xLightCone: An Algorithm for Perturbing an FLRW Spacetime on the Past Lightcone

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Perturbation on the past light cone

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INTRODUCTION

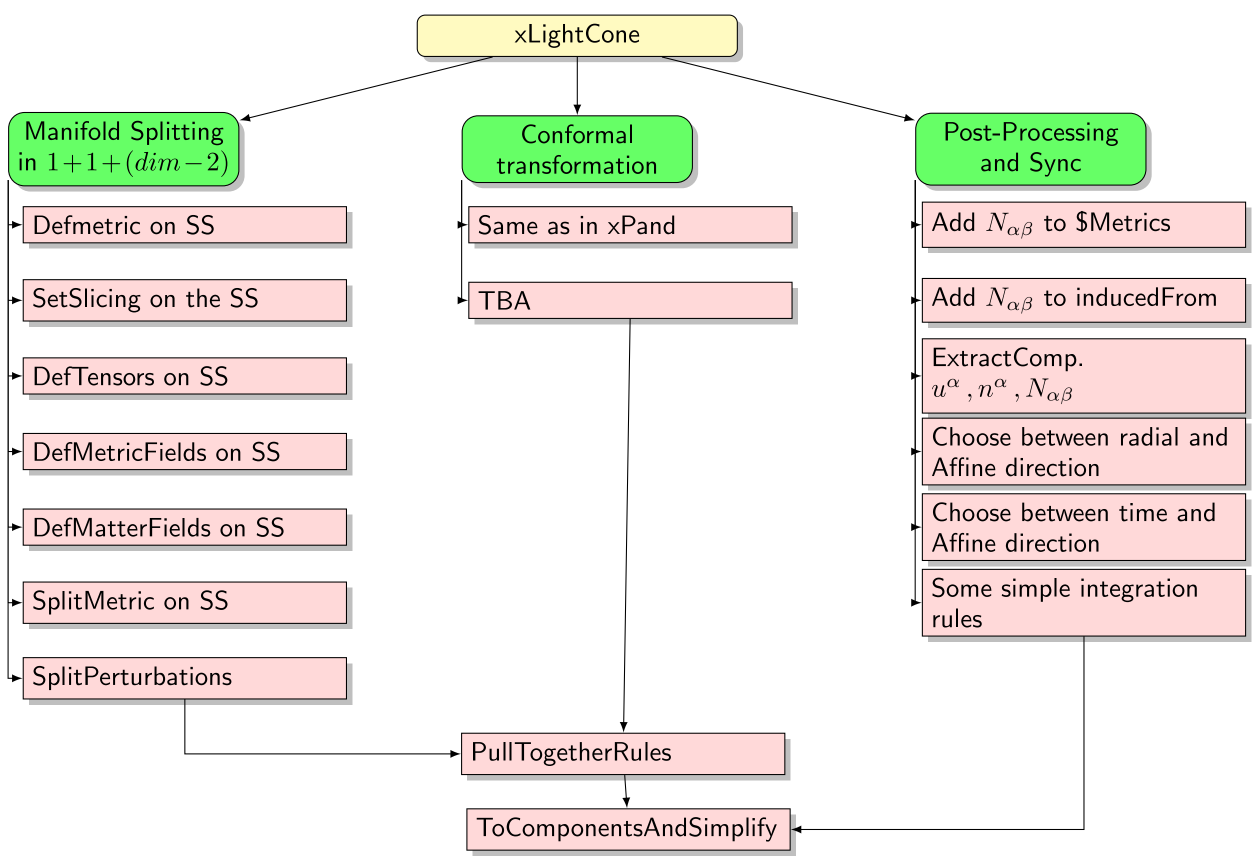
Advances in astronomical instrumentation is currently driving a precision induced revolution in Cosmology. On the observational side, things are getting increasingly more precise to the point that that there is an urgent demand for the theory to match the same level of precision.

This drive toward precision cosmology requires that the theoretical pundits calculates observables on geometry that are oriented in the same way as the observational set-up. Since we are restricted here on Earth to observe only the past of our null-cone/light-cone, this implies that on the theoretical side, that every observable should calculated within the geometry of the past light cone.

At the moment, I have been able to implement $1+1+2$ splitting by first decomposing in $3+1$ and splitting the spatial part further in $2+1$. Basically, I borrow all the rules of $3+1$ decomposition.

I. PACKAGE ARCHITECTURE

This is a preliminary architecture of the package "xLghtcone". The package is expected to build on xPand by adding additional functionalities that would enable further splitting of spacetime into radial and angular part in a very clean geometric way.



where SS stands for Screen Space and "dim" is the dimension of the remaining spatial section of the space. In four dimensions $dim - 2 = 2$ and it is the angular section of the spacetime.

A. Description:This is only for the Friedmann Cosmology

We now expatiate on the functionalities of various sections of the package.

- Manifold Decomposition
  - `DefScreenSpaceMetric`
    - AssignProperties:The following properties should be assign to the metric on the screen space.
  - `SetScreenSpaceSlicing`
  - `DefScreenSpaceProjectedTensors`
    - Label Indices: Every tensor will have atleast three label indices for:time, radial, and order of perturbation.
    - AssignProperties: Such as trace-free in both indices for rank two. How label indices should move when acted up.
  - `DefScreenSpaceProjectedMetricFields`
  - `DefScreenSpaceProjectedMatterfFields`
  - `ToInducedDerivativeScreenSpace:` This function performs irreducible decomposition of covariant derivative(s) of any projected tensor field.
  - `InducedDecompositionLightCone:` This function performs a similar action as xTensor's 'InducedDecomposition', the my major difference is that it has been extended to include decomposition on the screen space.
  - `Splitmetric`
  - `RemoveinducedDerivative`
  - `ToMetric`
  - `SplitPerturbations`
- Conformal Transformation
  - Borrow transformation rule from xPand
- Post-Processing
  - `Extract Components`
- Together
  - `ToLightConeFromRules`

II. BASIC TOOLS FOR $3+1$ DECOMPOSITION

A. Background Spacetime decomposition

In xPand, we decomposed the background manifold using the four vector $\bar{\boldsymbol{u}}$ which is time-like vector and it is normalized as $(\bar{u}^{\mu}\bar{u}_{\mu} = -1)$. The metric on $\overline{\mathcal{M}}$ is decomposed as

$$\bar{g}_{\mu\nu} = \bar{h}_{\mu\nu} - \bar{u}_{\mu}\bar{u}_{\nu}\,, \qquad \text{with} \qquad \bar{h}_{\mu\nu}\bar{u}^{\mu} = 0 \qquad \text{and} \qquad \bar{h}^{\mu}{}_{\rho}\bar{h}^{\rho}{}_{\nu} = \bar{h}^{\mu}{}_{\nu}\,, \tag{1}$$

where $\bar{\boldsymbol{h}}$ represents the induced metric of the spatial hypersuface

The covariant derivative of u^{μ}, (i.e $\nabla_{\mu}u_{\nu}$) may be decomposed into temporal part which describes the acceleration of u^{α} and the spatial part, which describes the Extrinsic curvature of the hypersurface.

