# A Fluid Approximation for the Single-leg Fare Allocation Problem with Nonhomogeneous Poisson Demand

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**Abstract**

Fare allocation for legs and O&D pairs plays a crucial role in airline revenue management. Despite a large number of dynamic pricing studies, there are only a few widely-adopted studies whose assumptions affect most tactical decisions with potentially large impacts on airline profitability. These decisions involve approximating future pricing schemes, allocation of fare classes, and setting booking limits. We propose a fare allocation model for a single leg in the presence of a realistic nonhomogeneous Poisson demand with an increasing rate. We aim to compute when and how to markup the price for an airfare product (switch to an upper fare class) to maximize the expected revenue. We study a fluid approximation of the underlying stochastic problem considering independent demand from each customer segment and examine different properties that lead to several important insights. Finally, we propose a dynamic look-ahead pricing scheme to compare our fluid approximation results against the well-known EMSRb heuristic and a dynamic programming solution on randomly generated booking request data. Numerical examples illustrate the effectiveness of our proposed approach.

**Keywords:** revenue management, airline operations, inventory allocation, nonhomogeneous Poisson process.

## 1 Introduction

The airline industry has been applying Revenue Management (RM) techniques for decades (Walczak et al., 2012). The main objective of using these techniques is to *dynamically* allocate seats for each fare class during the selling period such that the total revenue is maximized. From a passenger standpoint, this essentially translates into price markups for the same airfare product. Several widely-accepted RM techniques in the airline industry have proven success, showing significant increases in revenues (Talluri and Van Ryzin, 2006; Fiig et al., 2010; Weatherford and Khokhlov, 2012).

As far as the real-life airline operations are concerned, an RM team typically allocates initial inventories at the tactical level. Flights are initially assigned to RM analysts, depending on their experience with customer behaviors. Starting with an initial price set and associated price ranges, these analysts observe booking trends throughout the selling period. Through business intelligence

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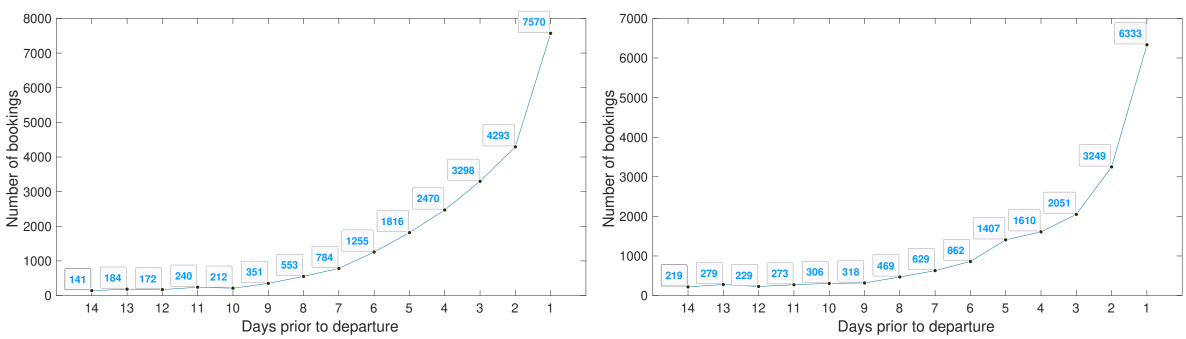
systems, these observations, combined with intuitive interactions, lead to actions in the form of price changes and interference to the fare class allocations. Policies that utilize both event-driven interference and random time observations are more efficient and widely used (Mukhopadhyay et al., 2007). That is, if a seat is sold at the offered price, the adopted model provides a new allocation if there is a better solution with increased expected revenue. At any point in time, not necessarily triggered by a seat sale, if another parameter/distribution happens to fit observed sales better, these inputs are updated, which might re-allocate fares. These input updates allow policies to better react to changes in the booking curve. In this regard, dynamic pricing for airline RM plays an important role, hence a large number of studies (den Boer, 2015).

Our study is motivated by somewhat typical customer behavior, which has not been addressed before in the widely-adopted studies. To illustrate, Figure 1 shows the booking trend for domestic flights scheduled to depart within June 2017 from Istanbul to two popular destinations, Izmir and Antalya, operated by Onur Air, a low-cost airline in Turkey. Scheduled seat capacity and departure times for both routes are homogenous through the days of the month. The booking curves show the sales for up to 14 days before departure. In this instance, we focus our attention on this timeframe because *for this airline and given O&D pairs*; (i) this timeframe contains approximately 70% of all bookings, (ii) bookings before this timeframe do not possess a particular structure, (iii) early *group* reservations (around 10% of all bookings) have different dynamics and trends. Booking curves for Izmir and Antalya flights in Figure 1 depict an increasing booking pattern with a rising pace towards departure time for 87.29% and 72.20% of total bookings, respectively.




(a) Istanbul-Izmir flights with 87.29% of sales within 14 days before departure

(b) Istanbul-Antalya flights with 72.20% of sales within 14 days before departure

Figure 1: Booking curves for two popular destinations for Onur Air.

We cannot generalize the booking trend in Figure 1 for all routes, such as those with a large rate of early bookings with more price-sensitive passengers. For instance, leisure travelers usually book early on international routes due to visa requirements, hotel booking, more extended traveling plans, and possibly more expensive fares with dramatic price markups experienced. For such courses, sales usually go up initially, reach a peak, and eventually decline towards the end of the planning horizon. Therefore, a revenue analyst should develop a different strategy for each set of routes with similar booking trends.

In this paper, we study a fare allocation policy for routes with a passenger tendency to have more requests towards departure, e.g., nonleisure passengers on short-haul routes, as shown in Figure 1. We focus on a realistic representation of tactical level decisions during *inventory allocation* process. Although the system is continuously open to price and allocation changes, a simplified model for the procedure helps identify critical decisions on strategic outputs such as the number of fare classes and booking limits. This approach is likely to provide a more stable plan and better explainability, reducing the risk of ambiguous changes during the selling period.

The remainder of this paper is organized as follows: We present studies from the literature that has a similar focus in Section 2. We formally define the single-leg inventory problem faced in the airline industry under realistic assumptions in Section 3. We present optimality conditions, insights, and a dynamic pricing scheme in Section 4. In Section 5, we compare our approach with two approaches from the literature using the dynamic pricing scheme on synthetic data. In Section 6, we provide concluding remarks and directions for future research.

## 2 Literature Review

There are two fundamental approaches to model fare allocations: static and dynamic. *Static* models consider the entire planning horizon and aim to maximize the overall revenue, whereas *dynamic* models frequently recompute decisions. There are two types of static models. The first type assumes the demand distribution for each fare class is known, independent, and static throughout the selling horizon. Solution of this category can be by means of linear programming (Wollmer, 1980; de Boer et al., 2002; Talluri and Van Ryzin, 2004). The second type of static models assumes the demand for different fare classes arrive independently and in non-overlapping time-periods. Littlewood (1972) and Richter (1982) study two fare class problems. Multiple fare classes are studied in (Belobaba, 1989; Wollmer, 1992; Brumelle and McGill, 1993; Robinson, 1995). As far as the dynamic pricing studies are concerned, there are mainly two approaches to modeling. The first one is *discretizing* time horizon $T$ and associating a probability of sale for each time segment depending on the trend and price (Zhang and Cooper, 2009). The other approach handles the problem in a *continuous manner* using Hamilton–Jacobi–Bellman equation to find an optimal policy (Liang, 1999; Gallego and Van Ryzin, 1994, 1997).

Some of the most widely-used RM approaches are based on inventory *protection* decisions for higher booking classes, which translates into a price markup at a certain point (Phillips, 2005; Talluri and Van Ryzin, 2006). This technique is first suggested by Littlewood (1972), who develops an optimal rule for two nested fare classes on a flight leg. This rule is extended to multiple nested fare classes by Belobaba (1987, 1989), who also introduce Expected Marginal Seat Revenue (EMSR) to the literature. This method is then improved to an EMSRb model (Belobaba and Group, 1992). The original EMSR is referred to as EMSRa after EMSRb is introduced. Both of these methods define decision rules for nested booking structures, and both are widely used by airline RM systems.

Some assumptions in these classical approaches do not reflect the real-world dynamics (Phillips, 2005). Below are three critical aspects of the fare allocation problem that should be considered:

- Demand for fare classes are not independent.
- Demand patterns might be predetermined for each fare (e.g., monotonically increasing in short-haul).
- Price sensitivity usually decreases over time.

Airlines realize that there are only buyers, who check different alternatives and select the best one which meets their needs, hence an increasing number of studies on customer-choice models (Xie et al., 2016). For instance, there are travelers with a willingness-to-pay that is high enough to book a full-fare but would be more than happy to purchase at lower fares, which leads to cannibalization. This violates the generally used fare class independence assumption. Most of the customers demand the full-fare booking only if the discount fare class is closed. Therefore, when discount fares are closed, this will generate some additional higher fare demand because the discount fare is not available anymore, and some of the customers who *missed* the discount fare are still willing to pay more for that flight. This phenomenon is known as *buy-up* or *sell-up* (Phillips, 2005). These customer behaviors are shown to have a large impact on revenues even on earlier simulations (Belobaba and Weatherford, 1996). The *decreased price sensitivity* of the inventoried items towards the end of the period is emphasized even in earlier studies (Pfeifer, 1989), but overseen in most of the approaches.

