QUANTUM MECHANICS LITERALLY: A FLUID OF TANGENT UNIVERSES

HOW QUANTIZATION CAN BE SEEN AS LETTING MANY CLASSICAL UNIVERSES INTERACT TANGENTIALLY

BY

Xabier Oianguren Asua

FOR
other Students who want to
"Understand", not just "Predict"

Universitat Autònoma de Barcelona

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A Continuum of Tangent Universes

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Contents

Preface - What is (Not) this?

First of all, let me explicitly state that this document is an informal guide (aimed for students around a third year Bachelor's in physics or some related field) to enlighten how one can indeed understand quantum mechanics without the need for qualitative conceptual steps nor "shut up and calculate"-s. It does not pretend to be a rigorous mathematical text, nor a thesis and even less a text book. I would call it rather a pedagogical-technical outreach essay for students with a confident background in first and second year calculus and algebra. I could not avoid some tedious mathematics at some points, and the appendices pretend to give enough qualitative insight to the student for a quasi-complete understanding of the text. However, the very technical parts can be diagonally read in a first read.

But why am I writing this? Well, the present is a summary of the understanding of quantum mechanics I developed during the Bachelor's studies, thanks to stimulating lectures, readings, conversations and coding. In front of the mainstream "incomprehensibility of quantum mechanics", with all its paradoxes and axioms that have no connection whatsoever with reality (that leave the mathematical structures with cryptic meaning), I found that this incomprehensibility is nothing more than an "opinion", a particular way to build the theory, but not at all the only narrative. When sharing the ideas that led me there with fellow students, I also found relief in their eyes, "finally a pedagogical narrative to build an intuition from!", said one of them. It was then that I felt the urge to leave these ideas written down, if only to help other students that were in our same situation of discomfort. Students who came to University looking for an understanding of the world, but were only given recipes and calculation insights, useful for prediction sure, but philosophically hazy.

Being nothing more than the amalgam of different sources, the present book introduces nothing new to physics, it is just a medley of other narratives already thought by great (but apparently ignored) thinkers. And yet, I have seen nowhere a complete treatise of the tangent Universe narrative, so I think the work might still be creative enough to deserve its own place in our drives.

The orthodox way to write about an alternative formulation of quantum mechanics is to focus it from a historical point of view, comparing it with the mainstream and other alternatives. Instead, here, we will just build the whole theory from zero, just assuming the classical theories that are well rooted in our intuitions, and only afterwards we will see how other narratives have inspired it.

Just take this reading as a philosophical amusement, where we will build quantum mechanics from zero, you and me, leaving all assumptions explicitly underlined.

With the disclaimer clear, dear reader, there is a single golden rule you should follow: whenever you open this treatise, forget everything you priorly knew about quantum mechanics.
Now, lets create a theory!

Guideline

We will start by (literally) questioning everything to arrive at a common philosophical point in our voyage towards a model of the Universe. From there, in part A, we will start the construction of our theory from the postulates of classical mechanics till the interaction between alternative Universes. Axiom by axiom, we will naturally arrive to the apparent randomness of quantum mechanics and end up deriving the Schrödinger Equation. In part B, we will see how the so-famous "collapse" of the wavefunction is simply an effective consequence of the interaction with a measuring apparatus, and we will see how measurement operators will naturally emerge. In part C, we will develop how the density matrix formulation fits in all of this, its usefulness and general quantum operations, all understood from our Tangent Universe approach. In part D, we will explain how general open quantum systems can naturally be described in our approach and we will derive the most general equations employed in non-relativistic quantum mechanics. Finally, in part E, we will understand what photons and other quasi-particles really are and even arrive to relativistic quantum mechanics with our Tangent Universe approach! Might we expand our knowledge from there? Well, you might be the one discovering it!

Philosophical Prolegomena

Lets Start from the Very Beginning

Focus your attention for a moment in that thing that is behind the eyes that are reading these lines and in front of the back of your head. Perhaps a bit upper than the eyes. There, in the center of your "world" or reality. We call it **consciousness**. It is you, the light that makes you. That thing that when a person expels the last breath, is no longer there [1]. That thing that when you look at the eyes of another person, you are (or at least expect to be) looking at. The relevance of that thing, the self-aware consciousness, is the following. Imagine you discover that the reality you considered as true and contingent and objectively material, is actually fake. For example, that your brain is actually in a liquid tank in some lab and it has connected to its visual-, auditive-, self-body internal perceptive- etc. cortices or brain regions, some electrodes that stimulate the exact same neural pulses that would be stimulated by a true perception in the visual neurons of a human eye that could be reading this, tactile neurons that could be wearing those clothes or internal nerves that would feel the body extension you feel. Well, there is absolutely no way to prove that this is not indeed the case. No way. Thus, there is no way either to prove that behind the eyes of the rest of the people there is a consciousness like yours. There is actually no way either to prove that your brain was not spontaneously arranged in the vacuum of space due to statistical fluctuations with these exact memories and current perceptions. Not either that it is not the product of a super-computer simulating each and every electron and fundamental particle in your brain to the quantum level, with these cortices being activated according to the algorithm of the machine, to make you feel this reality. There is no way either, in this line, to know that this consciousness should have any sort of ontologically material cause or foundation.

So yeah, everything could simply be "fake". Well, everything, except for one thing: that *you*, that light of self-awareness that is behind those eyes, that, which we call *you*, exists. The fact that *you* exist is necessarily true for you. It needs no prove, because from the moment in which you are able to think, its existence is a necessity. Pause and ponder it for a moment, and realize about its trueness. We are not talking about the rest of humanity, just you, in your awareness. The rest of humans could also be fake together with the perceptual reality you are placed in. But as long as you "think", you are: you exist. No possible discussion about it. *"Cogito ergo sum"*, I think, therefore I am, Descartes' famous statement.

Great, so now we are both on the same beginning of the story. Let us call that "self-aware-conscious-light" that is currently reading this, simply as "the *cogito*".

Now, lets make explicit the first assumption or postulate that any human being makes everyday: there exists an objective and materially contingent external reality, in which your *cogito* is placed as a cursor in a stage. Your *cogito* doesn't know directly about this scenario, else the stage itself would be part of the same mind. Instead, your *cogito* is attached to some organic sensors, like the neurons in your eardrum, which get activated in different ways according to how the external reality modulates the shape of the eardrum membrane. Accordingly it generates a signal that the neurons send to the acoustic cortex in your brain. The same happens for your visual neurons that are excited in different ways as a function of the energy of the light arriving to them. The same for the olfactory, gustatory, tactile, accelerometric (the inner ear liquid that lets you know the direction of gravity) or proprioceptive nerves[2]. Once all this information arrives to the brain, it is processed and each chunk is projected as perceptual feelings surrounding the center of your person, around your *cogito*. As such, the particular colors, tastes, sounds etc, you will feel are generated in that processing, presumably due to specific excitations in the sensor organs[3]. Let us give these concepts some names: the "external"

[1] It is no longer there when you are deep asleep either...

[2] Proprioception is the information we receive from muscles, tendons and joints that allow us to know where our body parts are at any given moment even if we close our eyes.

[3] Think about how unique those projections you feel are, that you will never be able to know if what you "see" and call "red" is not seen by another person as what you call "blue", just that you both agree it is called "red".

reality to your *cogito*, the reality "in itself", the one you will never know about, because you can only know through your perceptual sensors, is called **noumenon** (assuming of course, that such a thing objectively exists with the same certainty as your own *cogito*). Meanwhile, the set of projections where your *cogito* is centered, all that "virtual" surrounding that the processing of your perceptions produces around your *cogito*, is called the **phenomenon**. Of course, it is also convenient to accept the hypothesis that not only there is an objective *noumenon*, but that the rest of people have also behind those eyes, their own *cogito*, and that those *cogito*-s are also centered at a *phenomenon*, which is at least very similar to yours.

Now, note that what you know about the "external" reality is necessarily restricted to the particular way in which it is projected in your *phenomenon*! That is, about this *noumenon*, you will never know anything that cannot be directly or indirectly sensed and represented into your phenomenon. We are imprisoned in our phenomenon-s. To convince you even more about the tangential nature of the *phenomenon* and *noumenon*, let me remember you that whenever you are dreaming, you see and touch things or even feel your own body in configurations, that are really "not true". Your brain is generating directly in the visual, auditive and other cortices, the same neuronal excitations that would be generated if your eyes received that light, or your arms were in that disposition, when in reality your eyes are closed and your arms are still hugging your teddy bear. So yeah, dreaming is indeed the most immersive of the virtual realities. If you had a computer stimulating those regions with those same pulses, you would feel the same projections in the phenomenon. Dreaming just has the little inconvenience that the reasoning cortex is at least partially inhibited (together with the motor cortex -the one making your body actually move). That is why you feel more "instinctive" and less aware, like if your judgment was covered in a sort of veil. That is why you never confuse the oneiric- with the daylight- phenomenon. Clearly, if you made the inhuman experiment to avoid the rational cortex to be inhibited, you would entirely feel a dream, as real as right now, following the fact that if you avoid the inhibition of the motor cortex you will move in the reality as you are moving in the dream. Of course, "the plot" of that reality would be certainly confusing and inconsistent, like for Alice was the Wonderland, but the pehnomenon in itself would feel material. Think about it.

Extra Thoughts: Solipsism, Antropocentrism and a Cosmic Logos -Schrödinger's Arithmetic Paradox-

All this can easily lead you to *solipsism*, the philosophical posture where *you* are the only existing one, and as such, the reason of existence for all of your perceptions is your own existence. That is, *esse est percipi*, or whatever that *is*, can only *be*, if and while it is perceived by your *cogito*. That is, the Universe is as long as, or where, your *cogito*'s light illuminates it. That is, the Universe does not manifest its existence by itself. Instead, its "real"-ity is only as long as your *cogito* lights it. But sure, if you have accepted the postulate that there is such a thing as a *noumenon* (which you provide with existence through your phenomenon), why not accept also that there are more *cogito*-s lighting more *noumenon*-s?

At this step, you may realize that there is apparently a common *noumenon* for all of those *cogito*-s, because, the rest of *cogito*-s seem to report changes in their phenomenon-s that are identifiable with changes in your own phenomeon. It is this inter-personal agreement of coherence why we know dreams are not reality. Dreams do not alter this "common" *noumenon*. Only *you* feel the oneiric phenomenon. Schrödinger tries to manifest this dilemma in his *Mind and Matter* with the name "Arithmetic Paradox": How can there be multiple cogito-s, each giving existence with its light to the representations projected in each individual phenomenon, but even so, exist a single and shared *noumenon*?

Realize how the problem of this reality that requires to be part of a *phenomenon* for its existence to be manifested, but where not all the Universe is perceived by somebody at all times, could be solved by a rather simple idea.

The existence in itself of the Universe could be guaranteed without the need of having us, mammals, so casual results of evolution, which could have simply never come to live, by imagining a cosmic *cogito* that makes everything be explicitly manifested at all times. A *cogito* projecting the phenomenon of all the Universe at all times, and thus alleviating our fear of our Universe being a void theater if we were not here. We could identify this cosmic *cogito* with the pre-established harmony of Leibniz, perhaps a justified pretext for a greater existence or perhaps just the master algorithm that connects the perceptions that the computer simulating our brains needs to take care of. On this last idea, note that if we wanted to simulate a bunch of human brains -assuming a *cogito* can really emerge from non-organic computations- and make them believe they are "real" we would only need to simulate the atoms of their brains and the neural pulses generating their phenomenon, with the boundary condition of letting the different phenomenons influence each other. No need to simulate the whole Universe to make them believe they are true.

Notice however, before sanctifying the cosmic *cogito*, mother algorithm or pre-established harmony, that its existence is just one possible exit for the *impasse* of the arithmetic paradox over solipsism. We should never forget that the first lesson is ones *cogito* and that all the rest of concepts are based on postulates that build up one after the other over ones *cogito*. If suddenly a postulate of these erases the need of having the *"cogito ergo sum"* as basis for the existence (because there is an external reality or a cosmic *cogito*), then that is cheating the argumentation. Beneath this hypothesis, there will always be the one and only undeniable truth that needs no prove: *you exist*. Yeah, perhaps it is too anthropocentric. But it seems to be the least arbitrary and least unjustified conviction a human being could have.

Then, just realize that how *we* perceive sound or images or any other perception, is due exclusively to how our human brains process the information and manifest their corresponding projections on the phenomenon. That is, if there could be other *cogito*-s, evolved in different circumstances, they could have a different phenomenon to ours in front of the same stimuli or same *noumenon*. As an example, we know that dogs for instance, have a smaller variety of visual sensor neurons and thus end up having a phenomenon with a bi-chromatic range (they "see" in gray-scales). Now, let us call the rules that make us manifest the projections in the particular ways we do (or dogs do) as mental **categories**. These include two kinds: the purely perceptual ones -the axioms of raw perception- and the cognitive ones -the axioms of abstraction and comprehension-.

The **perceptual categories** are those that map, for instance, certain light energies to certain colors, certain frequencies of air oscillations to certain perceived sounds or certain kinds of molecules to certain smells. If one was to explain to a non-human intelligence what music is, would need to resort to these.

