INTRODUCTION

At the moment, I have been able to implement $1+1+2$ splitting by first decomposing in $3+1$ and splitting the spatial part further in $2+1$. Basically, I borrow all the rules of $3+1$ decomposition.

I. BASIC TOOLS FOR $3+1$ DECOMPOSITION

A. Background Spacetime decomposition

In xPand, we decomposed the background manifold using the four vector $\bar{\boldsymbol{u}}$ which is time-like vector and it is normalized as ($\bar{u}^\mu \bar{u}_\mu = -1$). The metric on $\overline{\mathcal{M}}$ is decomposed as

$$\bar{g}_{\mu\nu} = \bar{h}_{\mu\nu} - \bar{u}_\mu \bar{u}_\nu \,, \qquad \text{with} \qquad \bar{h}_{\mu\nu}\bar{u}^\mu = 0 \qquad \text{and} \qquad \bar{h}^\mu{}_\rho \bar{h}^\rho{}_\nu = \bar{h}^\mu{}_\nu \,, \tag{1}$$

where $\bar{\boldsymbol{h}}$ represents the induced metric of the spatial hypersuface

The covariant derivative of n^μ is also decomposed as

$$\nabla_\mu u_\nu = -\bar{a}_\nu \bar{u}_\mu + \bar{K}_{\mu\nu} \tag{2}$$

where

$$\bar{a}_\mu = \bar{u}^\rho\, \bar{\nabla}_\rho \bar{u}_\mu = \frac{\bar{D}_\mu \bar{\alpha}}{\bar{\alpha}} \,, \qquad \bar{K}_{\mu\nu} = \bar{h}^\rho{}_\mu \bar{h}^\sigma{}_\nu \bar{\nabla}_\rho \bar{u}_\sigma = \frac{1}{3}\bar{\Theta} h_{\mu\nu} + \bar{\sigma}_{\mu\nu} + \bar{\omega}_{\mu\nu} \,, \tag{3}$$

α is the lapse function. Here the trace of the extrinsic curvature vanishes: $\bar{K}^\mu{}_\mu = 0$ because the volume expansion of the background space-time is contained in our conformal factor a. But for for the general Bianchi cosmologies, the trace-free part is given by $\bar{K}_{\langle\mu\nu\rangle} = \bar{\sigma}_{\mu\nu}$. this is also called the shear of the Eulerian observers. The antisymmetry part, i.e the vorticity is vanishing for an irrotional hypersurface $\bar{\omega}_{\mu\nu} = \bar{h}^\rho{}_\mu \bar{h}^\sigma{}_\nu \bar{\nabla}_{[\rho} \bar{u}_{\sigma]} = 0 = \bar{K}_{[\mu\nu]}$

B. Decomposition of Tensor fields

For a rank-two covariant tensor $\boldsymbol{T}$, we have for instance:

$$T_{\mu\nu} = \bar{u}_\mu \bar{u}_\nu \left(\bar{u}^\rho \bar{u}^\sigma T_{\rho\sigma} \right) + 2\, \bar{u}_{(\mu} \left(\bar{u}^\rho\, \bar{h}^\sigma{}_{\nu)} T_{\rho\sigma} \right) + \left(\bar{h}^\rho{}_\mu \bar{h}^\sigma{}_\nu T_{\rho\sigma} \right) . \tag{4}$$

C. Covariant Derivative Decomposition

For general spatial tensors (namely, for spatial tensors defined within $\overline{\mathcal{M}}$ or defined within $\mathcal{M}$ then mapped onto $\overline{\mathcal{M}}$), the relation between the two derivatives reads:

$$\bar{\nabla}_\rho T_{\mu_1\ldots\mu_p} = -\bar{u}_\rho \dot{T}_{\mu_1\ldots\mu_p} + \bar{D}_\rho T_{\mu_1\ldots\mu_p} + \sum_{i=1}^{p} \bar{u}_{\mu_i}\, \bar{K}^\sigma{}_\rho\, T_{\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p} \,. \tag{5}$$