To the best of our knowledge, no study in the literature addresses the above issues in a fare allocation model. To achieve this, we assume each leg is independent and focus on the optimization of *one product type* in a single leg. Multiple fares call for product differentiation (e.g., baggage, fee/refund policies) in practice. However, most economy passengers only consider buying the most affordable available ticket among economy fares. Thus, we neglect minor product differentiations and assume *one product*, whose demand comes from multiple independent *customer segments*, each with a different *willingness to pay.*

## 3 Problem Definition

Before we formalize the mathematical model, we start with the critical assumptions in our derivations. First, we assume demand is unaffected by external factors, such as competitors' prices. In practice, airlines set prices based on product type, customer type, distribution channel, and competitors' decisions. It is difficult to accurately capture competition dynamics such as *pricing and shortage games* in our model. Thus, our model would be more useful for a dominant, price-maker airline in a monopoly or monopolistic market. The second fundamental assumption is the rigid behavior of such a dynamic and uncertain event. In practice, starting with estimators from previous sales and booking experience, posterior distribution parameters are dynamically re-estimated through *Bayes sequential analysis* after an event (a seat search, sale, or a certain length of a no-sale period). Our fluid approximation would inevitably deviate from reality in these wait-and-see departments. Both of these issues can be addressed but are out of the scope of this study.

We omit product differentiation for simplicity, assume each offered product is identical, and consider several independent customer segments with different willingness to pay. We adopt a *nonhomogeneous Poisson* demand for each *customer segment* to reflect the real-life demand pattern observed through real data. From this point forward, we calculate the demand for an offered fare class as the sum of demand from all customer segments who are willing to pay *at least* the price of that fare class. Therefore, the demand for a fare class is the sum of random demand variables of higher segments, also a nonhomogeneous Poisson random variable, whose rate is the sum of all higher customer segments' rates. This approach also ensures that a higher price would never attract more customers (i.e., from more customer segments). Consequently, regardless of the parameters, the arrival rate decreases momentarily with a price markup and increases gradually over time, perfectly aligned with real-life observations.

Our goal is to determine the price markup points (fare closing decisions) in such a way that the expected revenue is maximized. The demand intensity function for this process is $\lambda(t) = a_{p(t)} f(t)$, where $p(t)$ is the price at time $t$, $a_{p(t)}$ is the *intensity* for the price at $t$ and $f(\cdot)$ is a nondecreasing function that governs how demand rate increases. That implies demand increases when the flight time gets closer, and this holds for every price. We assume a linearly increasing demand rate function $f(t) = t$ for illustration; however, all the results we present in this section apply to any differentiable nondecreasing function $f(\cdot)$. Price can only take one of $k$ available values at any time, that is $p(t) \in \{p_1 \dots p_k\}$. Without loss of generality, available prices are indexed in descending order, i.e., $p_1 > p_2 > \dots > p_k$. Lower price always generates a higher demand, i.e., $a_{p'} > a_{p''} \Leftrightarrow p' < p''$. We assume the offered prices should be *monotonically increasing* over the given time horizon for a feasible solution, i.e., $p(t) \leq p(t+s), s \geq 0$. $T$ is the flight time, and $t = 0$ is the beginning of the planning period, which might be the time when seats for the flight are open for sale, or the time sporadic arrivals historically transition into an observable pattern.

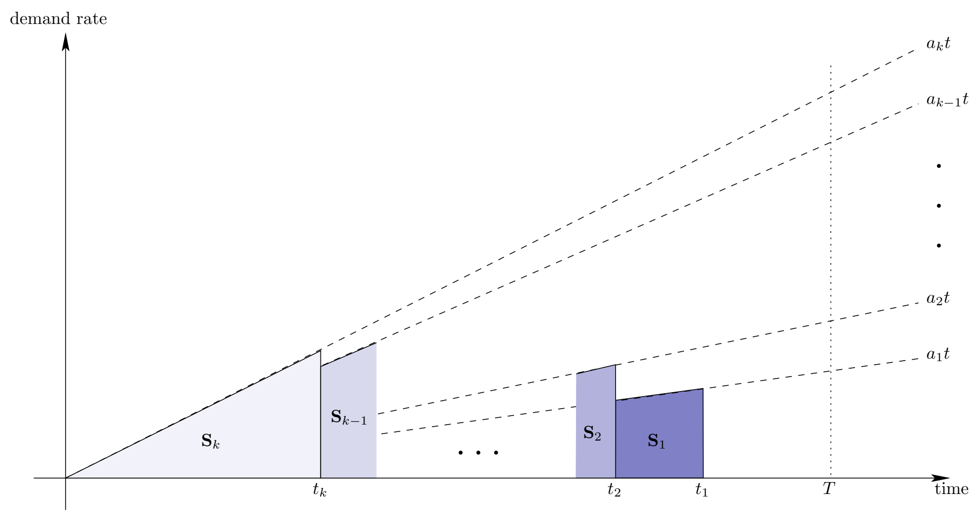



Figure 2: Multi-price model with expected rates for linearly increasing demand.

Figure 2 shows the generic structure of the demand intensity functions, where the area under each line segment represents the expected demand satisfied with that price. We show a sample pricing scheme for a markup pattern, where $t_i$ denotes the time price is marked up and price $i$ is not offered anymore. That does not necessarily mean price $i-1$ is offered next, because if $t_i = t_{i-1}$, that implies price $i-1$ is not offered at all. In the example illustrated in Figure 2, seats are expected to be sold out by time $t_1$, hence no sales in $(t_1, T]$. The difference between slopes of any two consecutive fare classes represents the arrival rate from the customer segment(s) that are willing to pay the lower price, but not the higher price.

## 3.1 Mathematical Model

The following notation is used throughout this section:

$k$: number of available prices (fare classes)

$p_i$: $i$-th highest price, $i = 1, \dots, k$

$T$: time of the flight

$S$: number of available seats

$\mathbf{S}_i$: random variable that denotes the number of seats sold with a price of $p_i$, $i = 1, \ldots, k$

$\mathbf{R}_i$: random variable that denotes the revenue generated from seats sold with price $p_i$, $i = 1, \ldots, k$

The set of decision variables for our model are the following:

$t_i$: ending time of time segment $i$ through which $i$-th price is offered, , $i = 1, \ldots, k$

We assume that there are $k$ non-overlapping time segments, each associated with a fare class/price. We aim to find the starting and ending times of time interval that a certain fare is open (i.e., price is offered). An interval might be of length zero, implying the price is never offered. For instance, price $i$ is not offered when $t_{i+1} = t_i$. Expected revenues can be calculated using the intensity functions of the corresponding prices for the determined time segments. The problem here is, determining a time segment will also affect the expected revenues of all other segments. The expected total revenue is maximized using the following formulation:

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{k} E[\mathbf{R}_i] = \sum_{i=1}^{k} E[\mathbf{S}_i] p_i & & \text{(1a)} \\
\text{s.t.} \quad & t_{i+1} \leq t_i & i = 1, \ldots, k \quad & \text{(1b)} \\
& t_{k+1} = 0 & & \text{(1c)} \\
& t_1 \leq T & & \text{(1d)} \\
& \mathbf{S}_i \sim \text{Poisson}\left( \int_{t_{i+1}}^{t_i} \lambda_i(t) dt \right) & i = 1, \ldots, k \quad & \text{(1e)} \\
& \sum_{i=1}^{k} \mathbf{S}_i \leq S & & \text{(1f)}
\end{aligned}
$$

In this formulation, $E[\mathbf{R}_i]$ denotes the expected revenue generated for the duration $p_i$ is applied. The constraints (1b) ensure no price mark-down is allowed. (1c) and (1d) indicate initial condition and final fare time restriction, respectively. (1e) is not a constraint but a definition for the distribution, and (1f) limits the number of seats sold.

### 3.2 A Fluid Approximation for the Linear Arrival Rate

In the linear rate case in Figure 2, $E[\mathbf{S}_i] = a_i \left(F(t_i) - F(t_{i+1})\right)$, where function $F(\cdot)$ satisfies $\frac{\partial F(t)}{\partial t} = f(t)$. We use a fluid approximation with the *realization* of that expectation for constraint (1f) and obtain the following deterministic model:

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{k} a_i p_i (F(t_i) - F(t_{i+1})) & & \text{(2a)} \\
\text{s.t.} \quad & t_{i+1} \leq t_i & i = 1, \ldots, k \quad & \text{(2b)} \\
& t_{k+1} = 0 & & \text{(2c)} \\
& t_1 \leq T & & \text{(2d)} \\
& \sum_{i=1}^{k} a_i (F(t_i) - F(t_{i+1})) \leq S & & \text{(2e)}
\end{aligned}
$$

For instance, for a linear increasing demand rate, $(F(t_i) - F(t_{i+1})) = (t_i^2 - t_{i+1}^2)/2$. We can work out other forms of demand rate functions similarly. This nonlinear programming formulation has a convex feasible region if intensities are nondecreasing (inversely correlated with price), i.e., $a_i \leq a_{i+1}, i = 1, \ldots k-1$. We cannot easily assess the concavity of the objective function though, as it depends on the products of intensities and prices, which we refer to as *intensity-price*. The first trick we apply here is substituting $F(t_i)$ with $t'_i$ and $F(T)$ with $T'$ to obtain the following linear program.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{k} a_i p_i (t'_i - t'_{i+1}) && && \text{(3a)}\\
\text{s.t.} \quad & t'_{i+1} \leq t'_i && i = 1, \ldots, k && \text{(3b)}\\
& t'_{k+1} = 0 && && \text{(3c)}\\
& t'_1 \leq T' && && \text{(3d)}\\
& \sum_{i=1}^{k} a_i (t'_i - t'_{i+1}) \leq S && && \text{(3e)}
\end{aligned}
$$

