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An EOQ Model with Deteriorating Items and Self-Selection Constraints

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Abstract

In this paper, we consider a store that sells two vertically differentiated items that might substitute each other. These items do not only differ in quality and price, but they also target two different customer segments. Items deteriorate over time and might require price adjustments to avoid *cannibalization*. We provide closed-form solutions for pricing and ordering of these items that lead to key managerial insights.

Keywords: Cannibalization, price markdown, deterioration.

1 Introduction

Companies in various industries generally offer differentiable lines of items within the same product category. These items can be differentiated horizontally or vertically, depending on whether the different variants of the item can be ranked based on the quality of the product. Consider a retailer that sells fresh produce, for instance. We say fuji and gala apples varieties are *horizontally differentiated* because the preference of consumers is not ordered even if both are available at the same price. Some people may inherently prefer fuji apples, whereas others may prefer gala apples. In contrast, *organic* gala apples and *conventional* gala apples are an example of *vertically differentiated* products. That is, if these variants of gala apples have the same price, consumers will buy higher quality organic apples over the conventional variety.

The treatment of the entire market as one homogeneous group, and providing all customers with the same offering is known as *mass marketing*. Though this provides economies of scale through mass production, distribution, and communication, it ignores the fact that different customers have varying needs, and consequently, not all of them can be satisfied with the same offering. Mass marketing may eventually lead to a loss in market share as competitors may offer products tailored to meet the needs of specific consumer groups. *Target marketing*, on the other hand, involves offering products with distinct features to meet the requirements of customers in different groups. This process of partitioning the market is called *market segmentation*. The idea behind market segmentation is to identify consumer groups that have shared preferences for products based on specific characteristics or attributes (price or quality).

Customer value-based pricing is a pricing strategy that considers how consumers in a group, or a segment, value the product. A commonly used variant is *perceived value pricing*, where

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the price of a product is governed by the customer's perception of the value or worth of that product. An extensive literature review on value-based pricing is available in Hinterhuber [7]. A value-based pricing case that is worth investigating is when the item in consideration deteriorates over time. Here, deterioration means a reduction in the quality of the item. In the literature, loss in quality is generally represented by an exponential or a linear function of time, i.e., either of the form $q(t) = q_0 e^{-\lambda t}$ or of the form $q(t) = q_0 - \lambda t$. Tsiros and Heilman [15] establish that for low product quality risk (PQR) items (fresh produce and dairy products) it is appropriate to assume that quality decreases linearly in time, whereas for high PQR items (meat products) exponential decrease is more appropriate. In an economic order quantity (EOQ) model, Fujiwara and Perera [6] use linear holding costs for items that lose their quality linearly, and exponential holding costs for items that deteriorate exponentially in time. The decrease in quality is accompanied by a reduction in the value of the item perceived by the consumers. Scudder and Blackburn [13] examine the supply chain design strategies for fresh food products whose values decrease exponentially over time. They define the value of a product as a function of time as $V(t) = V_0 e^{-\gamma t}$, where γ depends on the item and the temperature it is maintained at. When there is a decrease in the value, prices have to be adjusted so that the product still has some utility to the consumer. Wang and Li [16] propose a dynamic quality based pricing model where the quality may decrease linearly or exponentially in time. They investigate pricing policies where the price is either reduced once or multiple times during the lifetime of the item. Banerjee and Agrawal [1] propose a discounting policy where an item shows no loss in freshness (quality) for some time and then loses its freshness either linearly or exponentially in time. Liu et al. [9] propose a model where pricing decisions are based on the real-time value of items. This study is based on the supposition that radio-frequency identification (RFID) technology is available for instantaneous monitoring.

Moorthy and Png [10] propose a framework in the area of market segmentation and value-based pricing. They consider the problem of a seller who can supply an item with any requested quality value. The market the seller faces can be divided into two customer segments. Customers in one segment value quality more than the customers in the other segment. The seller has to decide on the quality of the products offered to both segments. However, the marginal cost of supplying the product increases as the quality of the item increases. The seller decides on the prices for products offered as well in a way to maximize the revenue avoiding cannibalization. Cannibalization occurs when customers of a particular segment consume products that target customers in a different segment.

Cannibalization is usually undesired, because it leads to shifted sales amongst products, making it harder to predict, market, and plan ahead of time. Stavrulaki [14] studies a retailer's inventory policy for two substitutable products and states that cannibalization is more significant when deterioration is involved. Hvolby and Steger-Jensen [8] consider a cannibalization scenario where stocks with different ages of a deteriorating product exist together on the shelves of the retailer. In such a case, consumers will have a tendency to consume the fresher item first, and older items may perish on the shelves. This causes the retailer to have high uncertainty in demand and quality of inventory. Hvolby and Steger-Jensen [8] state that this uncertainty can be reduced through the pricing of the products. Several other operations management studies incorporate the effect of cannibalization. Dan and Ding [4] examine the effect of cannibalization on a firm's profitability under different customer classifications. Parlaktürk [12] considers a firm that sells two vertically differentiated products. They compare the profits when the two products are on the market versus when there is one. A remanufactured product might also cannibalize the demand for the original product (see [11] and [5]).

In this paper, we introduce an EOQ model that incorporates the utility model in [10]. As in [10], we consider a retailer that faces a market with two customer segments. Customers in one segment value quality more. Hence, the retailer can offer a high-quality item with high prices to these customers. On the contrary, customers in the other segment do not value quality as much. Thus, the retailer is offering a lower quality item to these customers at lower prices. Unlike Moorthy and Png [10], we assume that the qualities of items offered to two customer segments are not decision variables, but input. Furthermore, unlike Moorthy and Png [10], we assume that these items deteriorate (lose quality) linearly over time. However, if the items deteriorate with different rates, the item that deteriorates slower can become attractive to both customer segments in time, unless a price adjustment is made. That is, an item can substitute the other in time, and cannibalization may occur. The retailer has to decide on the replenishment interval, the prices, and when to adjust prices. The prices are set at the beginning of the replenishment cycle and adjusted only once during the cycle. Because the values of the items decrease in time, prices have to be marked down so that items still have some utility to the consumers and that cannibalization does not occur. This is observed in retailers that produce and sell fresh produce, where linear quality loss assumption is appropriate. Both organic and conventional fruits (or vegetables) produced by the farm could exist together on the shelves. Even if all products are conventional, harnessing products of various qualities (sizes, tastes) may be inevitable. This implies that there might be a steady production of items of various qualities. Thus, the seller has to make sure that demand is also steady for those items to maximize his/her revenues. However, these items might also exhibit different deterioration rates. In time, the item that stays fresher might become more attractive to all customer segments. As a result, the retailer may observe stock-outs for one item, while the other item is never consumed and expires on the shelves without producing steady revenue.

There are several ordering and pricing studies for perishable items that are close to our area of study [3, 2, 17]. However, to the best of our knowledge, no ordering model has been studied that prevents *cannibalization* through pricing for products with decaying utilities.

The rest of the paper is organized as follows: We describe our model in detail and present a mathematical formulation in Section 2. We propose our solution method in Section 3. In Section 4, we perform sensitivity analysis and present managerial insights. Finally, we conclude the paper and discuss further research opportunities in Section 5.

2 Model Description

We consider a retail store that sells two vertically differentiated items that may substitute each other. These items differ in quality and price, and target two different customer segments. Throughout the paper, depending on their quality, we refer to the items as *high* or *low quality item*. Likewise, we refer to the two customer types these products target as *high* or *low quality (targeted) customers*. We use subscripts (superscripts) h and l on any function and variable that are associated respectively with high and low-quality items (customers). We assume that the retailer follows a replenishment policy, where the two items are replenished at the same time. As a result, stocks of two items follow repeating inventory cycles with equal replenishment intervals of length T. This can be the case when shipments are consolidated.

2.1 Quality and value functions

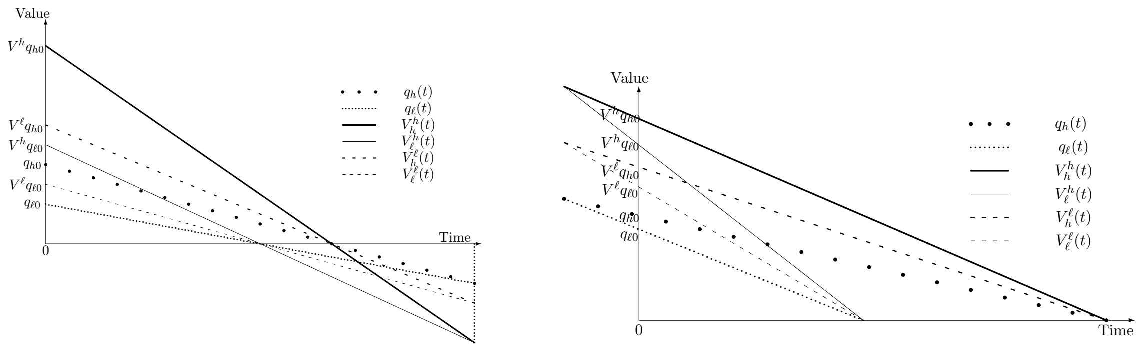
We assume that the quality of an item degrades linearly in time. We let quality function $q_i(t) = q_{i0} - m_i t$ define the quality of item $i \in \{h, \ell\}$ at time $t \in [0, T]$. Here, q_{i0} is the initial quality value at the beginning of the replenishment cycle, and m_i is the rate of decrease in quality of item $i \in \{h, \ell\}$. We assume that throughout an inventory cycle, the quality of a high quality item is always higher than the quality of a low quality item. That is $q_{h0} > q_{l0}$ and

$q_h(t) \geq q_l(t)$ for $t \in [0, T]$. Moreover, we assume that $q_i(t) \geq 0$ for $i \in \{h, \ell\}$ and $t \in [0, T]$.

A high-quality item does not necessarily have a lower rate of decay; that is, we might have $m_h > m_l$. In our analysis, we make a distinction between the two cases. In the first case, which we call Case 1, we assume that high-quality item degrades faster than the low-quality item (i.e., $m_h > m_\ell$). In the second case, which we call Case 2, we assume that the low-quality item degrades faster than the high-quality item (i.e., $m_\ell > m_h$). We deliberately ignore the $m_h = m_\ell$ case. Because if two items deteriorate at the same rate, then no item will be more attractive to the customer in the other segment in time, resulting in no risk of cannibalization. This discussion will be more apparent later in the paper.

We assume that customers from the two segments appraise quality differently. We let $V_i^j(t)$ be the value of item of quality $i \in \{h, \ell\}$ perceived by customer in segment $j \in \{h, \ell\}$ at time $t \in [0, T]$ and define it as $V_i^j(t) = V^j q_i(t) = V_{i0}^j - m_i^j t$, where $V_{i0}^j = V^j q_{i0}$, and $m_i^j = V^j m_i$. Here, factor V^j for each customer type $j \in \{h, \ell\}$ translates quality of an item to the perceived value of that item for that customer. By definition, high quality customers value quality more than low quality customers do, i.e., $V^h > V^l$.

Figures 1a and 1b show quality functions for high and low-quality items as well as the value functions of the two customer types. In both figures, the quality functions of high and low-quality items are illustrated with the thick and thin dotted lines, respectively. The values perceived by a high-quality customer are illustrated with thick and thin straight lines, respectively for high and low-quality items. The values of these two items perceived by a low-quality target customer are illustrated with the graphs drawn in thick and thin dashed lines.



(a) when $m_h > m_\ell$ (i.e., Case 1)

(b) when $m_\ell > m_h$ (i.e., Case 2)

Figure 1: Quality and value functions.

2.2 Mathematical formulation

We let $p_i(t)$ be the price of item of type $i \in \{h, \ell\}$ and define $U_i^j(t) = V_i^j(t) - p_i(t)$ as the utility of item $i \in \{h, \ell\}$ for the customer in segment $j \in \{h, \ell\}$ at time $t \in [0, T]$. The retailer sets prices in such a manner that the utility of an item for the customer type targeted by that item is always positive. We assume that if the two substitutable items are displayed together, customers buy the one that offers the highest utility. We also assume that, as long as cannibalization is prevented, high and low-quality items are consumed with constant demand rates of n_h and n_ℓ per unit time, respectively, regardless of the changes in their utilities.