The relation between $\mathcal{L}_{\bar{\boldsymbol{u}}}$ and $\bar{u}^\rho \bar{\nabla}_\rho$ is written[1]:

$$\mathcal{L}_{\bar{\boldsymbol{u}}} T_{\mu_1\ldots\mu_p} = \dot{T}_{\mu_1\ldots\mu_p} + \sum_{i=1}^{p} \bar{K}^\sigma{}_{\mu_i}\, T_{\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p} \,, \tag{6}$$

$$\bar{\nabla}_\rho T_{\mu_1\ldots\mu_p} = -\bar{u}_\rho\, \mathcal{L}_{\bar{\boldsymbol{u}}} T_{\mu_1\ldots\mu_p} \;+\; \bar{D}_\rho T_{\mu_1\ldots\mu_p} + 2\sum_{i=1}^{p} \bar{u}_{(\mu_i}\, \bar{K}^\sigma{}_{\rho)}\, T_{\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p} \,. \tag{7}$$

The commutation rule between the derivatives $\mathcal{L}_{\bar{\boldsymbol{u}}}$ and $\bar{\boldsymbol{D}}$ for general spatial tensors, it is given by

$$\mathcal{L}_{\bar{\boldsymbol{u}}} \left(\bar{D}_\rho T_{\mu_1\ldots\mu_p} \right) = \bar{D}_\rho \left(\mathcal{L}_{\bar{\boldsymbol{u}}} T_{\mu_1\ldots\mu_p} \right) + \sum_{i=1}^{p} \left(\bar{h}^{\sigma\zeta} \bar{D}_\zeta \bar{K}_{\rho\mu_i} - \bar{D}_\rho \bar{K}_{\mu_i}{}^\sigma - \bar{D}_{\mu_i} \bar{K}_\rho{}^\sigma \right) T_{\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p} \,, \tag{8}$$

[1] Note that for a spatial tensor $\boldsymbol{T}$, the quantity $\mathcal{L}_{\bar{\boldsymbol{u}}} T_{\mu_1\ldots\mu_p}$ is also spatial.

II. BASIC TOOLS FOR $2+1$ DECOMPOSITION

A. My Notations

For a general spatial tensor $T^{a\cdots}{}_{\cdots b}$, we can isolate the part lying in the screen space as

$$T_{\perp}{}^{a\cdots}{}_{\cdots b} = N^{a}{}_{c}\cdots N^{d}{}_{b}T^{c\cdots}{}_{\cdots d}\,, \tag{9}$$

and the part parallel to n^a as

$$T_{\parallel} = n_a\cdots n^b\,T^{a\cdots}{}_{\cdots b}\,. \tag{10}$$

We define the covariant angular derivative $\nabla_{\perp a}$ on the screen space and the derivative $\nabla_{\parallel}$ along the direction of observation:

$$\nabla_{\perp a}T^{b\cdots}{}_{\cdots c} = N^{d}{}_{a}N^{b}{}_{e}\cdots N^{f}{}_{c}\nabla_{d}T^{e\cdots}{}_{\cdots f}\,, \tag{11}$$

$$\nabla_{\parallel}T^{a\cdots}{}_{\cdots b} = n^{c}\nabla_{c}T^{a\cdots}{}_{\cdots b}\,. \tag{12}$$

B. Decomposition of background Spacetime

In section I, we have shown how a 4-d background spacetime and tensor fields that live on them may be decomposed using $3+1$ technique, now we are going to split further the spatial part (i.e the hyper-surface) of the 4-d spacetime using 1+2 decomposition technique.

First we need to define a space-like 3-vector, $\bar{n}^{\mu}$, which is normalized as $\bar{n}^{\mu}\bar{n}_{\mu}=1$ and an induced metric $N^{\mu,\nu}$ which lives on the 2-d sheet or the screen space.

$$\bar{h}^{\alpha\beta}\bar{n}_{\alpha}\bar{n}_{\beta}=1\,,\qquad \bar{N}_{\mu\nu}=\bar{h}_{\mu\nu}-\bar{n}_{\mu}\bar{n}_{\nu}=g_{\mu\nu}-\bar{u}_{\mu}\bar{u}_{\nu}+\bar{n}_{\mu}\bar{n}_{\nu}\,. \tag{13}$$