$$\nabla_{\mu}u_{\nu} = -\bar{a}_{\nu}\bar{u}_{\mu} + \bar{K}_{h\mu\nu} \tag{2}$$

where the acceleration of u^α is defined as $\bar{a}_\mu = \bar{u}^\rho\,\bar{\nabla}_\rho \bar{u}_\mu = \bar{D}_\mu\bar{\alpha}/\bar{\alpha}$, where α is the lapse function. The extrinsic curvature tensor on the hypersurface is given by

$$\bar{K_h}_{\mu\nu} = \bar{h}^\rho{}_\mu \bar{h}^\sigma{}_\nu \bar{\nabla}_\rho \bar{u}_\sigma = \frac{1}{3}\bar{\Theta} h_{\mu\nu} + \bar{\sigma}_{\mu\nu} + \bar{\omega}_{\mu\nu}\,, \tag{3}$$

where In line with the decomposition of the the time slicing four vector, decomposition of an other tensor fields living the manifold follows a similar technique, for example, vector is decomposed as

$$V_\mu = u_\mu(u^\alpha V_\alpha) + h^\alpha V_\alpha\,, \tag{4}$$

and for a rank-two tensor $\boldsymbol{T}$, we have:

$$T_{\mu\nu} = \bar{u}_\mu \bar{u}_\nu\,(\bar{u}^\rho \bar{u}^\sigma T_{\rho\sigma}) + 2\,\bar{u}_{(\mu}\big(\bar{u}^\rho\,\bar{h}^\sigma{}_{\nu)} T_{\rho\sigma}\big) + \big(\bar{h}^\rho{}_\mu \bar{h}^\sigma{}_\nu T_{\rho\sigma}\big)\,. \tag{5}$$

Decomposition of high rank tensor is simply axiomatic. The covariant Derivatives of these objects may also be decomposed following the same line of reasoning. For general spatial tensors (namely, for spatial tensors defined within $\overline{\mathcal{M}}$ or defined within $\mathcal{M}$ then mapped onto $\overline{\mathcal{M}}$), the relation between the two derivatives reads:

$$\bar{\nabla}_\rho T_{\mu_1\ldots\mu_p} = -\bar{u}_\rho \dot{T}_{\mu_1\ldots\mu_p} + \bar{D}_\rho T_{\mu_1\ldots\mu_p} + \sum_{i=1}^{p} \bar{u}_{\mu_i}\,\bar{K_h}{}^\sigma{}_\rho\, T_{\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p}\,. \tag{6}$$

The relation between $\mathcal{L}_{\bar{\boldsymbol{u}}}$ and $\bar{u}^\rho\bar{\nabla}_\rho$ is written[1]:

$$\mathcal{L}_{\bar{\boldsymbol{u}}} T_{\mu_1\ldots\mu_p} = \dot{T}_{\mu_1\ldots\mu_p} + \sum_{i=1}^{p} \bar{K_h}{}^\sigma{}_{\mu_i}\, T_{\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p}\,, \tag{7}$$

$$\bar{\nabla}_\rho T_{\mu_1\ldots\mu_p} = -\bar{u}_\rho\,\mathcal{L}_{\bar{\boldsymbol{u}}} T_{\mu_1\ldots\mu_p} + \bar{D}_\rho T_{\mu_1\ldots\mu_p} + 2\sum_{i=1}^{p} \bar{u}_{(\mu_i}\,\bar{K_h}{}^\sigma{}_{\rho)}\, T_{\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p}\,. \tag{8}$$

III. BASIC TOOLS FOR $2+1$ DECOMPOSITION

A. Decomposition of background Spacetime

In section II, we have shown how a 4-d background spacetime and tensor fields that live on them may be decomposed using $3+1$ technique, now we are going to split further the spatial part (i.e the hyper-surface) of the 4-d spacetime using 1+2 decomposition technique.

First we need to define a space-like 3-vector, $\bar{n}^\mu$, which is normalized as $\bar{n}^\mu\bar{n}_\mu = 1$ and an induced metric $N^{\mu,\nu}$ which lives on the 2-d sheet or the screen space.

$$\bar{h}^{\alpha\beta}\bar{n}_\alpha\bar{n}_\beta = 1\,, \qquad \bar{N}_{\mu\nu} = \bar{h}_{\mu\nu} - \bar{n}_\mu\bar{n}_\nu = g_{\mu\nu} - \bar{u}_\mu\bar{u}_\nu + \bar{n}_\mu\bar{n}_\nu\,. \tag{9}$$

Other properties of the projection tensor include

$$\bar{u}^\alpha\bar{N}_{\alpha\beta} = \bar{n}^\alpha\bar{N}_{\alpha\beta} = 0,\;\; \bar{k}^\alpha\bar{N}_{\alpha\beta} = 0,\;\; \bar{h}^{\alpha\beta}\bar{N}_{\alpha\beta} = g^{\alpha\beta}\bar{N}_{\alpha\beta} = 2,\;\; \bar{N}_{\alpha\beta}\bar{N}^{\alpha\beta} = 2,\;\; \bar{N}_{\alpha\gamma}\bar{N}^{\alpha\beta} = \bar{N}_{\beta\gamma}. \tag{10}$$

where $k^\alpha = \bar{E}\,(\bar{u}^\alpha \pm \bar{n}^\alpha)$ is the photon tangent vector. I have introduced it here because how to calculate it is central to what we plan to do.

Spatial covariant derivative of $\bar{n}^\alpha$ may be irreducibly decomposed as

$$\bar{\mathrm{D}}_\mu\bar{n}_\nu = \bar{n}_\mu\bar{\beta}_\nu + \bar{K_N}_{\mu\nu} \tag{11}$$

[1] Note that for a spatial tensor $\boldsymbol{T}$, the quantity $\mathcal{L}_{\bar{\boldsymbol{u}}} T_{\mu_1\ldots\mu_p}$ is also spatial.

where we have introduced an Extrinsic curvature (i.e $\bar{K}_{\perp\mu\nu}$) of on the screen space and it is decomposed as follows $\bar{K_{N\mu\nu}} = \frac{1}{2}\bar{K_{N\perp}}N_{\mu\nu} + \xi\bar{\varepsilon}_{\mu\nu} + \zeta_{\mu\nu}$. We have introduced the following notations

$$\bar{A_{n\alpha}} \equiv \bar{n}^{\gamma}N^{\beta}_{\alpha}\bar{D}_{\gamma}\bar{n}_{\beta} = N^{\beta}_{\alpha}\nabla_{\parallel}n_{a}, \qquad \text{Radial Acceleration} \tag{12}$$

$$\bar{K_{N\perp}} \equiv N^{\alpha\beta}\bar{D}_{\alpha}\bar{n}_{\beta}, \qquad \text{Trace of } \bar{K}^{\mu}_{\perp\mu} \tag{13}$$

$$\bar{\xi} \equiv \frac{1}{2}\bar{\varepsilon}^{\alpha\beta}N_{\alpha}{}^{\mu}N^{\nu}_{\beta}\bar{D}_{\perp\mu}\bar{n}_{\nu}, \qquad \text{Twist, the anti-symmetry part} \tag{14}$$

$$\bar{\zeta}_{\alpha\beta} \equiv N_{\alpha}{}^{\mu}N^{\nu}_{\beta}\bar{D}_{\perp\langle\mu}\bar{n}_{\nu\rangle}. \qquad \text{Shear, the symmetry part} \tag{15}$$