The retailer sets prices in such a manner that cannibalization does not occur. However, retailers usually avoid frequent price adjustments due to strategic and operational issues. Frequent price changes are known to be costly due to bookkeeping. Therefore, we assume that the retailer adjusts prices only once during the inventory cycle of the items. Furthermore, as the utilities of the items decrease in time, price adjustment implies a *price markdown*. We let t_m, $0 \leq t_m \leq T$, be the time of this single markdown. As the price is adjusted only once, we have two different prices that need to be determined in the replenishment cycle of each item. We let $p_{i,1}$ and $p_{i,2}$ be the price set for item $i \in \{h, \ell\}$ in the intervals $[0, t_m]$ and $[t_m, T]$, respectively.

Let K be the joint fixed cost of ordering and h_h and h_ℓ be the respective unit holding costs per unit time. Since demand rates for both items are constant, in an optimal solution, inventory levels at the end of each cycle are equal to zero. Thus, it is straightforward to calculate that the total ordering and inventory holding costs per unit time is equal to $\frac{K}{T} + \frac{T}{2}(n_h h_h + n_\ell h_\ell)$. The retailer's objective is to determine, (i) how long the replenishment interval should be (i.e., determine T), (ii) when to mark down prices (i.e., determine t_m, $0 \leq t_m \leq T$), and (iii) what prices should be set for the items in the intervals $[0, t_m]$ and $[t_m, T]$ (i.e., determine $p_{i,1}$ and $p_{i,2}$ for $i \in \{h, l\}$), in order to maximize profit per unit time. Letting (P) denote this problem, we provide a summary of the nomenclature and formulate it as follows.

Parameters:

- K: fixed ordering cost,
- h_i: unit holding costs per unit time for item of quality $i \in \{h, \ell\}$,
- n_i: demand rate for item of quality $i \in \{h, \ell\}$,
- $U_i^j(t)$: utility of item $i \in \{h, \ell\}$ for customer in segment $j \in \{h, \ell\}$ at time $t \in [0, T]$,
- V_{i0}^j: initial value of item of quality $i \in \{h, \ell\}$ perceived by customer in segment $j \in \{h, \ell\}$,
- m_i^j: slope of utility function $U_i^j(t)$.

Decision variables:

- T: length of the replenishment interval,
- t_m: time of price mark down,
- $p_{i,1}$: price in the interval $[0, t_m]$ for item of quality $i \in \{h, \ell\}$,
- $p_{i,2}$: price in the interval $[t_m, T]$ for item of quality $i \in \{h, \ell\}$.

$$
\text{(P)} \quad \max_{p_{\ell,1}, p_{\ell,2}, p_{h,1}, p_{h,2}, t_m, T} \frac{p_{\ell,1} n_\ell t_m + p_{\ell,2} n_\ell (T - t_m) + p_{h,1} n_h t_m + p_{h,2} n_h (T - t_m)}{T} - \frac{K}{T} - \frac{T}{2}(n_h h_h + n_\ell h_\ell) \quad \text{(1a)}
$$

$$
\begin{aligned}
\text{subject to} \quad & 0 \leq t_m \leq T && \text{(1b)}\\
& U_i^j(t) = V_{i0}^j - m_i^j t - p_{i,1} \qquad 0 \leq t \leq t_m, i \in \{h, \ell\}, j \in \{h, \ell\} && \text{(1c)}\\
& U_i^j(t) = V_{i0}^j - m_i^j t - p_{i,2} \qquad t_m \leq t \leq T, i \in \{h, \ell\}, j \in \{h, \ell\} && \text{(1d)}\\
& U_i^i(t) \geq U_{i'}^i(t) \qquad 0 \leq t \leq T, i \in \{h, \ell\}, i' \in \{h, \ell\}, i \neq i' && \text{(1e)}\\
& U_i^i(t) \geq 0 \qquad 0 \leq t \leq T, i \in \{h, \ell\} && \text{(1f)}\\
& T > 0 && \text{(1g)}
\end{aligned}
$$

Objective function (1a) represents the profit obtained per unit time. Constraint (1b) states that price markdown occurs at some time within the replenishment cycle. Constraints (1c) and (1d) ensure that utilities for both items are non negative throughout the cycle. These constraints ensure that prices are not set above the perceived values of the items. Constraints (1e) guarantee that the utility of a particular item to a customer targeted by that item should be higher than the utility of that item to a customer not targeted by that item. In other words, the utility of the high (low) quality item to the high (low) quality customer should be higher than the utility of the low (high) quality item to the high (low) quality customer. As a matter of fact, these are the constraints that prevent cannibalization. Following Moorthy and Png [10], we call these constraints "*self-selection constraints*". Constraints (1f) are customer satisfaction constraints, which state that utilities of items for customers targeted by those items should be nonnegative. Constraint (1g) states that the replenishment cycle should be greater than zero.

In addition to the constraints stated above, we have the following constraints:

$$T \leq \min\{\frac{q_{h0}}{m_h}, \frac{q_{\ell 0}}{m_\ell}\}, \tag{2}$$

$$T \leq \frac{q_{h0} - q_{\ell 0}}{m_h - m_\ell} \text{ (if } m_h > m_\ell). \tag{3}$$

Constraint (2) guarantees that qualities of the items do not drop below 0 (otherwise a negative price should be set in order to have positive utilities). Constraint (3) guarantees that quality of the "high quality" item is higher than the quality of the "low quality" item. Constraint (3) is included only if $m_h > m_\ell$, i.e, only in Case 1 (it automatically holds for Case 2). So if we let $T_{max,1} = \min\{\frac{q_{h0}}{m_h}, \frac{q_{\ell 0}}{m_\ell}, \frac{q_{h0}-q_{\ell 0}}{m_h-m_\ell}\}$, for Case 1, constraints (2) and (3) imply

$$T \leq T_{max,1} \tag{4}$$

Similarly, for Case 2, if we let $T_{max,2} = \min\{\frac{q_{h0}}{m_h}, \frac{q_{\ell 0}}{m_\ell}\}$, constraint (2) can be rewritten as

$$T \leq T_{max,2} \tag{5}$$

We call these constraints "cycle time bound constraints" as they imply a bound for the cycle time. In fact, we can incorporate into $T_{max,1}$ and $T_{max,2}$ any additional restrictions the retailers might have in their minds for their practical concerns. Throughout the paper, when we say "a feasible T", we mean a value for T that satisfies constraints (1b) to (1g), as well as constraint (4) for Case 1, and (5) for Case 2. We could also add constraints $p_{i,1}, p_{i,2} \geq 0$ for $i \in \{h, \ell\}$. But these constraints are readily satisfied at optimality, because the value functions are non-negative due to constraints (4) and (5), and because we maximize profit.

3 Solution Approach

The objective function contains terms where variables p, and t_m are multiplied. With no identified relation between these variables, it is neither concave nor convex, even when T is fixed. In order to solve (P) as it is, commercial solvers that implement all-purpose global optimization algorithms would be needed. Their solution time would depend on the parameters of the problem. However, through observation and further treatment, a constant time algorithm can be developed. In this section, we present the details of our solution approach, which proceeds as follows. We assume *feasible* T and t_m values are given in Sections 3.1 and 3.2. For given T and t_m, we derive expressions for optimal $p_{i,1}$ and $p_{i,2}$ for $i \in \{h, \ell\}$. Then, substituting these prices, we reformulate problem (P) in Section 3.3 and determine optimal T and t_m values.

3.1 Determining constant prices in a given time interval

Related to problem (P), let (SP(t_1, t_2)) denote a restricted version of (P) where, (i) cycle length T (which is feasible in (P)) is given, (ii) a single price p_i has to be set for $i \in \{h, \ell\}$ in some interval $[t_1, t_2]$, $0 \leq t_1 < t_2 \leq T$, (i.e., there is no price markdown, or equivalently, markdown time is fixed at $t_m = 0$), and (iii) the objective is to maximize revenues in the interval $[t_1, t_2]$ subject to constraints (1c)-(1f) of (P), which are adjusted to be valid only in $[t_1, t_2]$. Before presenting a formulation of (SP(t_1, t_2)), we state some important observations.

Due to customer satisfaction constraints (1f) of problem (P), we have that $p_i \leq V_i^i(t)$ for $t \in [t_1, t_2]$, $i \in \{h, \ell\}$ in problem (SP(t_1, t_2)). Since the value functions are decreasing in t, these constraints are translated to (SP(t_1, t_2)) as follows:

$$p_i \leq V_i^i(t_2) \text{ for } i \in \{h, \ell\}.$$

If there were no self-selection constraints (constraints (1e)), we could set $p_i = V_i^i(t_2)$. But there are self-selection constraints, which effectively restricts price choices. The effect of these constraints will be seen with the reformulation of (SP(t_1, t_2)). We can translate constraints (1e) of problem (P) to problem (SP(t_1, t_2)) as follows:

$$\begin{aligned} V_h^h(t) - p_h &\geq V_\ell^h(t) - p_\ell \text{ for } t \in [t_1, t_2], \\ V_\ell^\ell(t) - p_\ell &\geq V_h^\ell(t) - p_h \text{ for } t \in [t_1, t_2] \end{aligned}$$

We introduce the variable $x = p_h - p_\ell$. Then, recalling that $V_h^i(t) > V_\ell^i(t)$ for $i \in \{h, \ell\}$, $t \in [t_1, t_2]$, the self-selection constraints imply that

$$\begin{aligned} x = p_h - p_\ell &\leq V_h^h(t) - V_\ell^h(t) \text{ for } t \in [t_1, t_2] \\ x = p_h - p_\ell &\geq V_h^\ell(t) - V_\ell^\ell(t) > 0 \text{ for } t \in [t_1, t_2]. \end{aligned}$$

Then, (SP(t_1, t_2)) can be formulated as follows:

$$\begin{aligned}
(\mathrm{SP}(t_1, t_2)) \quad \max_{p_\ell, x} \quad & p_\ell n_\ell(t_1 - t_2) + (p_\ell + x) n_h(t_1 - t_2) && && \text{(6a)} \\
\text{subject to} \quad & x \leq V_h^h(t) - V_\ell^h(t) && \text{for } t \in [t_1, t_2] && \text{(6b)} \\
& V_h^\ell(t) - V_\ell^\ell(t) \leq x && \text{for } t \in [t_1, t_2] && \text{(6c)} \\
& p_\ell \leq V_\ell^\ell(t_2) && && \text{(6d)} \\
& p_\ell + x \leq V_h^h(t_2) && && \text{(6e)} \\
& p_\ell \geq 0
\end{aligned}$$

Note that self-selection constraints of problem (P) are translated to lower and upper bounds on the price difference between the two items in constraints (6b) and (6c). It is easy to see that without self-selection constraints, in an optimal solution to (SP(t_1, t_2)), we would have $p_i = V_i^i(t_2)$ for $i \in \{h, \ell\}$. Nevertheless, even with the self-selection constraints, (SP(t_1, t_2)) can still be solved by observation for Case 1 and Case 2 separately.