Other properties of the projection tensor include

$$\bar{u}^{\alpha}\bar{N}_{\alpha\beta}=\bar{n}^{\alpha}\bar{N}_{\alpha\beta}=0,\;\; \bar{k}^{\alpha}\bar{N}_{\alpha\beta}=0,\;\;\; \bar{h}^{\alpha\beta}\bar{N}_{\alpha\beta}=g^{\alpha\beta}\bar{N}_{\alpha\beta}=2,\;\;\; \bar{N}_{\alpha\beta}\bar{N}^{\alpha\beta}=2,\;\;\; \bar{N}_{\alpha\gamma}\bar{N}^{\alpha\beta}=\bar{N}_{\beta\gamma}. \tag{14}$$

where $k^{\alpha}=\bar{E}\left(\bar{u}^{\alpha}\pm\bar{n}^{\alpha}\right)$ is the photon tangent vector. I have introduced it here because how to calculate it is central to what we plan to do.

Spatial covariant derivative of $\bar{n}^{\alpha}$ may be irreducibly decomposed as

$$\bar{D}_{\mu}\bar{n}_{\nu}=\bar{n}_{\mu}\bar{\beta}_{\nu}+\bar{K}_{\perp\mu\nu} \tag{15}$$

where we have introduced an Extrinsic curvature (i.e $\bar{K}_{\perp\mu\nu}$) of on the screen space and it is decomposed as follows $\bar{K}_{\perp\mu\nu}=\frac{1}{2}\bar{K}_{\perp}N_{\mu\nu}+\xi\bar{\varepsilon}_{\mu\nu}+\zeta_{\mu\nu}$. We have introduced the following notations

$$\bar{\beta}_{\alpha}\equiv\bar{n}^{\gamma}\bar{D}_{\gamma}\bar{n}_{\alpha}=\nabla_{\parallel}n_{a},\qquad \text{Radial Acceleration} \tag{16}$$

$$\bar{K}_{\perp}\equiv\bar{D}_{\perp\alpha}\bar{n}^{\alpha},\qquad \text{Trace of } \bar{K}^{\mu}_{\perp\mu} \tag{17}$$

$$\bar{\xi}\equiv\frac{1}{2}\bar{\varepsilon}^{\alpha\beta}\bar{D}_{\perp\alpha}\bar{n}_{\beta},\qquad \text{Twist, the anti-symmetry part} \tag{18}$$

$$\bar{\zeta}_{\alpha\beta}\equiv\bar{D}_{\perp\langle\alpha}\bar{n}_{\beta\rangle}.\qquad \text{Shear, the symmetry part} \tag{19}$$

We also define the alternating Levi-Civita 2-tensor

$$\bar{\varepsilon}_{\alpha\beta}\equiv\bar{\varepsilon}_{\alpha\beta\gamma}\bar{n}^{\gamma}=\bar{u}^{\lambda}\bar{\eta}_{\lambda\alpha\beta\gamma}\bar{n}^{\gamma}, \tag{20}$$

so that $\bar{\varepsilon}_{\alpha\beta}\bar{n}^{\beta}=0=\bar{\varepsilon}_{(\alpha\beta}$, and

$$\bar{\varepsilon}_{\alpha\beta\gamma}=\bar{n}_{\alpha}\bar{\varepsilon}_{\beta\gamma}-\bar{n}_{\beta}\varepsilon_{\alpha\gamma}+\bar{n}_{\gamma}\varepsilon_{\alpha\beta}, \tag{21}$$

$$\bar{\varepsilon}_{\alpha\beta}\bar{\varepsilon}^{\gamma\lambda}=\bar{N}_{\alpha}{}^{\gamma}\bar{N}_{\beta}{}^{\lambda}-\bar{N}_{\alpha}{}^{\lambda}\bar{N}_{\beta}{}^{\gamma}, \tag{22}$$

$$\bar{\varepsilon}_{\alpha}{}^{\gamma}\bar{\varepsilon}_{\beta\gamma}=\bar{N}_{\alpha\beta},\quad \bar{\varepsilon}^{\alpha\beta}\bar{\varepsilon}_{\alpha\beta}=2. \tag{23}$$