We also define the alternating Levi-Civita 2-tensor

$$\bar{\varepsilon}_{\alpha\beta} \equiv \bar{\varepsilon}_{\alpha\beta\gamma}\bar{n}^{\gamma} = \bar{u}^{\lambda}\bar{\eta}_{\lambda\alpha\beta\gamma}\bar{n}^{\gamma}, \tag{16}$$

so that $\bar{\varepsilon}_{\alpha\beta}\bar{n}^{\beta} = 0 = \bar{\varepsilon}_{(\alpha\beta}$, and

$$\bar{\varepsilon}_{\alpha\beta\gamma} = \bar{n}_{\alpha}\bar{\varepsilon}_{\beta\gamma} - \bar{n}_{\beta}\varepsilon_{\alpha\gamma} + \bar{n}_{\gamma}\varepsilon_{\alpha\beta}\,, \quad \bar{\varepsilon}_{\alpha\beta}\bar{\varepsilon}^{\gamma\lambda} = \bar{N}_{\alpha}{}^{\gamma}\bar{N}_{\beta}{}^{\lambda} - \bar{N}_{\alpha}{}^{\lambda}\bar{N}_{\beta}{}^{\gamma}\,, \quad \bar{\varepsilon}_{\alpha}{}^{\gamma}\bar{\varepsilon}_{\beta\gamma} = \bar{N}_{\alpha\beta}, \quad \bar{\varepsilon}^{\alpha\beta}\bar{\varepsilon}_{\alpha\beta} = 2. \tag{17}$$

The spatial part of the tensor field that have been split using $3+1$ technique in section II may now be further split into the radial part and the angular part. The decomposition uses the same technique as in the case of $3+1$. For example for a 3-vector V^{a} can now be irreducibly split into a scalar, $V_{\parallel}$, which is the part of the vector parallel to $\bar{n}^{\alpha}$, and a vector, $V^{\alpha}_{\perp}$, lying in the sheet orthogonal to $\bar{n}^{\alpha}$;

$$V^{\alpha} = V_{\parallel}\bar{n}^{\alpha} + V^{\alpha}_{\perp}, \quad \text{where} \quad V_{\parallel} \equiv V_{\alpha}\bar{n}^{\alpha}, \quad \text{and} \quad V^{\alpha}_{\perp} \equiv \bar{N}^{\alpha\beta}V_{\beta}, \tag{18}$$

A rank two tensor field may also be decomposed into radial and screen space components according to

$$T_{\alpha\beta} = T_{\langle\alpha\beta\rangle} = T_{\parallel}\left(\bar{n}_{\alpha}\bar{n}_{\beta} - \frac{1}{2}N_{\alpha\beta}\right) + 2T_{\perp\parallel(\alpha}\bar{n}_{\beta)} + T_{\perp\langle\alpha\beta\rangle}, \tag{19}$$

where

$$T_{\parallel} \equiv \bar{n}^{\alpha}\bar{n}^{\beta}\psi_{\alpha\beta} = -\bar{N}^{\alpha\beta}T_{\alpha\beta}\,, \quad T_{\perp\alpha} \equiv \bar{N}_{\alpha}{}^{\alpha}\bar{n}^{\gamma}T_{\beta\gamma}\,, \quad T_{\perp\alpha\beta} \equiv T_{\langle\alpha\beta\rangle} \equiv \left(N_{(\alpha}{}^{\gamma}\bar{N}_{\beta)}{}^{\lambda} - \frac{1}{2}\bar{N}_{\alpha\beta}\bar{N}^{\gamma\lambda}\right)T_{\gamma\lambda}. \tag{20}$$

B. Decomposition of Covariant Derivatives

In general full decomposition of a spatial covariant derivative of any tensor field follows similar approach adopted in $1+3$ case. We find that a general tensor field may be decomposed as $T_{\mu_1\ldots\mu_p}$

$$\bar{\mathrm{D}}_{\rho}T_{\mu_1\ldots\mu_p} = \bar{n}_{\rho}\nabla_{\parallel}T_{\perp\mu_1\ldots\mu_p} + \sum_{i=1}^{p}\bar{n}_{\mu_i}\,\bar{K}^{\sigma}_{\perp\ \ \rho}\,T_{\perp\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p} + \bar{\nabla}_{\perp\rho}T_{\perp\mu_1\ldots\mu_p}\,. \tag{21}$$

If we use Lie derivative for the radial derivative, we will have to do the same thing we did for time

$$\mathcal{L}_{\bar{\boldsymbol{n}}}T_{\perp\mu_1\ldots\mu_p} = \nabla_{\parallel}T_{\mu_1\ldots\mu_p} + \sum_{i=1}^{p}\bar{K}^{\sigma}_{\perp\ \ \mu_i}\,T_{\perp\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p}\,, \tag{22}$$

Putting equation (22) into equation (21) gives

$$\bar{\bar{\mathrm{D}}}_{\rho}T_{\perp\mu_1\ldots\mu_p} = \bar{n}_{\rho}\,\mathcal{L}_{\bar{\boldsymbol{n}}}T_{\mu_1\ldots\mu_p} \;+\; \bar{\nabla}_{\perp\rho}T_{\mu_1\ldots\mu_p} + 2\sum_{i=1}^{p}\bar{n}_{(\mu_i}\,\bar{K}^{\sigma}_{\perp\ \ \rho)}\,T_{\perp\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p}\,. \tag{23}$$

IV. UNIFYING $1+3$ DECOMPOSITION WITH $1+2$ TO FORM $1+1+2$

Finally we can combine decomposition of $4-$ dimensional covariant derivative and the derivative on the hypersurface to arrive at

$$\bar{\nabla}_\rho T_{\mu_1\ldots\mu_p} = -\bar{u}_\rho\,\mathcal{L}_{\bar{\boldsymbol{u}}}T_{\mu_1\ldots\mu_p} + 2\sum_{i=1}^{p}\bar{u}_{(\mu_i}\,\bar{K}^{\sigma}{}_{\rho)}\,T_{\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p} + \bar{n}_\rho\,\mathcal{L}_{\bar{\boldsymbol{n}}}T_{\mu_1\ldots\mu_p} \tag{24}$$
$$+\bar{\nabla}_{\perp\rho}T_{\mu_1\ldots\mu_p} + 2\sum_{i=1}^{p}\bar{n}_{(\mu_i}\,\bar{K}^{\sigma}_{\perp}{}_{\rho)}\,T_{\perp\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p}\,.$$

For homogeneous cosmologies of interest, $\bar{K}^{\sigma}{}_{\rho)}$ is zero for FL cosmology but non-vanishing for some classes of Bianchi cosmologies.

