and τ^{BS} vectors. Then performed the same calculations to find the corresponding revenues and inventory holding costs in (P) and compared the results. There are two well known and effective MINLP solvers on the market. One is BARON: Brand and Reduce Optimization Navigator (see Tawarmalani and Sahinidis (2005)), and the other one is DICOPT: Discrete and Combinatorial Optimizer (see Duran and Grossmann (1986), Viswanathan and Grossmann (1990)). In our experiments, BARON could not find a feasible solution within one hour in the majority of the 10, 20 or 30 item instances of problem ($\tilde{\mathrm{P}}$). Furthermore, in the few instances where BARON was able to find a feasible solution, the objective values were worse than the ones found by DICOPT. Hence, we decided to compare the results of our algorithm to the results obtained by DICOPT, which proved quite effective in finding feasible integer solutions to problem ($\tilde{\mathrm{P}}$). DICOPT is a program designed specifically to solve MINLPs. The algorithm inside DICOPT solves an alternating sequence of nonlinear programs called *NLP subproblems* and mixed integer linear programs called *MIP master problems*. MIP master problems solve a linear

Table 1
Comparison of objective values when both algorithms use ($\tilde{P}$).

	Backr. Cap.		1		2		3	
	Variance		1	2	1	2	1	2
10 items	Disp. Cap.	1	1.016(6) (–)(0)	1.015(6) (–)(0)	1.001(6) (–)(0)	1.001(6) (–)(0)	1.003(6) (–)(0)	1.059(6) (–)(0)
		2	1.042(6) (–)(0)	1.452(6) (–)(0)	1.164(6) (–)(0)	1.112(6) (–)(0)	1.075(6) (–)(0)	1.018(6) (–)(0)
		3	1.136(6) (–)(0)	1.586(6) (–)(0)	1.002(6) (–)(0)	1.305(6) (–)(0)	1.057(6) (–)(0)	1.089(6) (–)(0)
20 items	Disp. Cap.	1	1.484(6) (–)(0)	3.665(6) (–)(0)	1.194(6) (–)(0)	5.142(6) (–)(0)	1.093(6) (–)(0)	3.401(6) (–)(0)
		2	1.274(6) (–)(0)	5.586(4) 1.192(2)	1.173(6) (–)(0)	3.227(6) (–)(0)	1.217(6) (–)(0)	3.977(5) 1.014(1)
		3	1.161(6) (–)(0)	4.215(6) (–)(0)	1.301(6) (–)(0)	3.842(6) (–)(0)	1.342(6) (–)(0)	1.073(6) (–)(0)
30 items	Disp. Cap.	1	2.045(6) (–)(0)	3.885(5) 1.292(1)	1.806(6) (–)(0)	1.471(4) 1.284(2)	1.991(6) (–)(0)	1.319(4) 1.341(2)
		2	1.769(6) (–)(0)	5.281(5) 1.066(1)	1.592(6) (–)(0)	9.906(5) 1.724(1)	1.078(6) (–)(0)	1.421(6) (–)(0)
		3	1.734(6) (–)(0)	8.086(5) 1.477(1)	1.331(6) (–)(0)	3.677(5) 1.401(1)	1.365(6) (–)(0)	1.451(6) (–)(0)

Table 2
Comparison of objective values in five item instances when BARON solves (P) with no approximation.

Backr. Cap.		1		2		3	
Variance		1	2	1	2	1	2
Disp. Cap.	1	3.374 (4)	1.275 (3)	4.045 (4)	13.572 (3)	5.386 (3)	13.084 (6)
	2	5.694 (2)	102.784 (3)	3.126 (4)	23.431 (3)	35.208 (1)	17.603 (2)
	3	16.196 (3)	40.885 (3)	2.156 (1)	1.018 (2)	3.362 (1)	4.315 (3)