Then, there are the **cognitive categories**, that are the basic essences through which we think and understand things (and through which we are doomed to understand them, as certainly as we are doomed to undeniably feel our own existence). For example, the idea of "unity" (of "one") as a concept, of "existence", or even "boundedness", which might be derived from having the idea of "unity" and "non existence" at the same time. These at the same time imply the category of "fragmentation": the reason why when you look at a scene you are able (and doomed to do so) of seeing and understanding individually delimited and separated things, as if they had independent existence from each other. You do not see everything as a single continuum, even if in the noumenon it could be that it is so. As a taste of how the nomenon could in fact be incompatible with the "fragmentation", notice that "physics" is essentially the study of the best model about the *noumenon* we can build, the best story to explain and predict the behavior of the *noumenon* and its consequences in the phenomenon. If our current models are good enough representations of the noumenon, then we know that in reality matter is a continuum of atoms that make really no distinction of whether they belong to a chair, to this paper or your own body, meaning that our "fragmentation" category is just the way we understand things, not something fundamental about reality. Most importantly however, these cognitive axioms are in essence, where mathematics are rooted on, the axioms that build them.

On Models, Interpretations and Narratives

"Being able to predict reality, does not make you understand it."

We can define a **physical model** or **theory**, as a mathematical construction that allows the prediction of reality (that is, of the events happening on the phenomenon).

A physical model must have two elements. One is a **mathematical model**, which is a self-consistent mathematical construction. The other one is a **basic interpretation** of the mathematical model, which can be defined as a set of links established between the mathematical entities of the model with phenomenological qualities that appear in our perceptions.

Now, such a physical model or theory, is enough from an operational point of view. It is enough for predicting the events we can perceive, which from an utilitarian view, is all physics should be about. However, we humans, have a natural tendency towards not only building physical models, as utilitarian mathematical tools, but for also seeking semantical or narrative models about reality. Stories that can aid us conceptually and intuitively understand what could be happening in the noumenon for the model to correctly predict the events we find in the phenomenon. We can define an **interpretation** or **narrative** of a physical model, as a self-consistent semantic story given to the mathematical structures and elements of the theory, that go beyond the set of concepts anchored to the phenomenon by the basic interpretation we defined earlier. That is, it is a conceptual representation of the noumenon, beyond the phenomenon. A conceptual representation that is consistent with each mathematical step given in the base physical model.

The fact that physical models or theories speak about phenomoenological events, qualities we immediately perceive, make them falsifiable. If a certain model fails in some prediction, the model will be false for a general case. However, interpretations or narratives talk about the noumenon, which by definition is "the external reality in itself, without us perceiving it through our senses and categories". This noumenon thus can never be known in itself, it is epistemologically inaccessible. Therefore, as long as a narrative is consistent with the mathematics of the model, it will be unfalsifiable as a model for the noumenon. This is, as long as the physical model it is grounded on is not proven to be incorrect.

It is this why one could say that working on an interpretation for a physical theory is a useless task. And it is indeed, from a purely productive or utilitarian point of view. However, since being able to predict reality does not make you understand it, there is place for scientists working on narratives. Intuitively understanding a theory, does not only allow a person to better make the questions about the model that can pull it to its limits, but it is also a way to better make it understand it (teach it), remember it and use it. Just as learning a text as part of a song makes it easier to digest it, or a moral is simpler to take when first known about through a tale, so is the case for a physical theory with a simple narrative, rather than the naked basic interpretation. It is a task that might not be as creative as creating the physical model itself, but is at least as imaginative and coloring for the community that reaches it. We could compare the physical theory with the outline of a picture, which has to be created "out of nothing", and thus has a big creative merit, while we could compare the interpretation or narrative with the coloring of the outlined picture. The painter is not "creating nothing usefully new", but, it gives body, consistency, appeal and digestibility to the overall picture.

In the present dissertation an illustrative narrative for the physical model comprehended under the name "Non-Relativistic Quantum Mechanics" will be exposed, as a means of introducing an intuitive explanation of the so called Open Quantum systems and their quandary.

Essentially Paraphrasing...

[1] David Bohm. *Wholeness and the Implicate Order.* Routledge, 1980.

[2] René Descartes. *Discourse on Method.* Harmondsworth, Penguin, 1968.

[3] Lisa Downing. George Berkeley. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy.* Metaphysics Research Lab, Stanford University, Fall 2021 edition, 2021.

[4] Immanuel Kant. *Critique of pure reason.* Willey Book Co., 1899.

[5] Mark Kulstad and Laurence Carlin. Leibniz's Philosophy of Mind. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy.* Metaphysics Research Lab, Stanford University, Winter 2020 edition, 2020.

[6] Nuccio Ordine. *The Usefulness of the Useless.* Philadelphia: Paul Dry Books, 2017.

[7] Erwin Schrödinger. *Mind and Matter.* Cambridge University Press, 1958.

Part A
The Axioms

*“ Quantum Mechanics is Ontologically
Deterministic but Epistemologically Stochastic.”*

Part A: The Axioms

Axioms or **postulates**[4] are the principles or assumptions from which the understanding of something is built up. They are principles that are rather unprovable or self-evident and are admitted as the first facts in the deductive creation of a conceptual framework. This means that if one keeps on asking *why*? within a certain framework, the last answers one can arrive to are these axioms over which all the rest is formulated.

As an example, the following ones are some of the main axioms or postulates of physics, as explained by Schrödinger in Refs. [10, 9]:

- The postulate that the *noumenon*, or at least its consequences in our *phenomenon*, are understandable for human cognition.[a]
- The postulate that this understandability, these rational models, can be formalized mathematically as a particular example of some mathematical framework, building what we called in the preface, a physical model or theory.
- The postulates that fix the degrees of freedom of these physical models to fit our phenomenologic observations. These are postulates about some direct or indirect experimental observations we find in the phenomenon: for example, the Planck constant is this particular number because we see/measure it to be so, or time is absolute or not because we experimentally see so etc.

It is interesting to note that in maths on the other hand, the most fundamental axioms are grounded on the *cognitive categories* we talked about in the prologue: unity, fragmentation, inclusion, boundedness etc. The mental categories that shape our mind. Even if it is true that over them, we shape other axioms to give more structure to the mathematical theories, note that there are no experimental postulates in mathematics. This is the reason why mathematics needs no experimental proof, and no matter how complicated a mathematical theory is, once one understands the formal proofs, their trueness is obvious and self-evident. This is because in its last resort, a mathematical statement is built on fundamental concepts with which you are doomed to understand the Universe (the cognitive categories). You are inevitably tied to those concepts as well as you are tied to your own existence by the *cogito ergo sum*.

[a] Note how this is a rather arbitrary assumption (as most axioms are). The inner workings of reality could have also been incomprehensible for human cognition! Yet, till date it looks like the Universe is understandable.

In much the same way, a narrative of a physical model, also has axioms or basic assumptions. Even if in general these are not explicited, they are a very pedagogical point to begin from, since these are the loose ends of the interpretation. If one keeps on asking *why?*, one arrives to these postulates or axioms.

In this first part of the book, we will delve with the basic postulates of the narrative that will be employed in the rest of the work. Along with the narrative concepts, we will also expose the basic mathematical structures representing them in the (non-relativistic) Quantum Theory. Thus, in this first part, we will cover most of the loose-ends of both the Quantum Theory and the present narrative.

[4] Historically, while the word “postulate” has had an “arbitrary supposition” connotation, the word “axiom” has had a more fundamental status, as a self-evident statement that “must” be true by its own virtue. In this work, as is done today in most texts, we will use them interchangeably since the postulates we will state to build quantum mechanics, are arbitrary decisions we will make (even if grounded in intuition), but at the same time, most will be assumptions that must be taken for the theory to hold.

A.1. The State of the Universe

A.1.1. Reviving Postulates of Classical Mechanics

If we are to provide an intuitive **narrative** for a general physical model, in line with the preface, it seems reasonable that we should choose an entity we can actually observe or measure in our phenomenon as the basic quantity that can be known about. That is, the epistemological[5] basis of the interpretation should be a Kantian phenomenological property we perceive. In the preface, we have defined as real, the set of observations of the phenomenon that all average humans agree on. The position of the objects in the physical three dimensional space is one such property we can have an agreement on. Phenomenologically, we find that there are three orthogonal degrees of freedom in which each object can be displaced. Thus, we could define an absolute orthogonal coordinate system $\mathbb{R}^3$ as a scenario, where all objects extend in a fixed range of space. This picture will be dynamical, meaning that it will have the option to be different at each point of a privileged degree of freedom we will call time, defined as a Universal absolute time-line denoted by t. At a given time, the range of space occupied by a certain object is what we will call its **position**. Following the fact that when we observe the smallest objects of reality, they all seem to be point-like, infinitesimal in spatial extension, instead of occupying a continuous range in space, we will postulate that all the objects of the Universe can be described in their last resort as cumulations of smaller objects that occupy only a single point in physical space. We will call them **particles** or fragments. Assuming there are n (arbitrary countable) particles in the Universe, we can model their position by stating the three numbers in the coordinate system of physical space that they occupy. As such, we could define the position of all the particles of the Universe, by specifying all these triplets. This means that the Universe as a whole has $3n = N$ **degrees of freedom**, which we can set together in a same tuple $\vec{x} = (x_1, ..., x_N) \in \mathbb{R}^N$. Each possible point $\vec{x}$ in $\mathbb{R}^N$ will then represent a possible configuration of the Universe. A possible arrangement of particles. It is this why we call this space, the **configuration space** of the Universe. Some configurations might not be realizable at every times, so we will restrict the configuration space to a subset $\Omega_t \subseteq \mathbb{R}^N$. Then, we let our Universe have an evolving configuration in time, $\vec{x}(t)$, which we will call the **trajectory** of the Universe.

The reason why we said the position was an easy perceptive quality to set in common was because it is a binary property of a particle per each point in space (unlike more convoluted phenomenic properties like colors or smells). But there is a yet more important reason. Since we base most of our knowledge about the world in the visual perception, it turns out that, whenever we take a quantitative measurement, we do it by looking at the position of a cursor in a screen or dial, or the position where the observed particle was detected, among others. That is, all measurements in a Lab can be made as measurements of the position of the particles composing the measuring devices.

But in this narrative, as is done in other interpretations, we will go even further and assume that not only will the configuration of the Universe in each time be the epistemological basis, but it will also be the ontological[6] foundation. That is, be it knowable, known or unknown, the position of each degree of freedom of the Universe at each time, will exist irrespective of whether they are part of the phenomenon of an observer or not.

An interesting point about the present narrative is that even if there are more phenomenological qualities of the entities we perceive than their position in space (like their colours, sounds, tastes, smells etc.), these other phenomenological properties will be derived from the positions of some particles composing them (like the spatial arrangement of the electrons and nuclei of the molecules arriving to our nose, their position interaction with our nerves etc.).

[5]Epistemology is the metaphysical study of what is knowable, of knowledge and know-ability.
[6]Ontology is the metaphysical study of what **is**, of existence and being.

Thus, the model we are building essentially suggests, as did classical particle mechanics, that knowing the configuration of the Universe in each time would be enough to predict any phenomenological perception.

A.1.2. A Fluid of Tangent Universes

All the postulates so far are identical to the ones assumed in interpretations of classical particle mechanics. However, we will now introduce the most fundamental difference that our Quantum Mechanics narrative will have with the classical intuition. Brace yourself. We will assume that each point in configuration space $\vec{x} \in \Omega_t \subseteq \mathbb{R}^N$ at each time, not only is a *possible* configuration for our Universe, but actually **is** a real physical Universe, among which one is our perceived Universe! Since there is a "continuously" infinite number of points in configuration space, we are effectively introducing here a **"fluid" of uncountably many Universes**.

Let us now formalize this continuum of uncountably infinite point-like Universes moving in configuration-space. As we said, assuming that a possible Universe is always observed in space as point-like in all its degrees of freedom[7], an $\mathbb{R}^N$ point $\vec{x}$ is a *possible* static configuration of the whole Universe. Now, as we also already said, a moving point in $\mathbb{R}^N$, $\vec{x}(t)$ could be seen as a possible Universe history. We could call it, the **trajectory** of a possible Universe. We will now define the **Fluid of Tangent Universes**, as the continuum of trajectories of possible/alternative Universes. Let us formalize this mathematically.

On the one hand, we will assume that the Universes *never collide*, because we never observe a particle in more than one position at the same time, meaning that each Universe at each time occupies a single configuration. At any given time t, this would allow us to label each Universe using a tuple $\vec{x} \in \mathbb{R}^N$ representing its position in configuration space. This means that the function assigning a configuration to each Universe in each time must be injective. In turn, this means that we could identify the Universe that at time t_0 was at $\vec{x} = \vec{\xi}$ for any time t with the same label $\vec{\xi}$. We will call it the **$\vec{\xi}$-th Universe or fluid element**.

On the other hand, since we seem to find that all the particles in our Universe move to nearby positions crossing all intermediate positions, we will assume that the trajectories of the Universes will be *continuous*. This, together with the fact that their trajectories never collide means that there is a continuous and bijective function $\vec{x}(\vec{\xi}, t) \equiv \vec{x}^{\,\xi}(t)$ that tells us where in Ω_t each fluid element $\vec{\xi} \in \Omega_0$ is at each time t.[8] This is what we will call the **trajectory** of the $\vec{\xi}$-th fluid element or Universe. Now, all this means that the function $\vec{x}(\vec{\xi}, t)$ must have an inverse function $\vec{\xi}(\vec{x}, t) \equiv \vec{x}^{\,-1}(\vec{\xi}, t)$, telling us which is the label of the fluid element crossing the configuration-space point $\vec{x}$ in time t. This inverse must be continuous as well, since the transformation $\vec{x}(\vec{\xi}, t)$ is required to be continuous in all its variables. Thus, we are asking that Ω_t is homeomorphic to Ω_0 at all times.