A. Commutation Relations for Screen Space Projected tensors

In order to obtain the commutation relation between involving Lie derivative and radial derivative,or Lie derivative and screen space projected angular derivative or radial derivative and screen space projected angular derivative, we will need the full spacetime Ricci identity for a generalized tensor

$$\nabla_\gamma\nabla_\delta T^{\alpha_1\cdots a_r}{}_{\beta_1\cdots\beta_s} - \nabla_\delta\nabla_\gamma T^{\alpha_1\cdots\alpha_r}{}_{\beta_1\cdots\beta_s} = -R^{\alpha_1}{}_{\rho\gamma\delta}T^{\rho\alpha_2\cdots\alpha_r}{}_{\beta_1\cdots\beta_s} - \cdots - R^{\alpha_r}{}_{\rho\gamma\delta}T^{\alpha_1\cdots\alpha_{r-1}}{}_{\beta_1\cdots\beta_s} \tag{25}$$
$$+R^{\sigma}{}_{\beta_1\gamma\delta}T^{\alpha_1\cdots\alpha_r}{}_{\beta_1\cdots\beta_{s-1}\sigma}$$

It is important to note that for a vector, we recover the well-known result

$$\nabla_\rho\nabla_\sigma A_\nu - \nabla_\sigma\nabla_\rho A_\nu = R^{\beta}{}_{\nu\rho\sigma}A_\beta\,. \tag{26}$$

The full spacetime Riemann tensor can also be decomposed completely at least with respect to the time slicing four vector using Gauss Codazzi relations :

$$\bar{R}_{\mu\nu\rho\sigma} = {}^3\bar{R}_{\mu\nu\rho\sigma} + 2\bar{K}_{\mu[\rho}\bar{K}_{\sigma]\nu} - 4\left(\bar{D}_{[\mu}\bar{K}_{\nu][\rho}\right)\bar{u}_{\sigma]} - 4\left(\bar{D}_{[\rho}\bar{K}_{\sigma][\mu}\right)\bar{u}_{\nu]} + 4\,\bar{u}_{[\mu}\,\bar{K}_{\nu]}{}^{\zeta}\,\bar{K}_{\zeta[\rho}\,\bar{n}_{\sigma]} + 4\,\bar{u}_{[\mu}\,\dot{\bar{K}}_{\nu][\rho}\bar{u}_{\sigma]}\,, \tag{27}$$

where ${}^3\bar{R}_{\mu\nu\rho\sigma}$ stands for the hypersurfaces projected Riemann curvature tensor, the over-dot the derivative along the hypersurface slicing four vector, i.e following our general notation for any tensor field $\boldsymbol{T}$, we have: $\dot{T}_{\mu_1\ldots\mu_p} = \bar{n}^\rho\bar{\nabla}_\rho T_{\mu_1\ldots\mu_p}$. It is well known that projecting on all the indices of the Riemann tensor with the metric on the hyper-surface leads to

$$\bar{h}^{\varphi}{}_{\mu}\bar{h}^{\upsilon}{}_{\nu}\bar{h}^{\xi}{}_{\rho}\bar{h}^{\zeta}{}_{\sigma}\,\bar{R}_{\varphi\upsilon\xi\zeta} = {}^3\bar{R}_{\mu\nu\rho\sigma} + 2\bar{K}_{\mu[\rho}\bar{K}_{\sigma]\nu}\,. \tag{28}$$

and when the fourth index is contracted with u^α and the remaining three indices still fully projected leads to

$$\bar{h}^{\varphi}{}_{\mu}\bar{h}^{\upsilon}{}_{\nu}\bar{h}^{\xi}{}_{\rho}\bar{u}^{\zeta}\,\bar{R}_{\varphi\upsilon\xi\zeta} = \bar{D}_\mu\bar{K}_{\nu\rho} - \bar{D}_\nu\bar{K}_{\mu\rho}\,, \tag{29}$$

In an FLRW universe, ${}^3\bar{R}_{\mu\nu\rho\sigma}$ has a simple form

$${}^3\bar{R}_{\mu\nu\rho\sigma} = 2k\,\bar{h}_{\rho[\mu}\,\bar{h}_{\nu]\sigma}\,,\qquad {}^3\bar{R}_{\mu\nu} = 2k\,\bar{h}_{\mu\nu}\,,\qquad {}^3\bar{R} = 6k\,, \tag{30}$$

where k is the mean spatial curvature of the hypersurface, it is zero in a flat FLRW universe, $+1$ is a closed universe and -1 is open universe.

Using the decomposition of the Riemann curvature given in equation (27), the commutation relation between Lie derivative $\mathcal{L}_{\bar{\boldsymbol{u}}}$ and projected spatial derivative on the on the hyper surface $\bar{\boldsymbol{D}}$ for general spatial tensors (i.e a tensor whose all indices are fully projected), it is given by

$$\mathcal{L}_{\bar{\boldsymbol{u}}}\left(\bar{D}_\rho T_{\mu_1\ldots\mu_p}\right) = \bar{D}_\rho\left(\mathcal{L}_{\bar{\boldsymbol{u}}}T_{\mu_1\ldots\mu_p}\right) + \sum_{i=1}^{p}\left(\bar{h}^{\sigma\zeta}\bar{D}_\zeta\bar{K}_{\rho\mu_i} - \bar{D}_\rho\bar{K}_{\mu_i}{}^{\sigma} - \bar{D}_{\mu_i}\bar{K}_{\rho}{}^{\sigma}\right)T_{\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p}\,, \tag{31}$$

This result is easily obtained using simple product rule with the Ricci identity in equation (25). For the commutation relation between Lie derivative for the conformal time and radial derivative, the projected angular derivative and radial derivative and the projected angular derivative and conformal time we find

$$\mathcal{L}_{\bar{\boldsymbol{u}}}\left(\bar{n}^{\beta}\bar{D}_{\beta}T_{\nu_1\ldots\nu_p}\right) = n^{\beta}\bar{D}_{\beta}\left(\mathcal{L}_{\bar{\boldsymbol{u}}}T_{\nu_1\ldots\nu_p}\right) + \bar{u}^{\alpha}\nabla_{\alpha}\bar{n}^{\beta}\bar{D}_{\beta}T_{\nu_1\ldots\nu_p} - \sum_{i=1}^{p}\left[\bar{n}^{\alpha}\bar{D}_{\alpha}K_{h\nu_1}^{\ \sigma}T_{\nu_2\ldots\sigma\cdots\nu_p}\right.$$
$$\left.+\bar{u}^{\alpha}\bar{n}^{\sigma}R_{\sigma\alpha\nu_1}{}^{\gamma}T_{\gamma\cdots\nu_p}\right], \tag{32}$$