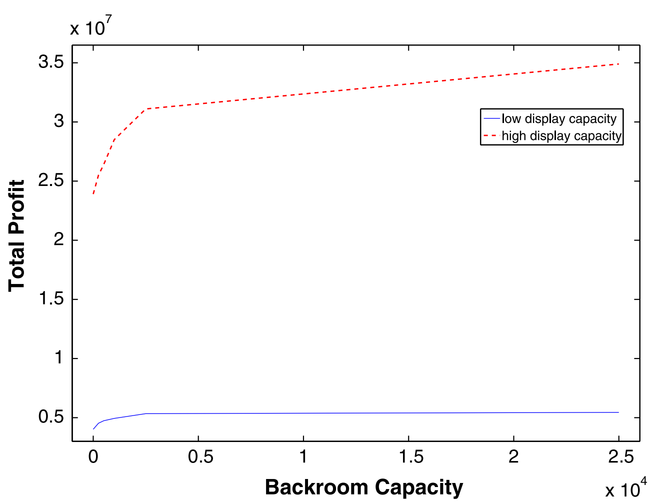

Fig. 3. Change in total profit when backroom capacity changes.

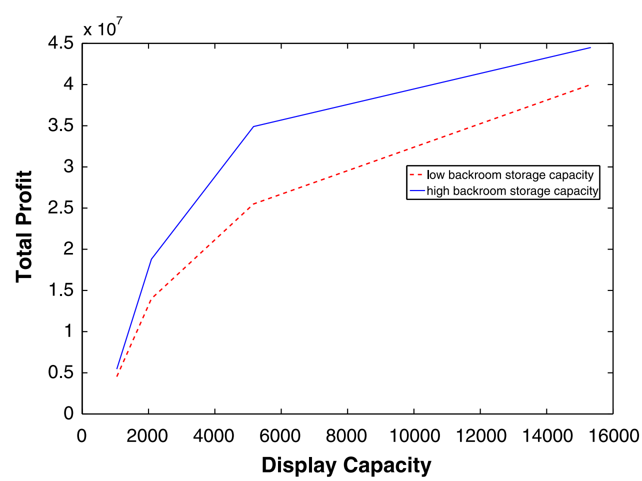

Fig. 4. Change in total profit when display area capacity changes.

approximation of the problem and predict the integer variables. NLP subproblems are then solved for these fixed integer variables. The algorithm alternates between the master problem and the subproblem until a stopping criterion is reached. The Master problem and the subproblem can be solved by any choice of MIP and NLP solver in the market. In our experiments we used CONOPT for the NLP subproblems and GUROBI for the MIP Master. With the default settings, DICOPT usually works very well for convex problems. However, since our problem is not convex, to improve the performance of the algorithm of DICOPT, we increased the iteration limits for the NLP and MIP subproblems and extended the search time for both problems. With this modification, it took approximately 15 min for DICOPT to solve a single problem instance, regardless of the group the instance is in.

Since we do not have reliable information on upper bounds, instead of comparing optimality gaps, we compared best profits generated by both algorithms. For any instance $\mathcal{I}$, let $O_1(\mathcal{I})$ and $O_2(\mathcal{I})$ be the objective values obtained by our heuristic approach and by DICOPT solver, respectively. Then, for every instance $\mathcal{I}$, we computed the ratio $\frac{O_1(\mathcal{I})}{O_2(\mathcal{I})}$ and the ratio $\frac{O_2(\mathcal{I})}{O_1(\mathcal{I})}$. Afterwards, within each group of instances, we counted the number of instances which yielded $\frac{O_1(\mathcal{I})}{O_2(\mathcal{I})} \geqslant 1$ (which corresponds to the case where DICOPT could not find a better objective value than our heuristic) and $\frac{O_2(\mathcal{I})}{O_1(\mathcal{I})} > 1$ (which corresponds to the case where DICOPT found

Table 3
Change in assortment when display area capacity changes.

