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Course: MATH712 - Numerical Methods II
Professor: Dr. P.G. Petropoulos

Homework #1
ID: YOUR ID # HERE
Term: Fall 2020
Due Date: 18 September, 2020

Problem 1
Use the Fourier Transform to solve the Initial Value Problem ($\delta(x)$ is the delta function):

$U_t + U_x + U_{xxx} = 0;\ -\infty < x < \infty,\ t > 0$
$U(x,0) = \delta(x);\ -\infty < x < \infty.$

Manipulate the solution to write it as $U(x,t) = \frac{1}{(3t)^{1/3}} Ai(\frac{x-t}{(3t)^{1/3}})$, where $Ai(z)$ is the Airy function of argument z. You will need to look up the integral representation of the Airy function. SHOW ALL WORK.
Then, do the following:

1. Produce graphs of U(x,t) for $t = 10, 20, 30$, with $-60 \leq x \leq 60$.
2. Explain how these graphs verify the speed of propagation ($= 1$ here), and the decay rate of $max\{U(\cdot,t)\}$ (note: $Ai(0)$ is finite). What can one say about the frequency of the oscillation at fixed t and $x \to -\infty$?
3. Find the dispersion relation, $\omega = W(k)$, and the phase velocity, $v_p(k)$, where ω is the frequency and k is the wave number, for this linear KdV partial differential equation.

Solution:

First we find the analytic solution to the linear Korteweg-de Vries initial value problem using the Fourier Transform pair. If we define the Fourier transform of U to be:

$$\hat{U}(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} U(x,t)dx, \tag{1}$$

and take the transform of the PDE of interest, then we obtain

$$\begin{cases} \hat{U}_t(\omega,t) + i\omega\hat{U} + (i\omega)^3\hat{U}(\omega,t) = 0, & (\omega,t) \in \mathbb{R}\times(0,\infty) \\ \hat{U}(\omega,0) = \hat{\delta}\omega = 1, & \omega \in \mathbb{R} \end{cases}$$

Note that this IVP corresponds to the linear kdV. Solving $\hat{U}_t - i\omega^3\hat{U} + i\omega\hat{U} = 0$ yields

$$\hat{U}(\omega,t) = \hat{\delta}(\omega)e^{i(\omega^3-\omega)t} = e^{i(\omega^3-\omega)t}. \tag{2}$$

Now we want to get from $\hat{U}$ back to our original solution U. To do this, we make use of the fact that

$$U(x,t) = \mathscr{F}^{-1}[\hat{U}(k,t)] = \mathscr{F}^{-1}[\hat{\delta}(\omega)e^{i(\omega^3-\omega)t}] = \delta(x) \star \mathscr{F}^{-1}[e^{i(\omega^3-\omega)t}] = \mathscr{F}^{-1}[e^{i(\omega^3-\omega)t}]. \tag{3}$$

Remark: $\mathscr{F}[Ai(x)] = e^{ik^3/3}$, where $Ai(x)$ is Airy function. Then

$$U(x,t) = \mathscr{F}^{-1}[e^{i(\omega^3-\omega)t}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(\omega^3-\omega)t} e^{i\omega x} d\omega, \tag{4}$$

We will make a change of variables $k = \omega \cdot \frac{1}{(3t)^{1/3}}$. Then we get

$$U(x,t) = \mathscr{F}^{-1}[e^{i(\omega^3-\omega)t}] = \frac{1}{2\pi} \frac{1}{(3t)^{1/3}} \int_{-\infty}^{\infty} e^{ik(\frac{x-t}{(3t)^{1/3}})} \cdot e^{ik^3/3} dk = \frac{1}{\sqrt[3]{3t}} \mathrm{Ai}\left(\frac{x-t}{\sqrt[3]{3t}}\right). \tag{5}$$