$$n^{\delta}\bar{D}_{\delta}\left(\nabla_{\perp\gamma}T_{\nu_1\ldots\nu_p}\right) = \nabla_{\perp\gamma}\left(n^{\delta}\bar{D}_{\delta}T_{\nu_1\ldots\nu_p}\right) - N^{\sigma}{}_{\gamma}\nabla_{\sigma}n^{\delta}\bar{D}_{\delta}T_{\nu_1\ldots\nu_p} - \sum_{i=1}^{p}\left[N^{\alpha}{}_{\gamma}h^{\sigma}{}_{\delta}\bar{n}^{\delta}R_{\sigma\alpha\nu}{}^{\mu}T_{\mu\nu_2\ldots\nu_{p-1}}\right], \tag{33}$$

$$\mathcal{L}_{\bar{\boldsymbol{u}}}\left(\nabla_{\perp\gamma}T_{\nu_1\ldots\nu_p}\right) = \nabla_{\perp\gamma}\mathcal{L}_{\bar{\boldsymbol{u}}}\left(T_{\nu_1\ldots\nu_p}\right) - \sum_{i=1}^{p}\left[N^{\delta}{}_{\gamma}K_{h}{}^{\beta}{}_{\delta}\nabla_{\beta}T_{\mu_1\cdots\mu_p} + N^{\delta}_{\gamma}\nabla_{\delta}K_{h}{}^{\sigma}{}_{\mu_1}T_{\mu_2\cdots\sigma\cdots\mu_p}\right.$$
$$\left.-N^{\alpha}{}_{\gamma}\bar{u}^{\beta}R_{\beta\alpha\mu_1}{}^{\sigma}T_{\mu_2\cdots\sigma\cdots\mu_p}\right].. \tag{34}$$

If we use Lie derivative in lieu of directional derivative along n^a, the first two equations for the commutations relation become

$$\mathcal{L}_{\bar{\boldsymbol{u}}}\left(\mathcal{L}_{\bar{\boldsymbol{n}}}T_{\nu_1\ldots\nu_p}\right) = \mathcal{L}_{\bar{\boldsymbol{n}}}\left(\mathcal{L}_{\bar{\boldsymbol{u}}}T_{\nu_1\ldots\nu_p}\right) + \bar{u}^{\alpha}\nabla_{\alpha}\bar{n}^{\beta}\bar{D}_{\beta}T_{\nu_1\ldots\nu_p} - \sum_{i=1}^{p}\left[\bar{n}^{\alpha}\bar{D}_{\alpha}K_{h\nu_1}^{\ \sigma}T_{\nu_2\ldots\sigma\cdots\nu_p}\right.$$
$$\left.+\mathcal{L}_{\bar{\boldsymbol{u}}}\left(K_{N}{}^{\sigma}{}_{\nu_1}\right)T_{\nu_2\ldots\sigma\cdots\nu_p} + \bar{u}^{\alpha}\bar{n}^{\sigma}R_{\sigma\alpha\nu_1}{}^{\gamma}T_{\gamma\cdots\nu_p}\right], \tag{35}$$

$$\mathcal{L}_{\bar{\boldsymbol{n}}}\left(\nabla_{\perp\gamma}T_{\nu_1\ldots\nu_p}\right) = \nabla_{\perp\gamma}\left(\mathcal{L}_{\bar{\boldsymbol{n}}}T_{\nu_1\ldots\nu_p}\right) - \sum_{i=1}^{p}\left[\nabla_{\perp\gamma}K_{N}{}^{\sigma}{}_{\mu_1}T_{\nu_2\cdots\sigma\cdots\nu_p}\right.$$
$$\left.+N^{\alpha}{}_{\gamma}h^{\sigma}{}_{\delta}\bar{n}^{\delta}R_{\sigma\alpha\nu_1}{}^{\mu}T_{\mu\nu_2\ldots\nu_{p-1}}\right], \tag{36}$$

where the full decomposition of the covariant derivative of $\nabla_{\mu}n_{\nu}$ is given by

$$\nabla_{\mu}\bar{n}_{\nu} = -\bar{n}_{\mu}\mathcal{L}_{\bar{\boldsymbol{u}}}\bar{n}_{\nu} - \bar{n}^{\alpha}\bar{u}_{\mu}\nabla_{\nu}\bar{u}_{\alpha} + K_{h\mu\beta}n^{\beta} + \bar{n}_{\mu}\bar{\beta}_{\nu} + \bar{K}_{N\mu\nu}\,. \tag{37}$$

The signs of the two derivatives

I was thinking we use "dot" for conformal time and "hat" for radial derivative. For example, given a tensor $T_{\nu_1\ldots\nu_p}$, time derivative will be $T'_{\nu_1\ldots\nu_p}$ and radial derivative $\hat{T}_{\nu_1\ldots\nu_p}$. When we have radial derivative first and conformal time second we write $\dot{\hat{T}}_{\nu_1\ldots\nu_p}$, if we have conformal time first and radial second we have $\hat{\dot{T}}_{\nu_1\ldots\nu_p}$

V. IMPLEMENTATION OF FLRW COSMOLOGY

The line element for the background FLRW cosmology is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}R^2}{1-KR^2} + R^2\mathrm{d}^2\Omega\right] = a^2(\eta)\left[-\mathrm{d}\eta^2 + \frac{\mathrm{d}R^2}{1-KR^2} + R^2\mathrm{d}^2\Omega\right] \tag{38}$$

where the angular part may further be split $\mathrm{d}\Omega = \mathrm{d}\theta + \sin\theta\mathrm{d}\phi$. On the past light cone, for some technical reasons, handling FLRW metric is much simplified in a curvature-normalized coordinate, where the spatial section is given by

$$\mathrm{d}\Sigma^2 = \mathrm{d}r^2 + f_k(r)^2\mathrm{d}^2\Omega\,, \tag{39}$$

where

$$f_k(r) = \begin{cases} r & \text{in the Euchidean case } k=0\text{, flat,}\\ \frac{1}{\sqrt{k}}\sin(\sqrt{k}r) & \text{in the Spherical case } k>0\text{, closed,}\\ \frac{1}{\sqrt{|k|}}\sinh(\sqrt{|k|}r) & \text{in the hyperbolic case } k<0\text{, open.}\end{cases} \tag{40}$$

The function $f(r)$ has a simple Taylor series expansion $f(r) = r - kr^3/6 + k^2r^5/(120) + \cdots$. Now consider a one parameter family of light-like geodesics, propagating towards an observer from the past of the observer current position, A tangent vector to the photon geodesic is given by k^{α}. Without loss of generality we consider photons propagation with a constant phase, S and k^{α} is defined as a covariant derivative of the phase, $k_{\alpha} = \nabla_{\alpha}S$. The tangent vector is light-like or null $k^\alpha k_\alpha = 0$. Considering only curves that converge at the observer with a minimum energy, implies that k^α must satisfy the geodesic equation and the curves will be invariant under affine transformation

$$k^\alpha \nabla_\alpha k^\beta = 0\,, \quad \text{and} \quad k^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\,, \tag{41}$$

where λ is the affine parameter. Given a time-slicing four vector u^α and direction of observation n^α, k^α may be decomposed in parts parallel to u^α and n^α:

$$k^\alpha = E\left(u^\alpha \pm n^\alpha\right)\,, \tag{42}$$

where E is the energy of the photon and it is defined as $E = -u_\alpha k^\alpha$. On an FLRW spacetime $E = 1/a(\eta)$ and in a conformal FLRW spacetime is unity. Without really working with components, we can show using the geodesic equation that the direction vector is constant along the affine parameter. Hence integrating in the opposite direction of the photon propagation, we find that the spatial position is given by

$$x^i = (\lambda_o - \lambda) n^i = \chi n^i\,, \tag{43}$$

where we have introduced the the comoving distance, χ.