Disp. Cap.	Assortment																													
Backroom capacity of 250																														
1000	0	0	0	1	0	1	0	1	1	0	0	0	0	1	1	1	0	0	1	0	0	0	1	1	0	0	0	0	0	0
2000	1	1	1	1	0	1	1	0	1	0	0	1	1	1	0	1	0	1	1	1	0	0	0	1	1	1	1	1	0	1
5000	1	1	1	1	0	1	1	0	1	0	0	1	1	1	1	1	0	1	1	1	0	0	0	1	1	1	1	1	0	1
15,000	1	1	1	1	0	1	1	0	1	0	0	1	1	1	1	1	0	1	1	1	0	0	0	1	1	1	1	1	0	1
Backroom capacity of 25,000																														
1000	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	1	0	0	1	0	0	0	1	1	0	0	0	0	0	0
2000	1	1	0	1	0	1	0	1	1	0	1	0	0	0	1	1	0	1	1	0	0	0	1	1	0	0	0	1	1	1
5000	1	1	0	1	0	1	0	1	1	0	1	1	0	0	1	1	0	1	1	0	0	1	1	1	0	0	1	1	1	1
15,000	1	1	0	1	0	1	0	1	1	1	1	1	0	0	1	1	0	1	1	0	0	1	1	1	0	0	1	1	1	1

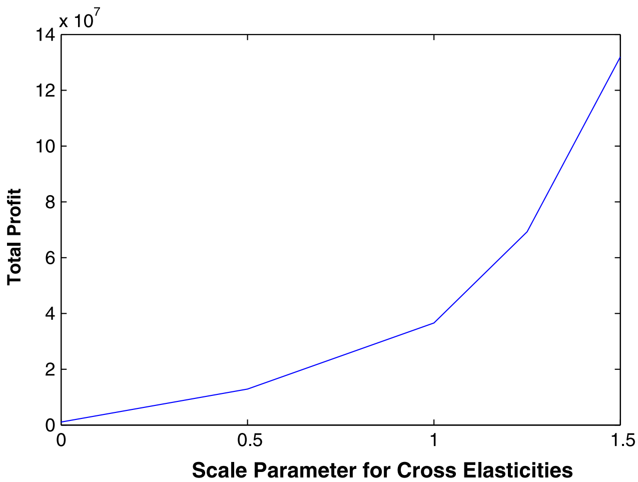

Fig. 5. Effect of cross dependencies on total profit.

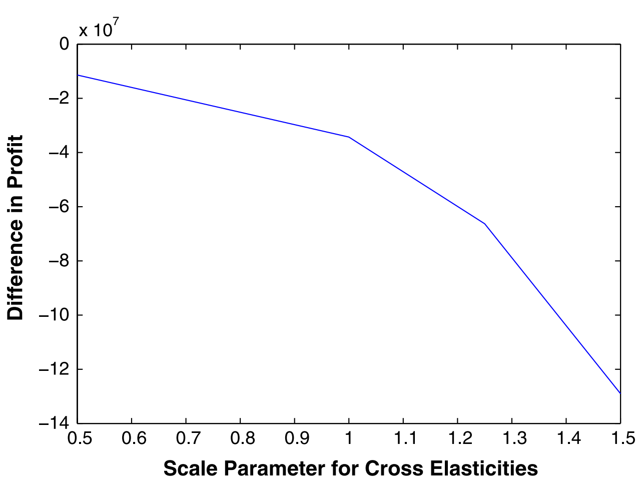

Fig. 6. Loss in profit when cross dependencies are ignored.

a better objective value than our heuristic). In every group of instances, we calculated the average of those ratios for both cases. These averages, together with the number of observations in both cases (shown in parenthesis next to the average values), are presented in Table 1. In the table, the group of instances with low, medium and high capacities are indicated by the numbers 1, 2 and 3. Likewise, the group of instances with low and high variance among cross elasticities are indicated by 1 and 2. To better explain the table, consider the results for the group of instances with 20 items, medium display capacity (2), low backroom capacity (1), and high variance among cross elasticities (2). In 4 out of 6 instances in that particular group, our heuristic approach found better (or equal) solution and the average of ratios of objective values for those instances turned out to be 5.586. In that group, DICOPT was able to find better solutions in two instances, which had an average ratio of 1.192. In other words, in 4 instances, our algorithm found solutions averaging more than 5 times the profit of the solutions found by DICOPT.

Looking at the table, we can easily say that, overall, our algorithm outperforms DICOPT. We can see that, towards the bottom left corner of the table, the ratios of objective values increase in favor of our heuristic algorithm. Therefore, we can say that our algorithm performs increasingly better as the number of products increases, display capacity increases, backroom capacity decreases and the variance among cross elasticities increases.