(3) has the same optimal solution with (2). The second trick we apply is substituting $t'_i - t'_{i+1}$ with a *nonnegative* variable $\Delta_i$ for $i = 1, \ldots, k$. We re-write (3) after the substitution as

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{k} a_i p_i \Delta_i && && \text{(4a)}\\
\text{s.t.} \quad & \sum_{i=1}^{k} \Delta_i \leq T' && && \text{(4b)}\\
& \sum_{i=1}^{k} a_i \Delta_i \leq S && && \text{(4c)}\\
& \Delta_i \geq 0 && i = 1, \ldots, k && \text{(4d)}
\end{aligned}
$$

$\Delta_i$ is not the duration for which price $i$ is applied but a somewhat related measure. $\Delta_i$ is the difference between the squares of two ending times for two consecutive prices $i$ and $i+1$. Consequently, $\Delta_i \geq 0$ guarantees constraint (3b) is satisfied, where $\Delta_i = 0$ implies price $i$ is not applied. This formulation is useful for a number of reasons. First of all, it can be used to find the optimal pricing scheme for any time interval, not necessarily starting at time zero. Until now, we assume the time begins at zero for the sake of brevity. However, in practice, we have to decide on booking classes in *real-time* that would maximize the expected revenue from that point forward, until the flight time. When we consider the optimization problem over an interval that starts at $T_0$, (3c) would be $t'_{k+1} = T_0$, leading to

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{k} a_i p_i \Delta_i && \text{(5a)}\\
\text{s.t.} \quad & \sum_{i=1}^{k} \Delta_i \leq T' - T'_0 && \text{(5b)}
\end{aligned}
$$

$$\sum_{i=1}^{k} a_i \Delta_i \leq S \tag{5c}$$

$$\Delta_i \geq 0 \qquad\qquad i = 1, \ldots, k \tag{5d}$$

The second reason formulation (4) and the generalization in (5) are useful is because a linear programming solver can solve these formulations using a rolling horizon approach and provide optimal real-time results. Another advantage of these linear programs is that they possess a unique structure that offers optimality conditions and several managerial insights.

# 4 Solution Approach

In this section, we first derive optimality conditions for formulation (5). Next, using these conditions for the fluid approximation, we propose a dynamic look-ahead pricing scheme that can be used in real-time.

## 4.1 Optimality Conditions and Insights

Using formulation (5), we prove the following result without any further imposition.

**Theorem 1.** *The optimal solution uses no more than two prices.*

*Proof.* A linear program with two functional constraints should have two basic variables at optimality. Therefore, the number of strictly positive $\Delta_i$ variables is at most two. □

The fact that *there are at most two prices (open fare classes) at optimality* is quite interesting from a revenue management standpoint. Next, we partition the fares into two and consider the cases where either (5b) or (5c) is not binding.

**Definition:** A fare class is *overselling/dilution* if all available seats are expected to be sold if that fare's price is applied during the planning horizon, that is $a_i(T' - T'_0) \geq S$, e.g., $a_i(T^2 - T_0^2) \geq 2S$ for the linear demand rates.

**Definition:** A fare class is *underselling/spoil* if unsold seats are expected if that fare's price is applied during the planning horizon, that is $a_i(T' - T'_0) < S$.

**Proposition 1.** *If all fare classes are overselling (underselling), a single fare with the highest price (largest intensity-price) is chosen at optimality.*

*Proof.* If all fare classes are overselling, that implies that the slack variable for constraint (5b) is basic, and constraint (5c) is binding at optimality. As there are two basic variables at optimality, only one $\Delta_i$ can be positive and known to satisfy $a_i\Delta_i = S$. This can be either one of the variables, but equality has to be satisfied. The best variable that can be selected is the one that multiplies that constant $a_i\Delta_i$ with the largest coefficient, hence the highest price.

If all fare classes are underselling, that implies that the slack variable for constraint (5c) is basic, and constraint (5b) is binding at optimality. As there are two basic variables at optimality, only one $\Delta_i$ can be positive and known to take value $T' - T'_0$. It is easy to see that among $k$ variables, the one with the largest objective coefficient (i.e., $a_i p_i$) is basic. □

As a consequence of these results, we have to clarify one crucial issue before our results on two prices.

**Proposition 2.** *Two fare classes chosen at optimality cannot be both overselling or underselling.*

*Proof.* The proof is immediate using the number of basic variables in the proof of Proposition 1. □

So far, we prove that either one of the following strategies can be optimal in general:

1. Among overselling fares, applying the one with the highest price
2. Among underselling fares, applying the one with the largest intensity-price
3. Applying two prices – an overselling fare is followed by an underselling fare

Finally, we focus on the most frequently observed case in practice, where there are a set of underselling and a set of overselling fare classes. We do not focus on the optimal timing of price markup for two prices, which can be solved easily. Instead, we aim to compute the price to be applied first. The same computation can be performed repetitively in real-time, which would indicate if there is a price markup, the timing, as well as the second price[1]. We start with a useful result.

**Proposition 3.** *In the presence of both underselling and overselling fares, applying one overselling fare only is suboptimal.*

*Proof.* The proof is by contradiction. Suppose that applying one overselling fare satisfies optimality conditions. If one overselling fare (say $o$) is applied only, then by the definition of an overselling fare, (5b) cannot be binding. Therefore, $\Delta_o$ and the slack variable for (5b) are basic. In this case, the optimal basis should provide all nonnegative reduced costs. The reduced cost of any underselling fare, say $u$, can be computed as

$$z_u = -a_u p_u + [a_o p_o 0] \begin{bmatrix} 1 & 1 \\ a_o & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ a_u \end{bmatrix}$$

$$= -a_u p_u + [a_o p_o 0] \begin{bmatrix} 0 & 1/a_o \\ 1 & -1/a_o \end{bmatrix} \begin{bmatrix} 1 \\ a_u \end{bmatrix} = -a_u p_u + a_o p_o a_u / a_o = a_u(p_o - p_u)$$

As $p_o < p_u$, we have a negative reduced cost, hence a contradiction.

□

Next, given an overselling fare, we identify the best complementary underselling fare that follows. We use the reduced cost identified in the proof above.

**Theorem 2.** *In the presence of both underselling and overselling fares, given an overselling fare $o$, the best underselling fare $u$ to be applied subsequently satisfies*

$$u = \arg\min_i a_i \frac{p_o - p_i}{a_o - a_i}. \tag{6}$$

[1]Although the look-ahead optimization problem leads to two prices, a rolling horizon approach might eventually lead to even more than two prices due to the stochastic nature of the demand.

*Proof.* Given the suboptimal basis of applying one overselling fare (say $o$), we aim to find the best neighbor vertex. Note that this point is not guaranteed to be optimal yet provides the best complementary price. We define the amount we can increase the underselling fare $i$ using the current right-hand side.

$$\begin{bmatrix} 0 & 1/a_o \\ 1 & -1/a_o \end{bmatrix} \begin{bmatrix} T' - T'_0 \\ S \end{bmatrix}$$

and the current coefficients for variable $\Delta_i$

$$\begin{bmatrix} 0 & 1/a_o \\ 1 & -1/a_o \end{bmatrix} \begin{bmatrix} 1 \\ a_i \end{bmatrix}.$$

Considering the reduced costs of $a_i(p_o - p_i)$ and the fact that current underselling variable cannot be leaving the basis, we determine that the best variable $u$ as follows:

$$u = \arg\min_i a_i(p_o - p_i) \frac{T' - T'_0 - S/a_o}{1 - a_i/a_o} = a_i(p_o - p_i) \frac{a_o T' - a_o T'_0 - S}{a_o - a_i} \tag{7}$$

As the nominator of the last multiplier is fixed for a given overselling fare, the proof is complete. □

Using the coefficients in the proof of Theorem 2, we can compute the durations for each fare in the two fare case as follows:

$$\Delta_o = \frac{a_o T' - a_o T'_0 - S}{a_o - a_u}, \Delta_u = T' - T'_0 - \Delta_o$$

Thus far, we do not enforce the convexity condition for the feasible region, i.e., inversely correlated prices and intensities. We need that for the main theorem below, $a_i \leq a_{i+1}, i = 1, \ldots k-1$, which is in fact quite realistic. Knowing that an underselling (higher) price is certainly applied at optimality, we identify the corresponding overselling fare to be applied first.

**Theorem 3.** *Given an underselling fare $u$ that is known to be applied at optimality, the best overselling fare that precedes it must satisfy*

$$a_o p_o > a_u p_u. \tag{8}$$

*Proof.* Suppose that in the current basis, we apply one underselling fare $u$ only. In that case, $\Delta_u$ and the slack variable for (5c) are basic, and the reduced cost of an overselling fare $i$ is

$$z_i = -a_i p_i + [a_u p_u 0] \begin{bmatrix} 1 & 0 \\ a_u & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ a_i \end{bmatrix} = -a_i p_i + [a_u p_u 0] \begin{bmatrix} 1 & 0 \\ -a_u & 1 \end{bmatrix} \begin{bmatrix} 1 \\ a_i \end{bmatrix} = -a_i p_i + a_u p_u$$

In order for an overselling fare to enter the basis, the reduced cost has to be negative, and the proof is complete.