1. *Extrinsic Curvature on Conformal sub-manifolds*

On the conformal FLRW spacetime, the trace of the extrinsic curvature is vanishing: $\bar{K}^\mu{}_\mu = 0$ because the effect of volume expansion is contained in the scalar factor or the conformal factor a. The antisymmetry part and the project symmetric trace-free para is vanishing $\bar{\omega}_{\mu\nu} = \bar{h}^\rho{}_\mu \bar{h}^\sigma{}_\nu \bar{\nabla}_{[\rho}\bar{u}_{\sigma]} = 0 = \bar{K}_{h[\mu\nu]} = \bar{K}_{h\langle\mu\nu\rangle} = \bar{\sigma}_{\mu\nu}$.

Now to the decomposition of the extrinsic curvature on the screen space, we follow two independent approaches to ensure consistency of implementation. First using equation (43) on the conformal FLRW spacetime, we find that the trace of the extrinsic curvature of the screen space is given by

$$K_{N\perp} = g^{\mu\nu} N^\alpha{}_\mu N^\beta{}_\nu \bar{\mathrm{D}}_\alpha n_\beta = \frac{2}{f_k(r)}\,. \tag{44}$$

From the photon geodesic equation, we find that $A_n{}^\alpha = 0$. The vorticity and the shear due to the direction vector must be also be zero, $\bar{\zeta}_{\alpha\beta} = 0 = \bar{\xi}$. The covariant of the screen space metric ($\bar{N}_{\alpha\beta} = \bar{g}_{\alpha\beta} + \bar{u}_\alpha \bar{u}_\beta - \bar{n}_\alpha \bar{n}_\beta$) is given by

$$\bar{\nabla}_\mu N_{\alpha\beta} = 2u_{(\alpha}\nabla_\mu u_{\beta)} - 2n_{(\alpha}\nabla_\mu n_{\beta)} \tag{45}$$

One could verify that the result given in equation (44) is indeed valid by solution the full non-linear equation for the trace on the conformal FLRW spacetime, This equation is exactly the null Raychaudhuri equation and it is given by [1]

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -\frac{1}{2}\theta^2 - \Sigma_{\alpha\beta}\Sigma^{\alpha\beta} - R_{\alpha\beta}k^\alpha k^\beta\,. \tag{46}$$

where $\Sigma_{\alpha\beta}$ is the null shear and it is zero simply by symmetry of the FLRW spacetime. The Ricci tensor, $R_{\alpha\beta}$ is given by equation ((30)) and θ is the trace.

Space-time	*Extrinsic curvature* $K_{h\,\mu\nu}$	*Extrinsic curvature* $K_{N\,\mu\nu}$	*Curvature tensor* ${}^3\bar{R}_{\mu\nu\rho\sigma}$
`Minkowski`	Null	Eqns. (44)	Null
`FLFlat`	Null	Eqns. (44)	Null
`FLCurved`	Null	Eqns. (44)	Eqs. (30)

TABLE I. The table above contains the characteristics of different limits of conformal FLRW spacetime as implemented in `xLightCone`.

2. Specializing Commutation Relation to Conformal FLRW spacetime

Using table I, we find that the covariant derivative of n^α simplifies to

$$\nabla_\mu \bar{n}_\nu = \bar{K}_{N\mu\nu} = \frac{N_{\mu\nu}}{f_k(r)} = \frac{1}{f_k(r)}\left(\bar{h}_{\mu\nu} - \bar{n}_\mu \bar{n}_\nu\right) . \tag{47}$$

where we have used the fact that Lie derivative of $n\alpha$ along u^α is zero, i.e $\mathcal{L}_{\bar{\boldsymbol{u}}}\bar{n}_\nu = 0$. On a conformal FLRW spacetime, the commutation relations simplifies

$$\left[\mathcal{L}_{\bar{\boldsymbol{u}}}, \mathcal{L}_{\bar{\boldsymbol{n}}}\right] T_{\nu_1\ldots\nu_p} = 0\,, \quad \left[\mathcal{L}_{\bar{\boldsymbol{n}}}, \nabla_{\perp\gamma}\right] T_{\nu_1\ldots\nu_p} = 0\,, \quad \left[\mathcal{L}_{\bar{\boldsymbol{u}}}, \nabla_{\perp\gamma}\right] T_{\nu_1\ldots\nu_p} = 0\,. \tag{48}$$

and in terms of the directional derivative for n^α, we find

$$\left[\mathcal{L}_{\bar{\boldsymbol{u}}}, \bar{n}^\beta \bar{D}_\beta\right] T_{\nu_1\ldots\nu_p} = 0\,, \qquad \left[n^\delta \bar{D}_\delta, \nabla_{\perp\gamma}\right] T_{\nu_1\ldots\nu_p} = -K_{N\gamma}{}^{\delta}\bar{\nabla}_{\perp\delta} T_{\nu_1\ldots\nu_p}\,. \tag{49}$$

where we have made use of the information presented in Table I. Also we have that the space time covariant derivative of $\bar{K}^\sigma_{N\nu}$ is given by

$$\nabla_\mu \bar{K}_\perp = n_\mu \mathcal{L}_{\bar{\boldsymbol{n}}} \bar{K}_\perp\,, \tag{50}$$

where $\nabla_{\perp\mu}\bar{K}^\sigma_{N\nu} = 0$ for a homogeneous spacetime and $\mathcal{L}_{\bar{\boldsymbol{u}}}\bar{K}_\perp = 0$ because $\bar{K}_\perp$ depends on the radial coordinate and Gaussian curvature constant of the hypersurface.