For now, this seems to be just a convenience, but we will find it is a necessity within Quantum Mechanics. This will be so since we will achieve the description of the time evolution of the positions of the fluid elements through the integration of a continuous velocity field in configuration space. By the Picard-Lindelöf theorem, we will have that the transformation $\vec{x}(\vec{\xi}, t)$ must therefore be a spatial homeomorphism. What is more, it will need to be differentiable, meaning it must be a diffeomorphism.

We say that these alternative Universes are **Tangent Universes**, because they can be arbitrarily close to each other in configuration space (that is, the set of positions in physical space for their particles can be arbitrarily similar), but their trajectories never cross each other, meaning they never have the same configuration at the same time.

[7] This turns out to be our case, since we always observe fundamental particles to be point-like in space.

[8] Note that we call Ω_0 the subset of $\mathbb{R}^N$ where we consider there are fluid elements at the reference time $t = t_0$.

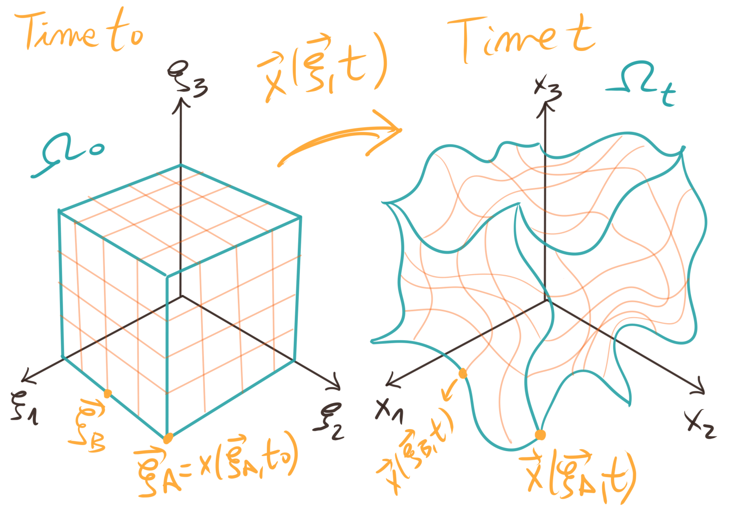

Figure 1: Depiction of the $N = 3$ case of a homeomorphism representing the trajectory ensemble of a fluid following the notation presented in the text.

Note now that if we were just given the continuous velocity field for the trajectories of the Universes $\vec{v}\Big(\vec{x}(\vec{\xi},t),t\Big) := \frac{\mathrm{d}\vec{x}(\vec{\xi},t)}{\mathrm{d}t}$, we could simply get the ensemble of trajectories by integration.

$$\vec{x}\Big(\vec{\xi},t\Big) = \int_{t_0}^{t} \vec{v}\Big(\vec{x}(\vec{\xi},t),t\Big)\ dt \tag{1}$$

using the initial condition $\vec{x}\Big(\vec{\xi},t_0\Big) = \vec{\xi}$ we have set by definition.

Thus, the velocity field seems to uniquely characterize the trajectory ensemble. This is why it will be one of the properties of the fluid we will look for when trying to define the state of the Tangent Universe Fluid.

There is yet another thing we need to assume to arrive to Quantum Mechanics. The relative amount of Universes in each configuration need not be homogeneous, meaning, we can define the ratio of Universes in a certain configuration, with respect to the total, as a (positive) density function $\rho(\vec{x},t)$, the **density of tangent Universes**. Since we define it to be normalized with respect to the total number of Universes, we assume it integrates to unity in configuration space.

$$\iint_{\Omega_t} \rho(\vec{x},t)\ dx_1\cdots dx_N = 1 \quad \forall t \tag{2}$$

Then, integrating this density in a certain volume $V \subseteq \mathbb{R}^N$ would give us the relative number of Universes with configurations in V, which by definition will be a positive real number in the range $[0,1]$.

These two fields in configuration space: the velocity of the trajectory ensemble $\vec{v}(\vec{x},t)$ (defining the ensemble $\vec{x}(\vec{\xi},t)$) and the relative tangent Universe density $\rho(\vec{x},t)$, will completely define the state of the Fluid of Universes at all times.

A.1.3. The Lagrangian and Eulerian Frames

We can now realize that we will be able to describe a state variable of the fluid, both by reference to a specific configuration-space position $\vec{x}$ at a certain time t or by reference to a specific fluid element (or possible Universe) $\vec{\xi}$ at a certain time. These will be respectively the Eulerian and the Lagrangian frames.

Given a property f of the fluid (like the density, the velocity or any other derived one), it will be described in the Lagrangian frame if its values are given as seen from the $\vec{\xi}$-th Universe, that is, along the trajectory of that fluid element $f(\vec{x}(\vec{\xi},t),t)$. Alternatively, we could see the value of the properties at each time from a particular fixed Universe configuration $\vec{x}$, as $f(\vec{x},t)$, which would be the Eulerian frame.

A.1.4. The Measurement Axiom

If we knew $\vec{\xi} \in \mathbb{R}^N$ was the position of every particle of our Universe at an arbitrary single time, say at time t_0, we could then deterministically predict the time evolution of our Universe using the trajectory ensemble $\vec{x}(\vec{\xi},t)$. However, knowning the position of every particle in the Universe at the same time is far from being possible. Therefore, we cannot exactly know which of the tangent Universes is ours. From our point of view, the configuration of our Universe is thus an unknown, a random variable.

Since a priori, we have no information about which of the Universes is ours, we can refer to the tangent Universes as **possible Universes**. A priori then, the best guess we can make is giving the same probability to each of the possible Universes to be ours. As the relative amount of Universes is given by the density $\rho(\vec{x},t)$, which we conveniently decided to normalize to unity, this would mean that we could identify the probability density of the configuration of our Universe with the density of tangent Universes (assigning a same probability to each possible Universe, we will have a weight due to their relative amount). That is, for a domain $V \subseteq \mathbb{R}^N$:

$$P(\text{our Universe has its configuration in } V \text{ at time } t) = P(\vec{x} \in V \mid t) := \iint_V \rho(\vec{x},t)\ dx_1 \cdots dx_N \quad (3)$$

If instead we are only interested on the probability that a subset of the particles of our Universe $\vec{y} = (x_1, ..., x_m)$ such that $m < N$, have their configuration in $V_y \subseteq \mathbb{R}^m$, we would simply use the marginal probability density derived from the global prior density:

$$P(\vec{y} \in V_y \mid t) := \iint_{V_y} \left(\iint_{\Omega_t} \rho(\vec{x},t)\ dx_{m+1} \cdots dx_N \right) dx_1 \cdots dx_m$$

Note that for the introduction of the narrative, we have been saying that ontologically our Universe has a discrete trajectory $\vec{x}(\vec{\xi},t)$, however, since in practice it is not possible to know the position of a particle with infinite precision (since we are dealing with continuous variables and not a variable with a countable support), the tangent Universes we should consider ontologically are those ones weighted by the density $\rho(\vec{\xi},t)$ and not the actual discrete ones. That is to say, the density of Universes will not need to be preserved along the trajectories. Instead they will need to be preserved in subsets of R^N that move along with the trajectories! This is related to the typical discussion on the meaninglessness of observing a discrete value for a continuous probability density variable. We will see the relevance of this when we talk about the local conservation of the amount of Universes in the next section.

It is interesting however that this observation opens the question of whether quantum theory is just the continous limit of a more fundamental theory where there is actually a countable number of possible Universes! If their number is very big, most predictions would coincide at a big degree with the theory were there is an uncountable number of Universes. However, they could clearly have different predictions for some very fine experiments. Different predictions that could be perceived in the phenomenon and thus help falsify the underlying theory. It is here where we can see the relevance of a narrative given to a physical theory. It can open ways to search for alternative theories and test non-trivial limits of the current theory. Actually, such an alternative Discrete Tangent Universe Quantum Theory was discussed recently in Ref. [8] by Hall et al.

A.2. The Dynamics of the Universe

After understanding the way in which we can define the **state** of the Universal fluid, let us now derive the equations of motion for its defining fields: the density and the velocity fields. For this, we will first derive the equations of motion of the particles in the physical space of each Universe from symmetry considerations about the nature of physical space and time. Then we will introduce the possibility of interaction between different Universes, together with an axiom similar to the Second Law of Thermodynamics, an axiom for the conservation of Universes and one for the total energy. All these together, will lead us directly to the well known Schrödinger Equation, evolving both the configuration-space velocity field and the Universe density at once.

A.2.1. The Stationary Action Axiom within each Universe

Let us first focus on a single Universe of the ensemble of Tangent Universes and the particles observed within the physical space of this single Universe.

Let us denote the degrees of freedom of the k-th particle as $\vec{q}_k = (x_{3k-2},\ x_{3k-1},\ x_{3k})$ with $k \in \{1, ..., n\}$, such that we denote the degrees of freedom of all the particles together as $\vec{x} = (x_1, ..., x_N) \equiv (\vec{q}_1, ..., \vec{q}_n)$.

The **Stationary Action Axiom within each Universe** states that given the boundary values $\vec{x}(t_0) = \vec{x}_0$ and $\vec{x}(t_f) = \vec{x}_f$, for some fixed $t_0 < t_f \in \mathbb{R}$, $\vec{x}_0, \vec{x}_f \in \mathbb{R}^N$ for the trajectory $\vec{x}(t)$ of all the particles in a single Universe, there exists some function $L(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t})$, called the **Lagrangian** of the Universe, such that, the trajectory taken by the Universe, $\vec{x}(t)$, is that yielding a critical point of the so called **action functional**. This is the integral of the Lagrangian along the positions and velocities taken by the trajectory:

$$\mathbb{S}[\vec{x}(t);\ t_0, t_f, x_0, x_f] = \int_{t_0}^{t_f} L\left(t, \vec{x}(t), \frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t}\right)\, dt \tag{4}$$

By a critical point, we mean a trajectory $\vec{x}(t)$ which if perturbed to the valid trajectories closest to it (the ones satisfying the boundaries), yields an action that is a critical point (typically a minimum). That is, we seek for trajectories for which the variation of the action is zero in the locality of the trajectory, yielding a stationary action.

As we prove in the next gray box, a necessary condition for such a trajectory is to obey the N, so-called, Euler-Lagrange equations

$$\frac{\partial L(\vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}, t)}{\partial x_j}\Bigg|_{\substack{\vec{x}=\vec{x}(t) \\ \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}=\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L(\vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}, t)}{\partial\left(\frac{\mathrm{d}x_j}{\mathrm{d}t}\right)}\right)\Bigg|_{\substack{\vec{x}=\vec{x}(t) \\ \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}=\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}}} = 0 \qquad \forall j \in \{1, ..., N\}. \tag{5}$$

These partial differential equations are called **the equations of motion** of the system, since they define the motion of the trajectory $\vec{x}(t)$.

Derivation of the Euler-Lagrange Equations

A.2.1.1. The Lagrangian of a Universe from symmetries of physical space and time

It turns out that the symmetries of the Lagrangian L are transferred to symmetries of the resulting differential equations. This means that if we construct a Lagrangian obeying the postulates about symmetries of the Universe that we prefer, then the resulting laws of motion will obey them as well.

A Single Universe and a Single Particle

Consider an isolated Universe (from the rest of Universes in the fluid) where there is only one particle, $\vec{x} = \vec{q}$. If there is only one particle (and one Universe), the only thing determining its motion should be the structure of the physical space and time in which it lies. This structure will be represented by what we will call **symmetries** of the equations of motion. A symmetry will be given by a transformation of the coordinate system that leaves the differential equation of motion invariant. Since the differential equations of motion are the mathematical expression of a "law of physics", this means that a symmetry is given by realizing under which frames of reference, we expect the laws of physics to be the same.

For example, first of all we will assume the Universal **time** axis is **homogeneous** with respect to the laws of physics, meaning that no matter where in time we are, or equivalently, no matter how much I shift the clock of our reference frame, the same equations of motion should describe the system: the laws of physics should be the same. A way to make sure that this is the case would be by imposing that the Lagrangian is the same (is invariant) for any time shift τ, meaning it does not depend explicitly on time

$$L\left(t+\tau,\ \vec{q},\ \frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right) = L\left(t,\ \vec{q},\ \frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right) \quad \forall t,\tau \in \mathbb{R} \Longrightarrow L\left(t,\ \vec{q},\ \frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right) = L\left(\vec{q},\ \frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right). \tag{6}$$

In general, the resulting equations of motion are the same for a Lagrangian and the same Lagrangian with an additive divergence term (with respect to the independent variables), in our case, a total time derivative. That is, the Lagrangian $L(t, \vec{q}, \frac{\mathrm{d}}{\mathrm{d}\vec{q}})$ and $L(t, \vec{q}, \frac{\mathrm{d}}{\mathrm{d}\vec{q}}) + \frac{\mathrm{d}}{\mathrm{d}t} f(t, \vec{q})$ for any "well behaved" function f of time and position, would yield the same equations of motion. We prove this in Appendix X.1. for the general case.

Thus, if we want the equations of motion to be the same at all times, instead of assuming it for the Lagrangian itself, we would have had enough if under the time shift, the Lagrangian was the same up to "a total time derivative"

$$L\left(t+\tau,\ \vec{q}(t),\ \frac{\mathrm{d}\vec{q}(t)}{\mathrm{d}t}\right) = L\left(t,\ \vec{q},\ \frac{\mathrm{d}\vec{q}(t)}{\mathrm{d}t}\right) + \frac{\mathrm{d}}{\mathrm{d}t} f(t,\ \vec{q}(t)) \quad \forall \tau, t \tag{7}$$

for some "well behaved" function f of time and position.