3.1.1 Solving (SP(t_1, t_2)) for Case 1

For Case 1, both $V_h^h(t) - V_\ell^h(t)$ and $V_h^\ell(t) - V_\ell^\ell(t)$ are decreasing in t as $m_h > m_\ell$. Hence, (6b) and (6c) can be replaced by

$$V_h^\ell(t_1) - V_\ell^\ell(t_1) \leq x \leq V_h^h(t_2) - V_\ell^h(t_2) \tag{7}$$

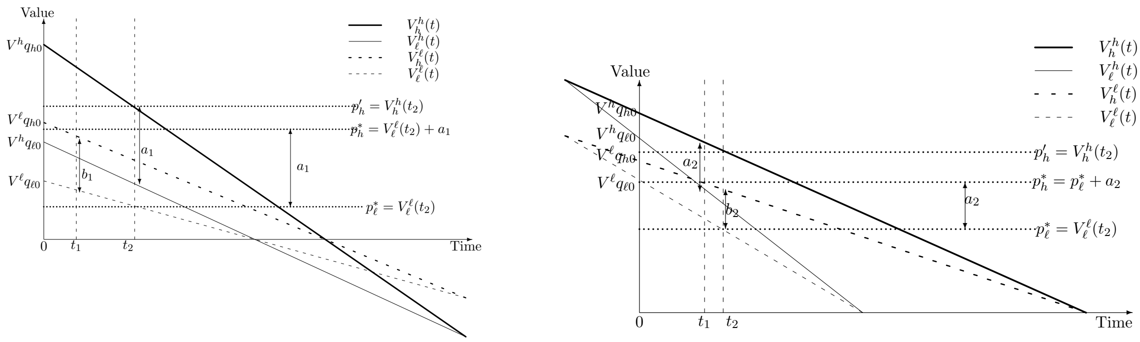
Note also that we should set p_ℓ as high as possible to maximize the objective function (6a). Assume we set $p_\ell = V_\ell^\ell(t_2)$, the highest value that p_ℓ can take. Then, we have, due to (6e),

$$x \leq V_h^h(t_2) - p_\ell = V_h^h(t_2) - V_\ell^\ell(t_2). \tag{8}$$

However, inequality (8) is already satisfied if (7) and (6c) are satisfied, because $V_h^h(t_2) - V_\ell^h(t_2) \leq V_h^h(t_2) - V_\ell^\ell(t_2)$ since $V_\ell^h(t_2) \geq V_\ell^\ell(t_2)$. This is the case for any choice of p_ℓ that satisfies (6d). That is, constraint (6e), which ties p_ℓ and x, is redundant, and hence it can be removed from the formulation. This implies that the problems of determining p_ℓ and x are separable. Furthermore, since this is a maximization problem, we set p_ℓ and x at their upper bounds. Therefore, $p_\ell^* = V_\ell^\ell(t_2)$, $x^* = V_h^h(t_2) - V_\ell^h(t_2)$, which implies that $p_h^* = V_\ell^\ell(t_2) + V_h^h(t_2) - V_\ell^h(t_2)$, as long as the problem is feasible. For problem (SP(t_1, t_2)) to be feasible, since the problem is separable for p_ℓ and x, it is sufficient to have that feasible regions for the two variables are non empty. Note that the set of feasible values of p_ℓ is never empty, i.e., there is always a p_ℓ that satisfies $0 \leq p_\ell \leq V_\ell^\ell(t_2)$, because, by our assumption, the non negativity constraints are implicitly satisfied in (P) (for any feasible T), and $V_\ell^\ell(t_2) \geq 0$ for $t_2 \leq T$. For the set of feasible values of x to be non empty, lower and upper bounds for x, as defined in (7) and (6c), should be appropriate. That is, for feasibility of problem (SP(t_1, t_2)) in Case 1, it is sufficient to have

$$V_h^\ell(t_1) - V_\ell^\ell(t_1) \leq V_h^h(t_2) - V_\ell^h(t_2). \tag{9}$$

See Figure 2a for a visual representation of these bounds and the optimal prices. Upper and lower bounds on x are denoted by the lengths $a_1 = V_h^h(t_2) - V_\ell^h(t_2)$ and $b_1 = V_h^\ell(t_1) - V_\ell^\ell(t_1)$ in the figure respectively. Condition 9 states that $b_1 \leq a_1$ for (SP(t_1, t_2)) to have a nonempty feasible region. Because both the lower and the upper bounds on x are decreasing in time, feasibility condition (9) implies that t_1 and t_2 should not be far apart from each other for feasibility. The figure also shows p_h^*, p_ℓ^*, in comparison to the $p_h' = V_h^h(t_2)$, which would be the price to set for item h if cannibalization were not an issue.



(a) when $m_h > m_\ell$ (i.e., Case 1)

(b) when $m_\ell > m_h$ (i.e., Case 2)

Figure 2: Value functions.

3.1.2 Solving (SP(t_1, t_2)) for Case 2

For Case 2, since $m_\ell > m_h$, both $V_h^h(t) - V_\ell^h(t)$ and $V_h^\ell(t) - V_\ell^\ell(t)$ are increasing in t. Hence, (6b) and (6c) can be replaced by

$$V_h^\ell(t_2) - V_\ell^\ell(t_2) \leq x \leq V_h^h(t_1) - V_\ell^h(t_1) \tag{10}$$

Following similar arguments as in Case 1, for feasibility, it is sufficient to have that

$$V_h^\ell(t_2) - V_\ell^\ell(t_2) \leq V_h^h(t_1) - V_\ell^h(t_1). \tag{11}$$

See Figure 2b for a visual representation of these bounds. Similar to Case 1, upper and lower bounds on x are denoted by the lengths $a_2 = V_h^h(t_1) - V_\ell^h(t_1)$ and $b_2 = V_h^\ell(t_2) - V_\ell^\ell(t_2)$ in the figure respectively, and condition (11) states that $b_2 \leq a_2$. If we set $p_\ell = V_\ell^\ell(t_2)$, we have the bound (8), as in Case 1 due to (6e). But, inequality (8) is already satisfied if (10) is satisfied because we can show that $V_h^h(t_1) - V_\ell^h(t_1) \leq V_h^h(t_2) - V_\ell^\ell(t_2)$ (see also Figure 2b).

That is, constraint (6e), which ties p_ℓ and x, is redundant for Case 2 as well. As in Case 1, we conclude that the problems of determining p_ℓ and x are separable, and we have, $p_\ell^* = V_\ell^\ell(t_2)$, $x^* = V_h^h(t_1) - V_\ell^h(t_1)$, which implies that, for this interval, we have $p_h^* = V_\ell^\ell(t_2) + V_h^h(t_1) - V_\ell^h(t_1)$, as long as condition (11) is satisfied. Optimal prices p_h^* and p_ℓ^*, as well as the price $p_h' = V_h^h(t_2)$, that could be set for item h if cannibalization did not exist, are also shown in Figure 2b.

3.2 Determining prices when cycle length and markdown time are given

Suppose that feasible values of t_m and T are both given. Note that, the prices set in either of the intervals ($[0,\, t_m]$ or $[t_m,\, T]$) can not affect the prices to be set in the other interval. That is, once t_m and T are given, prices in both intervals are independently determined. Also, note that, since T is given, inventory related costs are fixed. Then, if feasible values of t_m and T are both given, problem (P) reduces down to the problem of determining prices $p_{\ell,1}$ and $p_{h,1}$ in the interval $[0,\, t_m]$; and $p_{\ell,2}$ and $p_{h,2}$ in the interval $[t_m,\, T]$, such that revenues in both intervals are maximized and cannibalization is prevented. In other words, to solve problem (P) for a given value of t_m, and a given value of T, we can solve problem (SP$(0, t_m)$) to determine $p_{i,1}$ for $i \in \{\ell, h\}$; and we can solve (SP(t_m, T)) to determine $p_{i,2}$ for $i \in \{\ell, h\}$. Following the optimal prices we found for problem (SP(t_1, t_2)) in the previous section, this immediately implies that, for given values of t_m and T, we have the following prices under Case 1:

$$p_{\ell,1}^* = V_\ell^\ell(t_m) = V^\ell q_{\ell 0} - m_\ell^\ell t_m, \tag{12}$$

$$\begin{aligned} p_{h,1}^* &= V_\ell^\ell(t_m) + V_h^h(t_m) - V_\ell^h(t_m) = V^\ell q_{\ell 0} - m_\ell^\ell t_m + V^h q_{h0} - m_h^h t_m - \left(V^h q_{\ell 0} - m_\ell^h t_m\right), \\ &= \kappa + (m_\ell^h - m_\ell^\ell - m_h^h) t_m = \kappa + m' t_m \end{aligned} \tag{13}$$

$$p_{\ell,2}^* = V_\ell^\ell(T) = V^\ell q_{\ell 0} - m_\ell^\ell T, \tag{14}$$

$$\begin{aligned} p_{h,2}^* &= V_\ell^\ell(T) + V_h^h(T) - V_\ell^h(T) = V^\ell q_{\ell 0} - m_\ell^\ell T + V^h q_{h0} - m_h^h T - \left(V^h q_{\ell 0} - m_\ell^h T\right), \\ &= \kappa + \left(m_\ell^h - m_\ell^\ell - m_h^h\right) T = \kappa + m' T, \end{aligned} \tag{15}$$

where, $\kappa = V^\ell q_{\ell 0} + V^h q_{h0} - V^h q_{\ell 0} = V^\ell q_{\ell 0} + V^h (q_{h0} - q_{\ell 0}) > 0$, and $m' = m_\ell^h - m_\ell^\ell - m_h^h = V^h(m_\ell - m_h) - V^\ell m_\ell < 0$. These prices are valid only if problem (SP$(0, t_m)$) and (SP(t_m, T)) are both feasible. For feasibility of both problems, in addition to $T > 0$ and $0 \le t_m \le T$, t_m and T should be such that feasibility condition (9) defined for Case 1 should hold for (SP$(0, t_m)$) as stated in (16), and (SP(t_m, T)) as stated in (17):

$$V_h^\ell(0) - V_\ell^\ell(0) \le V_h^h(t_m) - V_\ell^h(t_m), \tag{16}$$

$$V_h^\ell(t_m) - V_\ell^\ell(t_m) \le V_h^h(T) - V_\ell^h(T). \tag{17}$$

After substituting for the value functions, conditions (16) and (17) reduce down to the following relation between t_m and T for Case 1:

$$\frac{V^h}{V^\ell} T - \frac{\left(V^h - V^\ell\right)}{V^\ell} \frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell} \le t_m \le \frac{\left(V^h - V^\ell\right)}{V^h} \frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell}. \tag{18}$$

Similarly, for Case 2, we have

$$
\begin{aligned}
p^*_{\ell,1} &= V^\ell_\ell(t_m) = V^\ell q_{\ell 0} - m^\ell_\ell t_m, && (19)\\
p^*_{h,1} &= V^\ell_\ell(t_m) + V^h_h(0) - V^h_\ell(0) = V^\ell q_{\ell 0} - m^\ell_\ell t_m + V^h q_{h0} - V^h q_{\ell 0}\\
&= \kappa - m^\ell_\ell t_m, && (20)\\
p^*_{\ell,2} &= V^\ell_\ell(T) = V^\ell q_{\ell 0} - m^\ell_\ell T, && (21)\\
p^*_{h,2} &= V^\ell_\ell(T) + V^h_h(t_m) - V^h_\ell(t_m) = V^\ell q_{\ell 0} - m^\ell_\ell T + V^h q_{h0} - m^h_h t_m - \left(V^h q_{\ell 0} - m^h_\ell t_m\right),\\
&= \kappa - m^\ell_\ell T + \left(m^h_\ell - m^h_h\right) t_m, && (22)
\end{aligned}
$$

where, κ is as defined above. In this case, in addition to $T > 0$ and $0 \le t_m \le T$, feasibility condition (11) defined for Case 2 should hold for both (SP$(0, t_m)$) and (SP(t_m, T)) as stated in (23) and (24) respectively:

$$V^\ell_h(t_m) - V^\ell_\ell(t_m) \le V^h_h(0) - V^h_\ell(0), \tag{23}$$

$$V^\ell_h(T) - V^\ell_\ell(T) \le V^h_h(t_m) - V^h_\ell(t_m). \tag{24}$$

After substituting for the value functions, conditions (23) and (24) reduce down to the following relation between t_m and T for Case 2:

$$\frac{V^\ell}{V^h}T - \frac{\left(V^h - V^\ell\right)}{V^h}\frac{(q_{h0} - q_{\ell 0})}{m_\ell - m_h} \le t_m \le \frac{\left(V^h - V^\ell\right)}{V^\ell}\frac{(q_{h0} - q_{\ell 0})}{m_\ell - m_h}. \tag{25}$$

3.3 Determining optimal markdown time and cycle length

In Section 3.2, we determined that for any given feasible values of t_m and T, optimal prices can be immediately determined through linear relations of t_m and T as in (12)–(15) for Case 1, and as in (19)–(22) for Case 2. In this section, we substitute those linear expressions in to problem (P) and attempt to determine optimal T and t_m values.