3. Perturbations of the metric

In cosmological perturbation theory, the full metric of the spacetime $\boldsymbol{g}$, may be expanded to any order [2]

$$\boldsymbol{g} = \bar{\boldsymbol{g}} + \sum_{m=1}^{\infty} \frac{{}^{\{m\}}\delta[\bar{\boldsymbol{g}}]}{m!}\,. \tag{51}$$

The metric perturbation $\delta^m[\bar{\boldsymbol{g}}]$, may be decomposed in $1+3$ using u^α and $h^{\alpha,\beta}$

$${}^{\{m\}}\delta g^{\alpha\beta} = \left({}^{\{m\}}\delta g^{\gamma\lambda} u_\gamma u_\lambda\right) u^\alpha u^\beta - 2u^{(\alpha}h^{\beta)}{}_{\lambda}\left({}^{\{m\}}\delta g^{\lambda\gamma} u_\gamma\right) + h^{(\alpha}{}_{\gamma} h^{\beta}{}_{\lambda}\,{}^{\{m\}}\delta g^{\gamma\lambda} \tag{52}$$

By performing the standard Scalar-Vector-Tensor (SVT) decomposition [2] of the metric perturbations yields the general expressions:

$$\bar{u}^\rho \bar{u}^\sigma\,{}^{\{m\}}\delta g_{\rho\sigma} = -2\,{}^{\{m\}}\phi\,, \tag{53}$$
$$\bar{u}^\rho \bar{h}^\sigma{}_\nu\,{}^{\{n\}}\delta g_{\rho\sigma} = -\bar{D}_\nu\,{}^{\{m\}}B - {}^{\{m\}}B_\nu\,, \tag{54}$$
$$\bar{h}^\rho{}_\mu \bar{h}^\sigma{}_\nu\,{}^{\{m\}}\delta g_{\rho\sigma} = 2\left(\bar{D}_\mu\bar{D}_\nu\,{}^{\{m\}}E + \bar{D}_{(\mu}\,{}^{\{m\}}E_{\nu)} + {}^{\{m\}}E_{\mu\nu} - {}^{\{m\}}\psi\,\bar{h}_{\mu\nu}\right)\,. \tag{55}$$

In order to extend the decomposition above to include splitting in n^α direction and on the screen, we need to perform further decomposition of equation (52) so that our irreducible quantities live on the screen space.

$$\begin{aligned}{}^{\{m\}}\delta g^{\alpha\beta} = &\left({}^{\{m\}}\delta g^{\gamma\lambda}u_\gamma u_\lambda\right)u^\alpha u^\beta - 2u^{(\alpha}n^{\beta)}\left({}^{\{m\}}\delta g^{\lambda\gamma}u_\gamma n_\lambda\right) - 2u^{(\alpha}N^{\beta)}{}_{\lambda}\left({}^{\{m\}}\delta g^{\lambda\gamma}u_\gamma\right)\\ &+\left({}^{\{m\}}\delta g^{\gamma\lambda}n_\gamma n_\lambda\right)n^\alpha n^\beta + 2n^{(\alpha}N^{\beta)}{}_{\lambda}\left({}^{\{m\}}\delta g^{\lambda\gamma}n_\gamma\right) + N^{(\alpha}{}_{\gamma}N^{\beta}{}_{\lambda}\,{}^{\{m\}}\delta g^{\gamma\lambda}\end{aligned} \tag{56}$$

Since SVT decomposition has already been done in equations (82), (84) and (88), we simply need to split the perturbation variables and its derivatives into components parallel and orthogonal to n^a:

$$\bar{\mathrm{D}}_\alpha T = \bar{n}_\alpha \nabla_\parallel T + \nabla_{\perp\alpha} T\,, \tag{57}$$
$$\bar{\mathrm{D}}_\alpha T_\beta = \bar{\mathrm{D}}_\alpha\left[T_\parallel \bar{n}_\beta + T_{\perp\beta}\right]\,, \tag{58}$$
$$= \bar{n}_\alpha\bar{n}_\beta\nabla_\parallel T_\parallel + \bar{n}_\beta\nabla_{\perp\alpha}T_\parallel + \bar{n}_\alpha\nabla_\parallel T_{\perp\beta} + T_\parallel \bar{K}_{N\alpha\beta} - \bar{\bar{n}}_\beta\,\bar{K}_{N\alpha}{}^{\gamma}T_{\perp\gamma} + \nabla_{\perp\alpha}T_{\perp\beta}\,, \tag{59}$$
$$\bar{\mathrm{D}}_\beta\bar{\mathrm{D}}_\alpha T = \bar{\mathrm{D}}_\beta\left[\nabla_\parallel T\bar{n}_\alpha + \nabla_{\perp\alpha}T\right] \tag{60}$$
$$= \bar{\bar{n}}_\alpha\bar{\bar{n}}_\beta\nabla_\parallel\nabla_\parallel T + \bar{\bar{n}}_\alpha\nabla_{\perp\beta}\nabla_\parallel T + \bar{n}_\beta\nabla_\parallel\nabla_{\perp\alpha}T + \bar{K}_{N\beta\alpha}\nabla_\parallel T - \bar{n}_\alpha\bar{K}_{N\beta}{}^{\gamma}\nabla_{\perp\gamma}T + \nabla_{\perp\beta}\nabla_{\perp\alpha}T\,. \tag{61}$$

where the SVT trace-free condition $\bar{\mathrm{D}}_\alpha E^\alpha = 0$ implies that $\nabla_{\parallel} E_{\parallel} - E_{\parallel} K_{N\perp} + \nabla_{\perp\alpha} E^\alpha_\perp = 0$ and $\bar{\mathrm{D}}^\mu E_{\mu\nu} = 0$ gives xxx. The induced metric on the hypersurface may be replaced with the induced metric on the screen space and the directional vector as $\bar{h}^{\sigma\nu} = \bar{N}^{\sigma\nu} + \bar{n}^\sigma \bar{n}^\nu$ to obtain:

$$\bar{u}^\rho \bar{u}^\sigma\,{}^{\{m\}}\delta g_{\rho\sigma} = -2\,{}^{\{m\}}\phi\,, \tag{62}$$
$$\bar{u}^\rho \bar{n}^\sigma\,{}^{\{m\}}\delta g_{\rho\sigma} = -\bar{n}^\beta D_\beta\,{}^{\{m\}}B - {}^{\{m\}}B_{v_\parallel}\,, \tag{63}$$
$$\bar{u}^\rho \bar{N}^\sigma{}_\nu\,{}^{\{n\}}\delta g_{\rho\sigma} = -\bar{\nabla}_{\perp\nu}\,{}^{\{m\}}B - {}^{\{m\}}B_{\perp\nu}\,, \tag{64}$$
$$\bar{n}^\rho \bar{n}^\sigma\,{}^{\{m\}}\delta g_{\rho\sigma} = 2\left(\bar{\nabla}_\parallel \bar{\nabla}_\parallel\,{}^{\{m\}}E + \bar{\nabla}_\parallel\,{}^{\{m\}}E_\parallel + {}^{\{m\}}E_{\parallel\parallel} - {}^{\{m\}}\psi\right)\,, \tag{65}$$
$$\bar{n}^\rho \bar{N}^\sigma{}_\nu\,{}^{\{n\}}\delta g_{\rho\sigma} = 2\left[\bar{\nabla}_{\perp\nu}\bar{\nabla}_\parallel\,{}^{\{m\}}E + \bar{\nabla}_{\perp\nu}\,{}^{\{m\}}E_\parallel - \bar{K}_{N\perp}\left({}^{\{m\}}E_{\perp\nu} + \bar{\nabla}_{\perp\nu}\,{}^{\{m\}}E\right) + {}^{\{m\}}E_{\parallel\perp\nu}\right]\,, \tag{66}$$
$$\bar{n}^\sigma \bar{N}^\rho{}_\nu\,{}^{\{n\}}\delta g_{\rho\sigma} = 2\left[\bar{\nabla}_{\perp\nu}\bar{\nabla}_\parallel\,{}^{\{m\}}E + \bar{\nabla}_\parallel\,{}^{\{m\}}E_{\perp\nu} - \bar{K}_{N\perp}\left(\bar{\nabla}_{\perp\nu}\,{}^{\{m\}}E\right) + {}^{\{m\}}E_{\parallel\perp\nu}\right]\,, \tag{67}$$
$$\bar{N}^\rho{}_\mu \bar{N}^\sigma{}_\nu\,{}^{\{m\}}\delta g_{\rho\sigma} = 2\left(\bar{\nabla}_{\perp\mu}\bar{\nabla}_{\perp\nu}\,{}^{\{m\}}E + \bar{\nabla}_{\perp(\mu}\,{}^{\{m\}}E_{\perp\nu)} + {}^{\{m\}}E_{\perp\mu\nu} + \left[\bar{K}_{N\perp}\left({}^{\{m\}}E_\parallel + {}^{\{m\}}E\right) - {}^{\{m\}}\psi\right]\bar{N}_{\mu\nu}\right)\,. \tag{68}$$