However, if by Occam's razor[a] we choose the simplest Lagrangian that yields the symmetry (time shift invariance) for the equations of motion, we can obtain this by further assuming if possible, that the Lagrangian itself is invariant under such a symmetry. This is the case if $f \equiv 0$, as assumed in (6). We will assume this for all symmetries but the last one.

The same happens for a global multiplicative constant in the Lagrangian. The resulting equations of motion will be the same irrespective of the multiplied global constant.

[a] If there is a simpler yet equivalent explanation for a same phenomeonon, the simplest one should be preferred. Thus, if a simpler Lagrangian (in this case meaning the one with the least terms) gives us the same equations of motion, we will prefer it over the rest.

Now, similarly, **physical space** should be **homogeneous**, meaning that wherever we center the coordinate system (apply a shift to the positions), the laws of physics should look the same. This means that we can impose the Lagrangian to avoid an explicit dependence on the position of the particle:

$$L\left(\vec{q}+\vec{c}, \frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right) = L\left(\vec{q}, \frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right) \quad \forall \vec{c}, \vec{q} \in \mathbb{R}^3 \Longrightarrow L\left(\vec{q}, \frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right) = L\left(\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right) \tag{8}$$

We will also assume the natural axiom that the **physical space** is **isotropic**, meaning that if we rotate our coordinate system, the equations of motion should look the same. Assuming the Lagrangian as well should be invariant, this will avoid the Lagrangian to depend on the direction of the velocities $\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}$, leaving it only to depend potentially on their magnitude[9] $||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||^2 = \frac{\mathrm{d}x_1}{\mathrm{d}t}^2 + \frac{\mathrm{d}x_1}{\mathrm{d}t}^2 + \frac{\mathrm{d}x_1}{\mathrm{d}t}^2$. Denoting a rotation of angle ϕ about an axis $\vec{u} \in \mathbb{R}^3$ for a vector in $\mathbb{R}^3$ as $Rot(\cdot;\phi,\vec{u})$, we are imposing:

$$L\left(Rot\left(\frac{\mathrm{d}\vec{q}}{\mathrm{d}t};\phi,\vec{u}\right)\right) = L\left(\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right) \quad \forall\phi \in \mathbb{R}\ \forall\vec{u} \in \mathbb{R}^3 \Longrightarrow L\left(\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right) = L\left(\left|\left|\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right|\right|^2\right) \tag{9}$$

Finally, we will assume **Galilean relativity**. That the equations of motion should be invariant under a constant velocity transformation for the coordinate system. That is, any observer moving at a constant velocity with respect to one another, should observe the same physical laws (differential equations of motion). This will lead us to conclude that $L\left(||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||^2\right) = C||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||^2$ for some real constant C.

To see why, consider an infinitesimal Galilean transformation of the coordinate system by a constant velocity $\vec{\varepsilon}$, with $\vec{\varepsilon}$ in an $\mathbb{R}^3$ ball centered at $\vec{0}$ with an arbitrarily small radious $\delta > 0$. Defining $\vec{v} := \frac{\mathrm{d}\vec{q}}{\mathrm{d}t} + \vec{\varepsilon}$, let us expand $L\left(||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t} + \vec{\varepsilon}||^2\right) = L(||\vec{v}||^2)$ in power series, centered at $\vec{\varepsilon} = \vec{0}$:

$$L(||\vec{v}||^2) = L\left(\left|\left|\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right|\right|^2\right) + \frac{\partial L(||\vec{v}||^2)}{\partial||\vec{v}||^2}\Big|_{\vec{v}=\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}}\left(||\vec{v}||^2 - \left|\left|\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right|\right|^2\right) + o\left(\left(||\vec{v}||^2 - \left|\left|\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right|\right|^2\right)^2\right) \tag{10}$$

Denoting by $\cdot$ the scalar dot product between two vectors and using that:

$$||\vec{v}||^2 - \left|\left|\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right|\right|^2 = \left|\left|\frac{\mathrm{d}\vec{q}}{\mathrm{d}t} + \vec{\varepsilon}\ \right|\right|^2 - \left|\left|\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right|\right|^2 = \left|\left|\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right|\right|^2 + ||\vec{\varepsilon}||^2 + 2\vec{q}\cdot\vec{\varepsilon} - \left|\left|\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right|\right|^2 = 2\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\cdot\vec{\varepsilon} + ||\vec{\varepsilon}||^2 \tag{11}$$

we get that:

$$L(||\vec{v}||^2) = L\left(\left|\left|\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right|\right|^2\right) + \frac{\partial L\left(||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||^2\right)}{\partial||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||^2}\left(\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\cdot\vec{\varepsilon}\right) + o\big(||\vec{\varepsilon}\ ||^2\big) \tag{12}$$

Since we could choose an arbitrarily small $||\vec{\varepsilon}\ ||^2$, we could take the expression to first order, such that:

$$L\left(\left|\left|\frac{\mathrm{d}\vec{q}}{\mathrm{d}t} + \vec{\varepsilon}\ \right|\right|^2\right) = L\left(\left|\left|\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right|\right|^2\right) + \frac{\partial L\left(||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||^2\right)}{\partial||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||^2}\left(\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\cdot\vec{\varepsilon}\right) \tag{13}$$

As it was explained in the last gray-box, this Lagrangian for the Galilean transformation of the coordinate system will lead to the same equations of motion as $L(||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||^2)$ if the additional term is a total time derivative. That is, if:

$$\frac{\partial L\left(||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||^2\right)}{\partial||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||^2}\left(\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\cdot\vec{\varepsilon}\right) = \frac{\mathrm{d}f(\vec{q}(t),t)}{\mathrm{d}t} \tag{14}$$

for some arbitrary "well behaved" function f of time and position. If we denote the gradient vector with respect to the variables $\vec{q}$ symbolically as:

$$\vec{\nabla}_{\vec{q}} \equiv \left(\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \frac{\partial}{\partial q_3}\right) \tag{15}$$

along the trajectory:

$$\frac{\partial L\left(||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||^2\right)}{\partial||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||^2}\Big|_{\frac{\mathrm{d}\vec{q}(t)}{\mathrm{d}t}}\left(\vec{\varepsilon}\cdot\frac{\mathrm{d}\vec{q}(t)}{\mathrm{d}t}\right) = \frac{\partial f(\vec{q},t)}{\partial t}\Big|_{\vec{q}(t)} + \vec{\nabla}_{\vec{q}}f(\vec{q},t)\Big|_{\vec{q}(t)}\cdot\frac{\mathrm{d}\vec{q}(t)}{\mathrm{d}t} \tag{16}$$

[9] Since $||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||$ has its image in $[0,\infty) \subset \mathbb{R}$, and $f(s) = s^2$ is a homeomorphism from $[0,\infty) \subset \mathbb{R}$ to itself, we can equivalently consider the dependence on $||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||$ or on $||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||^2$ with a little notational abuse for the Lagrangian L.

The only explicit variable in the left hand side is $\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}$, no explicit dependence on t nor $\vec{q}$ beyond it. Therefore, the right hand side will need to behave so, meaning $\frac{\partial f(\vec{q},t)}{\partial t}$ must be a constant a, since a partial derivative in time, of a function of t and $\vec{q}$, can never yield a term as a function of $\frac{\mathrm{d}\vec{q}(t)}{\mathrm{d}t}$. For the same reason, $\vec{\nabla}_q f(\vec{q},t)$ will also need to be a constant $\vec{b}$. This means that the left hand side should at most be linear in $\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}$, as is the right hand side. Since the $\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}$ term is already present in the left hand side, $\frac{\partial L(||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||^2)}{\partial ||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||^2}$ will need to be constant. This means that:

$$\frac{\partial L\left(||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||^2\right)}{\partial ||\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}||^2} = C \Longrightarrow L\left(\Big|\Big|\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\Big|\Big|^2\right) = C\Big|\Big|\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\Big|\Big|^2 + c \tag{17}$$

for some constants $C, c \in \mathbb{R}$. Since the constant c can be regraded as a total time derivative as well, the resulting equations of motion will be the same setting $c = 0$. This would leave the Lagrangian for a free particle as:

$$L\left(t, \vec{q}, \frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\right) = C\Big|\Big|\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}\Big|\Big|^2 \tag{18}$$

We call this term the **kinetic energy** of the particle.

A Single Universe and Many Non-Interacting Particles

Now, if we had more free particles in this Universe without interaction between them (lets say n), we could repeat the space-time symmetry arguments for each of their degrees of freedom and would get that the Lagrangian of the Universe would be a sum of their kinetic energies. The subtle point would be that the constant C would not need to be the same for each particle (we do not have any symmetry telling us so). Thus, the Universe would have a Lagrangian:

$$L\left(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right) = \sum_{k=1}^{n} C_k \Big|\Big|\frac{\mathrm{d}\vec{q}_k}{\mathrm{d}t}\Big|\Big|^2 \tag{19}$$

Since a Lagrangian will provide the same equations of motion for any non-zero multiplicative global factor, we could establish one of the $C_k = 1$ and the rest measure them relative to that one. That is, we could use the constant of one of the particles as a standard C and scale the rest accordingly. This would result in a very particular choice of units, that we will not follow. Instead, for historical reasons, we will write these constants as $C_k = m_k/2$, and call m_k the **mass** of the k-th particle. It will represent the restriction of the particle posed on its displacement.

A Single Universe and Many Interacting Particles

Now, if there was any kind of spatial influence that the position of one particle could exert on the position of the rest, we could model this influence of the k-th particle on the j-th one as a constraint between their positions, which can in general be modeled by an arbitrary function $V_{kj}(\vec{q}_k, \vec{q}_j)$ added to the free particle Lagrangian. Note that we do not allow an explicit time dependence on this influence term, since time should be homogeneous (the law of their interaction should be constant in time). For reasons that will become clear in a few paragraphs, we will choose to write these arbitrary constraint functions with a negative sign. Such interaction terms for every pair of particles would leave a Universal Lagrangian:

$$L\left(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right) = \sum_{k=1}^{n} \frac{1}{2} m_k \Big|\Big|\frac{\mathrm{d}\vec{q}_k}{\mathrm{d}t}\Big|\Big|^2 - \sum_{k<j} V_{kj}(\vec{q}_k, \vec{q}_j) \tag{20}$$

We could further simplify it by recurring back to the axioms. Rearranging the position variables of any particle pair $\vec{q}_k$, $\vec{q}_j$ as:

$$\vec{w}_{kj} := \frac{m_k\vec{q}_k + m_j\vec{q}_j}{m_k + m_j} \quad \vec{r}_{kj} := \vec{q}_k - \vec{q}_j \tag{21}$$

where $\vec{w}_{kj}$ is the center of mass between the particles and $\vec{r}_{kj}$ is their relative position, we could have abusing notation, the influence functions written as $V_{kj}(\vec{w}_{kj}, \vec{r}_{kj})$. Now, translational invariance due to homogeneity of space requires that if the coordinate system is displaced by $\vec{c} \in \mathbb{R}^3$, the Lagrangian is left invariant. Such a translation of the coordinate system would leave the relative position of the particles $\vec{r}_{kj}$ the same, but would displace the gravicenter to $\vec{w}_{kj}+\vec{c}$. This means we would be requiring for the homogeneity of space that $V_{kj}(\vec{w}_{kj}+\vec{c}, \vec{r}_{kj}) = V_{kj}(\vec{w}_{kj}, \vec{r}_{kj})$ $\forall \vec{c} \in \mathbb{R}^3$, which implies $V_{kj}(\vec{w}_{kj}, \vec{r}_{kj})$ cannot be a function of $\vec{w_{kj}}$. Leaving $V_{kj}(\vec{w}_{kj}, \vec{r}_{kj}) = V_{kj}(\vec{r}_{kj})$.

Finally, due to the rotational invariance of the Lagrangian required by the isotropy of space, $V_{kj}(Rot(\vec{r}_{kj}; \phi, \vec{u})) = V_{kj}(\vec{r}_{kj})$ for any rotation angle ϕ and axis $\vec{u}$. This could only be true if the mutual restriction V_{kj} was exclusively a function of the magnitude $||\vec{r}_{kj}||$ and not its direction. Thus the only allowed potential interaction term should be as a function of the relative distance of the particles $V(||\vec{q}_k - \vec{q}_j||)$. Leaving a total Universe Lagrangian as:

$$L\left(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right) = \sum_k \frac{1}{2} m_k \Big|\Big| \frac{\mathrm{d}\vec{q}_k}{\mathrm{d}t} \Big|\Big|^2 - \sum_{k<j} V_{kj}(\ ||\vec{q}_k - \vec{q}_j||\ ) \tag{22}$$

Substituting this Lagrangian in the Euler-Lagrange equations (5) would leave that the physical trajectory followed by the Universe would need to be given by the system of partial differential equations:

$$m_k \frac{\mathrm{d}^2\vec{q}_k(t)}{\mathrm{d}t^2} = -\sum_{j\neq k} \vec{\nabla}_{\vec{q}_k} V_{kj}(\ ||\vec{q}_k(t) - \vec{q}_j(t)||\ ) \qquad \forall k \in \{1, ..., n\} \tag{23}$$

which is known as **Newton's Second Law** for historical reasons. As such, we identify the interaction terms $V_{kj}(||\vec{q}_k(t) - \vec{q}_j(t)||)$ as the so called **potential energy** terms. The minus sign we introduced was such that we can view the dynamics of this system as the acceleration vector of each particle $\frac{\mathrm{d}^2\vec{q}_k(t)}{\mathrm{d}t^2}$ pointing towards the minimum of the total potential energy (instead of the maximum). In fact, in classical mechanics, the gradient $\vec{\nabla}_{\vec{q}_k} V_{kj}(\ ||\vec{q}_k(t) - \vec{q}_j(t)||\ ) =: \vec{F}_{kj}(\vec{q}_k(t);\ \vec{q}_j(t))$ is called the **force** exerted by the particle j on the k-th.