Since the prices obtained in Section 3.2 satisfy constraints (1c)–(1f) of problem (P) (for given T and t_m), these price values guarantee that cannibalization is prevented as long as t_m and T are feasible to (P). We also state in Section 3.2 that, for feasibility, t_m and T should satisfy condition (18) for Case 1 and (25) for Case 2. In addition, they should satisfy $T > 0$, $0 \le t_m \le T$, and cycle time bound constraints (i.e., constraints (4) for Case 1, and (5) for Case 2). For the time being, however, we omit the cycle time bound constraints. In Section 3.3.1, we present how these constraints can be incorporated. Then, we can reformulate (P) solely with decision variables t_m and T for Case 1 as follows.

$$
\begin{aligned}
\text{(P1)} \quad & \max_{t_m, T} \; F_1(T, t_m) && (26a)\\
\text{subject to} \quad & \frac{V^h}{V^\ell}T - \frac{\left(V^h - V^\ell\right)}{V^\ell}\frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell} \le t_m \le \frac{\left(V^h - V^\ell\right)}{V^h}\frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell} && (26b)\\
& 0 \le t_m \le T && (26c)\\
& T > 0.
\end{aligned}
$$

Likewise, a reformulation tailored for Case 2 is as follows.

$$
\begin{aligned}
\text{(P2)} \quad & \max_{t_m, T} \; F_2(T, t_m) && (27a)\\
\text{subject to} \quad & \frac{V^\ell}{V^h}T - \frac{\left(V^h - V^\ell\right)}{V^h}\frac{(q_{h0} - q_{\ell 0})}{m_\ell - m_h} \le t_m \le \frac{\left(V^h - V^\ell\right)}{V^\ell}\frac{(q_{h0} - q_{\ell 0})}{m_\ell - m_h} && (27b)
\end{aligned}
$$

$$0 \leq t_m \leq T \tag{27c}$$
$$T > 0.$$

In these formulations,

$$F_1(T, t_m) = \frac{\left(V^\ell q_{\ell 0} - m_\ell^\ell t_m\right) n_\ell t_m + \left(V^\ell q_{\ell 0} - m_\ell^\ell T\right) n_\ell (T - t_m)}{T} + \frac{\left(\kappa + m' t_m\right) n_h t_m + \left(\kappa + m' T\right) n_h (T - t_m)}{T} - \frac{K}{T} - \frac{T}{2}\left(n_h h_h + n_\ell h_\ell\right),$$

$$F_2(T, t_m) = \frac{\left(V^\ell q_{\ell 0} - m_\ell^\ell t_m\right) n_\ell t_m + \left(V^\ell q_{\ell 0} - m_\ell^\ell T\right) n_\ell (T - t_m)}{T} + \frac{\left(\kappa - m_\ell^\ell t_m\right) n_h t_m + \left(\kappa - m_\ell^\ell T + \left(m_\ell^h - m_h^h\right) t_m\right) n_h (T - t_m)}{T} - \frac{K}{T} - \frac{T}{2}\left(n_h h_h + n_\ell h_\ell\right)$$

are obtained by substituting equalities (12)–(15) for Case 1, and by substituting equalities (19)–(22) for Case 2 in the objective function of (P).

Problems (P1) and (P2) are optimization problems with convex feasible regions defined on the two dimensional (T, t_m) space. Figure 3a and Figure 3b illustrate the feasible regions. Let LB_1 and UB_1 denote the lines that bound t_m (from below and from above) in constraint (26b) in problem (P1). That is, LB_1 is the line $t_m = \frac{V^h}{V^\ell}T - \frac{\left(V^h - V^\ell\right)}{V^\ell}\frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell}$, and UB_1 is the line $t_m = \frac{\left(V^h - V^\ell\right)}{V^h}\frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell}$. Similarly, let LB_2 and UB_2 denote the lines that bound t_m in constraint (27b) in problem (P2). That is LB_2 is the line $t_m = \frac{V^\ell}{V^h}T - \frac{\left(V^h - V^\ell\right)}{V^h}\frac{(q_{h0} - q_{\ell 0})}{m_\ell - m_h}$, and UB_2 is the line $t_m = \frac{\left(V^h - V^\ell\right)}{V^\ell}\frac{(q_{h0} - q_{\ell 0})}{m_\ell - m_h}$. We will refer to these two lines by LB and UB when reference to the specific case is not necessary. In the figures, lines representing LB and UB, as well as the line $t_m = T$ are drawn in dotted lines. Feasible (T, t_m) pairs lie below the line UB, above the line $t_m = 0$, to the left of the line LB, and to the right of the line $t_m = T$. The figures also indicate the $t_m = \frac{T}{2}$ line and the following points: $T_{1A} = \frac{\frac{\left(V^h - V^\ell\right)}{V^\ell}\frac{q_{h0} - q_{\ell 0}}{m_h - m_\ell}}{\frac{V^h}{V^\ell} - \frac{1}{2}}$, $T_{2A} = 2\frac{\left(V^h - V^\ell\right)}{V^\ell}\frac{q_{h0} - q_{\ell 0}}{m_\ell - m_h}$, $T_{1B} = \frac{\left(V^h + V^\ell\right)}{V^h}\frac{\left(V^h - V^\ell\right)}{V^h}\frac{q_{h0} - q_{\ell 0}}{m_h - m_\ell}$, $T_{2B} = \frac{\left(V^h + V^\ell\right)}{V^\ell}\frac{\left(V^h - V^\ell\right)}{V^\ell}\frac{(q_{h0} - q_{\ell 0})}{m_\ell - m_h}$. As the figures illustrate, T_{1A} and T_{2A} are the T values at the intersection of the line $t_m = \frac{T}{2}$ and the lines LB_1 and UB_2 respectively. On the other hand, T_{1B} and T_{2B} are the T values at the intersection of the lines UB and LB in problems (P1) and (P2) respectively. From the figures, we see that no feasible solution exists to problems (P1) and (P2) for $T > T_{1B}$ and $T > T_{2B}$ respectively. The figures also show two other time points: $T_{1C} = 2\frac{\left(V^h - V^\ell\right)}{V^h}\frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell}$, and $T_{2C} = \frac{\frac{\left(V^h - V^\ell\right)}{V^h}\frac{(q_{h0} - q_{\ell 0})}{m_\ell - m_h}}{\frac{V^\ell}{V^h} - \frac{1}{2}}$. T_{1C} is at the intersection of $t_m = \frac{T}{2}$ and UB_1 in problem (P1), whereas T_{2C} is at the intersection of $t_m = \frac{T}{2}$ and LB_2 in problem (P2). Lemmas 1 and 2 describe some properties of these points and their relation to problem (P). We use these properties in our arguments in subsequent theorems. The proofs can be found in Appendices A and B, respectively.

Lemma 1. *We have $0 < T_{1A} < T_{1B} < T_{1C}$. If $\frac{V^\ell}{V^h} = \frac{1}{2}$, T_{2C} does not exist. Else, if $\frac{V^\ell}{V^h} < \frac{1}{2}$, then $T_{2C} < 0$. Otherwise, we have $0 < T_{2A} < T_{2B} < T_{2C}$.*

Lemma 2. *There exists no feasible (T, t_m) with $T > T_{1B}$ for problem (P1). Similarly, there exists no feasible (T, t_m) with $T > T_{2B}$ for problem (P2).*

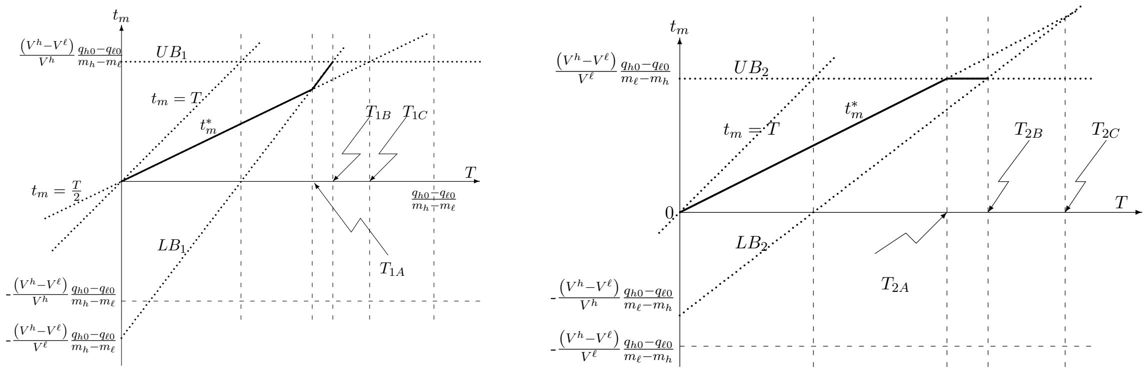


Figure 3: Graph of optimal value of t_m (t_m^*) as a function of T.

Besides having convex feasible regions, as we state in Lemma 3, objective functions of both problems are strictly concave (See Appendix C for the proof).

Lemma 3. *$F_1(T, t_m)$ and $F_2(T, t_m)$ are strictly concave for all (T, t_m) with $T > 0$, $t_m \geq 0$.*

Now let $(T_1', t_{m,1}')$ and $(T_2', t_{m,2}')$ be such that derivatives of F_1 and F_2 are equal to zero w.r.t. T and t_m. That is, $\frac{\partial F_1(T,t_m)}{\partial T}\Big|_{(T_1',t_{m,1}')} = \frac{\partial F_1(T,t_m)}{\partial t_m}\Big|_{(T_1',t_{m,1}')} = 0$; and $\frac{\partial F_2(T,t_m)}{\partial T}\Big|_{(T_2',t_{m,2}')} = \frac{\partial F_2(T,t_m)}{\partial t_m}\Big|_{(T_2',t_{m,2}')} = 0$. We can easily determine $(T_1', t_{m,1}')$ and $(T_2', t_{m,2}')$ as follows (if they exist). First, we take the derivative of F_1 w.r.t. t_m and equate it to zero, and we get

$$\frac{\partial F_1(T, t_m)}{\partial t_m} = V^\ell q_{\ell 0} n_\ell - V^\ell q_{\ell 0} n_\ell + \kappa n_h - \kappa n_h + 2m' n_h t_m - 2m_\ell^\ell n_\ell t_m + m_\ell^\ell n_\ell T - m' n_h T = 0,$$

which implies $t_m = \frac{T}{2}$. If we substitute $t_m = \frac{T}{2}$ in F_1, we get $F_{11}(T)$:

$$F_{11}(T) = \frac{A_1 T^2 + A_2 T}{T} - \frac{K}{T} - \frac{T}{2}\left(n_h h_h + n_\ell h_\ell\right).$$

Here, $A_1 = \frac{-3m_\ell^\ell n_\ell + 3m' n_h}{4}$, and $A_2 = V^\ell q_{\ell 0} n_\ell + \kappa n_h$. If we take the derivative of F_{11} w.r.t. T and equate it to 0, we get

$$\frac{\partial F_{11}(T)}{\partial T} = A_1 + \frac{K}{T^2} - \frac{(n_h h_h + n_\ell h_\ell)}{2} = 0 \to T = \sqrt{\frac{2K}{(n_h h_h + n_\ell h_\ell) - 2A_1}}.$$

That is $t_{m,1}' = \frac{T_1'}{2}$, and $T_1' = \sqrt{\frac{2K}{(n_h h_h + n_\ell h_\ell) - 2A_1}}$. We note that $A_1 < 0$, and hence $T_1' > 0$ always exists. Similarly, if we take the derivative of F_2 w.r.t. t_m and equate it to zero, we get

$$\frac{\partial F_2(T, t_m)}{\partial t_m} = 2(m_h^h n_h - m_\ell^\ell n_\ell - m_\ell^\ell n_h - m_\ell^h n_h)t_m - (m_h^h n_h - m_\ell^\ell n_\ell - m_\ell^\ell n_h - m_\ell^h n_h)T = 0$$

which, again, implies $t_m = \frac{T}{2}$. Then, if we substitute $t_m = \frac{T}{2}$ in F_2, we get $F_{21}(T)$:

$$F_{21}(T) = \frac{A_6 T^2 + A_7 T}{T} - \frac{K}{T} - \frac{T}{2}\left(n_h h_h + n_\ell h_\ell\right),$$

where $A_6 = \frac{(m_\ell^h - m_h^h - 3m_\ell^\ell)n_h}{4} - \frac{3m_\ell^\ell n_\ell}{4}$, and $A_7 = V^\ell q_{\ell 0} n_\ell + \kappa n_h$. If we take the derivative of F_{21} w.r.t. T and equate it to 0, we get

$$\frac{\partial F_{21}(T)}{\partial T} = A_6 + \frac{K}{T^2} - \frac{(n_h h_h + n_\ell h_\ell)}{2} = 0 \to T = \sqrt{\frac{2K}{(n_h h_h + n_\ell h_\ell) - 2A_6}}.$$

That is $t'_{m,2} = \frac{T'_2}{2}$, and $T'_2 = \sqrt{\frac{2K}{(n_h h_h + n_\ell h_\ell) - 2A_6}}$. Here, we note that T'_2 exists only if $A_6 < \frac{(n_h h_h + n_\ell h_\ell)}{2}$. However, if that is not the case, $\frac{\partial F_{21}(T)}{\partial T} > 0$ for all $T > 0$. We list without proof some elementary consequences of concavity of F in Lemma 4. Then, letting $(T_1^*, t_{m,1}^*)$ and $(T_2^*, t_{m,2}^*)$ be optimal solutions to (P1) and (P2) respectively, we present our main theorems.