□

**Corollary 1.** *One underselling fare $u$ is applied at optimality with no preceeding overselling fare if*

$$u = \arg\max_i a_i p_i,$$

*Proof.* The proof is immediate following the simplex iterations from the proof of Theorem 3. □

In light of these results, we can conclude the following in the presence of both overselling and underselling fares:

1. Among all fares, if the fare with the largest intensity-price is an underselling fare, this is the only fare applied at optimality.
2. Among all fares, if the fare with the largest intensity-price is an overselling fare, two prices are applied at optimality.

In the latter case, we also know that the overselling fare applied has a larger intensity-price than that of all underselling fares. However, we still have to compute the expected revenue of such combinations to find the best solution. Instead, we conclude with the following closed-form optimality condition for the two-fare case.

**Theorem 4.** *If two fares (underselling $u$ and overselling $o$) are known to be applied at optimality, the following condition must hold:*

$$a_i p_i \leq \frac{a_u p_u (a_o - a_i) + a_o p_o (a_i - a_u)}{a_o - a_u}, \quad \forall i \tag{9}$$

*Proof.* When an underselling fare $u$ and an overselling fare $o$ are in the basis, the reduced cost can be computed as

$$z_i = -a_i p_i + [a_u p_u a_o p_o] \begin{bmatrix} 1 & 1 \\ a_u & a_o \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ a_i \end{bmatrix}$$

$$= -a_i p_i + [a_u p_u a_o p_o] \frac{1}{a_u - a_o} \begin{bmatrix} -a_o & 1 \\ a_u & -1 \end{bmatrix} \begin{bmatrix} 1 \\ a_i \end{bmatrix} = -a_i p_i + \frac{a_o a_u (p_o - p_u) + a_i (a_u p_u - a_o p_o)}{a_u - a_o}$$

and the condition for that reduced cost to be nonnegative completes the proof. □

This theorem actually presents the optimal two fare scheme. The right-hand side of equation (9) is the combination of chosen intensity-price over the intensity axis. The result highlights that no other price should exceed the line combining the two fares over the two-dimensional space of intensity and intensity-price.

Consider the following toy example: Suppose $T = 90$ and $S = 180$ and we have 4 fares available. The intensities are $a_1 = 0.02, a_2 = 0.04, a_3 = 0.06, a_4 = 0.08$ and rates are linearly increasing. The prices are $p_1 = \$320, p_2 = \$200, p_3 = \$150, p_4 = \$120$.

For $T_0 = 0$, we can decide if a fare is an overselling/underselling using the critical value

$$\hat{a}(T^2 - T_0^2) = 2S \rightarrow \hat{a} \approx 0.044.$$

As fares 1 and 2 have smaller intensities than $\hat{a}$, they are underselling. On the contrary, fares 3 and 4 have larger intensities, hence overselling.

Figure 3 shows the graph of 4 fares with respect to price and intensity-price. The largest intensity-price is an overselling fare (fare 4). Therefore, due to Corollary 1, two fares are applied. Using Theorem 4, the two fares that are applied at optimality can be identified. As we elaborated above, the expression in Theorem 4 implies no other price should exceed the line combining the two fares over the two-dimensional space of intensity and intensity-price. The graph helps us visually

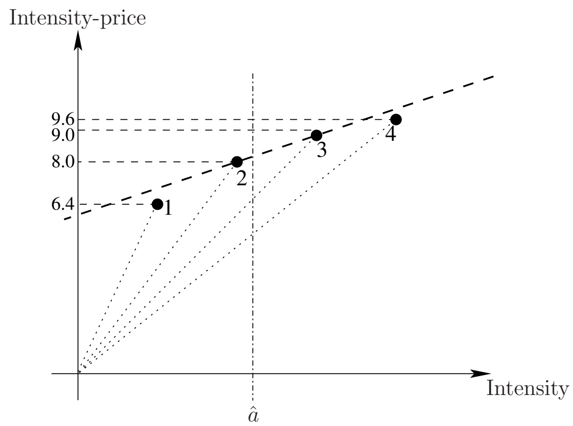



Figure 3: Optimal solution for the toy example with 4 available fares. The two fares that are applied at optimality are fares 2 and 3.

identify the two such fares easily, which are fares 2 and 3. In practice, this implies fare 3 is applied first, followed by 2. Note that the selected two fares at optimality have neither the highest price (fare 1) nor the largest intensity-price (fare 4). This analysis allows further results regarding the allowable ranges for certain parameters. For instance, everything else being the same, the price of fare 1 should be at least \$350, so that it is applied. In that case, fare 2 is not going to be applied. Likewise, with the initial parameter values, the price of fare 4 should be at least \$125, so that it is offered first, instead of fare 2.

### 4.2 A Dynamic Pricing Scheme

We show that expected revenue is maximized with a fluid approximation using at most two booking classes if the demands for predefined booking classes are known to follow certain patterns. This result can inspire a new methodology to be applied in a real-life stochastic environment. In this section, we present a scheme, through which we generate random booking requests and process them using any approach that decides on which fares to keep open. We illustrate the numerical comparison of different approaches in the next section. Hereby we propose Algorithm 1, which sets an active booking class (best buy) at each time a booking request occurs. Since initial fare allocation is a static method, it is not appropriate to use it during the whole booking period. Certainly, there will be deviations from the expected arrival rates, and this will generate some potential disadvantages for any approach. Therefore protection levels for higher fare classes are calculated each time a booking demand arrives, and a price markup decision is generated based on this protection level.

## 5 Numerical Experiments

To validate our approach, we synthetically create passenger arrivals and compare our policy, which we refer to as KKS, against two methods: (i) the widely-used EMSRb (Belobaba and Group, 1992) and (ii) dynamic programming (DP) (Fiig et al., 2010). We segment the arriving passengers so that each type $i$ passenger has an associated willingness to pay $p_i$ to purchase the ticket, which is

**INPUT:** $p$ (active price), $Prices$, $Demands$, $RemainingSeats$, $Revenue$, $RequestTime, RequestBookingClass$ (booking request information)

$CurrentTimePoint \leftarrow RequestTime$
Find the current best price that is not smaller than $p$ using a fare allocation method.
$p_{Best} = BestPrice(p, Prices, Demands, RemainingSeats)$
$p \leftarrow p_{Best}$
**if** $p_{RequestBookingClass} \geq p$ & $RemainingSeats > 0$ **then**
  Accept Request
  $RemainingSeats \leftarrow RemainingSeats - 1$
  $Revenue = Revenue + p$
**else**
  Reject Request
**end**

**OUTPUT:** $p$, $RemainingSeats$, $Revenue$

**Algorithm 1:** Processing booking requests dynamically

the maximum amount that passengers can pay. A type $i$ passenger purchases a ticket on sale for a price lower than or equal to $p_i$; otherwise, that passenger is lost. This structure allows overlapping demand structures, i.e., a lower price is affordable for higher classes. Therefore, the rates of the intensity functions for NHP demands of different fare classes are considered in a cumulative manner.

We randomly generate the data for six experiments to compare the three methods that are applied dynamically within Algorithm 1: our KKS solutions, EMSRb, and DP. For all experiments, the booking horizon is set to $T = 90$ days, and the number of available seats is $S = 180$. The slopes of intensity functions for NHP demand distributions and associated prices for each experiment are shown in Table 1 and Table 2, respectively. It should also be noted that, for a fair comparison, random booking requests with the associated willingness to pay is generated, and the same set of requests in each simulation instance are considered for all methods. At any point within the given time horizon, we can compute the upcoming pricing scheme (booking limits) using any approach that has a seat sale. That is, given the number of available seats and the rates above, we can compute the expected upcoming arrival scheme and make fare closing decisions.

| **Experiment** | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.02 | 0.04 | 0.06 | 0.08 | | | | |
| 2 | 0.02 | 0.03 | 0.04 | 0.06 | 0.07 | 0.08 | | |
| 3 | 0.02 | 0.025 | 0.03 | 0.04 | 0.06 | 0.07 | 0.075 | 0.08 |
| 4 | 0.02 | 0.03 | 0.035 | 0.04 | 0.055 | 0.06 | 0.065 | 0.08 |
| 5 | 0.02 | 0.04 | 0.06 | 0.08 | | | | |
| 6 | 0.02 | 0.04 | 0.06 | 0.08 | | | | |

Table 1: Slopes of NHP demand for experiments

In the EMSRb application, we first compute the expected demand for each fare using slope differences in Table 1. Formally, at time $t$, fare $i$ has an expected demand of $(a_i - a_{i-1})(T^2 - t^2)/2$.

| **Experiment** | **Available Prices** | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 350 | 200 | 150 | 120 | | | | |
| 2 | 350 | 275 | 200 | 150 | 135 | 120 | | |
| 3 | 350 | 325 | 300 | 200 | 150 | 130 | 125 | 120 |
| 4 | 350 | 250 | 225 | 200 | 160 | 150 | 140 | 120 |
| 5 | 350 | 200 | 130 | 125 | | | | |
| 6 | 220 | 200 | 130 | 125 | | | | |

Table 2: Price parameters for experiments

Due to Poisson properties, this is also the variance of demand. Instead of assuming another (normal) distribution with that mean and variance, we use the more accurate cumulative distribution function of a Poisson for the fare closing decisions in EMSRb. Using demands incrementally, we avoid double counting, but the fully undifferentiated structure in our study is not in line with the class independence required by EMSRb. Consequently, EMSRb might suffer from upper fare class requests deteriorating over time. However, due to the dynamic update mechanism of protection levels, EMSRb usually fine-tunes itself upon each arrival and relaxes an overaggressive strategy.

