xLightCone: An Algorithm for Perturbing an FLRW Spacetime on the Past Lightcone

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Perturbation on the past light cone

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INTRODUCTION

At the moment, I have been able to implement $1+1+2$ splitting by first decomposing in $3+1$ and splitting the spatial part further in $2+1$. Basically, I borrow all the rules of $3+1$ decomposition.

I. BASIC TOOLS FOR $3+1$ DECOMPOSITION

A. Background Spacetime decomposition

In xPand, we decomposed the background manifold using the four vector $\bar{\boldsymbol{u}}$ which is time-like vector and it is normalized as ($\bar{u}^{\mu}\bar{u}_{\mu}=-1$). The metric on $\overline{\mathcal{M}}$ is decomposed as

$$\bar{g}_{\mu\nu}=\bar{h}_{\mu\nu}-\bar{u}_{\mu}\bar{u}1_{\nu}\,,\qquad \text{with}\qquad \bar{h}_{\mu\nu}\bar{u}^{\mu}=0\qquad \text{and}\qquad \bar{h}^{\mu}{}_{\rho}\bar{h}^{\rho}{}_{\nu}=\bar{h}^{\mu}{}_{\nu}\,, \tag{1}$$

where $\bar{\boldsymbol{h}}$ represents the induced metric of the spatial hypersuface

The covariant derivative of n^{μ} is also decomposed as

$$\nabla_{\mu}u_{\nu}=-\bar{a}_{\nu}\bar{u}_{\mu}+\bar{K}_{\mu\nu} \tag{2}$$

where

$$\bar{a}_{\mu}=\bar{u}^{\rho}\,\bar{\nabla}_{\rho}\bar{u}_{\mu}=\frac{\bar{D}_{\mu}\bar{\alpha}}{\bar{\alpha}}\,,\qquad \bar{K}_{\mu\nu}=\bar{h}^{\rho}{}_{\mu}\bar{h}^{\sigma}{}_{\nu}\bar{\nabla}_{\rho}\bar{u}_{\sigma}=\frac{1}{3}\bar{\Theta}h_{\mu\nu}+\bar{\sigma}_{\mu\nu}+\bar{\omega}_{\mu\nu}\,, \tag{3}$$

α is the lapse function. Here the trace of the extrinsic curvature vanishes: $\bar{K}^{\mu}{}_{\mu}=0$ because the volume expansion of the background space-time is contained in our conformal factor a. But for for the general Bianchi cosmologies, the trace-free part is given by $\bar{K}_{\langle\mu\nu\rangle}=\bar{\sigma}_{\mu\nu}$. this is also called the shear of the Eulerian observers. The antisymmetry part, i.e the vorticity is vanishing for an irrotional hypersurface $\bar{\omega}_{\mu\nu}=\bar{h}^{\rho}{}_{\mu}\bar{h}^{\sigma}{}_{\nu}\bar{\nabla}_{[\rho}\bar{u}_{\sigma]}=0=\bar{K}_{[\mu\nu]}$

B. Decomposition of Tensor fields

For a rank-two covariant tensor $\boldsymbol{T}$, we have for instance:

$$T_{\mu\nu}=\bar{u}_{\mu}\bar{u}_{\nu}\left(\bar{u}^{\rho}\bar{u}^{\sigma}T_{\rho\sigma}\right)+2\,\bar{u}_{(\mu}\big(\bar{u}^{\rho}\,\bar{h}^{\sigma}{}_{\nu)}T_{\rho\sigma}\big)+\big(\bar{h}^{\rho}{}_{\mu}\bar{h}^{\sigma}{}_{\nu}T_{\rho\sigma}\big)\,. \tag{4}$$

C. Covariant Derivative Decomposition

For general spatial tensors (namely, for spatial tensors defined within $\overline{\mathcal{M}}$ or defined within $\mathcal{M}$ then mapped onto $\overline{\mathcal{M}}$), the relation between the two derivatives reads:

$$\bar{\nabla}_{\rho}T_{\mu_1\dots\mu_p}=-\bar{u}_{\rho}\dot{T}_{\mu_1\dots\mu_p}+\bar{D}_{\rho}T_{\mu_1\dots\mu_p}+\sum_{i=1}^{p}\bar{u}_{\mu_i}\,\bar{K}^{\sigma}{}_{\rho}\,T_{\mu_1\dots\mu_{i-1}\sigma\mu_{i+1}\dots\mu_p}\,. \tag{5}$$