Lemma 4. *F_1 and F_2 are strictly increasing (decreasing) in t_m for $t_m \le \frac{T}{2}$ ($t_m \ge \frac{T}{2}$). F_{11} is strictly increasing (decreasing) in T for $T < T'_1$ ($T > T'_1$). Likewise, F_{21} is strictly increasing (decreasing) in T for $T < T'_2$ ($T > T'_2$) if T'_2 exists. Otherwise, F_{21} is strictly increasing in T.*

Theorem 1. *If $0 < T'_1 \le T_{1A}$, then $(T'_1, t'_{m,1})$ is feasible to (P1), and we have $(T_1^*, t_{m,1}^*)$=$(T'_1, t'_{m,1})$. Similarly, if $0 < T'_2 \le T_{2A}$, then $(T'_2, t'_{m,2})$ is feasible to (P2), and we have $(T_2^*, t_{m,2}^*)$=$(T'_2, t'_{m,2})$.*

Proof. See Appendix D. □

Theorem 2. *If $T'_1 > T_{1A}$, then $(T'_1, t'_{m,1})$ is not feasible to (P1), and we have $T_1^* \in [T_{1A}, T_{1B}]$ and $t_{m,1}^* = \frac{V^h}{V^\ell}T_1^* - \frac{(V^h - V^\ell)}{V^\ell}\frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell}$. Similarly, if $T'_2 > T_{2A}$, or if T'_2 does not exist, then $T_2^* \in [T_{2A}, T_{2B}]$ and $t_{m,2}^* = \frac{(V^h - V^\ell)}{V^\ell}\frac{(q_{h0} - q_{\ell 0})}{m_\ell - m_h}$.*

Proof. See Appendix E. □

Lemma 4 restates the fact that for any choice of T, F_1 and F_2 achieve their maximum values at $t_m = \frac{T}{2}$. As a matter of fact, $(T'_1, t'_{m,1})$ and $(T'_2, t'_{m,2})$ (if it exists) are both on the $t_m = \frac{T}{2}$ line. As shown in Figures 3a and 3b, $t_m = \frac{T}{2}$ line intersects the feasible region of problem (P1) for $T \le T_{1A}$, and problem (P2) for $T \le T_{2A}$. Given this, Theorem 1 states a condition to easily check whether $(T'_1, t'_{m,1})$ and $(T'_2, t'_{m,2})$ are feasible to problems (P1) and (P2) respectively. So if $T'_1 \le T_{1A}$, then $(T'_1, t'_{m,1})$ is feasible and hence $T_1^* = T'_1$ and $t_{m,1}^* = \frac{T'_1}{2}$. Likewise, if $T'_2 \le T_{2A}$, then $(T'_2, t'_{m,2})$ is feasible and hence $T_2^* = T'_2$ and $t_{m,2}^* = \frac{T'_2}{2}$. As shown in Figures 3a and 3b, $t_m = \frac{T}{2}$ line leaves the feasible regions of (P1) and (P2) for $T > T_{1A}$ and $T > T_{2A}$ respectively. In such a case, i.e., if $T > T_{1A}$ or $T > T_{2A}$, Theorem 2 states that optimal (T, t_m) pair must be somewhere on the boundary of the feasible region defined by LB_1 or UB_2 for $T > T_{1A}$ and $T > T_{2A}$ for problems (P1) and (P2) respectively. In particular, it states that T_1^*, must be in the interval $[T_{1A}, T_{1B}]$ if $T'_1 > T_{1A}$, and T_2^*, must be in the interval $[T_{2A}, T_{2B}]$ if $T'_2 > T_{2A}$. In such a case, Theorem 2 states that, for any $T_1^* \in [T_{1A}, T_{1B}]$, we have $t_{m,1}^* = \frac{V^h}{V^\ell}T_1^* - \frac{(V^h - V^\ell)}{V^\ell}\frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell}$, and for any $T_2^* \in [T_{2A}, T_{2B}]$ $t_{m,2}^* = \frac{(V^h - V^\ell)}{V^\ell}\frac{(q_{h0} - q_{\ell 0})}{m_\ell - m_h}$. In Figures 3a and 3b, dark straight line indicates optimal t_m for any choice of T.

According to Theorem 2, if $T'_1 > T_{1A}$ ($T'_2 > T_{2A}$), we don't know the exact value, but only the range of possible values for T_1^* (T_2^*). To determine T_1^* in that case, we substitute $t_m = \frac{V^h}{V^\ell}T - \frac{(V^h - V^\ell)}{V^\ell}\frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell} = v'T - Q_1$ in the definition of function F_1, and obtain $F_{12}(T)$:

$$F_{12}(T) = \frac{A_3 T^2 + A_4 T + A_5}{T} - \frac{K}{T} - \frac{T}{2}(n_h h_h + n_\ell h_\ell),$$

where, $A_3 = \left(-m_\ell^\ell n_\ell + m' n_h\right)\left(v'^2 - v' + 1\right)$, $A_4 = V^\ell q_{\ell 0} n_\ell + 2m_\ell^\ell Q_1 n_\ell v' - m_\ell^\ell Q_1 n_\ell + \kappa n_h - 2m' v' n_h Q_1 + Q_1 n_h m'$, and $A_5 = m' Q_1^2 n_h - m_\ell^\ell Q_1^2 n_\ell$. Likewise, to determine T_2^*, we substitute $t_m = \frac{\left(V^h - V^\ell\right)}{V^\ell}\frac{(q_{h0}-q_{\ell 0})}{m_\ell - m_h} = Q_2$ in F_2, and obtain $F_{22}(T)$:

$$F_{22}(T) = \frac{A_8 T^2 + A_9 T + A_{10}}{T} - \frac{K}{T} - \frac{T}{2}\left(n_h h_h + n_\ell h_\ell\right)$$

where $A_8 = -m_\ell^\ell\left(n_h + n_\ell\right)$, $A_9 = V^\ell q_{\ell 0} n_\ell + m_\ell^\ell n_\ell Q_2 + \kappa n_h + \left(m_\ell^h - m_h^h\right) Q_2 n_h + Q_2 n_h m_\ell^\ell$, and $A_{10} = -m_\ell^\ell n_\ell Q_2^2 - \left(m_\ell^\ell + m_\ell^h - m_h^h\right) n_h Q_2^2$. Since F_1 and F_2 are concave, both F_{12} and F_{22} are concave as well. Let T_{12}' and T_{22}' be the T values where the derivatives (w.r.t. T) of F_{12} and F_{22} are equal to 0. That is,

$$T_{12}' = \sqrt{\frac{2\left(K - A_5\right)}{\left(n_h h_h + n_\ell h_\ell\right) - 2A_3}}, \quad T_{22}' = \sqrt{\frac{2\left(K - A_{10}\right)}{\left(n_h h_h + n_\ell h_\ell\right) - 2A_8}}.$$

Note that T_{12}' and T_{22}' always exist because $A_3 < 0$, $A_5 < 0$, $A_8 < 0$, and $A_{10} < 0$. Given this, we can extend Theorem 2, and determine the exact optimal value of T as follows.

Theorem 3. *If $T_1' > T_{1A}$, then*

(i) if $T_{12}' \leq T_{1A} \leq T_{1B}$, then $T_1^* = T_{1A}$

(ii) else if $T_{1A} \leq T_{12}' \leq T_{1B}$, then $T_1^* = T_{12}'$

(iii) else $T_1^* = T_{1B}$.

Similarly, if $T_2' > T_{2A}$, then

(i) if $T_{22}' \leq T_{2A} \leq T_{1B}$, then $T_2^* = T_{2A}$

(ii) else if $T_{2A} \leq T_{22}' \leq T_{2B}$, then $T_2^* = T_{22}'$

(iii) else $T_2^* = T_{2B}$.

Proof. The results follow because of the concavity of F_{12} and F_{22}. □

3.3.1 Incorporating cycle time bound constraints

In Sections 3.1 and 3.3, we assume feasible T and t_m values are given. As such we assume that they obey the cycle time bound constraints as well. In the beginning of Section 3.3, we ignore these constraints and determine optimal (T, t_m) values for problems (P1) and (P2). Addition of cycle time bound constraints can reduce feasible regions of (P1) and (P2, and potentially change the optimal values for T and t_m. Clearly, if $T_1^* \leq T_{max,1}$, or $T_2^* \leq T_{max,2}$, then $(T_1^*, t_{m,1}^*)$ and $(T_2^*, t_{m,2}^*)$ are feasible and optimal to problems (P) for Case 1 and Case 2, respectively. However, if $T_1^* > T_{max,1}$, or $T_2^* > T_{max,2}$, we adjust T_1^* and T_2^* as state in Theorem 4 and obtain optimal solution to (P) for Case 1 and Case 2, respectively.

Theorem 4. *For Case 1, if $T_1^* \leq T_{max,1}$, $(T_1^*, t_{m,1}^*)$ is optimal to (P). Else, we set $T_1^* = T_{max,1}$. If $T_{max,1} \leq T_{1A}$, then we set $t_{m,1}^* = \frac{T_{max,1}}{2}$, otherwise $t_{m,1}^* = \frac{V^h}{V^\ell} T_{max,1} - \frac{\left(V^h - V^\ell\right)}{V^\ell}\frac{(q_{h0}-q_{\ell 0})}{m_h - m_\ell}$. For Case 2, if $T_2^* \leq T_{max,2}$, $(T_1^*, t_{m,1}^*)$ is optimal to (P). Else, we set $T_2^* = T_{max,2}$. If $T_{max,2} \leq T_{2A}$, then we set $t_{m,2}^* = \frac{T_{max,2}}{2}$, otherwise $t_{m,2}^* = \frac{\left(V^h - V^\ell\right)}{V^\ell}\frac{(q_{h0}-q_{\ell 0})}{m_\ell - m_h}$*

Proof. See Appendix F. □

Using the properties discussed so far, we present Algorithm 1 and 2 to solve problem (P) for Case 1 and Case 2, respectively. Both are constant time algorithms.

Algorithm 1 Solution algorithm for problem (P) for Case 1.