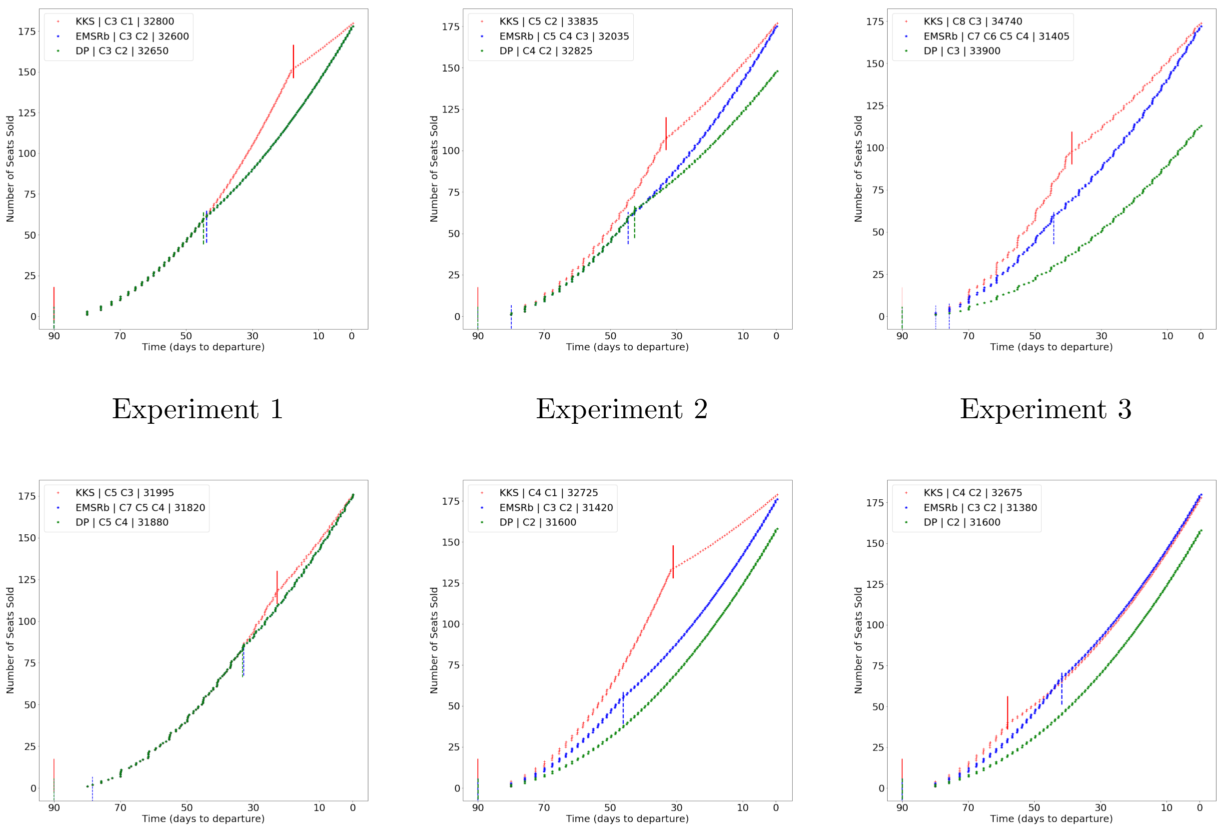
For the DP, at any point there is a sale, we order the fares in decreasing marginal revenue. Note that we consider the remaining demand until the end of the time horizon. The lowest fare we consider is the one that is expected to exceeds the remaining available capacity. Here, we do not need to transform the fare structure as we use the slope differences in Table 1 to compute the requests from independent populations with different willingness to pay. That is, at time $t$, fare $i$ has an expected demand of $(a_i - a_{i-1})(T^2 - t^2)/2$. Next, we choose the fare that has a nonnegative marginal revenue, that is no less than the current fare.

For verification, we first present results on deterministic experiments. In these scenarios, all requests appear at the expected time, considering the demand rates of the distribution – except infinitesimal offsets for each class to avoid simultaneous arrivals. Table 3 show the revenues for three approaches.

| **Experiment** | **KKS** | **EMSRb** | **DP** |
|---|---|---|---|
| 1 | **32,800** | 32,600 | 32,650 |
| 2 | **33,835** | 32,035 | 32,825 |
| 3 | **34,740** | 31,405 | 33,900 |
| 4 | **31,995** | 31,820 | 31,880 |
| 5 | **32,725** | 31,420 | 31,600 |
| 6 | **32,675** | 31,380 | 31,600 |

Table 3: Revenue comparison for expected booking requests

As expected, KKS yields the maximum revenue under the scenario of expected requests. The primary shortcoming of EMSRb is the inability to handle class dependence, and DP is a lack of information on upcoming demand changes. We illustrate the booking curves for these three approaches in Figure 4. KKS, as expected, can stick to the two price policies due to no stochasticity, taking advantage of high volume sales from the best possible low fare and profits of the best possible high fare.




Experiment 1 Experiment 2 Experiment 3

Experiment 4 Experiment 5 Experiment 6

Figure 4: Booking curve for experiments under expected requests.

Next, we perform 1000 random generations of each experiment setup, and the summary of the results is shown in Figure 5.

We observe that algorithms behave differently for each experimental setup. EMSRb outperforms other approaches only in experiment 1 (on average and in terms of median), but the differences are subtle. In experiment 2, KKS performs the best by a margin in terms of mean, median, and deviation in revenues. In experiment 3, DP performs the best on average, primarily due to a few outlier cases with outstanding revenues. In experiments 4, 5, and 6, despite similar averages, KKS performs slightly better than DP in terms of the median with a smaller deviation of results. For a better analysis, next, we provide the instances where KKS performs best/worst against EMSRb and DP.

We discuss some interesting instances in this section, and the remaining pairs of comparisons are in the Appendix. For reference, Figure 4 shows what these algorithms aim for unless there is an unexpectedly high or low arrival rate. We start with KKS vs. DP results on experiment 1, presented in Figure 6. In both extremes, algorithms stick to their initial strategies until the end of the horizon. Both KKS and DP start with fare class 3; thus, early fast/slow sales equally affect both approaches. In fact, considering the expected time for price markup for DP (around 45 days to

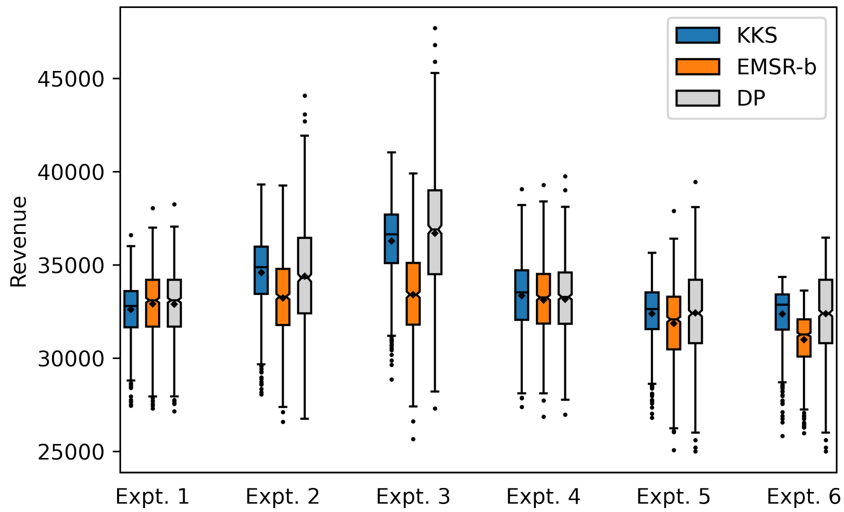
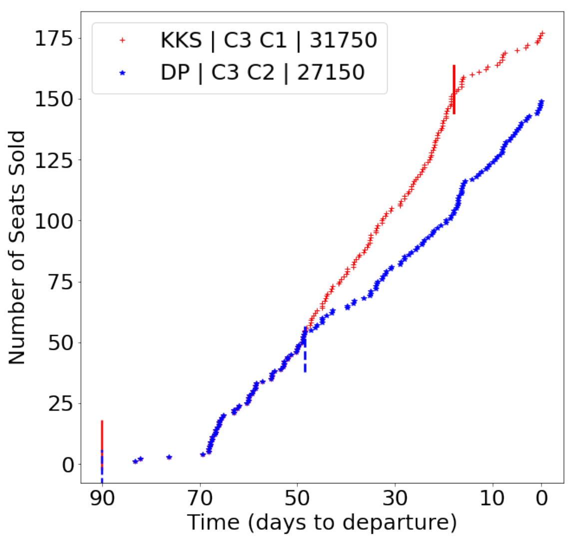
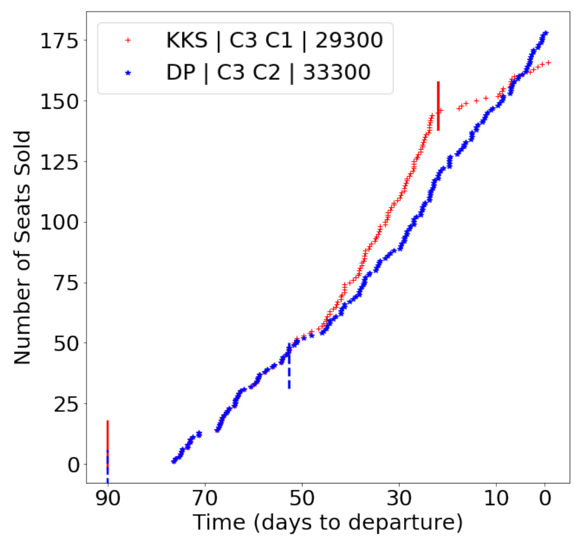



Figure 5: Box plots for revenues generated using dynamic pricing via KKS, EMSRb, and DP. Diamonds show the average of each algorithm for each experiment.