C. Decomposition of Tensor fields

The spatial part of the tensor field that have been split using $3+1$ technique in section I may now be further split into the radial part and the angular part. For example for a 3-vector V^a can now be irreducibly split into a scalar, $V_{\parallel}$, which is the part of the vector parallel to $\bar{n}^{\alpha}$, and a vector, $V_{\perp}^{\alpha}$, lying in the sheet orthogonal to $\bar{n}^{\alpha}$;

$$V^{\alpha} = V_{\parallel}\bar{n}^{\alpha} + V_{\perp}^{\alpha}, \quad \text{where} \quad V_{\parallel} \equiv V_{\alpha}\bar{n}^{\alpha}, \quad \text{and} \quad V_{\perp}^{\alpha} \equiv \bar{N}^{\alpha\beta}V_{\beta}, \tag{24}$$

A rank two tensor field may also be decomposed into radial and screen space components according to

$$T_{\alpha\beta} = T_{\langle\alpha\beta\rangle} = T_{\parallel}\left(\bar{n}_{\alpha}\bar{n}_{\beta} - \frac{1}{2}N_{\alpha\beta}\right) + 2T_{\perp_{\parallel}(\alpha}\bar{n}_{\beta)} + T_{\perp\langle\alpha\beta\rangle}, \tag{25}$$

where

$$\begin{aligned} T_{\parallel} &\equiv \bar{n}^{\alpha}\bar{n}^{\beta}\psi_{\alpha\beta} = -\bar{N}^{\alpha\beta}T_{\alpha\beta}, \\ T_{\perp\alpha} &\equiv \bar{N}_{\alpha}{}^{\alpha}\bar{n}^{\gamma}T_{\beta\gamma} \\ T_{\perp\alpha\beta} &\equiv T_{\langle\alpha\beta\rangle} \equiv \left(N_{(\alpha}{}^{\gamma}\bar{N}_{\beta)}{}^{\lambda} - \frac{1}{2}\bar{N}_{\alpha\beta}\bar{N}^{\gamma\lambda}\right)T_{\gamma\lambda}. \end{aligned} \tag{26}$$

D. Decomposition of Covariant Derivatives

The decomposition of spatial covariant derivative follows similarly the covariant decomposition rules for the 4-d covariant derivatives, I will show starting a scalar:

$$\bar{\mathrm{D}}_{\alpha}\Psi = \nabla_{\parallel}\Psi\bar{n}_{a} + \nabla_{\perp a}\Psi\,, \tag{27}$$

$$\bar{\mathrm{D}}_{a}\Psi_{b} = -\bar{n}_{a}\bar{n}_{b}\Psi_{c}\nabla_{\parallel}\bar{n}^{c} + \bar{n}_{a}\hat{\Psi}_{\bar{b}} - \bar{n}_{b}\left[\frac{1}{2}\phi\Psi_{a} + \left[\xi\varepsilon_{ac} + \zeta_{ac}\right]\Psi^{c}\right] + \nabla_{\perp a}\Psi_{b}\,, \tag{28}$$

$$\bar{\mathrm{D}}_{a}\Psi_{bc} = -2\bar{n}_{a}\bar{n}_{(b}\Psi_{c)d}\nabla_{\parallel}n^{d} + \bar{n}_{a}\hat{\Psi}_{bc} - 2\bar{n}_{(b}\left[\frac{1}{2}\phi\Psi_{c)a} + \Psi_{c)}{}^{d}\left[\xi\varepsilon_{ad} + \zeta_{ad}\right]\right] + \nabla_{\perp a}\Psi_{bc}. \tag{29}$$

The first term on the RHS is zero because $\nabla_{\parallel}\bar{n}^{c} = 0$ for the type of homogeneous cosmology we consider. So in general full decomposition of a spatial covariant derivative of tensor will look like this:

$$\bar{\mathrm{D}}_{\rho}T_{\mu_1\ldots\mu_p} = \bar{n}_{\rho}\nabla_{\parallel}T_{\perp\mu_1\ldots\mu_p} + \sum_{i=1}^{p}\bar{n}_{\mu_i}\,\bar{K}_{\perp}^{\sigma}{}_{\rho}\,T_{\perp\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p} + \bar{\nabla}_{\perp\rho}T_{\perp\mu_1\ldots\mu_p}\,. \tag{30}$$