Input: m_ℓ, m_h, V^ℓ, V^h, $q_{\ell 0}$, q_{h0},n_ℓ, n_h, K, h_h, h_ℓ.
Compute T_{1A}, T_{1B}, $T_{max,1}$, T_1', $t'_{m,1}$, T'_{12}, v', Q_1, A_1–A_5.
if $0 < T_1' \leq T_{1A}$ **then**
  $T_1^* = T_1'$, $t^*_{m,1} = t'_{m,1}$.
else
  if $T'_{12} \leq T_{1A}$ **then** $T_1^* = T_{1A}$
  else if $T_{1A} < T'_{12} \leq T_{1B}$ **then** $T_1^* = T'_{12}$
  else $T_1^* = T_{1B}$ **end if**
  $t^*_{m,1} = \frac{V^h}{V^\ell} T_1^* - \frac{\left(V^h - V^\ell\right)}{V^\ell} \frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell}$.
end if
if $T_1^* > T_{max,1}$ **then**
  $T_1^* = T_{max,1}$
  if $T_{max,1} \leq T_{1A}$ **then** $t^*_{m,1} = \frac{T_{max,1}}{2}$
  else $t^*_{m,1} = \frac{V^h}{V^\ell} T_{max,1} - \frac{\left(V^h - V^\ell\right)}{V^\ell} \frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell}$ **endif**
end if
Set prices using (12)–(15) with $T = T_1^*$ and $t_m = t^*_{m,1}$.

Algorithm 2 Solution algorithm for problem (P) for Case 2.

Input: m_ℓ, m_h, V^ℓ, V^h, $q_{\ell 0}$, q_{h0},n_ℓ, n_h, K, h_h, h_ℓ.
Compute T_{2A}, T_{2B}, $T_{max,2}$, T_2', $t'_{m,2}$, T'_{22}, v', Q_2, A_6–A_{10}.
if $0 < T_2' \leq T_{2A}$ **then**
  $T_2^* = T_2'$, $t^*_{m,2} = t'_{m,2}$.
else
  if $T'_{22} \leq T_{2A}$ **then** $T_2^* = T_{2A}$
  else if $T_{2A} < T'_{22} \leq T_{2B}$ **then** $T_2^* = T'_{22}$
  else $T_2^* = T_{2B}$ **end if**
  $t^*_{m,2} = \frac{\left(V^h - V^\ell\right)}{V^\ell} \frac{(q_{h0} - q_{\ell 0})}{m_\ell - m_h}$.
end if
if $T_2^* > T_{max,2}$ **then**
  $T_2^* = T_{max,2}$
  if $T_{max,2} \leq T_{2A}$ **then** $t^*_{m,2} = \frac{T_{max,2}}{2}$
  else $t^*_{m,2} = \frac{\left(V^h - V^\ell\right)}{V^\ell} \frac{(q_{h0} - q_{\ell 0})}{m_\ell - m_h}$ **end if**
end if
Set prices using (19)–(22) with $T = T_2^*$ and $t_m = t^*_{m,2}$.

4 Sensitivity Analysis and Managerial Implications

In this section, we analyze how the optimal profit for (P) changes as problem parameters change. We are interested in understanding the effect of cannibalization. In particular, we would like to know how restrictive self-selection constraints are. To do that, we compare the optimal profit of (P) to the optimal profit that would be obtained if cannibalization did not exist.

Suppose cannibalization were not an issue for the retailer. Then, we would not have the self-selection constraints in the formulation of (P), and the problem would become a profit maximizing EOQ. In that case, we would set $p_i = V_i^i(t_2)$ for $i \in \{h, \ell\}$ in problem (SP(t_1, t_2)) regardless of the case. Then for any given t_m and T, we would have the following prices:

$p_{\ell,1} = V^\ell q_{\ell 0} - m_\ell^\ell t_m$, $p_{h,1} = V^h q_{h0} - m_h^h t_m$, $p_{\ell,2} = V^\ell q_{\ell 0} - m_\ell^\ell T$, and $p_{h,2} = V^h q_{h0} - m_h^h T$. Let $G(T, t_m)$ be the objective function obtained by substituting these prices, and let (t'_m, T') be the point that maximizes $G(T, t_m)$. Then, following the arguments in the paper, we would have $t'_m = \frac{T'}{2}$ and $T' = \sqrt{\frac{2K}{(n_h h_h + n_\ell h_\ell) + 2A_{11}}}$, where $A_{11} = \frac{3}{4}\left(n_h V^h m_h + n_\ell V^\ell m_\ell\right)$.

In what follows, in response to the changes in problem parameters, we observe the behaviors of $F_1(T_1^*, t_{m,1}^*)$, i.e., optimal profit of problem (P), and $G(T', t'_m)$, i.e., optimal profit that would be obtained if there were no self-selection constraints. Obviously, the two optimal solutions may differ in their choices of t_m and T. But even if they have the same values for those variables, they would have different prices due to restrictions imposed by the self-selection constraints. Therefore, to detect the differences solely on the restricting effect of self-selection constraints, we are also going to observe the behavior of $G(T_1^*, t_{m,1}^*)$, i.e., optimal profit with no self-selection constraints with the optimal t_m and T values found for problem (P). Clearly, for any given set of problem parameters, $G(T', t'_m) \geq G(T_1^*, t_{m,1}^*) \geq F_1(T_1^*, t_{m,1}^*)$.

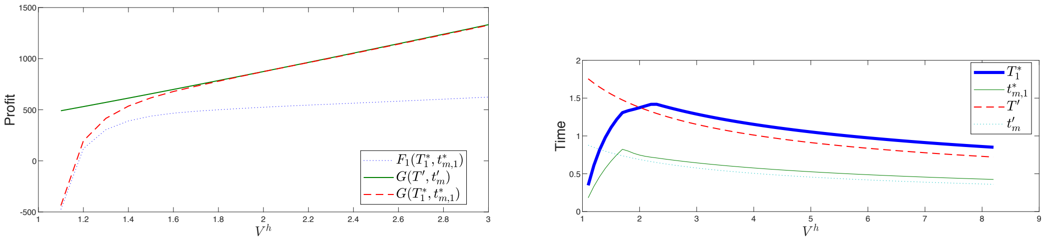
There are three factors that affect the solution to (P): (i) difference in quality perceptions of customer segments indicated by the difference in V^h and V^ℓ, (ii) difference in the qualities of items indicated by the difference in q_{h0} and $q_{\ell 0}$, and (iii) difference in deterioration rates indicated by the difference in m_h and m_ℓ. We analyze each factor separately.

4.1 Customers' quality perceptions

First, consider problem (P1). Figure 4a shows the change in optimal profits, as V^h is increased, while keeping everything else (in particular V^ℓ) constant. As the figure illustrates, and as we expect, increasing V^h increases net profits. Since V^h indicates the quality perception of the high-quality customer, an increase in V^h indicates that high-quality customers can pay more to the same quality. The figures also indicate that both for very high and very low values of V^h, restricting effects of self-selection constraints are amplified. This can be seen in the increasing gap between $F_1(T_1^*, t_{m,1}^*)$ and $G(T', t'_m)$. However, as the graph of $G(T_1^*, t_{m,1}^*)$ indicates, the difference is mostly due to choices in T and t_m for low values of V^h, whereas it is due to price restrictions for high values of V^h. This can be observed in Figure 4b as well. As the figure illustrates, as V^h increases, the choices in T and t_m converge to similar values. On the other hand, for low values of V^h, self-selection constraints severely restrain choices of T in (P1). This can be explained as follows. As the difference between two customer segments vanishes, the item with a lower deterioration rate might become attractive to both customer types very quickly. At that point, the retailer should adjust prices to prevent cannibalization. But since the retailer is allowed to adjust prices only once in a replenishment cycle, he/she is forced to have shorter replenishment intervals. In such cases, in order to have longer replenishment intervals, more than one price adjustment may be needed. We observe similar behavior in (P2) as seen in Figures 5a and 5b. However, in (P2), as V^h increases, optimal replenishment interval increases exponentially, because for high values of V^h, function F_{21} is an increasing function of T.

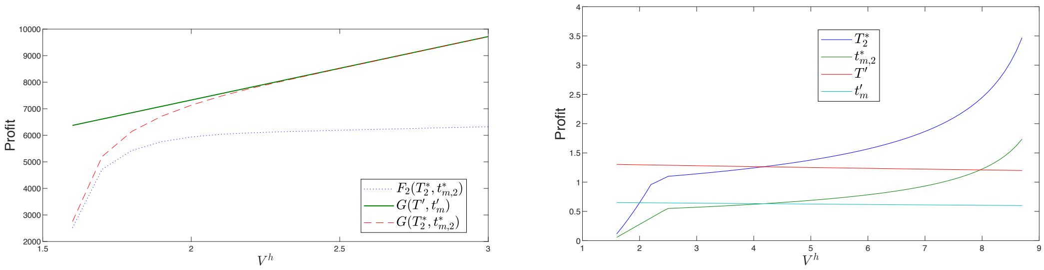
4.2 Quality difference in items

Figure 6a and Figure 6b illustrate, respectively, how optimal profits and replenishment intervals change in problem (P1) as q_{h0} is increased, while keeping everything else constant. As the figures illustrate, for low values of q_{h0}, the gap between $F_1(T_1^*, t_{m,1}^*)$ and $G(T', t'_m)$ is high. Looking at the graph of $G(T_1^*, t_{m,1}^*)$, we can say that this difference is mainly due to the choices of T and t_m. When the quality difference is low, self-selection constraints enforce short replenishment intervals, because for long replenishment intervals, single price adjustment is not sufficient in preventing cannibalization. Observations for (P2) are similar, as shown in Figures 7a and 7b.



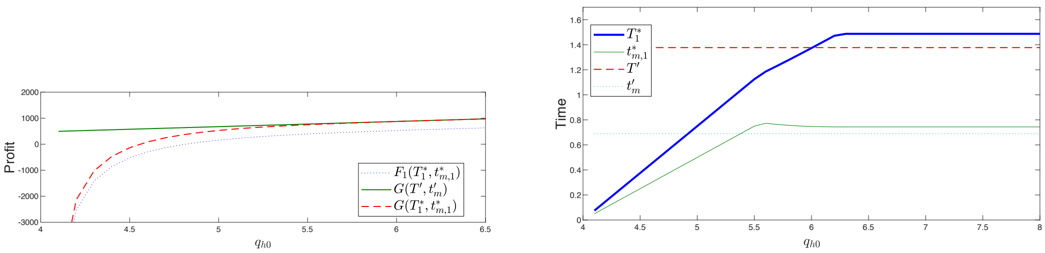
(a) Profit as a function of V^h (b) Cycle length as a function of V^h.

Figure 4: Effect of V^h in Case 1.



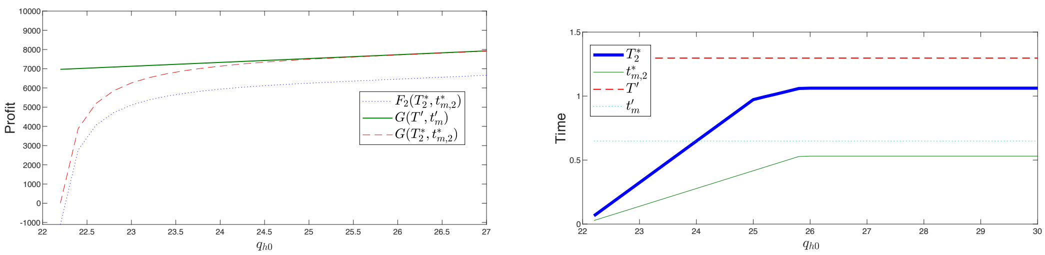
(a) Profit as a function of V^h (b) Cycle length as a function of V^h.

Figure 5: Effect of V^h in Case 2.



(a) Profit as a function of q_{h0} (b) Cycle length as a function of q_{h0}.

Figure 6: Effect of q_{h0} in Case 1.

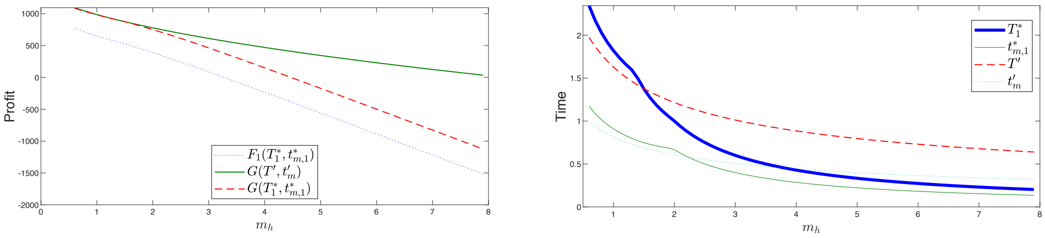


(a) Profit as a function of q_{h0} (b) Cycle length as a function of q_{h0}.