An influence in the motion of the Universe by external factors (like other Universes)

Let us now loosen the assumption that the dynamics of the Universe (the motion of its particles) is only due to the properties of its space-time and the positions of its particles, and introduce the option to have some external factor to this Universe affect the dynamics inside it. This external factor should affect the trajectory of the Universe as a function of where its particles are, and it could actually vary in time for more generality. We could include such an influence by an arbitrary function $-Q(\vec{x}, t)$ of the positions and time in the Lagrangian (the minus sign would be just a convenience for having Newton's second law allow us interpret it as a potential energy). This would leave the Universe Lagrangian as:

$$L\left(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right) = \sum_{k=1}^{n} \frac{1}{2} m_k \Big|\Big| \frac{\mathrm{d}\vec{q}_k}{\mathrm{d}t} \Big|\Big|^2 - \left[\sum_{k<j} V_{kj}(\ ||\vec{q}_k - \vec{q}_j||\ ) + Q(\vec{q}_1, ..., \vec{q}_N, t)\right] \tag{24}$$

Let us call $Q(\vec{x}, t)$ the **external potential energy**. The equations of motion given by (5) would then be:

$$m_k \frac{\mathrm{d}^2\vec{q}_k(t)}{\mathrm{d}t^2} = -\vec{\nabla}_{\vec{q}_k} \left[\sum_{j\neq k} V_{kj}(\ ||\vec{q}_k - \vec{q}_j||\ ) + Q(\vec{q}_1, ..., \vec{q}_N, t)\right]\Bigg|_{\vec{q}=\vec{q}(t)} \qquad \forall k \in \{1, ..., n\} \tag{25}$$

For simplicity we could actually define a **total potential energy** function U as:

$$U(\vec{x}, t) := \sum_{k<j} V_{kj}(\ ||\vec{q}_k - \vec{q}_j||\ ) + Q(\vec{q}_1, ..., \vec{q}_N, t) \tag{26}$$

Having the packs of degrees of freedom for a same particle $\vec{q}_k$ jiggling around is a bit cumbersome. For notational ease we will just write it all for each individual degree of freedom x_k. This would leave us the Lagrangian and equations of motion for the Universe as:

$$L\left(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right) = \sum_{k=1}^{N} \frac{1}{2} m_k \left(\frac{\mathrm{d}x_k}{\mathrm{d}t}\right)^2 - U(\vec{x}, t) \tag{27}$$

$$m_k \frac{\mathrm{d}^2 x_k(t)}{\mathrm{d}t^2} = -\frac{\partial}{\partial x_k} U(\vec{x}, t)\Big|_{\vec{x}=\vec{x}(t)} \qquad \forall k \in \{1, ..., N\} \tag{28}$$

where we identify the masses of the same particle $m_{3j} = m_{3j-1} = m_{2j-2}$ for $j \in \{1, ..., n\}$.

Note now that in reality the external potential could be seen as the potential for a bigger system in which we forget about the things that are not the Universe of interest. That is, imagine for instance, that we have two Universes that interact between them, an alpha Universe of trajectory $\vec{x}^\alpha(t) = (x_1^\alpha(t), ..., x_N^\alpha(t))$ and a beta Universe of trajectory $\vec{x}^\beta(t) = (x_1^\beta(t), ..., x_M^\beta(t))$, with N, M not necessarily the same numbers (allowing them to have a different number of particles $n = N/3$ and $m = M/3$). A potential of interaction between them, that depended on the positions of one and the other one alone, could be introduced in the joint Lagrangian for the two Universes as a function $Q(\vec{x}^\alpha, \vec{x}^\beta)$ of their positions. Since time is homogeneous, we assume no explicit time dependence on their interaction term. The total Lagrangian would be:

$$L\left(t, \vec{x}^\alpha, \frac{\mathrm{d}\vec{x}^\alpha}{\mathrm{d}t}, \vec{x}^\beta, \frac{\mathrm{d}\vec{x}^\beta}{\mathrm{d}t}\right) = \sum_{k=1}^{n} \frac{1}{2} m_k^\alpha \left|\left|\frac{\mathrm{d}\vec{q}_k^\alpha}{\mathrm{d}t}\right|\right|^2 + \sum_{k=1}^{m} \frac{1}{2} m_k^\beta \left|\left|\frac{\mathrm{d}\vec{q}_k^\beta}{\mathrm{d}t}\right|\right|^2 - \tag{29}$$

$$-\left[\sum_{k<j} V_{kj}^\alpha(\ ||\vec{q}_k^\alpha - \vec{q}_j^\alpha||\ ) + \sum_{k<j} V_{kj}^\beta(\ ||\vec{q}_k^\beta - \vec{q}_j^\beta||\ ) + Q(\vec{x}^\alpha, \vec{x}^\beta)\right]$$

With the equations of motion for each Universe $\vartheta \in \{\alpha, \beta\}$ being left as:

$$m_k \frac{\mathrm{d}^2\vec{x}^\vartheta(t)}{\mathrm{d}t^2} = -\vec{\nabla}_{\vec{x}^\vartheta} \left[\sum_{j<k} V^\vartheta kj(\ ||\vec{q}_k^\vartheta - \vec{q}_j^\vartheta||\ ) + Q(\vec{x}^\alpha, \vec{x}^\beta)\right]\Bigg|_{\substack{\vec{x}^\alpha = \vec{x}^\alpha(t) \\ \vec{x}^\beta = \vec{x}^\beta(t)}} \qquad \vartheta \in \{\alpha, \beta\} \tag{30}$$

Then, if we simply wanted to forget about the existence of the Universe β and only contemplate the dynamics of α, since the gradient is only on the degrees of freedom of the Universe of interest, we could directly evaluate the trajectory of the uninteresting Universe, such that:

$$\vec{\nabla}_{\vec{x}^\alpha} Q(\vec{x}^\alpha, \vec{x}^\beta)\Big|_{\substack{\vec{x}^\alpha = \vec{x}^\alpha(t) \\ \vec{x}^\beta = \vec{x}^\beta(t)}} = \vec{\nabla}_{\vec{x}^\alpha} Q(\vec{x}^\alpha, \vec{x}^\beta(t))\Big|_{\vec{x}^\alpha = \vec{x}^\alpha(t)} \tag{31}$$

where we could then write with a notational abuse:

$$Q(\vec{x}^\alpha, \vec{x}^\beta(t)) = Q(\vec{x}^\alpha, t) \tag{32}$$

Which is the "external influence" potential we introduced. Leaving an equation of motion for the α-th Universe as:

$$m_k \frac{\mathrm{d}^2\vec{x}^\alpha(t)}{\mathrm{d}t^2} = -\vec{\nabla}_{\vec{x}^\alpha} \left[\sum_{j<k} V^\vartheta kj(\ ||\vec{q}_k^\alpha - \vec{q}_j^\alpha||\ ) + Q(\vec{x}^\alpha, t)\right]\Bigg|_{\vec{x}^\alpha = \vec{x}^\alpha(t)} \tag{33}$$

which is exactly the equation of motion (and Lagrangian) we had in equation (25).

In fact, we could also introduce interaction potentials between an arbitrary number of Universes and we would be able to do the same trick of equations (31) and (32) to get the external potential shape. Not only that, but we could introduce a collective interaction between all the Universes, as a potential depending on the trajectories of all of them, even for a continuity of Universes, in a way that from the perspective of each single Universe, the influence would still look like a simple external potential in time $Q(\vec{x}^\alpha, t)$. It is this why we are not dropping the homogeneity of the global time axis t by introducing the external potential that can depend on time. It is the **laws of physics for the fluid of Universes that have the same shape under time translations** (not each single Universe!). It would happen the same if we only considered one particle in a Universe with more than one particle interacting with the considered one. From the perspective of the single particle there would be an external time dependent potential, making time seem to be inhomogeneous. It must be the time of the outermost system (the most global perspective) for which time is homogeneous.

Now, in order to have the velocities of the possible trajectories and their energies all gathered in a same function, we will need to introduce some additional notation delving around with the Lagrangian. This will be a convenient way in which we will have enough with a single function to define the dynamic state of all the Tangent Universes. It will be possible by introducing two alternatives to the raw Euler-Lagrange equations that gave us Newton's Second Law. One will be the momentum and the Hamiltonian formalism, and the other will be the Hamilton-Jacobi action function.

A.2.1.2. The Momentum and the Hamiltonian (the Total Energy of a Universe)

First of all, let us define what we will call the **momentum** of the k-th degree of freedom of the Universe. It is the partial derivative of the Lagrangian with respect to the velocity of the k-th degree of freedom. That is:

$$p_k\left(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right) := \frac{\partial L(t, x, \frac{\mathrm{d}x_k}{\mathrm{d}t})}{\partial \frac{\mathrm{d}x_k}{\mathrm{d}t}} \tag{34}$$

Evaluating the most general Lagrangian for a Universe that we got, equation (24) or (27), we see that the k-th momentum is only a function of the k-th velocity for this Lagrangian:

$$p_k\left(\frac{\mathrm{d}x_k}{\mathrm{d}t}\right) = m_k \frac{\mathrm{d}x_k}{\mathrm{d}t} \tag{35}$$

Now, since the Lagrangian is a quadratic function with respect to any of the velocities $\frac{\mathrm{d}x_k}{\mathrm{d}t}$ (the potentials do not depend on velocity explicitly), it has a positive second partial derivative with respect to velocity. This means that the first partial derivative of the Lagrangian with respect to these velocities is a monotonically increasing function (strictly increases), implying that there is a one to one (and continuous) correspondence between the derivative of the Lagrangian with respect to a velocity (the momentum p_k) and the value of the velocity $\frac{\mathrm{d}q_k}{\mathrm{d}t}$. That is, by fixing the rest of variables, for each momentum p_k there is a single velocity variable $\frac{\mathrm{d}x_k}{\mathrm{d}t}$ and vice-versa. This means that the relation in equation (35) can be used as a change of variables: there exists an expression to get the momentum from the velocity $p_k(\frac{\mathrm{d}x_k}{\mathrm{d}t})$ and an inverse expression $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}(p_k)$, to get the velocity from the value of the momentum. In our case the relation is very simple indeed:

$$p_k\left(\frac{\mathrm{d}x_k}{\mathrm{d}t}\right) = m_k \frac{\mathrm{d}x_k}{\mathrm{d}t} \qquad \frac{\mathrm{d}x_k}{\mathrm{d}t}(p_k) = \frac{1}{m_k} p_k \tag{36}$$

The momentum is just the velocity times the mass.

We could now define the Lagrangian as a function of the momentum by simply evaluating $\frac{\mathrm{d}x_k}{\mathrm{d}t} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}(p_k)$. Instead however, we are looking for a function $H(t, \vec{x}, \vec{p})$ such that the derivative with respect to the k-th momentum, evaluated in $p_k(\frac{\mathrm{d}x_k}{\mathrm{d}t})$ gives us back the velocity $\frac{\mathrm{d}x_k}{\mathrm{d}t}$. This is, we are looking for a sort of reciprocal function of the Lagrangian with respect to this transformation. The function fulfilling this property is the so called **Hamiltonian** of the Universe:

$$H(t, x_1, ..., x_N, p_1, ..., p_N) := \sum_{k=1}^{N} p_k \frac{\mathrm{d}x_k}{\mathrm{d}t}(p_k) - L(t, x_1, ..., x_N, \frac{\mathrm{d}x_1}{\mathrm{d}t}(p_1), ... \frac{\mathrm{d}x_N}{\mathrm{d}t}(p_N)) \tag{37}$$

You can check with the chain rule that this and the Lagrangian are the functions relating the momenta and the velocity by their slopes. That is:

$$\frac{\partial H}{\partial p_k} = \frac{\mathrm{d}x_k}{\mathrm{d}t} \quad \text{and} \quad \frac{\partial L}{\partial \frac{\mathrm{d}x_k}{\mathrm{d}t}} = p_k\left(\frac{\mathrm{d}x_k}{\mathrm{d}t}\right) \tag{38}$$

We call this transformation that gives us the Hamiltonian from the Lagrangian and vice-versa, the **Legendre transformation** with respect to the velocity or the momentum. Note that this transformation is defined such that it is its own inverse: the Hamiltonian is the Legendre transformation of the Lagrangian with respect to the velocities, and the Lagrangian is the Legendre transformation of the Hamiltonian with respect to the momentum. The interest of such a transformation is that the Hamiltonian formalism with the momentum and positions instead of velocities and positions, gives us an alternative set of equations of motion to those ones offered by the Euler-Lagrange equations as follows.