The relation between $\mathcal{L}_{\bar{\boldsymbol{u}}}$ and $\bar{u}^\rho\bar{\nabla}_\rho$ is written[1]:

$$\mathcal{L}_{\bar{\boldsymbol{u}}}T_{\mu_1\ldots\mu_p} = \dot{T}_{\mu_1\ldots\mu_p} + \sum_{i=1}^{p}\bar{K}^{\sigma}{}_{\mu_i}\,T_{\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p}\,, \tag{6}$$

$$\bar{\nabla}_\rho T_{\mu_1\ldots\mu_p} = -\bar{u}_\rho\,\mathcal{L}_{\bar{\boldsymbol{u}}}T_{\mu_1\ldots\mu_p} \;+\; \bar{D}_\rho T_{\mu_1\ldots\mu_p} + 2\sum_{i=1}^{p}\bar{u}_{(\mu_i}\,\bar{K}^{\sigma}{}_{\rho)}\,T_{\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p}\,. \tag{7}$$

II. BASIC TOOLS FOR $2+1$ DECOMPOSITION

A. My Notations

For a general spatial tensor $T^{\alpha\cdots}{}_{\cdots\beta}$, we can isolate the part lying in the screen space as

$$T_\perp{}^{\alpha\cdots}{}_{\cdots\beta} = N^\alpha{}_\gamma\cdots N^\mu{}_\beta T^{\gamma\cdots}{}_{\cdots\mu}\,, \tag{8}$$

and the part parallel to n^α as

$$T_\parallel = n_\alpha\cdots n^\beta\,T^{\alpha\cdots}{}_{\cdots\beta}\,. \tag{9}$$

We define the covariant angular derivative $\nabla_{\perp\alpha}$ on the screen space and the derivative $\nabla_\parallel$ along the direction of observation:

$$\nabla_{\perp\alpha}T^{\beta\cdots}{}_{\cdots\gamma} = N^\lambda{}_\alpha N^\beta\quad\cdots N^\mu{}_\gamma\nabla_\lambda T\;{}^{\cdots}{}_{\cdots\mu}\,, \tag{10}$$

$$\nabla_\parallel T^{\alpha\cdots}{}_{\cdots\beta} = n^\gamma\nabla_\gamma T^{\alpha\cdots}{}_{\cdots\beta}\,. \tag{11}$$

B. Decomposition of background Spacetime

In section I, we have shown how a 4-d background spacetime and tensor fields that live on them may be decomposed using $3+1$ technique, now we are going to split further the spatial part (i.e the hyper-surface) of the 4-d spacetime using 1+2 decomposition technique.

First we need to define a space-like 3-vector, $\bar{n}^\mu$, which is normalized as $\bar{n}^\mu\bar{n}_\mu = 1$ and an induced metric $N^{\mu,\nu}$ which lives on the 2-d sheet or the screen space.

$$\bar{h}^{\alpha\beta}\bar{n}_\alpha\bar{n}_\beta = 1\,,\qquad \bar{N}_{\mu\nu} = \bar{h}_{\mu\nu} - \bar{n}_\mu\bar{n}_\nu = g_{\mu\nu} - \bar{u}_\mu\bar{u}_\nu + \bar{n}_\mu\bar{n}_\nu\,. \tag{12}$$

Other properties of the projection tensor include

$$\bar{u}^\alpha\bar{N}_{\alpha\beta} = \bar{n}^\alpha\bar{N}_{\alpha\beta} = 0,\;\; \bar{k}^\alpha\bar{N}_{\alpha\beta} = 0,\;\;\; \bar{h}^{\alpha\beta}\bar{N}_{\alpha\beta} = g^{\alpha\beta}\bar{N}_{\alpha\beta} = 2,\;\;\; \bar{N}_{\alpha\beta}\bar{N}^{\alpha\beta} = 2,\;\;\; \bar{N}_{\alpha\gamma}\bar{N}^{\alpha\beta} = \bar{N}_{\beta\gamma}. \tag{13}$$

where $k^\alpha = \bar{E}\left(\bar{u}^\alpha \pm \bar{n}^\alpha\right)$ is the photon tangent vector. I have introduced it here because how to calculate it is central to what we plan to do.