Figure 7: Effect of q_{h0} in Case 2.

4.3 Deterioration rates of items

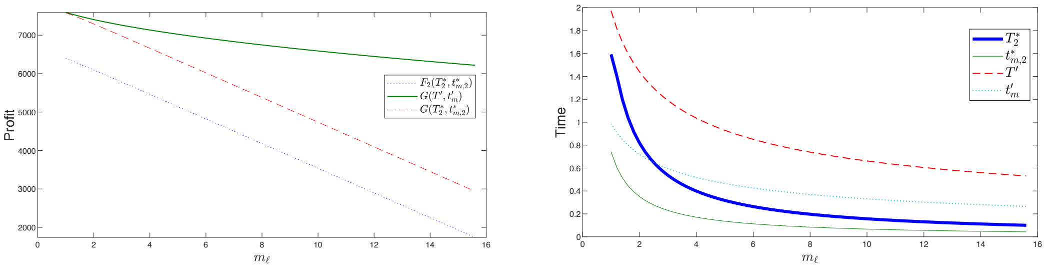
If items deteriorate faster, one would expect shorter replenishment intervals. Furthermore, following the discussions on the restricting effects of self-selection constraints, we expect to see sharp decrease in profit in (P1) and (P2) as deterioration rates increase. Figure 8a and Figure 9a verifies this for (P1) and (P2) respectively. As m_h is increased, while keeping everything else constant, the gap between $F_1(T_1^*, t_{m,1}^*)$ and $G(T', t_m')$ increases. As the graph of $G(T_1^*, t_{m,1}^*)$ suggests, this gap is mainly due to prices. The same is true as m_ℓ is increased in (P2). Figure 8b and Figure 9b illustrate the change in optimal replenishment intervals and price markdown times as m_h increases in (P1), and as m_ℓ increases in (P2), respectively. The figures are in agreement with our expectation that, as m_h or m_ℓ increases, optimal replenishment interval shortens. The effect of self-selection constraints are also visible in the figures. Replenishment intervals are more sensitive when the self-selection constraints are present.



(a) Profit as a function of m_h

(b) Cycle length as a function of m_h.

Figure 8: Effect of m_h in Case 1.



(a) Profit as a function of m_ℓ

(b) Cycle length as a function of m_ℓ.

Figure 9: Effect of increasing m_ℓ in Case 2.

5 Conclusion and Future Research

In this study, we analyze an EOQ model for two deteriorating items that share a common replenishment cycle. The items differ in quality and target two different customer types. If the two items have different deterioration rates, one item might become more attractive to both customer types and cannibalization might occur. Cannibalization is to be prevented via pricing decisions. In our model, prices of the items are adjusted only once over the replenishment cycles. We propose algorithms to solve two possible cases of the problem.

As we see in sensitivity analysis, prices and order frequencies are affected in order to avoid cannibalization. The effects are more severe, if (i) the two items are not much different in terms of quality, or (ii) the two customer types are similar in terms of the perception of quality.

In this paper, we assume that items lose their utility linearly in time. For future research, one could assume that items deteriorate exponentially over time. In such a case, for any given feasible T and t_m, we expect to obtain optimal prices as some (possibly) exponential functions of T and t_m, as in Section 3.3. Substituting these functions in problem (P), we might obtain similar concavity (or convexity) properties. It would also be interesting to work on a case where the retailer marks down prices more than once in a cycle. The number of price markdowns might be constrained, or some cost might be involved every time prices are adjusted.

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Appendices

A Proof of Lemma 1

All the time points mentioned in the lemma, except for T_{2C}, are positive. Then, proving $T_{1B} < T_{1C}$ is straightforward as $\frac{(V^h+V^\ell)}{V^h} < 2$. To prove that $T_{1A} < T_{1B}$, let $\frac{V^\ell}{V^h} = \alpha$ and note that we have $0 < \alpha < 1$. Then

$$\frac{T_{1B}}{T_{1A}} = \frac{(1+\alpha)(1-\alpha)}{\left(\frac{1}{\alpha}-1\right)/\left(\frac{1}{\alpha}-\frac{1}{2}\right)} = \frac{(1+\alpha)(2-\alpha)}{2} = \frac{2+\alpha-\alpha^2}{2} = 1 + \frac{\alpha-\alpha^2}{2} > 1.$$

Therefore we have $T_{1A} < T_{1B} < T_{1C}$. On the other hand, $T_{2A} < T_{2B}$, because $\frac{(V^h+V^\ell)}{V^h} < 2$. Also it is easy to see that if $\alpha < \frac{1}{2}$, $T_{2C} < 0$ and if $\alpha = \frac{1}{2}$, T_{2C} does not exist. Now assume $\alpha > \frac{1}{2}$, which makes $T_{2C} > 0$, and consider $\frac{T_{2B}}{T_{2C}}$:

$$\frac{T_{2B}}{T_{2C}} = \frac{\frac{(V^h+V^\ell)}{V^\ell}\frac{(V^h-V^\ell)}{V^\ell}\frac{(q_{h0}-q_{\ell 0})}{m_\ell-m_h}}{\frac{\frac{(V^h-V^\ell)}{V^h}\frac{(q_{h0}-q_{\ell 0})}{m_\ell-m_h}}{\frac{V^\ell}{V^h}-\frac{1}{2}}} = \frac{\frac{(1-\alpha)(1+\alpha)}{\alpha^2}}{\frac{1-\alpha}{\alpha-\frac{1}{2}}} = \frac{2\alpha^2+\alpha-1}{2\alpha^2} = 1 + \frac{1}{2\alpha}\left(1-\frac{1}{\alpha}\right) < 1.$$

Hence, the statement in the lemma follows. $\square$

B Proof of Lemma 2

Consider problem (P1). For all $T > 0$, we can always find $t_m \geq 0$ that satisfies $0 \leq t_m \leq T$, i.e., constraints (26c). On the other hand, LB_1 and UB_1 intersect at $T = T_{1B}$, because

$$LB_1 = UB_1 \to \frac{V^h}{V^\ell}T - \frac{(V^h-V^\ell)}{V^\ell}\frac{(q_{h0}-q_{\ell 0})}{m_h-m_\ell} = \frac{(V^h-V^\ell)}{V^h}\frac{(q_{h0}-q_{\ell 0})}{m_h-m_\ell}$$
$$\to T = \frac{(V^h+V^\ell)}{V^h}\frac{(V^h-V^\ell)}{V^h}\frac{q_{h0}-q_{\ell 0}}{m_h-m_\ell} = T_{1B}.$$

One can easily verify that for $T > T_{1B} > 0$ we have $LB_1 > UB_1$, that is $\frac{V^h}{V^\ell}T - \frac{(V^h-V^\ell)}{V^\ell}\frac{(q_{h0}-q_{\ell 0})}{m_h-m_\ell} > \frac{(V^h-V^\ell)}{V^h}\frac{(q_{h0}-q_{\ell 0})}{m_h-m_\ell}$ and there exists no $t_m \geq 0$ that satisfies constraint (27b). Similar arguments can be stated for problem (P2) and T_{2B}. $\square$

C Proof of Lemma 3

The Hessian matrix of the objective function $F_1(T, t_m)$ is as follows

$$\mathbf{H}(F_1(T,t_m)) = \begin{bmatrix} \frac{2\left(m'n_h - m_\ell^\ell n_\ell\right)t_m^2 - 2K}{T^3} & \frac{-2\left(m'n_h - m_\ell^\ell n_\ell\right)t_m}{T^2} \\ \frac{-2\left(m'n_h - m_\ell^\ell n_\ell\right)t_m}{T^2} & \frac{2\left(m'n_h - m_\ell^\ell n_\ell\right)}{T} \end{bmatrix}.$$

Since $\left(m'n_h - m_\ell^\ell n_\ell\right) < 0$, the first order principal minors $\left(\frac{2\left(m'n_h - m_\ell^\ell n_\ell\right)t_m^2 - 2K}{T^3}\right.$ and $\left.\frac{2\left(m'n_h - m_\ell^\ell n_\ell\right)}{T}\right)$ are negative for all $T > 0$, $t_m \geq 0$. Second order principal minor is

$$\frac{4\left(m'n_h - m_\ell^\ell n_\ell\right)^2 t_m^2 - 4K\left(m'n_h - m_\ell^\ell n_\ell\right)}{T^4} - \frac{4\left(m'n_h - m_\ell^\ell n_\ell\right)^2 t_m^2}{T^4} = \frac{-4K\left(m'n_h - m_\ell^\ell n_\ell\right)}{T^4} > 0.$$

Hence, objective function of (P1) is strictly concave for all (T, t_m) with $T > 0$, $t_m \geq 0$. Likewise,

$$\mathbf{H}(F_2(T,t_m)) = \begin{bmatrix} \frac{2Ct_m^2 - 2K}{T^3} & \frac{-2Ct_m}{T^2} \\ \frac{-2Ct_m}{T^2} & \frac{2C}{T} \end{bmatrix},$$

where $C = -m_\ell^\ell n_\ell - m_\ell^\ell n_h - \left(m_\ell^h - m_h^h\right) < 0$. As in Case 1, all the first order principal minors are negative and the second order principal minor is positive. Hence, the objective function of (P2) is also strictly concave for all (T, t_m) with $T > 0$, $t_m \geq 0$. $\square$

D Proof of Theorem 1

Due to concavity of F_1, if $(T_1', t_{m,1}')$ is feasible to (P1), we have $(T_1^*, t_{m,1}^*)=(T_1', t_{m,1}')$. With the same reasoning, the same statement is true for $(T_2', t_{m,1}')$ and (P2). We show that $(T_1', t_{m,1}')$ is feasible to (P1) if $0 < T_1' \leq T_{1A}$, and $(T_2', t_{m,2}')$ is feasible to (P2) if $0 < T_2' \leq T_{2A}$.

First, consider problem (P1). Note that the point $(T, \frac{T}{2})$ satisfies the constraints $0 \leq t_m \leq T$ (i.e., constraint (26c) of (P1)) for all $T \geq 0$. Now consider the line $t_m = \frac{T}{2}$ and the line LB_1. We can see that at $T = T_{1A}$, the two lines intersect, because

$$\frac{T}{2} = \frac{V^h}{V^\ell}T - \frac{\left(V^h - V^\ell\right)}{V^\ell}\frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell} \rightarrow T = \frac{\frac{\left(V^h - V^\ell\right)}{V^\ell}\frac{q_{h0} - q_{\ell 0}}{m_h - m_\ell}}{\frac{V^h}{V^\ell} - \frac{1}{2}} = T_{1A}.$$

Then it is easy to verify that we have $\frac{T}{2} \geq \frac{V^h}{V^\ell}T - \frac{\left(V^h - V^\ell\right)}{V^\ell}\frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell}$ for all $T \leq T_{1A}$, and $\frac{T}{2} < \frac{V^h}{V^\ell}T - \frac{\left(V^h - V^\ell\right)}{V^\ell}\frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell}$ for $T > T_{1A}$. That is, the point $(T, \frac{T}{2})$ does not violate the lower bound of the constraints (26b) for $T \leq T_{1A}$.

Now consider the line $t_m = \frac{T}{2}$ and the line UB_1. We can see that at $T = T_{1C}$, the two lines intersect, because

$$\frac{T}{2} = \frac{\left(V^h - V^\ell\right)}{V^h}\frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell} \rightarrow T = 2\frac{\left(V^h - V^\ell\right)}{V^h}\frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell} = T_{1C}.$$

It is easy to verify that $\frac{T}{2} \leq \frac{\left(V^h - V^\ell\right)}{V^h}\frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell}$ for $T \leq T_{1C}$, and $\frac{T}{2} > \frac{\left(V^h - V^\ell\right)}{V^h}\frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell}$ for $T > T_{1C}$. That is, the point $(T, \frac{T}{2})$ does not violate the upper bound of the constraints (26b) for $T \leq T_{1C}$. Since $T_{1A} < T_{1B} < T_{1C}$ due to Lemma 1, neither the lower nor the upper bound of the constraint (26b) is violated, for $T \leq T_{1A}$. Hence, we say that the point $(T, \frac{T}{2})$ satisfies the constraints (26b) of problem (P1) for $T \leq T_{1A}$. Together with the constraint that $T > 0$, we can state that $(T_1', t_{m,1}')$=$(T_1', \frac{T_1'}{2})$ is feasible to (P1) if $0 < T_1' \leq T_{1A}$.