So final solution for the linear kdV equation is $U(x,t) = \frac{1}{\sqrt[3]{3t}} \mathrm{Ai}\left(\frac{x-t}{\sqrt[3]{3t}}\right)$.
Remark:

$$Ai(x) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\frac{k^3}{3}+xk)} dk.$$

Now it remains to produce graphs and find the dispersion relation and plane velocity.
Want to find the dispersion relation for our linear kDV equation. To do this, we substitute the general plane wave solution:

$$U(x,t) = e^{i(kx+\omega t)}, k \in \mathbb{R} \tag{6}$$

into our PDE. Doing this yields:

$$U(x,t)(-i\omega + ik + (ik)^3) = 0 \tag{7}$$

which yields $-\omega i = ik + -ik^3$. Dividing both sides by $-i$, we obtain our dispersion relation:

$$\omega(k) = k^3 - k. \tag{8}$$

and our phase velocity is:

$$v_p(k) = \frac{\omega(k)}{k} = k^2 - 1. \tag{9}$$

So we see dispersive phenomena.

From the graphs on the next page, we make the following observations:
(i) The general trend is the quick decay of the solution as $x \to \infty$ for every case and a slightly damped oscillation as $x \to -\infty$. In other words, as $x \to -\infty$, we see an oscillation of the solution that decays but very slowly.
(ii) reduction in amplitude as x gets smaller and smaller. Wave is moving to the right when we compare peak of the wave front. Can confirm speed of propagation is 1 from Figure 4.