Spatial covariant derivative of $\bar{n}^\alpha$ may be irreducibly decomposed as

$$\bar{\mathrm{D}}_\mu\bar{n}_\nu = \bar{n}_\mu\bar{\beta}_\nu + \bar{K}_{\perp\mu\nu} \tag{14}$$

where we have introduced an Extrinsic curvature (i.e $\bar{K}_{\perp\mu\nu}$) of on the screen space and it is decomposed as follows $\bar{K}_{\perp\mu\nu} = \frac{1}{2}\bar{K}_\perp N_{\mu\nu} + \xi\bar{\varepsilon}_{\mu\nu} + \zeta_{\mu\nu}$. We have introduced the following notations

$$\bar{\beta}_\alpha \equiv \bar{n}^\gamma\bar{D}_\gamma\bar{n}_\alpha = \nabla_\parallel n_a,\qquad \text{Radial Acceleration} \tag{15}$$

$$\bar{K}_\perp \equiv \bar{D}_{\perp\alpha}\bar{n}^\alpha,\qquad \text{Trace of } \bar{K}^\mu_{\perp\mu} \tag{16}$$

$$\bar{\xi} \equiv \frac{1}{2}\bar{\varepsilon}^{\alpha\beta}\bar{D}_{\perp\alpha}\bar{n}_\beta,\qquad \text{Twist, the anti-symmetry part} \tag{17}$$

$$\bar{\zeta}_{\alpha\beta} \equiv \bar{D}_{\perp\langle\alpha}\bar{n}_{\beta\rangle}.\qquad \text{Shear, the symmetry part} \tag{18}$$

[1] Note that for a spatial tensor $\boldsymbol{T}$, the quantity $\mathcal{L}_{\bar{\boldsymbol{u}}}T_{\mu_1\ldots\mu_p}$ is also spatial.

We also define the alternating Levi-Civita 2-tensor

$$\bar{\varepsilon}_{\alpha\beta} \equiv \bar{\varepsilon}_{\alpha\beta\gamma}\bar{n}^{\gamma} = \bar{u}^{\lambda}\bar{\eta}_{\lambda\alpha\beta\gamma}\bar{n}^{\gamma}, \tag{19}$$

so that $\bar{\varepsilon}_{\alpha\beta}\bar{n}^{\beta} = 0 = \bar{\varepsilon}_{(\alpha\beta}$, and

$$\bar{\varepsilon}_{\alpha\beta\gamma} = \bar{n}_{\alpha}\bar{\varepsilon}_{\beta\gamma} - \bar{n}_{\beta}\varepsilon_{\alpha\gamma} + \bar{n}_{\gamma}\varepsilon_{\alpha\beta}, \tag{20}$$

$$\bar{\varepsilon}_{\alpha\beta}\bar{\varepsilon}^{\gamma\lambda} = \bar{N}_{\alpha}{}^{\gamma}\bar{N}_{\beta}{}^{\lambda} - \bar{N}_{\alpha}{}^{\lambda}\bar{N}_{\beta}{}^{\gamma}, \tag{21}$$

$$\bar{\varepsilon}_{\alpha}{}^{\gamma}\bar{\varepsilon}_{\beta\gamma} = \bar{N}_{\alpha\beta}, \quad \bar{\varepsilon}^{\alpha\beta}\bar{\varepsilon}_{\alpha\beta} = 2. \tag{22}$$

C. Decomposition of Tensor fields

The spatial part of the tensor field that have been split using $3+1$ technique in section I may now be further split into the radial part and the angular part. For example for a 3-vector V^a can now be irreducibly split into a scalar, $V_{\parallel}$, which is the part of the vector parallel to $\bar{n}^{\alpha}$, and a vector, $V_{\perp}^{\alpha}$, lying in the sheet orthogonal to $\bar{n}^{\alpha}$;

$$V^{\alpha} = V_{\parallel}\bar{n}^{\alpha} + V_{\perp}^{\alpha}, \quad \text{where} \quad V_{\parallel} \equiv V_{\alpha}\bar{n}^{\alpha}, \quad \text{and} \quad V_{\perp}^{\alpha} \equiv \bar{N}^{\alpha\beta}V_{\beta}, \tag{23}$$