6.2. Performance of approximate formulation

Using the approximate formulation ($\tilde{\text{P}}$) helps DICOPT as well. We can see that in 11 out of 324 instances, it found better solutions than our heuristic approach did. Also, especially in 10 item instances, the objective values obtained by our algorithm and DICOPT were quite close. On the other hand, with no use of approximation, DICOPT performed very poorly. As a matter of fact, the revenue and holding cost functions in (P) are so complex that DICOPT could not find feasible solution to formulation (P) in any of the 10, 20 or 30 item instances. Those functions are so complex that DICOPT could not even solve simple 5 item instances of problem (P). However, although BARON does not perform well with integer variables, it handles such complicated functions better than DICOPT does. Therefore, we also tested our approach in 5 item instances and let BARON solve formulation (P) with no

Table 4
Change in assortment with increasing shelf lives.

Shelf life	Assortment																													
Low	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	1	0	0	1	0	0	0	1	1	0	0	0	0	0	0
Medium	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	1	0	0	1	0	0	0	1	1	0	0	0	0	0	0
High	0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	1	0	0	1	0	0	0	1	1	0	0	0	0	0	0

Table 5
Change in assortment with increasing shelf lives when there are no cross dependencies.

Shelf life	Assortment																													
Low	0	0	0	1	0	0	1	0	0	1	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	1	0
Medium	0	1	0	1	0	1	1	1	0	1	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
High	0	1	0	1	0	1	1	1	0	1	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0

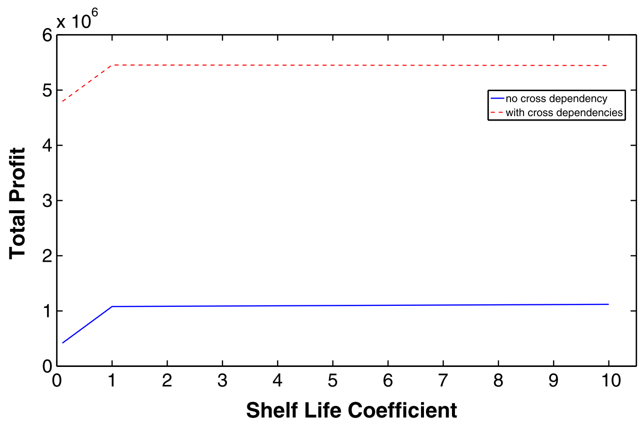

Fig. 7. Change in total profit for varying shelf life coefficients.

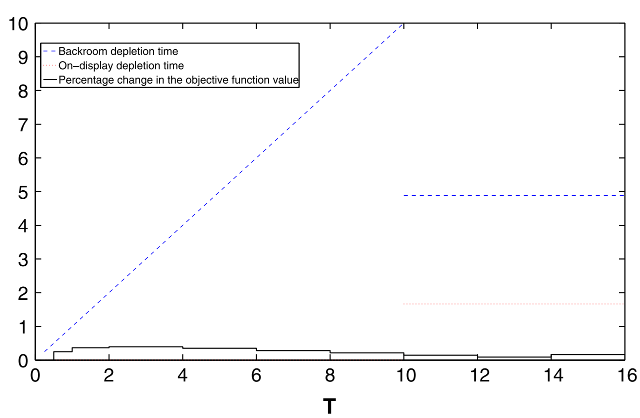

Fig. 8. Effect of shelf life for a single item on depletion time and change in the objective function.

approximation. We allowed BARON to run up to 1 h. The results with similar comparisons are summarized below in Table 2. In very few of the instances was BARON able to find a feasible solution, and in none of them it was able to find a better solution than our heuristic algorithm. In the table, the numbers in parenthesis shows the number of instances (out of 6) where BARON was able to find a feasible solution. The results presented in Tables 1 and 2 clearly show the effectiveness of our solution approach.

6.3. Managerial insights

In this section we analyze the effect of capacities, cross-dependencies, and shelf lives on the optimal assortment and the objective function value. To that end we performed several experiments on a representative 30 item instance.

6.3.1. Effect of capacities

Naturally, one would expect that increasing the capacities of either the backroom or the display areas increases the total profit. In line with our expectations, for our 30 item instance, Fig. 3 shows

Table 6
A summary of the decision variables and parameters of (P).