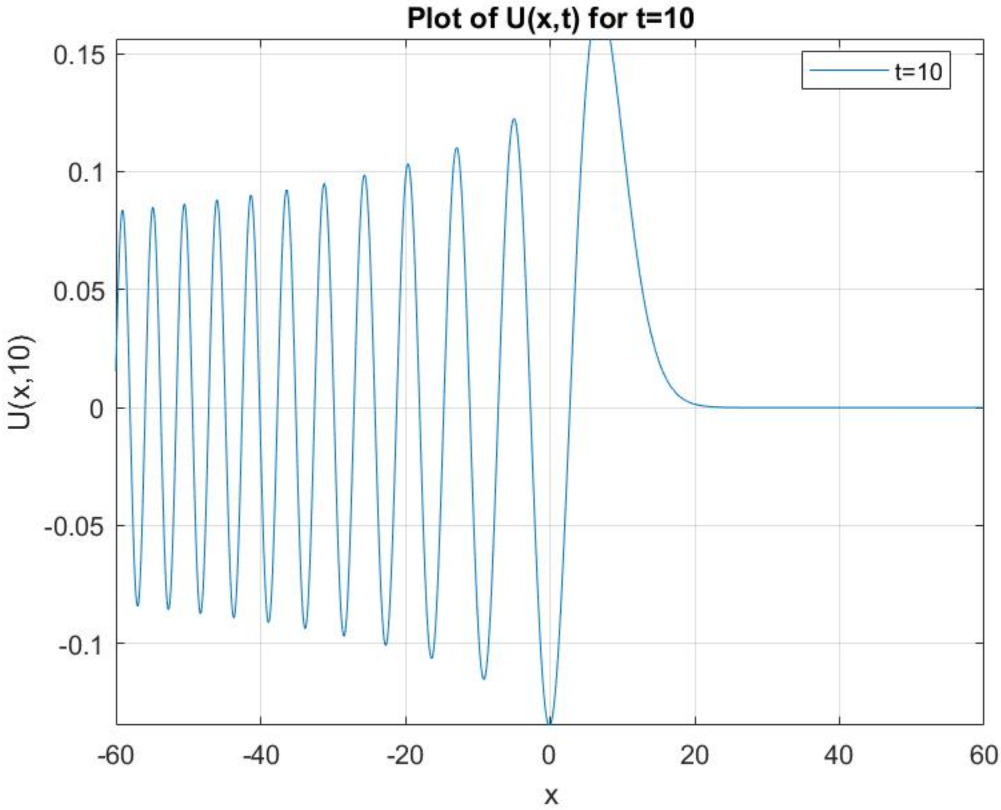

Figure 1: $U(x,t)$ behavior for $t = 10$

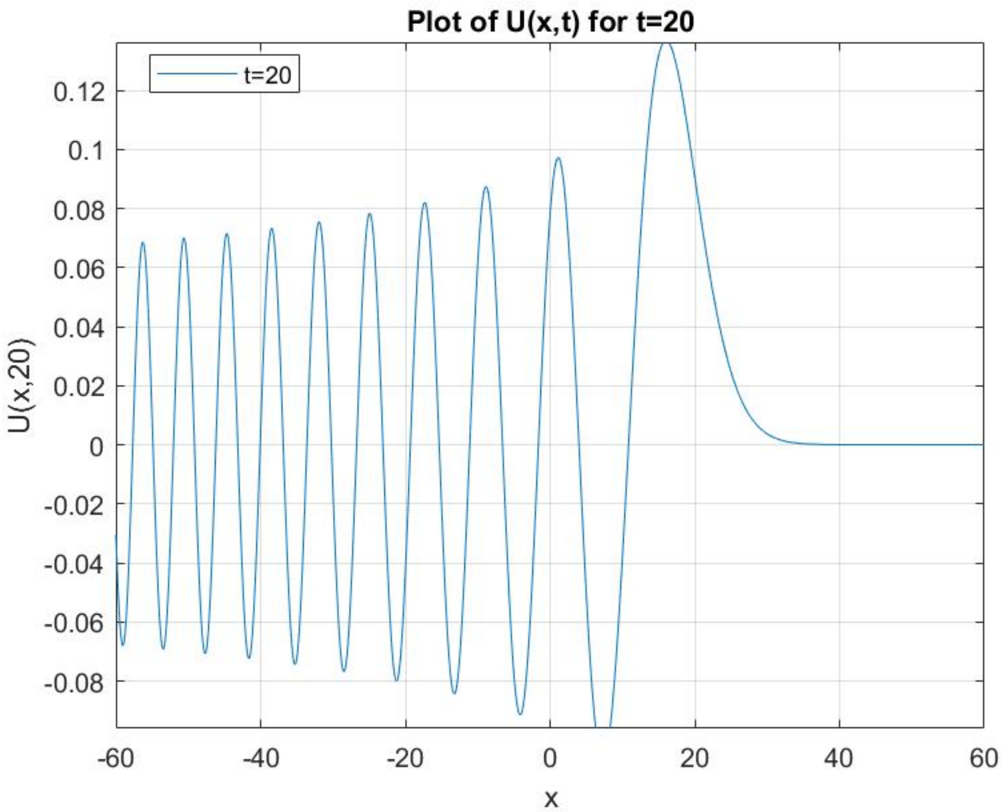

Figure 2: $U(x,t)$ behavior for $t = 20$

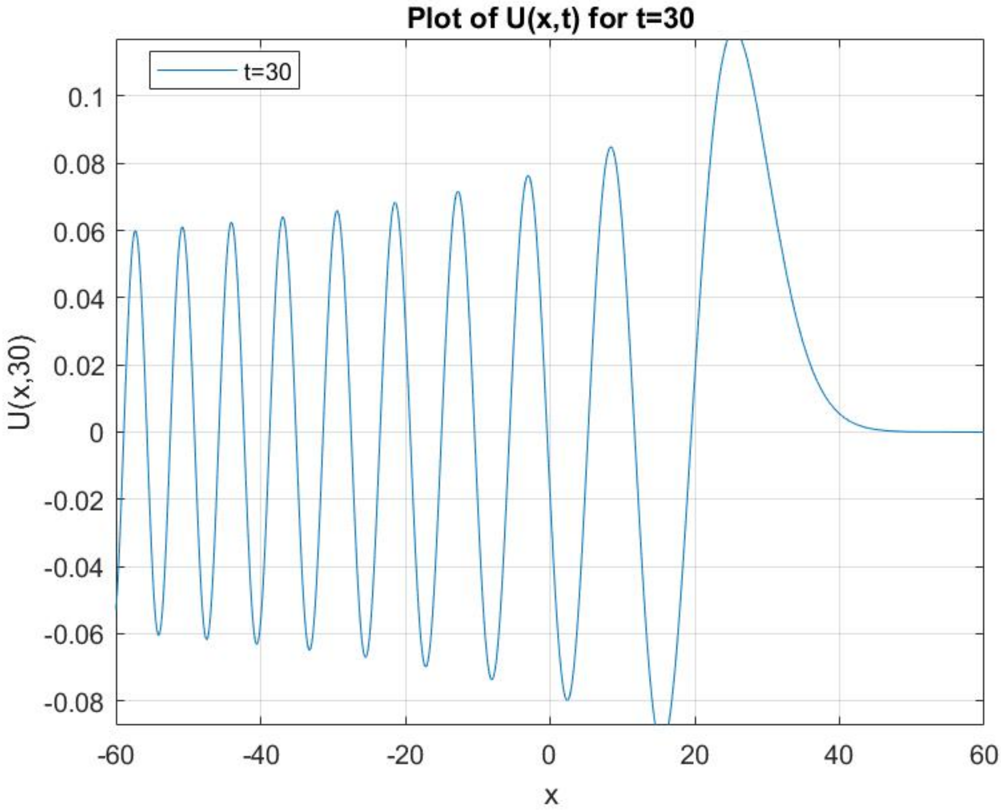

Figure 3: $U(x, t)$ behavior for $t = 30$

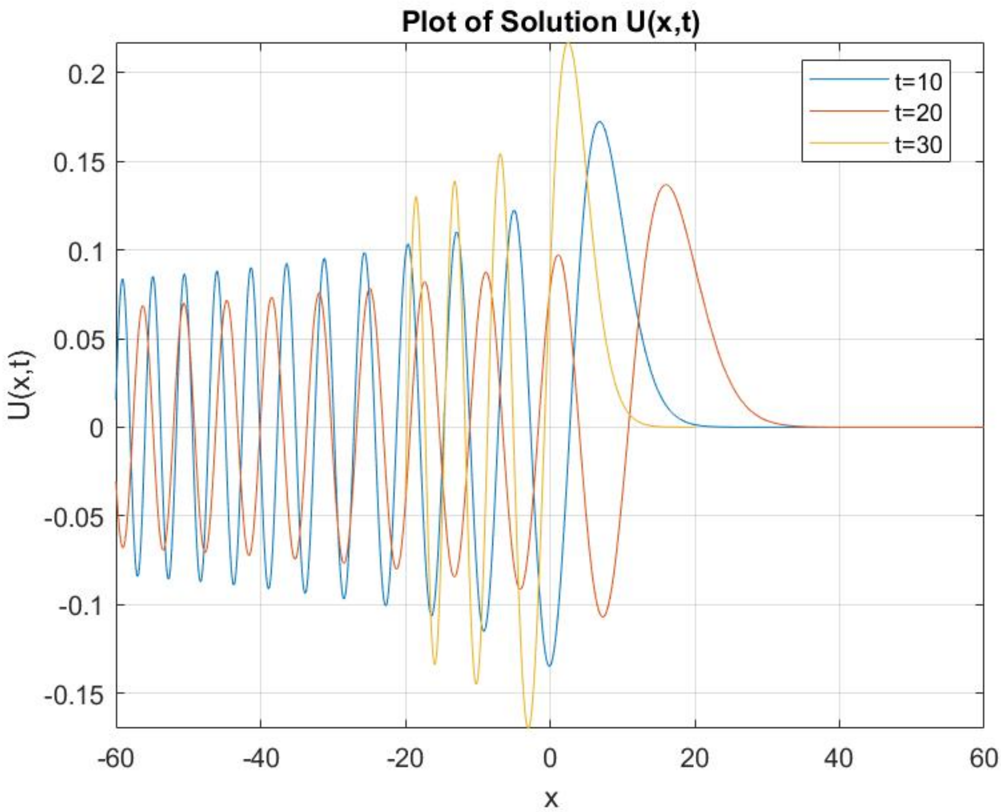

Figure 4: $U(x, t)$ behavior, all three plots

Problem 2
First, derive the centered $\mathcal{O}(h^4)$ accurate approximation to $u_x(x)$ that I gave in class, at a grid point $x_j = jh$, on an equispaced grid with step h, using Taylor series expansions (centered at x_j) at the grid points $x_{j\pm1},\ x_{j\pm2}$.

Next, consider $u(x) = \sin kx$ on a periodic domain of length $L = 2\pi$. Use the code *hw2a.f*, included in the directory with the homework TeX file, to verify the accuracy of the second- and fourth-order accurate approximations to u_x by producing one graph of the error for $k = 1$ and $k = 5$ (the graph should have four straight lines on it), and determine the slope of each line.