A rank two tensor field may also be decomposed into radial and screen space components according to

$$T_{\alpha\beta} = T_{\langle\alpha\beta\rangle} = T_{\parallel}\left(\bar{n}_{\alpha}\bar{n}_{\beta} - \frac{1}{2}N_{\alpha\beta}\right) + 2T_{\perp\parallel(\alpha}\bar{n}_{\beta)} + T_{\perp\langle\alpha\beta\rangle}, \tag{24}$$

where

$$\begin{aligned} T_{\parallel} &\equiv \bar{n}^{\alpha}\bar{n}^{\beta}\psi_{\alpha\beta} = -\bar{N}^{\alpha\beta}T_{\alpha\beta}, \\ T_{\perp\alpha} &\equiv \bar{N}_{\alpha}{}^{\alpha}\bar{n}^{\gamma}T_{\beta\gamma} \\ T_{\perp\alpha\beta} &\equiv T_{\langle\alpha\beta\rangle} \equiv \left(N_{(\alpha}{}^{\gamma}\bar{N}_{\beta)}{}^{\lambda} - \frac{1}{2}\bar{N}_{\alpha\beta}\bar{N}^{\gamma\lambda}\right)T_{\gamma\lambda}. \end{aligned} \tag{25}$$

D. Decomposition of Covariant Derivatives

The decomposition of spatial covariant derivative follows similarly the covariant decomposition rules for the 4-d covariant derivatives, I will show starting a scalar:

$$\bar{\mathrm{D}}_{\alpha}\Psi = \nabla_{\parallel}\Psi\bar{n}_{a} + \nabla_{\perp a}\Psi\,, \tag{26}$$

$$\bar{\mathrm{D}}_{a}\Psi_{b} = -\bar{n}_{a}\bar{n}_{b}\Psi_{c}\nabla_{\parallel}\bar{n}^{c} + \bar{n}_{a}\hat{\Psi}_{\bar{b}} - \bar{n}_{b}\left[\frac{1}{2}\phi\Psi_{a} + \left[\xi\varepsilon_{ac} + \zeta_{ac}\right]\Psi^{c}\right] + \nabla_{\perp a}\Psi_{b}\,, \tag{27}$$

$$\bar{\mathrm{D}}_{a}\Psi_{bc} = -2\bar{n}_{a}\bar{n}_{(b}\Psi_{c)d}\nabla_{\parallel}n^{d} + \bar{n}_{a}\hat{\Psi}_{bc} - 2\bar{n}_{(b}\left[\frac{1}{2}\phi\Psi_{c)a} + \Psi_{c)}{}^{d}\left[\xi\varepsilon_{ad} + \zeta_{ad}\right]\right] + \nabla_{\perp a}\Psi_{bc}. \tag{28}$$

The first term on the RHS is zero because $\nabla_{\parallel}\bar{n}^{c} = 0$ for the type of homogeneous cosmology we consider. So in general full decomposition of a spatial covariant derivative of tensor will look like this:

$$\bar{\mathrm{D}}_{\rho}T_{\mu_1\ldots\mu_p} = \bar{n}_{\rho}\nabla_{\parallel}T_{\perp\mu_1\ldots\mu_p} + \sum_{i=1}^{p}\bar{n}_{\mu_i}\,\bar{K}_{\perp}^{\sigma}{}_{\rho}\,T_{\perp\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p} + \bar{\nabla}_{\perp\rho}T_{\perp\mu_1\ldots\mu_p}\,. \tag{29}$$

If we use Lie derivative for the radial derivative, we will have to do the same thing we did for time

$$\mathcal{L}_{\bar{\boldsymbol{n}}}T_{\perp\mu_1\ldots\mu_p} = \nabla_{\parallel}T_{\mu_1\ldots\mu_p} + \sum_{i=1}^{p}\bar{K}_{\perp}^{\sigma}{}_{\mu_i}\,T_{\perp\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p}\,, \tag{30}$$

Finally the full decomposition will look like this.

$$\bar{\bar{\mathrm{D}}}_{\rho}T_{\perp\mu_1\ldots\mu_p} = \bar{n}_{\rho}\,\mathcal{L}_{\bar{\boldsymbol{n}}}T_{\mu_1\ldots\mu_p} + \bar{\nabla}_{\perp\rho}T_{\mu_1\ldots\mu_p} + 2\sum_{i=1}^{p}\bar{n}_{(\mu_i}\,\bar{K}_{\perp}^{\sigma}{}_{\rho)}\,T_{\perp\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p}\,. \tag{31}$$

III. BY-PASSING XPAND TO DO THE DECOMPOSITION

A. Perturbations of the metric

The SVT decomposition of the metric perturbations yields the general expressions:

$$\bar{u}^\rho \bar{u}^\sigma \, {}^{\{n\}}h_{\rho\sigma} = -2\, {}^{\{n\}}\phi \,, \tag{32}$$

$$\bar{u}^\rho \bar{h}^\sigma{}_\nu \, {}^{\{n\}}h_{\rho\sigma} = -\bar{D}_\nu \, {}^{\{n\}}B - \, {}^{\{n\}}B_\nu \,, \tag{33}$$

$$\bar{h}^\rho{}_\mu \bar{h}^\sigma{}_\nu \, {}^{\{n\}}h_{\rho\sigma} = 2\left(\bar{D}_\mu \bar{D}_\nu \, {}^{\{n\}}E + \bar{D}_{(\mu} \, {}^{\{n\}}E_{\nu)} + \, {}^{\{n\}}E_{\mu\nu} - \, {}^{\{n\}}\psi \, \bar{h}_{\mu\nu}\right) . \tag{34}$$