OU:I will need to check these results again to ensure accuracy

I think the idea is this: We do mode (SVT) decomposition on the hypersurface first, sort out the gauge issues the modes on the screen space. There may be a better way of doing this, but for now I have only implemented it this way.

- `Poisson Gauge:` in Poisson gauge, on the hypersurface we set $E = B = 0$ to obtain

$$\bar{u}^\rho \bar{u}^\sigma\,{}^{\{m\}}\delta g_{\rho\sigma} = -2\,{}^{\{m\}}\phi\,, \tag{69}$$
$$\bar{u}^\rho \bar{h}^\sigma{}_\nu\,{}^{\{n\}}\delta g_{\rho\sigma} = -\,{}^{\{m\}}B_\nu\,, \tag{70}$$
$$\bar{h}^\rho{}_\mu \bar{h}^\sigma{}_\nu\,{}^{\{m\}}\delta g_{\rho\sigma} = 2\left({}^{\{m\}}E_{\mu\nu} - {}^{\{m\}}\psi\,\bar{h}_{\mu\nu}\right)\,. \tag{71}$$

While on the Screen space we find

$$\bar{u}^\rho \bar{u}^\sigma\,{}^{\{m\}}\delta g_{\rho\sigma} = -2\,{}^{\{m\}}\phi\,, \tag{72}$$
$$\bar{u}^\rho \bar{n}^\sigma\,{}^{\{m\}}\delta g_{\rho\sigma} = -\,{}^{\{m\}}B_\parallel\,, \tag{73}$$
$$\bar{u}^\rho \bar{N}^\sigma{}_\nu\,{}^{\{n\}}\delta g_{\rho\sigma} = -\,{}^{\{m\}}B_{\perp\nu}\,, \tag{74}$$
$$\bar{n}^\rho \bar{n}^\sigma\,{}^{\{m\}}\delta g_{\rho\sigma} = 2\left({}^{\{m\}}E_{\parallel\parallel} - {}^{\{m\}}\psi\right)\,, \tag{75}$$
$$\bar{n}^\rho \bar{N}^\sigma{}_\nu\,{}^{\{n\}}\delta g_{\rho\sigma} = 2\,{}^{\{m\}}E_{\parallel\perp\nu}\,, \tag{76}$$
$$\bar{n}^\sigma \bar{N}^\rho{}_\nu\,{}^{\{n\}}\delta g_{\rho\sigma} = 2\,{}^{\{m\}}E_{\perp\nu\parallel}\,, \tag{77}$$
$$\bar{N}^\rho{}_\mu \bar{N}^\sigma{}_\nu\,{}^{\{m\}}\delta g_{\rho\sigma} = 2\left({}^{\{m\}}E_{\perp\mu\nu} - {}^{\{m\}}\psi\,\bar{N}_{\mu\nu}\right)\,. \tag{78}$$

- `Co-moving Gauge`

$$\bar{u}^\rho \bar{u}^\sigma\,{}^{\{m\}}\delta g_{\rho\sigma} = -2\,{}^{\{m\}}\phi\,, \tag{79}$$
$$\bar{u}^\rho \bar{h}^\sigma{}_\nu\,{}^{\{n\}}\delta g_{\rho\sigma} = -\bar{D}_\nu\,{}^{\{m\}}B - {}^{\{m\}}B_\nu\,, \tag{80}$$
$$\bar{h}^\rho{}_\mu \bar{h}^\sigma{}_\nu\,{}^{\{m\}}\delta g_{\rho\sigma} = 2\left({}^{\{m\}}E_{\mu\nu} - {}^{\{m\}}\psi\,\bar{h}_{\mu\nu}\right)\,. \tag{81}$$

Projecting onto the screen space, we find

$$\bar{u}^\rho \bar{u}^\sigma\,{}^{\{m\}}\delta g_{\rho\sigma} = -2\,{}^{\{m\}}\phi\,, \tag{82}$$
$$\bar{u}^\rho \bar{n}^\sigma\,{}^{\{m\}}\delta g_{\rho\sigma} = -\bar{n}^\beta D_\beta\,{}^{\{m\}}B - {}^{\{m\}}B_{v_\parallel}\,, \tag{83}$$
$$\bar{u}^\rho \bar{N}^\sigma{}_\nu\,{}^{\{n\}}\delta g_{\rho\sigma} = -\bar{\nabla}_{\perp\nu}\,{}^{\{m\}}B - {}^{\{m\}}B_{\perp\nu}\,, \tag{84}$$
$$\bar{n}^\rho \bar{n}^\sigma\,{}^{\{m\}}\delta g_{\rho\sigma} = 2\left({}^{\{m\}}E_{\parallel\parallel} - {}^{\{m\}}\psi\right)\,, \tag{85}$$
$$\bar{n}^\rho \bar{N}^\sigma{}_\nu\,{}^{\{n\}}\delta g_{\rho\sigma} = 2\,{}^{\{m\}}E_{\parallel\perp\nu}\,, \tag{86}$$
$$\bar{n}^\sigma \bar{N}^\rho{}_\nu\,{}^{\{n\}}\delta g_{\rho\sigma} = 2\,{}^{\{m\}}E_{\parallel\perp\nu}\,, \tag{87}$$
$$\bar{N}^\rho{}_\mu \bar{N}^\sigma{}_\nu\,{}^{\{m\}}\delta g_{\rho\sigma} = 2\left({}^{\{m\}}E_{\perp\mu\nu} - {}^{\{m\}}\psi\,\bar{N}_{\mu\nu}\right)\,. \tag{88}$$

[1] O. Umeh, C. Clarkson, and R. Maartens, (2014), arXiv:1402.1933 [astro-ph.CO].
[2] C. Pitrou, X. Roy, and O. Umeh, (2013), arXiv:1302.6174 [astro-ph.CO].