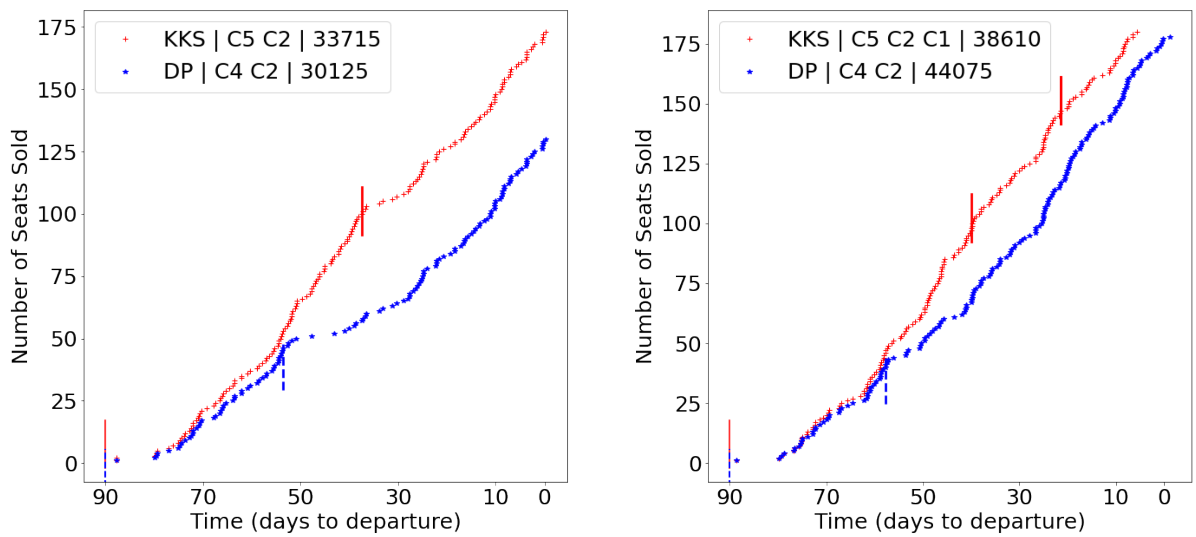

KKS outperforms DP




DP outperforms KKS

Figure 6: Largest difference between KKS and DP for experiment 1

departure), both extreme instances seem to have reasonably high demands. However, there is one major difference between the two random instances. After DP switches to class 2, if class 2 closely follows 3, that is, most of the arrivals buying from 3 are already willing to pay for 2, as in the figure on the right, the sales are unexpectedly well for DP, leading to a better solution. Furthermore, until KKS switches to class 1, the requests are still aligned with expectations; however, class 1 sales afterward are unexpectedly low in the last 20 days. On the other hand, if class 3 and 2 demand are somewhat differentiated as in the expected scenario, and class 1 sales are reasonably close to expectations as in the figure on the left, KKS is superior. These two instances are also the ones with the largest differences between KKS and EMSRb, and the fares of EMSRb are the same with DP with subtle differences in time or markup and revenue (see the Appendix).




KKS outperforms DP

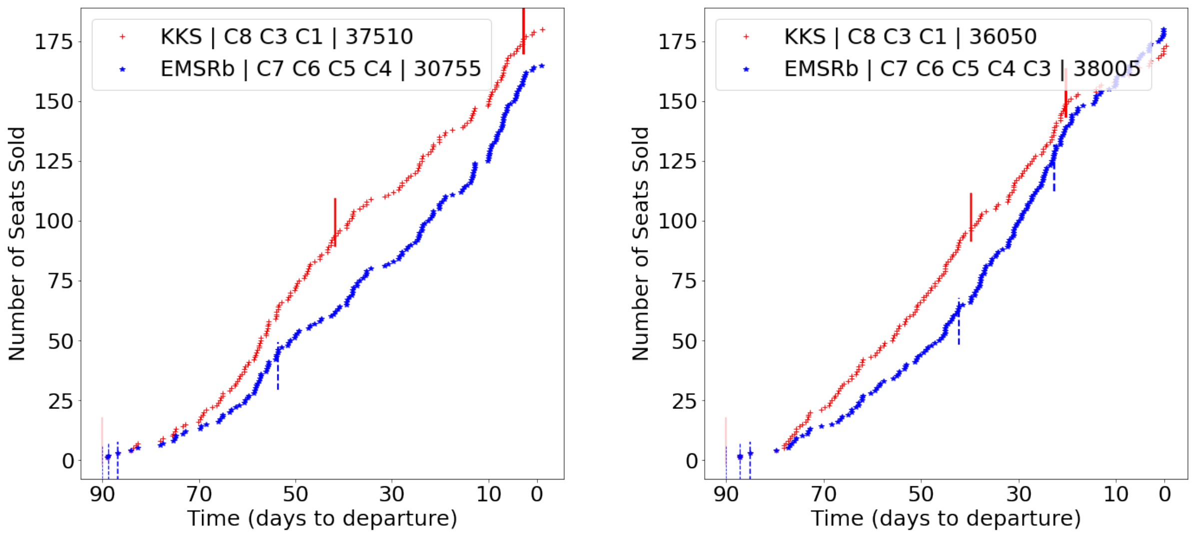
DP outperforms KKS

Figure 7: Largest difference between KKS and DP for experiment 2

Figure 7 shows the KKS vs. DP results in experiment 2. Here KKS enjoys a larger arrival rate due to opening a lower fare in the beginning. A lower fare is advantageous for near-expected or less-than-expected request scenarios, as seen on the left. However, this strategy leads to a sell-out in high demand scenarios. In this case, such as the one on the right plot, KKS switches to an even higher fare later but cannot make up for the loss. We observe seats are sold out around 10 days to departure, where demand is highest, and significant revenue potential is missed. This high demand scenario does not often occur, and KKS yields a better average revenue in experiment 2.

For experiment 3, presented in Figure 8, we compare KKS and EMSRb, because DP applies only fare 3 until the end of the planning period unless there is an unexpectedly high request rate. That strategy might seem successful in the light of 5; however, the extreme cases in the Appendix show that the performance is inferior, especially when the demand is lower than expected. Going back to KKS against EMSRb, both start with lower fares and adapt to booking requests' pace. Relatively lower demand on fares 7 through 4 is disadvantageous for EMSRb on the left, whereas early markup to class 1 followed by unexpectedly low demand seems to disrupt KKS on the right.

All experiments lead to similar observations. We witness that KKS typically starts with a lower fare class, primarily because booking requests will speed up later. We see this behavior in the majority of cases in the Appendix as well. Starting with a lower fare is advantageous and results in




KKS outperforms EMSRb

EMSRb outperforms KKS

Figure 8: Largest difference between KKS and EMSRb for experiment 3

better average revenues with less variability. Even in cases where KKS performs poorly, we observe that the price markup points are often not significantly off compared to the planned times. The higher expectations for KKS towards departure time is crucial in these cases: if booking requests do not speed up towards departure time as expected, KKS performs poorly. Another formulation can address that shortcoming, considering risk minimization, as well as expected revenue maximization. Regarding our benchmark approaches, DP also performs quite successfully; however, its low performance and high variability on some data sets are noteworthy. EMSRb, on the other hand, performs poorly on average and under demand deviations.

# 6 Concluding Remarks

In this study, we address three crucial aspects that most of the studies oversee in airline RM. We address the issue of *fare dependency* and cannibalization, using independent customer segments. This way, we also ensure a lower price can never have a lower demand rate for the same product. We model the *increasing demand patterns* and provide real booking data from an airline company as evidence.

If we solely focus on maximizing the expected revenue statically, over a fixed time period, our fluid approximation shows that the airline is expected to (i) sell all seats and (ii) use one or two prices. The first result is somewhat expected. However, the second result is quite impressive, as this is in contrast with the usual RM assumption of targeting each segment with the best possible price. In practice, dynamic pricing may offer several prices due to demand dynamics. Nevertheless, if the demand can be assumed a nonhomogeneous Poisson process (e.g., monopolistic market), it is interesting to show offering a small number of prices is expected to cultivate maximum revenue, unless there is a significant deviation from the expected demand pattern. Consequently, this might transform customers' willingness to pay, which might lead to future RM studies.

The optimality conditions for the proposed approach leads to a practical solution algorithm that is shown to outperform two well-established procedures under suitable conditions. During the presented simulation study, dynamically considering the remaining time horizon also addresses the inherent stochasticity by *offering more than two prices* from time to time. That illustrates one of the shortcomings of the fluid approximation, which does not consider the risks associated with stochasticity of demand during price markups that cannot be reverted.

There are a number of directions for future work in the area of dynamic pricing with the given demand patterns. Stochastic optimization models can be studied under different scenarios to provide a structural understanding. A probabilistic dynamic programming approach would be useful under the presented demand behavior. A fluid approximation and a dynamic programming scheme would also benefit from risk measures, alongside revenue maximization.