Each of the spatial derivative above would be replaced with the following:

$$\bar{\mathrm{D}}_\alpha E = \bar{n}_\alpha \nabla_\parallel E + \nabla_{\perp\alpha} E \,, \tag{35}$$

$$\bar{\mathrm{D}}_\alpha E_\beta = \bar{\mathrm{D}}_\alpha \left[E_\parallel \bar{n}_\beta + E_{\perp\beta}\right] , \tag{36}$$

$$= \bar{n}_\alpha \bar{n}_\beta \nabla_\parallel E_\parallel + \bar{n}_\beta \nabla_{\perp\alpha} E_\parallel + \bar{n}_\alpha \nabla_\parallel E_{\perp\beta} + E_\parallel \bar{K}_{\perp\alpha\beta} + \bar{n}_\beta \, \bar{K}_{\perp\alpha}{}^\gamma E_{\perp\gamma} + \nabla_{\perp\alpha} E_{\perp\beta} \,, \tag{37}$$

$$\bar{\mathrm{D}}_\beta \bar{\mathrm{D}}_\alpha \Psi = \bar{\mathrm{D}}_\beta \left[\nabla_\parallel \Psi \bar{n}_\alpha + \nabla_{\perp\alpha}\Psi\right] \tag{38}$$

$$= \bar{n}_\alpha \bar{n}_\beta \nabla_\parallel \nabla_\parallel \Phi + \bar{n}_\alpha \nabla_{\perp\beta} \nabla_\parallel \Psi + \bar{n}_\beta \nabla_\parallel \nabla_{\perp\alpha} \Psi + K_{\perp\beta\alpha} \nabla_\parallel \Psi + \bar{n}_\alpha K_{\perp\beta}{}^\gamma \nabla_{\perp\gamma} \psi + \nabla_{\perp\beta} \nabla_{\perp\alpha} \Psi \,. \tag{39}$$

B. From Space time covariant derivative to Screen Space Derivative

$$\begin{aligned}
\bar{\nabla}_\rho T_{\mu_1 \ldots \mu_p} = & -\bar{u}_\rho \, \mathcal{L}_{\bar{\boldsymbol{u}}} T_{\mu_1 \ldots \mu_p} + 2 \sum_{i=1}^{p} \bar{u}_{(\mu_i} \, \bar{K}^\sigma{}_{\rho)} \, T_{\mu_1 \ldots \mu_{i-1} \sigma \mu_{i+1} \ldots \mu_p} + \bar{n}_\rho \, \mathcal{L}_{\bar{\boldsymbol{n}}} T_{\mu_1 \ldots \mu_p} \\
& + \bar{\nabla}_{\perp\rho} T_{\mu_1 \ldots \mu_p} + 2 \sum_{i=1}^{p} \bar{n}_{(\mu_i} \, \bar{K}^\sigma_{\perp \ \rho)} \, T_{\perp \mu_1 \ldots \mu_{i-1} \sigma \mu_{i+1} \ldots \mu_p} \,.
\end{aligned} \tag{40}$$

For homogeneous cosmologies of interest, $\bar{K}^\sigma{}_{\rho)}$ is zero for FL cosmology but non-vanishing for some classes of Bianchi cosmologies.

C. Commutation Relations for Screen Space Projected tensors

In order to obtain the commutation relation between involving Lie derivative and radial derivative,or Lie derivative and screen space projected angular derivative or radial derivative and screen space projected angular derivative, we will need the full spacetime Ricci identity for a generalized tensor

$$\begin{aligned}
\nabla_\gamma \nabla_\delta T^{\alpha_1 \cdots a_r}{}_{\beta_1 \cdots \beta_s} - \nabla_\delta \nabla_\gamma T^{\alpha_1 \cdots \alpha_r}{}_{\beta_1 \cdots \beta_s} = & -R^{\alpha_1}{}_{\rho\gamma\delta} T^{\rho\alpha_2 \cdots \alpha_r}{}_{\beta_1 \cdots \beta_s} - \cdots - R^{\alpha_r}{}_{\rho\gamma\delta} T^{\alpha_1 \cdots \alpha_{r-1}}{}_{\beta_1 \cdots \beta_s} \\
& + R^\sigma{}_{\beta_1 \gamma\delta} T^{\alpha_1 \cdots \alpha_r}{}_{\beta_1 \cdots \beta_{s-1} \sigma}
\end{aligned} \tag{41}$$