Then, comment out the *dimension* line, uncomment the line immediately below it, and redo the graph.

With these two graphs, explain

1. why the two graphs are different (hint: precision of number representation in a numerical code).
2. why at least for $nmax = 1500$ (smallest $h = 2\pi/1499$) the fourth-order method breaks down first (change nmax to a larger value and you should see the second-order method to break down too).

It seems that the fourth-order method breaks down at a fixed value of the product kh (the same should happen with the second-order method at a larger nmax): Can you think of a reason ?

Solution:
Idea is to obtain a 5-point finite difference approximation for the first derivative using the points $u_{j-2}, u_{j-1}, u_j, u_{j+1}, u_{j+2}$. First consider Taylor series expansions about the point $u_j = u(x_j)$ centered at x_j:

$$u_{j-2} = u_j - (2h)u_{x,j} + \frac{1}{2}(2h)^2 u_{xx,j} - \frac{1}{6}(2h)^3 u_{xxx,j} + \frac{1}{24}(2h)^4 u_{xxxx,j} + \mathcal{O}(h^5). \tag{10}$$

$$u_{j-1} = u_j - hu_{x,j} + \frac{1}{2}h^2 u_{xx,j} - \frac{1}{6}h^3 u_{xxx,j} + \frac{1}{24}h^4 u_{xxxx,j} + \mathcal{O}(h^5). \tag{11}$$

$$u_j = u_j \tag{12}$$

$$u_{j+1} = u_j + hu_{x,j} + \frac{1}{2}h^2 u_{xx,j} + \frac{1}{6}h^3 u_{xxx,j} + \frac{1}{24}h^4 u_{xxxx,j} + \mathcal{O}(h^5). \tag{13}$$

$$u_{j+2} = u_j + (2h)u_{x,j} + \frac{1}{2}(2h)^2 u_{xx,j} + \frac{1}{6}(2h)^3 u_{xxx,j} + \frac{1}{24}(2h)^4 u_{xxxx,j} + \mathcal{O}(h^5). \tag{14}$$

To find the centered difference approx. for first-order partial derivative u_x, we take -(14)+8(13)-8(11)+(10). ((10), (11), (13), (14) are u_{j-2}, etc. as labeled earlier). Then we obtain

$$-u_{j+2}+8u_{j+1}-8u_{j-1}+u_{j-2}=0u_j+12hu_j'+0h^2u_j''+0h^3u_j'''+0h^4u_j^{(4)}-\frac{2}{5}h^5u_j^{(5)}. \tag{15}$$