Decision variables	
S_i	Display area allocated for item i
p_i	Price assigned for item i
$d_i(0)$	Demand rate of item i at the beginning of its replenishment cycle
τ_i^{BS}	Time until the backroom inventory of item i is depleted
τ_i^{DA}	Time between τ_i^{BS} and the end of cycle time of item i
NP_i	Auxiliary variable to represent the net profit of item i
Parameters	
B^{cap}	Backroom capacity
S^{cap}	Display area capacity
T_i	Maximum lifetime of item i
S_i^{min}	Minimum display area that needs to be allocated to item i
S_i^{max}	Maximum display area that can be allocated to item i
p_i^{min}	Minimum price that needs to be assigned to item i
p_i^{max}	Maximum price that can be assigned to item i
Rev_i^{max}	Maximum revenue that can be obtained by the sale of item i (calculated)
α_i	Scale parameter of the demand function of item i
β_i	Space elasticity parameter of the demand function of item i
γ_i	Price elasticity parameter of the demand function of item i
μ_{ij}, δ_{ij}	Cross-price and cross-space elasticity parameters between items i and j

how total profit increases with increased backroom capacity for low and high display capacities of 1000 and 5000, respectively. Likewise, Fig. 4 shows how total profit changes with increased display area capacity for low and high backroom capacities of 250 and 25,000, respectively.

Both figures show that total profit increases with capacity in a concave fashion. Interestingly, display capacity has a more significant effect on total profit than the backroom capacity does. This is mostly due to the fact that as the display capacity changes, more items are added to the assortment list. This, in turn, increases the demand rates of items in the assortment list, which eventually increases revenues and profits. Table 3 illustrates how the assortment changes as the display capacity increases. In the table, 1 denotes an item is in the assortment and 0 denotes otherwise. We observe that as the display capacity increases, more items are added to the assortment instead of increasing the shelf spaces of the items that are already in the assortment. However, once an ideal set of items (items that complement each other) are included in the assortment, we observe that shelf spaces start to increase as display capacity is increased.

6.3.2. Effect of cross dependencies

To investigate the effect of cross dependencies, we performed the following experiment on our 30 item instance. We multiplied the cross-price and cross-space elasticity parameters by 0, 0.5, 1, 1.25, and 1.5 respectively, and observed optimal assortment and total profit in each case. Multiplying cross elasticities in such a manner creates instances with reduced or amplified cross dependencies among items.

Fig. 5 shows that if the assortment is determined taking into account these cross dependencies, total profit increases exponentially with increasing cross dependencies among items. On the other hand, assume that an optimal assortment is determined without taking into account the cross dependencies (i.e., assuming that cross-price and cross-space elasticity parameters are all equal to zero). Assuming such an assortment is used, Fig. 6 illustrates the amount of loss in profits with increasing values of real cross dependencies among items. As the figure suggests, there are drastic differences in solution quality when cross elasticities are taken into account, which highlights the importance of these parameters.

6.3.3. Effect of shelf lives

First, we aim to understand how the optimal assortment and total profit change if the items had lower or higher shelf lives. For that purpose, we performed the following experiment. We set the display and the backroom capacities of our representative 30 item instance to 1000 and 25,000 units, respectively. Then, by multiplying the original shelf lives of the items by 0.1, 1 and 10 respectively, we created instances with very low, medium, and very high shelf lives. Table 4 lists the assortments for each case. As illustrated in the table, the assortment does not change as the shelf lives get longer. On the other hand, we performed another set of experiments, where we set all cross elasticity parameters equal to 0 (such that there are no cross dependencies). Table 5 shows the corresponding assortments in such a case. As we see, when there are no cross dependencies, increased shelf lives result in different assortment. This shows that shelf lives might affect the choice of assortment, but the cross dependencies among items (since they ultimately affect the demand rates) have a stronger effect than the shelf lives on the choice of assortment. Fig. 7 shows how the total profit changes with increased shelf lives. Clearly, with the assumed demand model, shelf lives have very mild effect on total profit.