It is important to note that for a vector, we recover the well-known result

$$\nabla_\rho \nabla_\sigma A_\nu - \nabla_\sigma \nabla_\rho A_\rho = R^\beta{}_{\nu\rho\sigma} A_\beta \,. \tag{42}$$

The full spacetime Riemann tensor can also be decomposed completely at least with respect to the time slicing four vector using Gauss Codazzi relations :

$$\bar{R}_{\mu\nu\rho\sigma} = {}^3\bar{R}_{\mu\nu\rho\sigma} + 2\bar{K}_{\mu[\rho} \bar{K}_{\sigma]\nu} - 4\left(\bar{D}_{[\mu} \bar{K}_{\nu][\rho}\right) \bar{u}_{\sigma]} - 4\left(\bar{D}_{[\rho} \bar{K}_{\sigma][\mu}\right) \bar{u}_{\nu]} + 4\, \bar{u}_{[\mu} \, \bar{K}_{\nu]}{}^\zeta \, \bar{K}_{\zeta[\rho} \, \bar{n}_{\sigma]} + 4\, \bar{u}_{[\mu} \, \dot{\bar{K}}_{\nu][\rho} \bar{u}_{\sigma]} \,, \tag{43}$$

where ${}^3\bar{R}_{\mu\nu\rho\sigma}$ stands for the hypersurfaces projected Riemann curvature tensor, the over-dot the derivative along the hypersurface slicing four vector, i.e following our general notation for any tensor field $\boldsymbol{T}$, we have: $\dot{T}_{\mu_1 \ldots \mu_p} = \bar{n}^\rho \bar{\nabla}_\rho T_{\mu_1 \ldots \mu_p}$. It is well known that projecting on all the indices of the Riemann tensor with the metric on the hyper-surface leads to

$$\bar{h}^\varphi{}_\mu \bar{h}^\upsilon{}_\nu \bar{h}^\xi{}_\rho \bar{h}^\zeta{}_\sigma \, \bar{R}_{\varphi\upsilon\xi\zeta} = {}^3\bar{R}_{\mu\nu\rho\sigma} + 2\bar{K}_{\mu[\rho} \bar{K}_{\sigma]\nu} \,. \tag{44}$$

and when the fourth index is contracted with u^α and the remaining three indices still fully projected leads to

$$\bar{h}^{\varphi}{}_{\mu}\bar{h}^{\upsilon}{}_{\nu}\bar{h}^{\xi}{}_{\rho}\bar{n}^{\zeta}\,\bar{R}_{\varphi\upsilon\xi\zeta} = \bar{D}_\mu \bar{K}_{\nu\rho} - \bar{D}_\nu \bar{K}_{\mu\rho}\,, \tag{45}$$

In an FLRW universe, ${}^3\bar{R}_{\mu\nu\rho\sigma}$ has a simple form

$${}^3\bar{R}_{\mu\nu\rho\sigma} = 2k\,\bar{h}_{\rho[\mu}\,\bar{h}_{\nu]\sigma}\,, \qquad {}^3\bar{R}_{\mu\nu} = 2k\,\bar{h}_{\mu\nu}\,, \qquad {}^3\bar{R} = 6k\,, \tag{46}$$

where k is the mean spatial curvature of the hypersurface, it is zero in a flat FLRW universe, $+1$ is a closed universe and -1 is open universe.