So the fourth order centered difference approx. for u_x is:

$$u_{x,j}\approx\frac{-u_{j+2}+8u_{j+1}-8u_{j-1}+u_{j-2}}{12h}. \tag{16}$$

Similarly, for fourth order centered difference approx. for u_{xx}, we can take the coefficients of each term and construct a matrix. Namely, observe that the finite difference approximation has the form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} & 2 \\ -\frac{8}{6} & -\frac{1}{6} & 0 & \frac{1}{6} & \frac{8}{6} \\ \frac{2}{3} & \frac{1}{24} & 0 & \frac{1}{24} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} \alpha_{j-2} \\ \alpha_{j-1} \\ \alpha_j \\ \alpha_{j+1} \\ \alpha_{j+2} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

where each $\alpha_{j-2},\alpha_{j-1},\alpha_j,\alpha_{j+1},\alpha_{j+2}$ are corresponding coefficients. Solving this system yields:

$$\begin{pmatrix} \alpha_{j-2} \\ \alpha_{j-1} \\ \alpha_j \\ \alpha_{j+1} \\ \alpha_{j+2} \end{pmatrix}=\begin{pmatrix} -\frac{1}{12} \\ \frac{4}{3} \\ -\frac{5}{2} \\ \frac{4}{3} \\ -\frac{1}{12} \end{pmatrix}.$$

Thus the fourth-order accurate centered difference formula using the points $(u_{j-2},u_{j-1},u_j,u_{j+1},u_{j+2})$ is:

$$u_{xx,j}\approx\frac{-u_{j-2}+16u_{j-1}-30u_j+16u_{j+1}-u_{j+2}}{12h}. \tag{17}$$

for second-order partial derivative u_{xx}.
Slope for each line is equal to the order of accuracy in bottom graphs.

1. Why the two graphs are different?
Mainly round-off error: We get a loss of significant figures(precision) with the subtraction of nearby numbers. 2. Why at least for $nmax = 1500$, the 4th-order method breaks down first?
4th-order method requires fewer grid points than a 2nd order method to attain a predetermined accuracy. Because of high rate of convergence of higher order methods, 4th order method error decreases with a faster slope than the 2nd order method.