Similarly, for problem (P2), for all $T \geq 0$, the point $(T, \frac{T}{2})$ satisfies the constraints $0 \leq t_m \leq T$ (i.e., constraint (27c) of (P2)). Consider the line $t_m = \frac{T}{2}$ and the line UB_2. We can see that at $T = T_{2A}$, the two lines intersect, because

$$\frac{T}{2} = \frac{(V^h - V^\ell)}{V^\ell}\frac{(q_{h0} - q_{\ell 0})}{m_\ell - m_h} \rightarrow T = 2\frac{(V^h - V^\ell)}{V^\ell}\frac{q_{h0} - q_{\ell 0}}{m_\ell - m_h} = T_{2A}.$$

Then it is easy to verify that we have $\frac{T}{2} \leq \frac{(V^h-V^\ell)}{V^\ell}\frac{q_{h0}-q_{\ell 0}}{m_\ell - m_h}$ for all $T \leq T_{2A}$, and $\frac{T}{2} > \frac{(V^h-V^\ell)}{V^\ell}\frac{q_{h0}-q_{\ell 0}}{m_\ell - m_h}$ for all $T > T_{2A}$. That is, the point $(T, \frac{T}{2})$ does not violate the upper bound of the constraints (26b) for $T \leq T_{2A}$.

Now consider the line $t_m = \frac{T}{2}$ and the line LB_2. At $T = T_{2C}$, the two lines intersect, because

$$\frac{T}{2} = \frac{V^\ell}{V^h}T - \frac{(V^h - V^\ell)}{V^h}\frac{(q_{h0} - q_{\ell 0})}{m_\ell - m_h} \rightarrow T = \frac{\frac{(V^h-V^\ell)}{V^h}\frac{(q_{h0}-q_{\ell 0})}{m_\ell - m_h}}{\frac{V^\ell}{V^h} - \frac{1}{2}} = T_{2C}.$$

Note that if $\frac{V^\ell}{V^h} = \frac{1}{2}$, T_{2C} does not exist and we have $\frac{T}{2} \geq \frac{V^\ell}{V^h}T - \frac{(V^h-V^\ell)}{V^h}\frac{(q_{h0}-q_{\ell 0})}{m_\ell - m_h}$ for all T. Else if $\frac{V^\ell}{V^h} < \frac{1}{2}$, we have $T_{2C} < 0$ and $\frac{T}{2} \geq \frac{V^\ell}{V^h}T - \frac{(V^h-V^\ell)}{V^h}\frac{(q_{h0}-q_{\ell 0})}{m_\ell - m_h}$ for all $T \geq 0 \geq T_{2C}$. Otherwise (if $\frac{V^\ell}{V^h} > \frac{1}{2}$), we have $\frac{T}{2} \geq \frac{V^\ell}{V^h}T - \frac{(V^h-V^\ell)}{V^h}\frac{(q_{h0}-q_{\ell 0})}{m_\ell - m_h}$ for $T \leq T_{2C}$. Considering all the cases for $\frac{V^\ell}{V^h}$, since we should have $T \leq T_{2B}$ for feasibility, and since $T_{2B} < T_{2C}$, we can say that $\frac{T}{2} \geq \frac{(V^h-V^\ell)}{V^h}\frac{(q_{h0}-q_{\ell 0})}{m_h - m_\ell}$ for $0 \leq T \leq T_{2B}$. That is, for $0 \leq T \leq T_{2B}$, the point $(T, \frac{T}{2})$ does not violate the lower bound of the constraints (27b) of problem (P2). Therefore, together with the constraint that $T > 0$, and since $T_{2A} < T_{2B}$ due to Lemma 1, we can state that the point $(T, \frac{T}{2})$ satisfies all the constraints of problem (P2) for $0 < T \leq T_{2A}$. And we conclude that the point $(T_2', t_{m,2}')$=$(T_2', \frac{T_2'}{2})$ is feasible to (P2) if $0 < T_2' \leq T_{2A}$. □

E Proof of Theorem 2

First consider problem (P1). We showed in the proof of Theorem 1 that, for $T > T_{1A}$, the point $(T, \frac{T}{2})$ violates constraint (26b) in (P1). Therefore, if $T_1' > T_{1A}$, then $(T_1', t_{m,1}')$=$(T_1', \frac{T_1'}{2})$ is not feasible to (P1). If $(T_1', t_{m,1}')$ is not feasible to (P1), then we have $T_1^* \geq T_{1A}$. This follows because, F_{11} is increasing in T for $T \leq T_{1A} < T_1'$ and hence we have $F_{11}(T_{1A}) = F_1(T_{1A}, \frac{T_{1A}}{2}) > F_{11}(T) = F_1(T, \frac{T}{2}) \geq F_1(T, t_m)$ for all feasible (T, t_m) with $0 < T < T_{1A}$. Therefore $T_1^* \geq T_{1A}$. Since there exists no feasible (T, t_m) for (P1) with $T > T_{1B}$, we can claim $T_1^* \in [T_{1A}, T_{1B}]$. Consider again constraint (26b). For any given (T, t_m) pair with $T \in [T_{1A}, T_{1B}]$, we have

$$\frac{T}{2} \leq \frac{V^h}{V^\ell}T - \frac{(V^h - V^\ell)}{V^\ell}\frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell} \leq t_m \leq \frac{(V^h - V^\ell)}{V^h}\frac{(q_{h0} - q_{\ell 0})}{m_h - m_\ell},$$

where the leftmost equality holds only at $T = T_{1A}$. The above inequalities imply that for all feasible (T, t_m) pairs with $T \in [T_{1A}, T_{1B}]$, we have $t_m \geq \frac{T}{2}$ (with equality only at $T = T_{1A}$). We also know that for any given T, function F_1 is increasing in t_m for $t_m < \frac{T}{2}$, and decreasing in t_m for $t_m > \frac{T}{2}$. Hence, among all (T, t_m) pairs with $T \in [T_{1A}, T_{1B}]$ that satisfy constraint (26b), objective function F_1 is maximum at $t_m = \frac{V^h}{V^\ell}T - \frac{(V^h - V^\ell)}{V^\ell}\frac{(q_{h0}-q_{\ell 0})}{m_h - m_\ell}$. It is easy to verify that for any given $T \in [T_{1A}, T_{1B}]$, the point $(T, \frac{V^h}{V^\ell}T - \frac{(V^h - V^\ell)}{V^\ell}\frac{(q_{h0}-q_{\ell 0})}{m_h - m_\ell})$ satisfies all the rest of the constraints of problem (P1), and is feasible to (P1). Therefore, if $T_1^* \in [T_{1A}, T_{1B}]$, we should have $t_{m,1}^* = \frac{V^h}{V^\ell}T_1^* - \frac{(V^h - V^\ell)}{V^\ell}\frac{(q_{h0}-q_{\ell 0})}{m_h - m_\ell}$.

Now consider problem (P2). We showed in the proof of Theorem 1 that for $T > T_{2A}$, the point $(T, \frac{T}{2})$ violates constraint (27b) in (P2). Therefore, if $T_2' > T_{2A}$, then $(T_2', t_{m,2}')$=$(T_2', \frac{T_2'}{2})$ is not feasible to (P2). If $(T_2', t_{m,2}')$ is not feasible to (P2), then we have $T_2^* \geq T_{2A}$. This follows because, F_{21} is increasing in T for $T \leq T_{2A} < T_2'$ if T_2' exists, and F_{21} is increasing in T for $T > 0$ if T_2' does not exist. Hence, regardless of the existence of T_2', we have $F_{21}(T_{2A}) = F_2(T_{2A}, \frac{T_{2A}}{2}) > F_{21}(T) = F_2(T, \frac{T}{2}) \geq F_2(T, t_m)$ for all feasible (T, t_m) with $0 < T < T_{2A}$. Therefore $T_2^* \geq T_{2A}$. Since there exists no feasible (T, t_m) for (P2) with $T > T_{2B}$, we can claim $T_2^* \in [T_{2A}, T_{2B}]$. Consider again constraint (27b). For any given (T, t_m) pair with $T \in [T_{2A}, T_{2B}]$, we have

$$\frac{T}{2} \geq \frac{(V^h - V^\ell)}{V^\ell}\frac{(q_{h0} - q_{\ell 0})}{m_\ell - m_h} \geq t_m \geq \frac{V^\ell}{V^h}T - \frac{(V^h - V^\ell)}{V^h}\frac{(q_{h0} - q_{\ell 0})}{m_\ell - m_h},$$

where the leftmost equality holds only at $T = T_{2A}$. The above inequalities imply that for all feasible (T, t_m) pairs with $T \in [T_{2A}, T_{2B}]$, we have $t_m \leq \frac{T}{2}$ (with equality only at $T = T_{2A}$). We also know that for any given T, F_2 is increasing in t_m for $t_m < \frac{T}{2}$ and decreasing in t_m for $t_m > \frac{T}{2}$. Hence, among all (T, t_m) pairs with $T \in [T_{2A}, T_{2B}]$ that satisfy constraint (27b), F_2 is maximum at $t_m = \frac{(V^h - V^\ell)}{V^\ell}\frac{(q_{h0}-q_{\ell 0})}{m_\ell - m_h}$. It is easy to verify that for any given $T \in [T_{2A}, T_{2B}]$, the point $(T, \frac{(V^h - V^\ell)}{V^\ell}\frac{(q_{h0}-q_{\ell 0})}{m_\ell - m_h})$ satisfies all the rest of the constraints as well, and is feasible to (P2). This implies that if $T_2^* \in [T_{2A}, T_{2B}]$, we have $t_{m,2}^* = \frac{(V^h - V^\ell)}{V^\ell}\frac{(q_{h0}-q_{\ell 0})}{m_\ell - m_h}$. $\square$

F Proof of Theorem 4

Consider Case 1 (Case 2 is similar). If $T_1^* \leq T_{max,1}$, then $(T_1^*, t_{m,1}^*)$ is feasible and optimal to problem (P). Therefore, assume $T_1^* > T_{max,1}$, and let $(\mathcal{T}_1^*, \tau_{m,1}^*)$ be the optimal (T, t_m) for problem (P). Assume on the contrary that $\mathcal{T}_1^* < T_{max,1}$, and consider the direction $\mathbf{d}$=$(\mathcal{T}_1^*, \tau_{m,1}^*)$-$(T_1^*, t_{m,1}^*)$. Since F_1 is strictly concave, F_1 is strictly decreasing along direction $\mathbf{d}$(otherwise $(T_1^*, t_{m,1}^*)$ would not be an optimal solution to (P1)). This implies that $F_1\left((T_1^*, t_{m,1}^*) + \lambda\mathbf{d}\right) > F_1\left((T_1^*, t_{m,1}^*) + \mathbf{d}\right) = F_1\left(\mathcal{T}_1^*, \tau_{m,1}^*\right)$ for $0 \leq \lambda < 1$. The point with with $T = T_{max,1}$ is at $(T_1^*, t_{m,1}^*)$+$\lambda\mathbf{d}$ for some $0 \leq \lambda < 1$, and such a point has a higher objective value than the point $(\mathcal{T}_1^*, \tau_{m,1}^*)$. This is a contradiction and hence we should have $\mathcal{T}_1^* = T_{max,1}$. Furthermore, we have already established in the proofs of Theorems 1 and 2 that, for any given $T \leq T_{1A}$, optimal value of $F1$ occurs at $t_m = \frac{T}{2}$, and for any given $T_{1A} < T \leq T_{1B}$, optimal value of $F1$ occurs at $t_m = \frac{V^h}{V^\ell}T - \frac{(V^h - V^\ell)}{V^\ell}\frac{(q_{h0}-q_{\ell 0})}{m_h - m_\ell}$. Hence the result follows. $\square$