Using the decomposition of the Riemann curvature given in equation (43), the commutation relation between Lie derivative $\mathcal{L}_{\bar{\boldsymbol{u}}}$ and projected spatial derivative on the on the hyper surface $\bar{\boldsymbol{D}}$ for general spatial tensors (i.e a tensor whose all indices are fully projected), it is given by

$$\mathcal{L}_{\bar{\boldsymbol{u}}}\left(\bar{D}_\rho T_{\mu_1\ldots\mu_p}\right) = \bar{D}_\rho\left(\mathcal{L}_{\bar{\boldsymbol{u}}} T_{\mu_1\ldots\mu_p}\right) + \sum_{i=1}^{p}\left(\bar{h}^{\sigma\zeta}\bar{D}_\zeta \bar{K}_{\rho\mu_i} - \bar{D}_\rho \bar{K}_{\mu_i}{}^{\sigma} - \bar{D}_{\mu_i}\bar{K}_{\rho}{}^{\sigma}\right) T_{\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p}\,, \tag{47}$$

This result is easily obtained using simple product rule with the Ricci identity in equation (41). For the commutation relation between Lie derivative for the conformal time and radial derivative, the projected angular derivative and radial derivative and the projected angular derivative and conformal time we find

$$\begin{aligned}
\mathcal{L}_{\bar{\boldsymbol{u}}}\left(\bar{n}^\beta \bar{D}_\beta T_{\nu_1\ldots\nu_p}\right) &= n^\beta \bar{D}_\beta\left(\mathcal{L}_{\bar{\boldsymbol{u}}} T_{\nu_1\ldots\nu_p}\right) + \bar{u}^\alpha \nabla_\alpha \bar{n}^\beta \bar{D}_\beta T_{\nu_1\ldots\nu_p} - \sum_{i=1}^{p}\Big[\bar{n}^\alpha \bar{D}_\alpha K_{h}{}^{\sigma}_{\mu_1} T_{\nu_1\ldots\sigma\cdots\nu_p} && (48)\\
&\quad + \bar{u}^\alpha \bar{n}^\sigma R_{\sigma\alpha\mu_1}{}^{\gamma} T_{\gamma\cdots\nu_p}\Big]\,,\\
n^\delta \bar{D}_\delta\left(\nabla_{\perp\gamma} T_{\nu_1\ldots\nu_p}\right) &= \nabla_{\perp\gamma}\left(n^\delta \bar{D}_\delta T_{\nu_1\ldots\nu_p}\right) - \nabla^\delta n_\gamma \bar{D}_\gamma T_{\nu_1\ldots\nu_p} - \sum_{i=1}^{p}\left[N^\alpha{}_\gamma h^\sigma{}_\delta \bar{n}^\delta R_{\sigma\alpha\mu}{}^{\gamma} T_{\gamma\nu_2\ldots\nu_{p-1}}\right]\,, && (49)\\
\mathcal{L}_{\bar{\boldsymbol{u}}}\left(\nabla_{\perp\gamma} T_{\nu_1\ldots\nu_p}\right) &= \nabla_{\perp\gamma}\mathcal{L}_{\bar{\boldsymbol{u}}}\left(T_{\nu_1\ldots\nu_p}\right) - \sum_{i=1}^{p}\Big[N^\delta{}_\gamma K_{h}{}^{\beta}{}_{\delta}\nabla_\beta T_{\mu_1\cdots\mu_p} + N^\delta_\gamma \nabla_\delta K_{h}{}^{\sigma}{}_{\mu_1} T_{\mu_2\cdots\sigma\cdots\mu_p} && (50)\\
&\quad - N^\alpha{}_\gamma \bar{u}^\beta R_{\beta\alpha\mu_1}{}^{\sigma} T_{\mu_2\cdots\sigma\cdots\mu_p}\Big]\,..
\end{aligned}$$

where the full decomposition of the covariant derivative of $\nabla_\mu n_\nu$ is given by