If we take the partial derivative of the Hamiltonian with respect to the k-th position, we find:

$$\frac{\partial H}{\partial x_k} = \frac{\partial}{\partial x_k}\left(\sum_{k=1}^{N} p_k \frac{\mathrm{d}x_k}{\mathrm{d}t}(p_k) - L\left(t, x_1, ..., x_N, \frac{\mathrm{d}x_1}{\mathrm{d}t}(p_1), ...\frac{\mathrm{d}x_N}{\mathrm{d}t}(p_N)\right)\right) = -\frac{\partial L}{\partial x_k} \tag{39}$$

Now, the physical trajectory (the one extremizing the action) needed to obey the Euler-Lagrange equations (5), where we can find what $\frac{\partial L}{\partial x_k}$ must be equal to for that. Introducing it in the last expression and applying the definition of k-th momentum we get:

$$\frac{\partial H}{\partial x_k} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial(\frac{\mathrm{d}x_k}{\mathrm{d}t})} = -\frac{\mathrm{d}}{\mathrm{d}t}p_k \tag{40}$$

Gathering equations (38) (40), we have that instead of using the Euler-Lagrange equations to evolve the trajectory in time $\vec{x}(t)$, we could use the so called **Hamiltonian equations**, that will provide us an equivalent trajectory in the so called phase space $(\vec{x}(t), \vec{p}(t))$:

$$\frac{\partial H(t, \vec{x}(t), \vec{p}(t))}{\partial p_k} = \frac{\mathrm{d}x_k(t)}{\mathrm{d}t} \qquad \frac{\partial H(t, \vec{x}(t), \vec{p}(t))}{\partial x_k} = -\frac{\mathrm{d}p_k(t)}{\mathrm{d}t} \qquad k \in \{1, ..., N\} \tag{41}$$

The practical advantage of these equations in front of the Euler-Lagrange equations (which should yield the same solutions), is that while the Euler-Lagrange equations are N second order differential equations, Hamilton's equations are $2N$ first order differential equations, which might provide a better insight in some problems. In particular, if a coordinate x_k does not appear in the Hamiltonian, its corresponding momentum is conserved in time by the second Hamilton's equation of (41): thus the coordinate can be ignored in the other equations, effectively reducing the number of equations to $2N-2$. In the Lagrangian formalism on the other hand, if some momentum is conserved, the velocities in the Lagrangian might still occur, avoiding it to reduce the number of equations to be solved.

Aside from this practical point, the Hamiltonian has also a very interesting feature: it is the definition of the **total energy** of the system in a given time, configuration and momentum. For the most general Lagrangian we got from equation (27), we have that the Hamiltonian is:

$$H(t, \vec{x}, \vec{p}\,) = \sum_{k=1}^{N} \frac{p_k}{2m_k}^2 + U(\vec{x}, t) = \sum_{k=1}^{N} \frac{p_k}{2m_k}^2 + \sum_{j>k} V_{kj}(\ ||\vec{q}_k - \vec{q}_j||\ ) + Q(\vec{x}, t) \tag{42}$$

which using that $p_k = m_k \frac{\mathrm{d}x_k}{\mathrm{d}t}$ can be seen to be the sum of the total kinetic and potential energies (inter-particle and external ones). That is why we called those terms as "energies" in the first place!

A.2.1.3. Introducing the Hamilton-Jacobi Action function

It turns out that even if the action functional for an optimal path (4) is complicated to be computed analytically, we can still compute the variation of the action in front of variations of the end point of the physical trajectory in space or time (the point $\vec{x}_f$ or t_f).

Let us name the physical trajectory for the end-points $\vec{x}(t_0) = \vec{x}_0$ and $\vec{x}(t_f) = \vec{x}_f$ as $\vec{x}_{old}(t)$. Let us now consider what happens if we slightly **vary the end position** of the trajectory $\vec{x}_f$ to $\vec{x}_f + \Delta\vec{x}$, with $\Delta\vec{x}_f = (\Delta x_{1_f}, ..., \Delta x_{N_f})$ a vector of small displacements in configuration space $\mathbb{R}^N$. If we keep fixed the starting point $\vec{x}(t_0) = \vec{x}_0$, the new trajectory for the Universe will be the previous physical one $\vec{x}_{old}(t)$ plus a variation $\delta\vec{x}(t) = (\delta x_1(t), ..., \delta x_N(t))$ at each time, such that the new trajectory $\vec{x}(t) = \vec{x}_{old} + \delta\vec{x}(t)$ will have $\delta\vec{x}(t_0) = 0$ and $\delta\vec{x}(t_f) = \Delta\vec{x}$. If we now expand the Lagrangian in Taylor series for the trajectory $\vec{x}(t) = \vec{x}_{old}(t) + \delta\vec{x}(t)$ (such that $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{x}_{old}}{\mathrm{d}t}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{x}(t)$), centered at the trajectory $\vec{x}_{old}(t)$, we have that:

$$L\left(t, \vec{x}(t), \frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t}\right) = L\left(t, \vec{x}_{old}(t), \frac{\mathrm{d}\vec{x}_{old}(t)}{\mathrm{d}t}\right) + \sum_{k=1}^{N} \frac{\partial L\left(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right)}{\partial x_k}\Big|_{\vec{x}_{old}(t)} (x_k(t) - x_{k_{old}}(t)) + \tag{43}$$

$$+\sum_{k=1}^{N} \frac{\partial L\left(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right)}{\partial \frac{\mathrm{d}x_k}{\mathrm{d}t}}\Big|_{\vec{x}_{old}(t)} \left(\frac{\mathrm{d}}{\mathrm{d}t}x_k(t) - \frac{\mathrm{d}}{\mathrm{d}t}x_{k_{old}}(t)\right) + o\left(||\vec{x}(t) - \vec{x}_{old}(t)||^2 + ||\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}(t) - \frac{\mathrm{d}\vec{x}_{old}}{\mathrm{d}t}(t)||^2\right) =$$

$$= L\left(t, \vec{x}_{old}(t), \frac{\mathrm{d}\vec{x}_{old}(t)}{\mathrm{d}t}\right) + \sum_{k=1}^{N} \left\{ \frac{\partial L\left(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right)}{\partial x_k}\Big|_{\vec{x}_{old}(t)} \delta x_k(t) + \frac{\partial L\left(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right)}{\partial \frac{\mathrm{d}x_k}{\mathrm{d}t}}\Big|_{\vec{x}_{old}(t)} \frac{\mathrm{d}}{\mathrm{d}t}\delta x_k(t) \right\}$$

$$+o\left(||\delta\vec{x}(t)||^2 + ||\frac{\mathrm{d}\delta\vec{x}(t)}{\mathrm{d}t}||^2\right)$$

Since the displacement of the end-point $\Delta\vec{x}_f$ can be taken arbitrarily small, the variation in the action between the old and the new trajectory (differing in their end points) will then be given to first order by:

$$\Delta\mathbb{S} := \mathbb{S}[\vec{x}(t)] - \mathbb{S}[\vec{x}_{old}(t)] = \int_{t_0}^{t_f} L\left(t, \vec{x}(t), \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right) dt - \int_{t_0}^{t_f} L\left(t, \vec{x}_{old}(t), \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right) dt = \tag{44}$$

$$= \int_{t_0}^{t_f} \sum_{k=1}^{N} \left\{ \frac{\partial L\left(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right)}{\partial x_k}\Big|_{\vec{x}_{old}(t)} \delta x_k(t) + \frac{\partial L\left(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right)}{\partial \frac{\mathrm{d}x_k}{\mathrm{d}t}}\Big|_{\vec{x}_{old}(t)} \frac{\mathrm{d}}{\mathrm{d}t}\delta x_k(t) \right\} dt$$

Since $\vec{x}_{old}(t)$ is a physical trajectory, it follows the Euler-Lagrange equations (5), meaning we can evaluate $\frac{\partial L}{\partial x_k}$ to get:

$$\Delta\mathbb{S} = \int_{t_0}^{t_f} \sum_{k=1}^{N} \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L\left(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right)}{\partial \frac{\mathrm{d}x_k}{\mathrm{d}t}}\Big|_{\vec{x}_{old}(t)} \right) \delta x_k(t) + \frac{\partial L\left(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right)}{\partial \frac{\mathrm{d}x_k}{\mathrm{d}t}}\Big|_{\vec{x}_{old}(t)} \frac{\mathrm{d}}{\mathrm{d}t}\delta x_k(t) \right\} dt = \tag{45}$$

$$= \sum_{k=1}^{N} \int_{t_0}^{t_f} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L\left(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right)}{\partial \frac{\mathrm{d}x_k}{\mathrm{d}t}}\Big|_{\vec{x}_{old}(t)} \delta x_k(t) \right) dt =$$

$$= \sum_{k=1}^{N} \left[\frac{\partial L\left(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right)}{\partial \frac{\mathrm{d}x_k}{\mathrm{d}t}}\Big|_{\vec{x}_{old}(t)} \delta x_k(t) \right]_{t_0}^{t_f} = \sum_{k=1}^{N} \frac{\partial L\left(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right)}{\partial \frac{\mathrm{d}x_k}{\mathrm{d}t}}\Big|_{\vec{x}(t_f)} \Delta x_{k_f}$$

where we used the derivative of a product rule and the fact that $\delta\vec{x}(t_f) = \Delta\vec{x}_f$ and $\delta\vec{x}(t_0) = 0$.

Now, choosing the displacement to be $\Delta x_{j_f} = 0 \ \forall j \neq k$:

$$\frac{\Delta\mathbb{S}(\vec{x}_f)}{\Delta x_{k_f}} = \frac{\partial L\left(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right)}{\partial \frac{\mathrm{d}x_k}{\mathrm{d}t}}\Big|_{\vec{x}(t_f)} =: p\left(\frac{\mathrm{d}\vec{x}(t_f)}{\mathrm{d}t}\right) \tag{46}$$

As we make the limit $\Delta x_{k_f} \to 0$, we get that the first order approximation we made will be an exact equality and that:

$$\frac{\partial S(\vec{x}, t)}{\partial x_k} = p_k(t) \tag{47}$$

where we defined the **Hamilton-Jacobi action function** as:

$$S(\vec{x}, t) := \int_{t_0;\ \vec{x}_0}^{t;\ \vec{x}} L\left(\tau, \vec{x}(\tau), \frac{\mathrm{d}\vec{x}(\tau)}{\mathrm{dt}}\right)\, d\tau \tag{48}$$

with t_0 and $\vec{x}_0$ are fixed undetermined constants and $\vec{x}, t$ representing the final end-point of the trajectory.

Equation (47) means that the variation in the action with respect to space, gives the momentum field (and thus velocity field) for the physical trajectories.

Finally, let us understand what happens if we **vary the end time** of the trajectory by an arbitrarily small amount Δt (ensuring that the same point $\vec{x}_f$ is crossed at time t_f):

$$\Delta\mathbb{S}[t_f] := \mathbb{S}[\vec{x}(t); \vec{x}(t_f), t_f] - \mathbb{S}[\vec{x}(t); \vec{x}(t_f+\Delta t), t_f+\Delta t] = \int_{t_0}^{t_f+\Delta t} L\left(t, \vec{x}(t), \frac{\mathrm{d}\vec{x}}{\mathrm{dt}}\right) dt - \int_{t_0}^{t_f} L\left(t, \vec{x}(t), \frac{\mathrm{d}\vec{x}}{\mathrm{dt}}\right) dt \tag{49}$$

$$\lim_{\Delta t\to 0} \Delta\mathbb{S}[t_f] = \lim_{\Delta t\to 0} \int_{t_f}^{\Delta t} L\left(t, \vec{x}(t), \frac{\mathrm{d}\vec{x}}{\mathrm{dt}}\right) dt = \lim_{\Delta t\to 0} L\left(t_f, \vec{x}(t_f), \frac{\mathrm{d}\vec{x}}{\mathrm{d}t_f}\right)\Delta t$$

where we used the definition of integral as a limit of Riemann sums. Then, these would leave:

$$\lim_{\Delta t\to 0}\left(\frac{\Delta\mathbb{S}[t_f]}{\Delta t}\right) = L\left(t_f, \vec{x}(t_f), \frac{\mathrm{d}\vec{x}}{\mathrm{d}t_f}\right) \tag{50}$$

such that:

$$\frac{\mathrm{d}S(\vec{x}, t)}{\mathrm{d}t} = L(t, \vec{x}, \frac{\mathrm{d}\vec{x}}{\mathrm{dt}}) \tag{51}$$

Now, using that along the trajectory $\vec{x}(t)$ by the chain rule:

$$\frac{\mathrm{d}S(\vec{x}(t), t)}{\mathrm{dt}} = \frac{\partial S(\vec{x}(t), t)}{\partial t} + \sum_{k=1}^{N} \frac{\partial S(\vec{x}, t)}{\partial x_k}\bigg|_{\vec{x}(t)} \frac{\mathrm{d}x_k(t)}{\mathrm{dt}} \tag{52}$$

and using that if the trajectory $\vec{x}(t)$ is a physical trajectory, following the Euler-Lagrange equations, then equations (47) and (51) hold, we can evaluate them here to get:

$$L\left(t, \vec{x}(t), \frac{\mathrm{d}\vec{x}(t)}{\mathrm{dt}}\right) = \frac{\partial S(\vec{x}(t), t)}{\partial t} + \sum_{k=1}^{N} p_k\left(\frac{\mathrm{d}x_k(t)}{\mathrm{dt}}\right)\frac{\mathrm{d}x_k(t)}{\mathrm{dt}} \tag{53}$$

By writing the velocity as a function of the momentum $\frac{\mathrm{d}\vec{x}}{\mathrm{dt}}(\vec{p})$ and rearranging terms, we can see that suddenly the Hamiltonian emerges back:

$$-\frac{\partial S(\vec{x}(t), t)}{\partial t} = \sum_{k=1}^{N} p_k \frac{\mathrm{d}x_k}{\mathrm{dt}}(p_k(t)) - L\left(t, \vec{x}(t), \frac{\mathrm{d}\vec{x}(\vec{p}(t))}{\mathrm{dt}}\right) = H(t, \vec{x}(t), \vec{p}(t)) \tag{54}$$

Thus leaving that:

$$\frac{\partial S(\vec{x}, t)}{\partial t} = -H(t, \vec{x}, \vec{p}) \tag{55}$$

which means that the variation of the action if we move a time step along the trajectory is (minus) the energy of the Universe at that time.