Next, we observe the changes in the assortment and the objective function value when we increase the shelf life (T_i) of a particular item i that was not included in the assortment in the solution of the original instance (i.e., the instance with medium shelf lives, display and backroom capacities of 1000 and 25,000 units respectively, and with cross dependencies). Fig. 8 shows how backroom depletion time (τ_i^{BS}), on-display depletion time (τ_i^{DA}), and total profit change as we increase the shelf life of the item in consideration. Initially, i.e., until T_i reaches 0.5, the assortment and the total profit do not change. From that point on, the item is included in the assortment, which causes an improvement in the objective function value. As a matter of fact, the increase in shelf life improves the objective function value by approximately 0.5%. As the shelf life of the item is increased further, backroom depletion time (τ_i^{BS}) set for the item increases linearly. In fact, we have $T_i = \tau_i^{BS}$, and $\tau_i^{DA} = 0$ for T_i values less than 10. Recall that, in a replenishment cycle, during the time interval $[0, \tau_{BS}]$, the demand rate is the highest since the display area is always full in this interval. To keep the demand rate as high as possible, as the shelf life is increased, it is more profitable to increase τ_{BS}, which also corresponds to increased order quantities. However, having higher order quantities results in lower profits at some point. As seen in Fig. 8, the improvement in profit slowly decreases for shelf life values between 4 and 10. As we increase the shelf life of the item higher than 10, the τ_{BS} drops down sharply. We get $\tau_i^{BS} < T_i$ and $\tau_i^{DA} > 0$. This is because of the fact that, the loss in profits by having higher order quantities (higher τ_i^{BS}), exceeds the gains in profits by increased demand rates.

7. Concluding remarks

In this paper, we consider a product assortment problem for a single retail store with separate and capacitated storage and display areas. The retailer has to decide on an assortment of items to sell in the store. We assume that demand rate of an item in the assortment may be affected (either positively or negatively) on prices and shelf spaces allocated for the other items in the assortment.

We develop a mixed integer nonlinear programming model for this problem. Then we propose a Tabu Search based heuristic algorithm. We validate the proposed approach through several numerical examples and report our results, which prove that our approach is quite effective in finding good quality solutions in a reasonable amount of time. We also perform sensitivity analysis to determine the effects of capacities, cross dependencies and shelf lives on the optimal assortment and total profit.

For future research, innovative heuristic approaches that use some other approximation methods can be considered. It would also be interesting to see how the analysis of the same problem changes under several other demand models.

Appendix A. Decision variables and parameters

Appendix B. Data generation

Tables 7–9 show the distributions we utilized to create our problem instances:

Table 7
Demand parameters.

α	δ, μ (low variance)	δ, μ (high variance)	β	γ
U(1000, 3000)	U(−0.05, 0.05)	U(−0.1, 0.1)	U(0.1, 0.4)	U(−1, 0)

Table 8
Cost parameters.

c	c^{BS}	c^{DA}	h	K
U(10, 25)	U(0.5, 2.5)	U(1, 10)	$c \times 0.2$	U(50, 400)

Table 9
Revenue and other parameters.

r	p^{max}	S^{max}	p^{min}	S^{min}	T
$c \times 0.1$	U(100, 250)	U(50, 200)	$p^{max} \times 0.8$	$S^{max} \times 0.8$	U(15, 25)

After the parameters above are determined, we set S^{cap} equal to either $\frac{1}{3}\sum_{i=1}^{n} S_i^{min}, \frac{2}{3}\sum_{i=1}^{n} S_i^{min}$ or $\sum_{i=1}^{n} S_i^{min}$ to make the instance either a low, medium or a high display capacity instance. Likewise we set B^{cap} equal to $\frac{1}{200}\sum_{i=1}^{n} Q_i^{max}, \frac{1}{100}\sum_{i=1}^{n} S_i^{min}$ or $\frac{1}{50}\sum_{i=1}^{n} S_i^{min}$ to make the instance either a low, medium or a high backroom capacity instance. In these calculations $Q_i^{max} = T_i d_i(0)^{max}$ is the maximum order quantity that would result in zero deteriorated item in the backroom storage assuming maximum possible demand rate for the item throughout its life cycle.

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