If we use Lie derivative for the radial derivative, we will have to do the same thing we did for time

$$\mathcal{L}_{\bar{\boldsymbol{n}}}T_{\perp\mu_1\ldots\mu_p} = \nabla_{\parallel}T_{\mu_1\ldots\mu_p} + \sum_{i=1}^{p}\bar{K}_{\perp}^{\sigma}{}_{\mu_i}\,T_{\perp\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p}\,, \tag{31}$$

Finally the full decomposition will look like this.

$$\bar{\bar{\mathrm{D}}}_{\rho}T_{\perp\mu_1\ldots\mu_p} = -\bar{n}_{\rho}\,\mathcal{L}_{\bar{\boldsymbol{n}}}T_{\mu_1\ldots\mu_p} + \bar{\nabla}_{\perp\rho}T_{\mu_1\ldots\mu_p} + 2\sum_{i=1}^{p}\bar{n}_{(\mu_i}\,\bar{K}^{\sigma}{}_{\rho)}\,T_{\perp\mu_1\ldots\mu_{i-1}\sigma\mu_{i+1}\ldots\mu_p}\,. \tag{32}$$

III. BY-PASSING XPAND TO DO THE DECOMPOSITION

A. Perturbations of the metric

The SVT decomposition of the metric perturbations yields the general expressions:

$$\bar{u}^{\rho}\bar{u}^{\sigma}\,{}^{\{n\}}h_{\rho\sigma} = -2\,{}^{\{n\}}\phi\,, \tag{33}$$

$$\bar{u}^{\rho}\bar{h}^{\sigma}{}_{\nu}\,{}^{\{n\}}h_{\rho\sigma} = -\bar{D}_{\nu}\,{}^{\{n\}}B - {}^{\{n\}}B_{\nu}\,, \tag{34}$$

$$\bar{h}^{\rho}{}_{\mu}\bar{h}^{\sigma}{}_{\nu}\,{}^{\{n\}}h_{\rho\sigma} = 2\left(\bar{D}_{\mu}\bar{D}_{\nu}\,{}^{\{n\}}E + \bar{D}_{(\mu}\,{}^{\{n\}}E_{\nu)} + {}^{\{n\}}E_{\mu\nu} - {}^{\{n\}}\psi\,\bar{h}_{\mu\nu}\right)\,. \tag{35}$$

Each of the spatial derivative above would be replaced with the following:

$$
\begin{aligned}
\bar{\mathrm{D}}_\alpha E &= \bar{n}_\alpha \nabla_\parallel E + \nabla_{\perp\alpha} E \,, &(36)\\
\bar{\mathrm{D}}_\alpha E_\beta &= \bar{\mathrm{D}}_\alpha \left[E_\parallel \bar{n}_\beta + E_{\perp\beta} \right] \,, &(37)\\
&= \bar{n}_\alpha \bar{n}_\beta \nabla_\parallel E_\parallel + \bar{n}_\beta \nabla_{\perp\alpha} E_\parallel + \bar{n}_\alpha \nabla_\parallel E_{\perp\beta} + E_\parallel \bar{K}_{\perp\alpha\beta} + \bar{n}_\beta \, \bar{K}_{\perp\alpha}{}^\gamma E_{\perp\gamma} + \nabla_{\perp\alpha} E_{\perp\beta} \,, &(38)\\
\bar{\mathrm{D}}_\beta \bar{\mathrm{D}}_\alpha \Psi &= \bar{\mathrm{D}}_\beta \left[\nabla_\parallel \Psi \bar{n}_\alpha + \nabla_{\perp\alpha} \Psi \right] &(39)\\
&= \bar{n}_\alpha \bar{n}_\beta \nabla_\parallel \nabla_\parallel \Phi + \bar{n}_\alpha \nabla_{\perp\beta} \nabla_\parallel \Psi + \bar{n}_\beta \nabla_\parallel \nabla_{\perp\alpha} \Psi + K_{\perp\beta\alpha} \nabla_\parallel \Psi + \bar{n}_\alpha K_{\perp\beta}{}^\gamma \nabla_{\perp\gamma} \psi + \nabla_{\perp\beta} \nabla_{\perp\alpha} \Psi \,. &(40)
\end{aligned}
$$