The two equations (47) and (55), lead us to a third way to get the motion of the Universe:

$$\frac{\partial S(\vec{x}, t)}{\partial t} = -H\left(t, \vec{x}, \vec{p} = \vec{\nabla}_x S(\vec{x}, t)\right) \tag{56}$$

which lets us evolve the field $S(\vec{x}, t)$ given an initial relative action field $S(\vec{x}, t_0) = s_0(\vec{x})$. This is called the **Hamilton-Jacobi equation**.

Note that when solving this equation, one cannot simply evolve a single trajectory, as allowed us the Euler-Lagrange (Newton's law) and Hamilton's equations. Instead, we here evolve a field in configuration space $S(\vec{x}, t)$, the spatial derivative of which will provide us the momentum and thus velocity field to evolve the trajectories. That is, solving the Hamilton-Jacobi equation means getting all the possible trajectories of the system provided a relative initial action field. This is why it is not typically employed as a practical tool for calculations, since normally we are only interested in getting a single trajectory for the system, and not all the possible trajectories. It turns out however, that this will be the perfect tool for evolving all the trajectories of the different Universes at once! Thus the relevance of its introduction for us to arrive to quantum mechanics.

A.2.2. The Stationary Action Axiom at the Universe Fluid level

The equations until now assumed that if there were more than one Universes they were not able to interact with each other. We have found that within each of these non-interacting Universes, the trajectory of their particles was predictable by just knowing the position of the particles at any particular time and integrating any of the shapes for the equations of motion we derived (most conveniently the Euler-Lagrange or Hamilton's equations).

Now, let us explicitly introduce the assumption that in the locality of any possible configuration $\vec{x} \in \Omega_t \subseteq \mathbb{R}^N$, there are $\rho(\vec{x}, t)dx_1 \cdots dx_N$ actual physical Universes with the **same number of particles and relative interaction potentials**, with their physical spaces having the **same symmetries** as the ones required to derive the equations of motion of the previous sections. If all these conditions were the same for each of the Universes, the equations of motion for all of them would be the same as well, and would only depend on their initial configurations. This would also make them all have the same Lagrangian and action functional, and all of them would have **the same Hamilton-Jacobi action** $S(\vec{x}, t)$. The derivative $\frac{\partial S(\vec{x},t)}{\partial x_k}$ would yield **the same** field of momentums and velocities for their possible physical trajectories, by equation (47). The derivative $-\frac{\partial S(\vec{x},t)}{\partial t}$ would yield **the same** total energy for a trajectory at $\vec{x}$ and time t, of any of the Universes, due to equation (55).

Now, if we assume that each Universe has a different and defined configuration $\vec{\xi}$ at a time t_0, we will have that the single **same** action function $S(\vec{x}, t)$ will encode the velocity and total energy fields (and thus the trajectories) of all the different Universes. As was stated in the introduction, we will express these trajectories with a function $\vec{x}(\vec{\xi}, t)$, where the label $\vec{\xi}$ tags the trajectory that was at $\vec{\xi}$ at time t_0. That is, using that $\vec{x}(\vec{\xi}, t_0) = \vec{\xi}$.

Since all Universes have the same $S(\vec{x}, t)$, we will be able to get the different trajectories by integrating the same velocity field for each of the different $\vec{\xi}$. We can get this velocity field by inverting the relation between the momentum and the velocity:

$$\frac{\mathrm{d}x_k(\vec{\xi},\, t)}{\mathrm{d}t} = v_k(\vec{x}(\vec{\xi}, t), t) = \frac{1}{m_k} p_k\left(\frac{\mathrm{d}x_k(\vec{\xi},\, t)}{\mathrm{d}t}\right) = \frac{1}{m_k}\frac{\partial S(\vec{x}, t)}{\partial x_k}\Big|_{\vec{x}=\vec{x}(\vec{\xi},t)} \quad \forall k \in \{1, ..., N\} \tag{57}$$

Defining the velocity field for the fluid of Universes as:

$$v_k(\vec{x}(\vec{\xi}, t), t) := \frac{1}{m_k}\frac{\partial S(\vec{x}, t)}{\partial x_k}\Big|_{\vec{x}=\vec{x}(\vec{\xi},t)} \tag{58}$$

In short, we will have enough with a single field $S(\vec{x}, t)$ to encode the trajectories of all the Universes.

If we leave a null external potential $Q(\vec{x},t) \equiv 0$, the narrative till this point would be a valid interpretation of classical mechanics. This interpretation would be a bit convoluted because it would be saying that there are infinite **parallel** Universes (**not tangent**, since they do not interact and thus could cross each other like ghosts). The only interest of this narrative would be that since we do not know which exactly are our "initial" conditions, computing all the possible trajectories $\vec{x}(\vec{\xi},t)$ would be helpful. To reflect this uncertainty we could introduce a probability density, that according to this interpretation could be seen as a density of parallel Universes $\rho(\vec{x},t)$. Evolving the density with a continuity equation, we would have a stochastic classical mechanics theory.

However, since the density would not be able to affect the trajectories of the Universes, as it appears no-where in the equations of motion, the "only one version of reality" interpretation would work as fine as the parallel Universe interpretation. So there would be no need to say that the probability density is in reality the density of parallel Universes. If we see no effect of the parallel Universes in the motion of ours, why to assume their existence? By Occam's Razor, we would clearly prune this overcomplicated interpretation.

It will turn out in quantum mechanics however, that the density will affect contingently each trajectory, such that from any of the Universes the pushing effect of the rest could be physically perceived. It is this why in quantum mechanics, a **tangent** Universe interpretation will be the most natural one, against an interpretation where "possible but non-existent" realities affect the perceived one, which is clearly non-sense.

Note how for now nothing seems to avoid the trajectories $\vec{x}(\vec{\xi},t)$ to get multivalued (to cross each other at the same time). The "univalued-ness" will be ensured by the fact that we will assume S **is differentiable with continuous first derivatives**[10], such that the momentum and thus the velocity field $\vec{v}(\vec{x},t)$ for the trajectories, defined in equation (58), will be **continuous**. Then, by the Picard-Lindelöf theorem for the existence and uniqueness of the initial value problem (57), the trajectories will exist and will be unique for a given initial condition. This means that just a single trajectory will cross each configuration-space and time point at the same time.

Finally, let us explicitly introduce the possibility for the trajectories of the particles in each Universe to be affected by the positions of the other Universes' particles. That is, let the density of Universes ρ affect the action $\mathbb{S}$. We will introduce an interaction between the Tangent Universes via the extra potential energy term $Q(\vec{x},t)$ we left for influences that could come from outside each single Universe. Its shape will be determined by the last three axioms of the narrative.

A.2.2.1 The "Second Law of Thermodynamics" for the Fluid of Universes

We will force the density of Universes to tend towards a homogeneous distribution as time goes forward. That is, we will introduce a constraint such that the natural tendency of the fluid of Universes is towards having the same number of Universes in each possible configuration. This, from our perspective would mean that the a priori knowledge about which of them is our Universe would be each time more uncertain, a priori attributing to all of them the same likelihood and thus using the density of Universes as probability density of our Universe. This could be translated in the gradual loss of some information measure, such as an increase in Shannon entropy of the density $\rho(\vec{x},t)$, or decrease of its Fisher information.

We have the Shannon entropy for the density as:

$$s[\rho(\vec{x},t);\ t] = \iint_{\Omega_t} \rho(\vec{x},t)\ log\left(\frac{1}{\rho(\vec{x},t)}\right) dx_1 \cdots dx_N \tag{59}$$

[10]Lipschitz-continous in $\vec{x}$ and continous in t.

while the Fisher Information would be:

$$I[\rho(\vec{x},t);\ t] = \iint_{\Omega_t} \rho(\vec{x},t)\ \left|\left|\vec{\nabla}_x log\left(\frac{1}{\rho(\vec{x},t))}\right)\right|\right|^2 dx_1\cdots dx_N \tag{60}$$

They both have a similar look, just that in the Fisher information we are integrating the absolute value of the **local variation** of the information in each support point, while in the Shannon entropy we integrate the information in the whole support globally[11].

Both metrics are extremized with the most uncertain of the densities, which would be a uniform probability density, where every "point" in the support is equally likely. As such, an increase in entropy or decrease in Fisher information could be contemplated as an approach of the density towards a uniform distribution. However, in the case of Shannon entropy, this statement is only loosely held, since it only gives us a global perception of the proximity of the density to a uniform distribution. That is, it is not sensitive to small intervals of the support getting closer to uniformity: it is only a global metric in this sense. Unlike entropy, since Fisher information can be rewritten as:

$$I[\rho(\vec{x},t);\ t] = \iint_{\Omega_t} \frac{1}{\rho(\vec{x},t)}\ \sum_{k=1}^{N}\left(\frac{\partial\rho(\vec{x},t)}{\partial x_k}\right)^2 dx_1\cdots dx_N = \iint_{\Omega_t}\frac{1}{\rho(\vec{x},t)}\ \left|\left|\vec{\nabla}_x\rho(\vec{x},t)\right|\right|^2 dx_1\cdots dx_N \tag{61}$$

we can see that it measures precisely how flat the density is in each locality, by weighting the integral in each volume element by the absolute value of the gradient of density at that point. The flatter the density in *all* directions, the smaller the magnitude of the gradient will be in this locality (the gradient measures the directional derivative in the maximum increasing direction), meaning if there is even a single direction in which the density is not flat, the gradient's magnitude will capture it. Then the flatter the locality of a volume element, it will contribute less. Being the continuous sum of the contribution of each locality to the global flatness, it provides us a more local metric of uniformity of the density. Note that this sum is weighted by the inverse of the density at that locality as a means of normalization for its local variation.

Following this line of reasoning, we will add the axiom that given the boundary density functions $\rho(\vec{x},t_0) = w_0(\vec{x})$ and $\rho(\vec{x},t_f) = w_f(\vec{x})$ with $t_0 < t_f$, **the density will need to evolve in a way that minimizes its total Fisher information**. That is, we will look for the minimization of the functional:

$$\mathscr{M}[\rho(\vec{x},t);\ t_0,t_f,w_0(\vec{x}),w_f(\vec{x})] = \int_{t_0}^{t_f} I[\rho(\vec{x},t);t]\ dt = \tag{62}$$

$$= \int_{t_0}^{t_f}\iint_{\Omega_t}\frac{1}{\rho(\vec{x},t)}\ \sum_{k=1}^{N}\left(\frac{\partial\rho(\vec{x},t)}{\partial x_k}\right)^2 dx_1\cdots dx_N\ dt$$

Now, instead of considering the axiom from an information theoretic point of view, we could have also introduced it as a claim on the behaviour of the Tangent Universes. We could have posed the axiom in these other words: the Tangent Universes evolve in a repulsive manner between each other, such that they try to avoid their agglomeration and instead tend towards homogenizing the number of Universes in each patch of configuration-space. They tend to get relatively dispersed as much as possible. Then, we would contemplate the magnitude of the gradient of the density of Universes as a measure of the discomfort of the Universes in a certain locality of configuration-space, as a measure of the possibility for the Universes in that locality to get more evenly dispersed. We would try to minimize it normalized by the relative amount of Universes in that locality. If so, we would arrive to the same functional to be minimized: equation (62).

[11]Note that the information density is defined as $i[\rho(\vec{x})]) = log\left(\frac{1}{\rho(\vec{x})}\right)$. This gives the logarithm of the relative uncertainty cleared out when the outcome of the random variable is known to be $\vec{x}$ (well, technically the integral in a domain gives that, since it is a density). In the discrete analogy, if an event has probability $P = 2/3$, then it happens in 2 out of 3 alternative Universes. If we knew that this event has happened, we would have gained $3/2 = 1.5$ Universes worth information.

Now, this would be the case if all the particles were exactly the same. However, it turns out that not every particle has a same mobility in front of a same impulse, being the restriction measured by the mass constant m_k. This will mean that how the Tangent Universe discomfort acts in each particle will not need to be the same for all the particles. Instead, we will allow the contribution to the discomfort by each particle to be weighted by its mass, in a way where the bigger the mass, the less discomfort will be provided by the particle in front of a same agglomeration. We will represent this by adding a dividing mass m_k weighting the contribution of the particle to the discomfort of the Universe in a certain configuration $\vec{x}$. This would give us a Fisher Information where the variable x_k is weighted by its mass m_k, which would leave equation (62) in its general form as:

$$\mathscr{M}[\rho(\vec{x},t);\ t_0, t_f, w_0(\vec{x}), w_f(\vec{x}), m_k] = \tag{63}$$

$$= \int_{t_0}^{t_f} \iint_{\Omega_t} \frac{1}{\rho(\vec{x},t)} \sum_{k=1}^{N} \frac{1}{m_k} \left(\frac{\partial \rho(\vec{x},t)}{\partial x_k}\right)^2 dx_1 \cdots dx_N \ dt$$

Before we jump into the last axiom, note how subtlely we have here introduced a very big assumption about time. Until here time was isotropic, meaning that the laws of physics were the same in any of the directions of time we could look at, both forwards or backwards, implying that the fundamental laws of physics did not have any preference for our perceived flow direction of time. However, we have here introduced the postulate that **time goes in the direction in which the Fisher Information of the fluid of Tangent Universes is minimized**, which turns out to be the direction of time we are used to. A direction in which energy gets irreversibly dispersed (as we will see in the next sub-section).!!!!!!!!! Habrá que ver si la variación del haimltoniano along a trajectory es negativa en el tiempo o qué.