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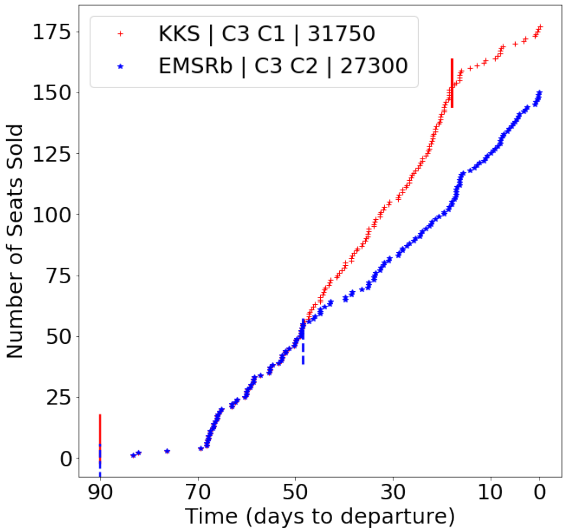
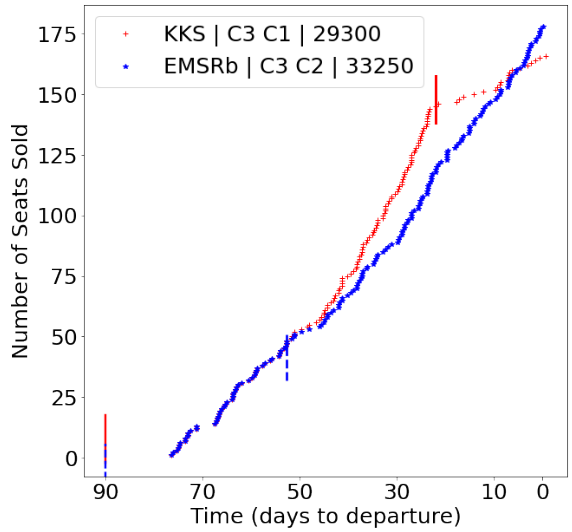
# Appendix

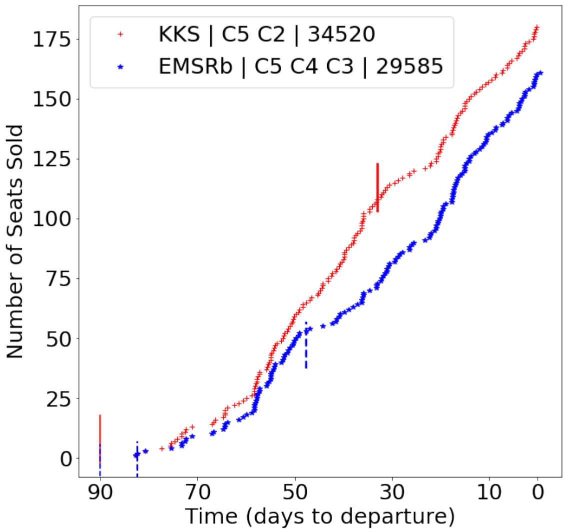



KKS outperforms DP

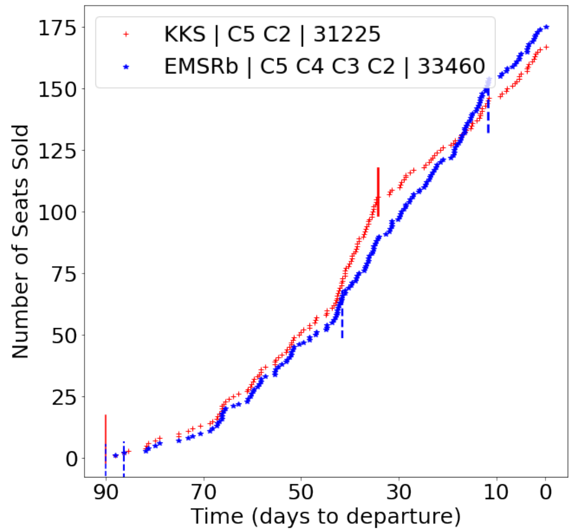



DP outperforms KKS

Figure 9: Largest difference between KKS and EMSRb for experiment 1

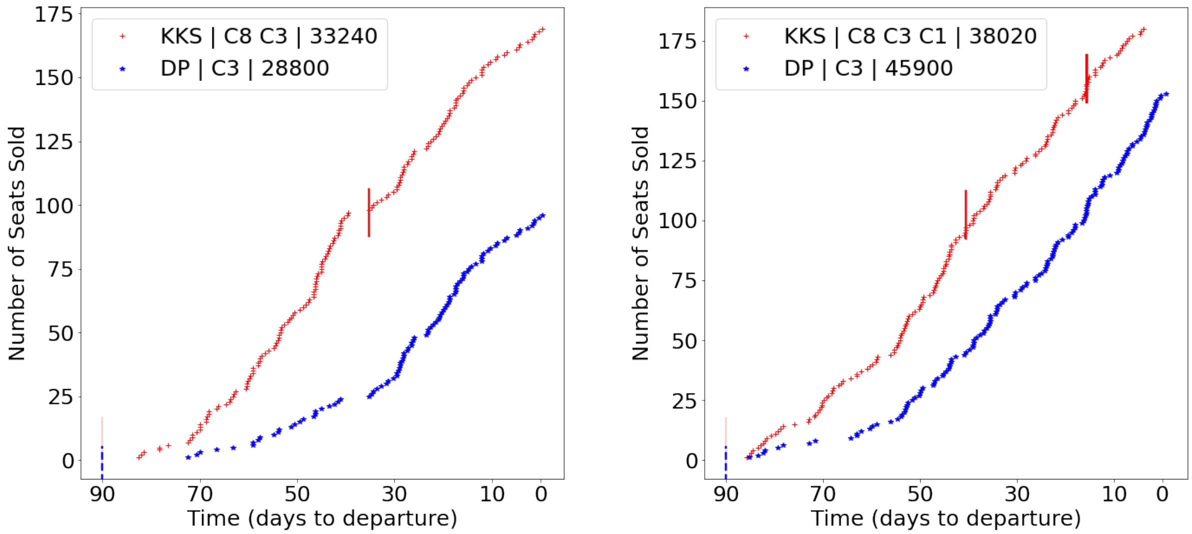



KKS outperforms EMSRb




EMSRb outperforms KKS

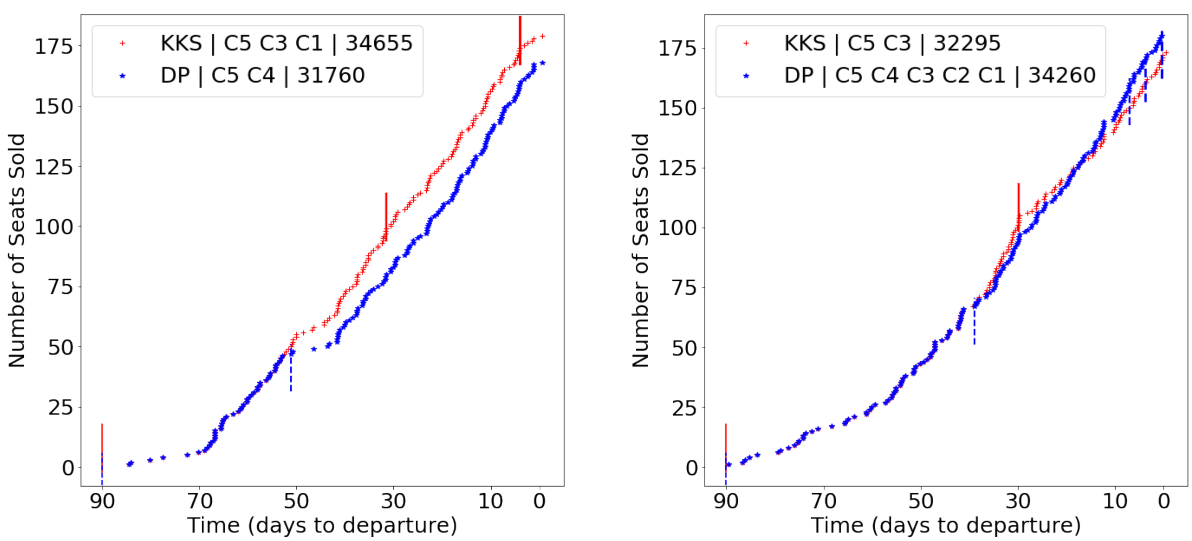
Figure 10: Largest difference between KKS and EMSRb for experiment 2




KKS outperforms DP

DP outperforms KKS

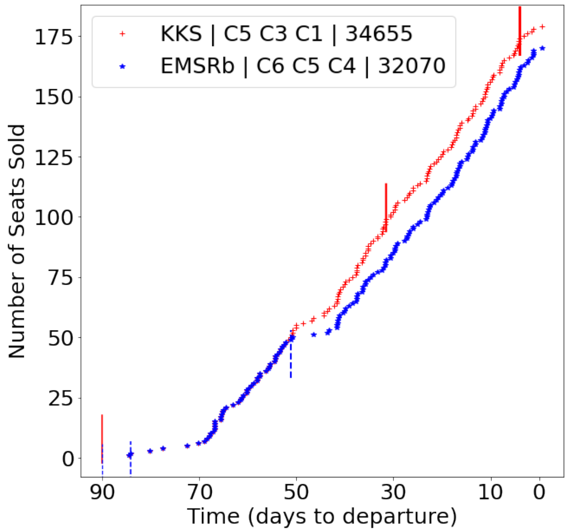
Figure 11: Largest difference between KKS and DP for experiment 3