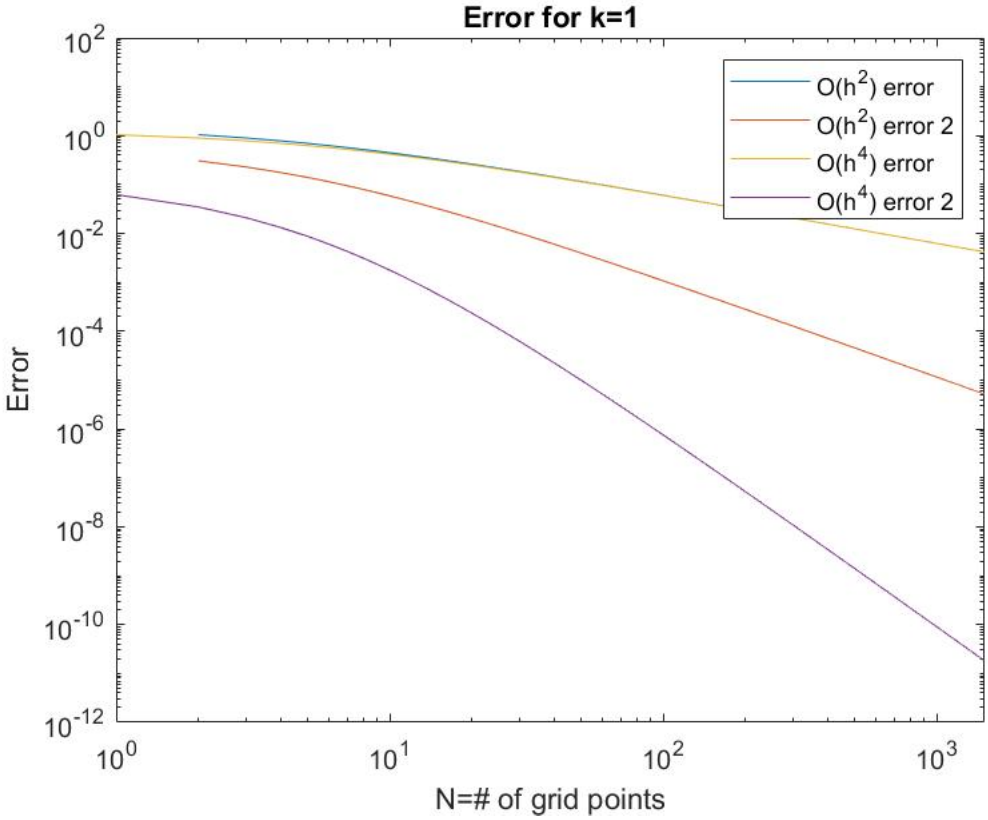

Figure 5: LogLog Plot of Errors (Centered Difference Approximation of u_x for $k = 1$: $N = \frac{2\pi}{h}$ vs Error

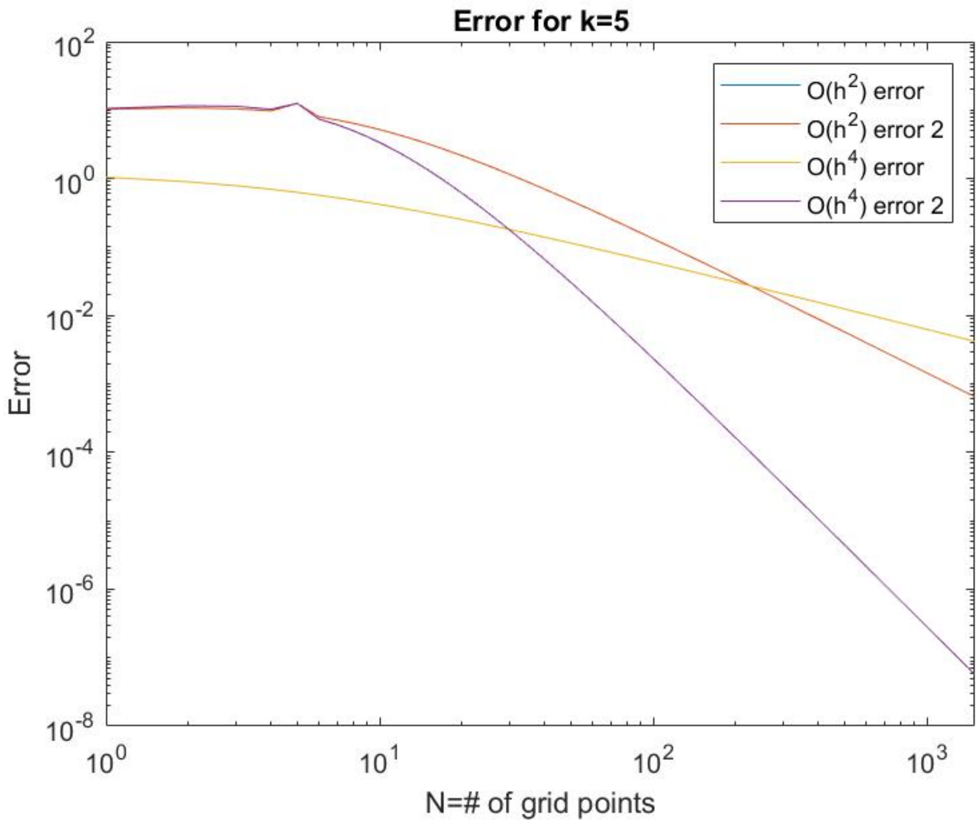

Figure 6: LogLog Plot of Errors (Centered Difference Approximation of u_x) for $k = 5$: $N = \frac{2\pi}{h}$ vs Error