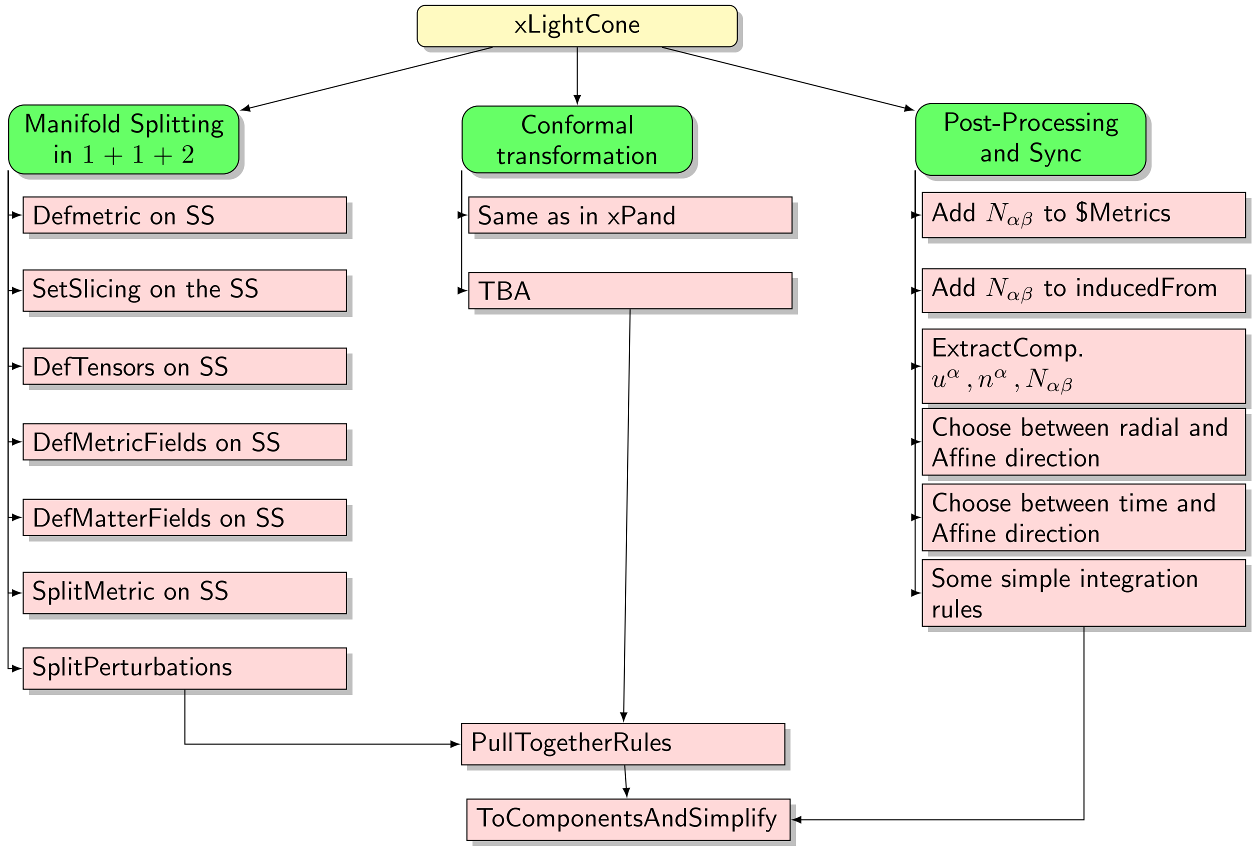
$$\nabla_\mu \bar{n}_\nu = -\bar{n}_\mu \mathcal{L}_{\bar{\boldsymbol{u}}}\bar{n}_\nu - \bar{n}^\alpha \bar{u}_\mu \nabla_\nu \bar{u}_\alpha + K_{h\mu\beta} n^\beta + \bar{n}_\mu \bar{\beta}_\nu + \bar{K}_{N\mu\nu}\,. \tag{51}$$

The signs of the two derivatives

I was thinking we use "dot" for conformal time and "hat" for radial derivative. For example, given a tensor $T_{\nu_1\ldots\nu_p}$, time derivative will be $T'_{\nu_1\ldots\nu_p}$ and radial derivative $\hat{T}_{\nu_1\ldots\nu_p}$. When we have radial derivative first and conformal time second we write $\dot{\hat{T}}_{\nu_1\ldots\nu_p}$, if we have conformal time first and radial second we have $\hat{\dot{T}}_{\nu_1\ldots\nu_p}$

IV. PACKAGE ARCHITECTURE

This is a preliminary architecture of the package "xLghtcone". The package is expected to build on xPand by adding additional functionalities that would enable further splitting of spacetime into radial and angular part in a very clean geometric way.



where SS stands for Screen Space.

A. Description:This is only for the Friedmann Cosmology

We now expatiate on the functionalities of various sections of the package.

- Manifold Decomposition
  - `DefScreenSpaceMetric`
    - AssignProperties:The following properties should be assign to the metric on the screen space.
  - `SetScreenSpaceSlicing`
  - `DefScreenSpaceProjectedTensors`
    - Label Indices: Every tensor will have atleast three label indices for:time, radial, and order of perturbation.
    - AssignProperties: Such as trace-free in both indices for rank two. How label indices should move when acted up.
  - `DefScreenSpaceProjectedMetricFields`
  - `DefScreenSpaceProjectedMatterfFields`
  - `Splitmetric`
  - `RemoveinducedDerivative`
  - `ToMetric`
  - `SplitPerturbations`

- Conformal Transformation
  - Borrow transformation rule from xPand
- Post-Processing
  - `Extract Components`
- Together
  - `ToLightConeFromRules`