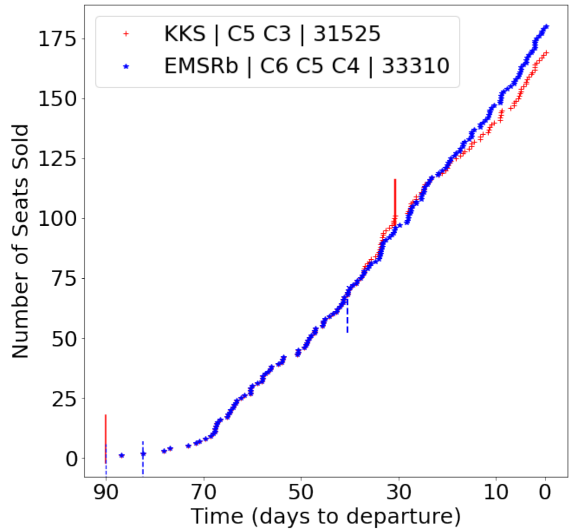

KKS outperforms DP

DP outperforms KKS

Figure 12: Largest difference between KKS and DP for experiment 4

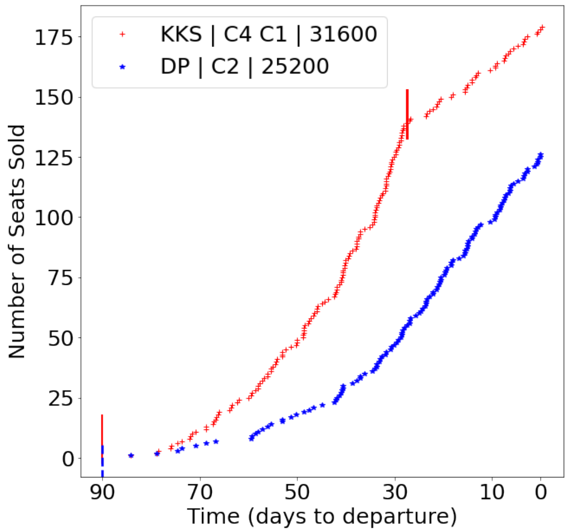



KKS outperforms EMSRb

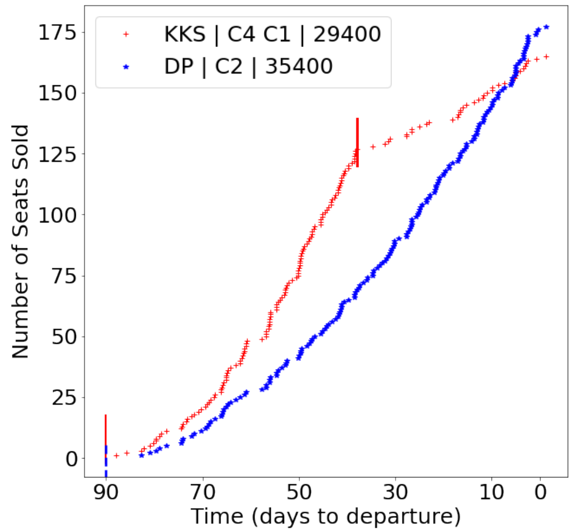



EMSRb outperforms KKS

Figure 13: Largest difference between KKS and EMSRb for experiment 4

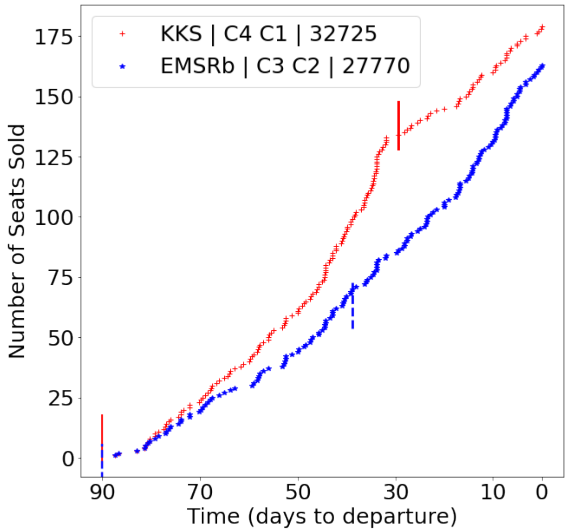



KKS outperforms DP

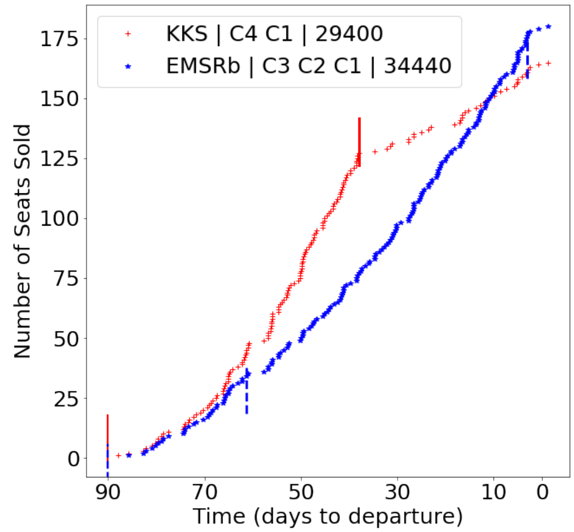



DP outperforms KKS

Figure 14: Largest difference between KKS and DP for experiment 5

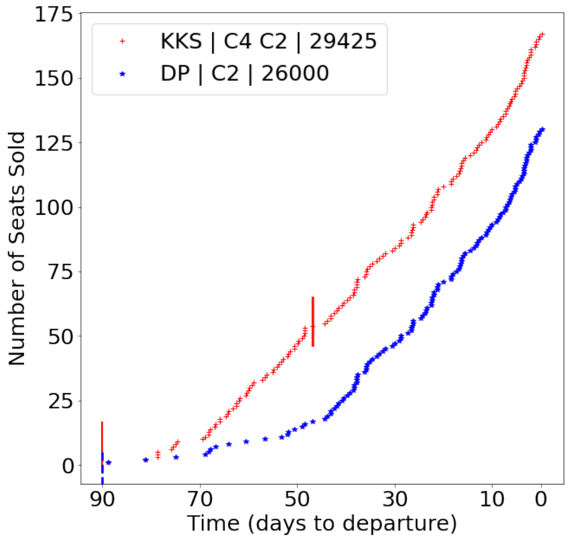



KKS outperforms EMSRb

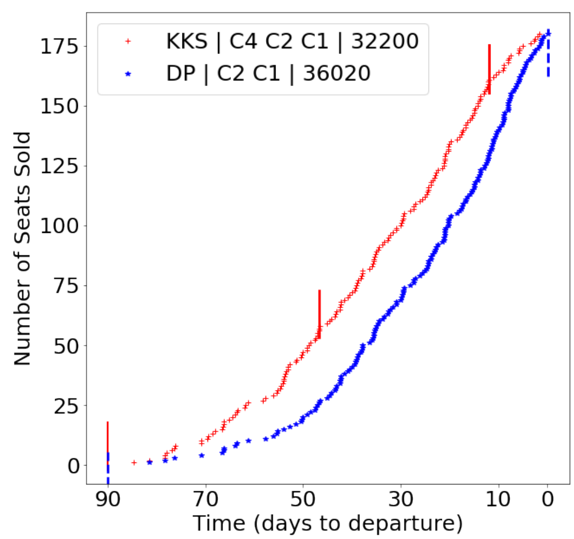



EMSRb outperforms KKS

Figure 15: Largest difference between KKS and EMSRb for experiment 5

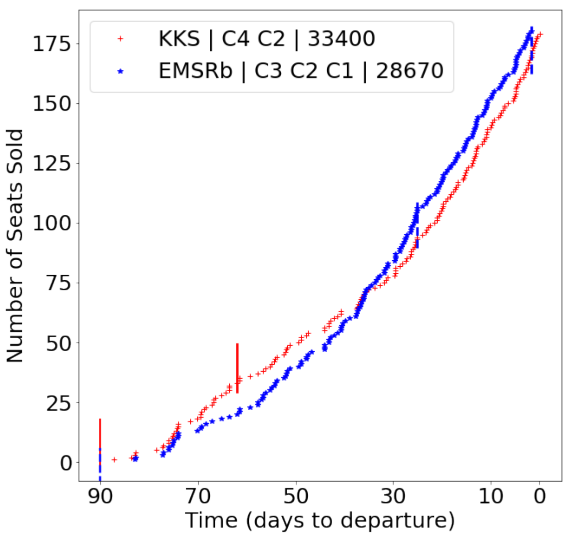



KKS outperforms DP

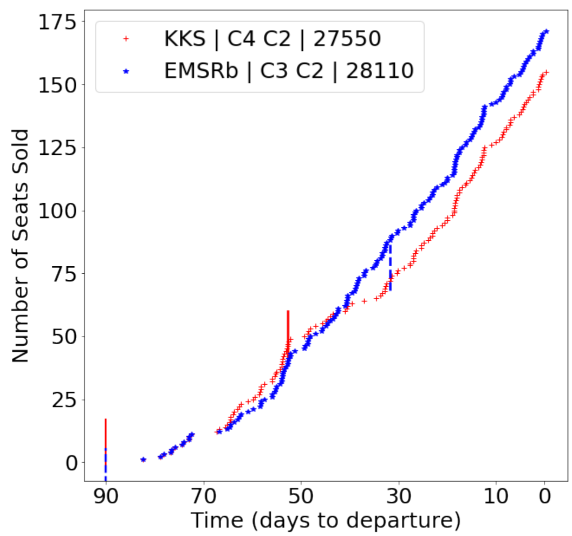



DP outperforms KKS

Figure 16: Largest difference between KKS and DP for experiment 6




KKS outperforms EMSRb




EMSRb outperforms KKS

Figure 17: Largest difference between KKS and EMSRb for experiment 6