At this point it is clear the analogy of this axiom with the Second Law of Thermodynamics. In fact, since thermodynamics is a theory for the emergent macro-world states from the underlying micro-world dynamics, the Second Law could possibly be related to this axiom ruling the dispersive evolution of the Universes in a more fundamental level.

A.2.2.2 Universes are not Created nor Destroyed

Given an arbitrary configuration-space volume $V_0 \subseteq \Omega_0 \subseteq \mathbb{R}^N$, we can know the (relative) number of Universes in that volume at the reference time t_0 by simply integrating the density of Universes over it:

$$\iint_{V_0} \rho(\vec{x}, t=t_0) dx_1 \cdots dx_N = \iint_{V_0} \rho(\vec{\xi}, t=t_0) d\xi_1 \cdots d\xi_N \tag{64}$$

Note how we could employ directly both the density in the Lagrangian and the Eulerian frames for this integration, since we defined that $\vec{x}(\vec{\xi}, t=t_0) = \vec{\xi}$.

Now, we would like to get the condition for the next axiom to hold: **the number of Universes in any locality of the fluid should be conserved**, that is, Universes are never created nor destroyed, they can move to adjacent points in configuration-space, but not disappear nor appear out of the blue.

This is equivalent to asking that the number of Universes in the arbitrary volume V_0 at time t_0 and the number of Universes in the volume $\vec{x}(V_0, t)$ following the trajectories of the Universes in V_0, should be the same at any time t. Mathematically this could be written as follows:

$$\iint_{V_0} \rho(\vec{\xi}, t=t_0) d\xi_1 \cdots d\xi_N = \iint_{\vec{x}(V_0,t)} \rho(\vec{x}, t) dx_1 \cdots dx_N \quad \forall t \tag{65}$$

If we use the change of variables for integrals (**??**) explained in Appendix β, we can rewrite it as:

$$\iint_{V_0} \rho(\vec{\xi}, t=t_0) d\xi_1 \cdots d\xi_N = \iint_{V_0} \rho(\vec{\xi}, t) J(\vec{\xi}, t) d\xi_1 \cdots d\xi_N \quad \forall t \tag{66}$$

Manipulating it we get:

$$\iint_{V_0} \left[\rho(\vec{\xi}, t=t_0) - \rho(\vec{\xi}, t)J(\vec{\xi}, t)\right] d\xi_1 \cdots d\xi_N = 0 \tag{67}$$

Since this must be true for any arbitrary V_0, it must be true that:

$$\rho(\vec{\xi}, t_0) = \rho(\vec{\xi}, t)J(\vec{\xi}, t) \tag{68}$$

Or equivalently if we evaluate $\vec{\xi} = \vec{\xi}(\vec{x}, t)$:

$$\rho(\vec{x}, t_0) = \rho(\vec{x}, t)J(\vec{x}, t) \tag{69}$$

As the left hand side is time independent in both previous equations, if we derivate each side in time, we get for the first that:

$$0 = \frac{\partial}{\partial t}\Big(\rho(\vec{\xi}, t)J(\vec{\xi}, t)\Big) \tag{70}$$

which implies that along any given Universe trajectory $\vec{\xi}$, the quantity $\rho(\vec{\xi}, t)J(\vec{\xi}, t)$ must be a constant of motion if we want the local conservation of the number of Universes. For the second one we get:

$$0 = \frac{\partial}{\partial t}(\rho(\vec{x}, t)J(\vec{x}, t)) \tag{71}$$

which implies that even if we fix a configuration space point, the product of the Jacobian of the trajectories and the density should be constant in time for the conservation of the number of Universes.

Note in particular that the local conservation implies that the norm of the density in the whole configuration space will also be preserved in time. This is perfect for the epistemological interpretation we gave it as a probability density!

A.2.2.3 The Total Energy in the Fluid of Universes is Preserved (Energy is never Created nor Destroyed)

Note before anything that the variation of the total energy (the Hamiltonian) of a Universe along its trajectory in phase space $\vec{x}(t), \vec{p}(t)$ is:

$$\frac{\mathrm{d}}{\mathrm{d}t}H(t, \vec{x}(t), \vec{p}(t)) = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x_k}\Big|_{\vec{x}(t);\ \vec{p}(t)} \frac{\mathrm{d}x_k(t)}{\mathrm{d}t} + \frac{\partial H}{\partial p_k}\Big|_{\vec{x}(t);\ \vec{p}(t)} \frac{\mathrm{d}p_k(t)}{\mathrm{d}t} \tag{72}$$

where using Hamilton's equations of motion for the physical trajectory, equations (41), two of its terms cancel each other out to leave:

$$\frac{\mathrm{d}}{\mathrm{d}t}H(t, \vec{x}(t), \vec{p}(t)) = \frac{\partial H}{\partial t} - \frac{\mathrm{d}p_k(t)}{\mathrm{d}t}\frac{\mathrm{d}x_k(t)}{\mathrm{d}t} + \frac{\mathrm{d}x_k(t)}{\mathrm{d}t}\frac{\mathrm{d}p_k(t)}{\mathrm{d}t} = \frac{\partial H(t, \vec{x}(t), \vec{p}(t))}{\partial t} \tag{73}$$

For our most general Lagrangian (and thus Hamiltonian) for a Universe (24), this would mean that only the explicitly time dependent terms would survive. In particular, this could only be the external potential energy:

$$\frac{\mathrm{d}}{\mathrm{d}t}H(t, \vec{x}(t), \vec{p}(t)) = \frac{\partial}{\partial t}H(t, \vec{x}(t), \vec{p}(t)) = \frac{\partial}{\partial t}Q(\vec{x}(t), t) \tag{74}$$

Notice that if there was no external potential energy $Q(\vec{x}, t) \equiv 0$, the total energy of a single Universe would be a constant of motion (energy would not be created nor destroyed, just transferred between the particles within the Universe). This is the case in classical mechanics, where $Q(\vec{x}, t) \equiv 0$. There, the total energy in the Universe is constant in time, whcih is actually a direct consequence of the time within each Universe being homogeneous.

However, we can see that when an external potential $Q(\vec{x},t) \neq 0$ is introduced (unlike in classical mechanics), its time dependence would act as a varying source or sink of the total energy of this Universe in time.

At any given time t, the total energy of the set of Universes around the configuration $\vec{x}$ will be the Hamiltonian at that point times the number of Universes in that locality $H(t,\vec{x},\vec{p})\rho(\vec{x},t)dx_1\cdots dx_N$. Thus, we could compute the total amount of energy in the whole configuration space by adding up the energy contribution of each Universe as:

$$E_{fluid}(t) = \iint_{\Omega_t} H\ \rho(\vec{x},t)dx_1\cdots dx_N \tag{75}$$

where we employed equation (55).

We can change the variables to the label space using (**??**) and integrate the energy in the same volume at all times. This will be convenient to compute the variation in time of the energy, since it will allow us to introduce the time derivative inside the integral. Otherwise, because the domain itself is changing in time, we would need to struggle a bit more. With respect to the reference domain Ω_0, we will have that:

$$E_{fluid}(t) = \iint_{\Omega_0} H\Big|_{\vec{x}(\vec{\xi},t)}\rho(\vec{x}(\vec{\xi},t),t)J(\vec{\xi},t)d\xi_1\cdots d\xi_N \tag{76}$$

Now, asking that the total energy in the fluid of Universes, along the trajectories is constant means that the time derivative of the fluid energy must be zero:

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{fluid}(t) = \iint_{\Omega_0} -\frac{\mathrm{d}}{\mathrm{d}t}\left[H\Big|_{\vec{x}(\vec{\xi},t)}\rho(\vec{x}(\vec{\xi},t),t)J(\vec{\xi},t)\right]d\xi_1\cdots d\xi_N = 0 \tag{77}$$

As we anticipated, now that the integral is over a constant domain, assuming regularity for the Hamiltonian, the density and the Jacobian, we can introduce the derivative into the integral. Developing, we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{fluid}(t) = \iint_{\Omega_0} \Big[\rho(\vec{\xi},t)J(\vec{\xi},t)\frac{\mathrm{d}}{\mathrm{d}t}H(t,\vec{x}(\vec{\xi},t),\vec{p}(\vec{\xi},t)) + \tag{78}$$
$$+H(t,\vec{x}(\vec{\xi},t),\vec{p}(\vec{\xi},t))\frac{\partial}{\partial t}\Big(\rho(\vec{\xi},t)J(\vec{\xi},t)\Big)\Big]d\xi_1\cdots d\xi_N$$

Two things are simplified here. On the one hand, we can use the same reasoning as in equation (74), to get that the total derivative of the Hamiltonian (the variation along a trajectory) is equal to the partial derivative of Q in time. On the other hand, we can use the axiom by which Universes cannot be created nor destroyed, which was summarized by equation (70). This would leave:

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{fluid}(t) = \iint_{\Omega_0} \rho(\vec{\xi},t)J(\vec{\xi},t)\frac{\partial}{\partial t}Q(\vec{x},t)\Big|_{\vec{x}(\vec{\xi},t)}d\xi_1\cdots d\xi_N \tag{79}$$

Finally, we can write a reverse derivative of a product to get:

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{fluid}(t) = \iint_{\Omega_0} \left\{\frac{\partial}{\partial t}[\rho(\vec{x},t)J(\vec{x},t)Q(\vec{x},t)]\Big|_{\vec{x}=\vec{x}(\vec{\xi},t)} - \frac{\partial}{\partial t}(\rho(\vec{x},t)J(\vec{x},t))\Big|_{\vec{x}(\vec{\xi},t)}Q(\vec{x},t)\Big|_{\vec{x}(\vec{\xi},t)}\right\}d\xi_1\cdots d\xi_N \tag{80}$$

where using again the conservation of Universes, this time in the Eulerian frame (71), we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{fluid}(t) = \iint_{\Omega_0} \frac{\partial}{\partial t}[\rho(\vec{x},t)J(\vec{x},t)Q(\vec{x},t)]\Big|_{\vec{x}=\vec{x}(\vec{\xi},t)}d\xi_1\cdots d\xi_N = \tag{81}$$
$$= \frac{\partial}{\partial t}\iint_{\Omega_0} \rho(\vec{\xi},t)J(\vec{\xi},t)Q(\vec{x},t)\Big|_{\vec{x}(\vec{\xi},t)}d\xi_1\cdots d\xi_N$$

Then, returning the equation back to the Eulerian frame using (**??**), we get that:

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{fluid}(t) = \frac{\partial}{\partial t}\iint_{\Omega_t} \rho(\vec{x},t)Q(\vec{x},t)dx_1\cdots dx_N \tag{82}$$

The conservation of the total energy of the fluid of Universes would then be equivalent to saying that the total external potential is constant in time:

$$\iint_{\Omega_t} \rho(\vec{x},t)Q(\vec{x},t)dx_1\cdots dx_N = \mu \tag{83}$$

for some constant μ.

Finally, in order to write the potential Q as a function of the Hamilton-Jacobi action $S(\vec{x},t)$ (which will be convenient for the next section), we could rewrite it as the difference between the total energy and the classical kinetic and potential energies due to the particles within the Universe (just isolating Q in the definition of total energy (**??**)):

$$Q(\vec{x},t) = H(t,\vec{x},\vec{p}) - \left(\sum_{k=1}^{N}\frac{1}{2m_k}p_k^2 + \sum_{j>k} V_{kj}(\ ||\vec{q}_k - \vec{q}_j||\ )\right) \tag{84}$$

If we define for the sake of notational simplicity:

$$V(\vec{x}) := \sum_{j>k} V_{kj}(\ ||\vec{q}_k - \vec{q}_j||\ ) \tag{85}$$

without forgetting that the dependence is on the relative positions, we can re-write equation (83) as:

$$\iint_{\Omega_t} \rho(\vec{x},t)\left[H(t,\vec{x},\vec{p}) - \left(\sum_{k=1}^{N}\frac{1}{2m_k}p_k^2 + V(\vec{x},t)\right)\right]dx_1\cdots dx_N = \mu \tag{86}$$

where using equations (47) and (55) to introduce S, we get:

$$\iint_{\Omega_t} \rho(\vec{x},t)\left[\frac{\partial S(\vec{x},t)}{\partial t} + \sum_{k=1}^{N}\frac{1}{2m_k}\left(\frac{\partial S(\vec{x},t)}{\partial x_k}\right)^2 + V(\vec{x},t)\right]dx_1\cdots dx_N = -\mu = \nu \tag{87}$$

Note the redefinition of the constant to absorb a negative sign.

This last equation imposes in a **same constraint** that the **total energy of the fluid is conserved and the fact that the number of Universes should be conserved.**

A.2.2.4. The Functional of the Fluid of Universes

Finally, we have arrived to the peak of the cooking process.

Let us fix the boundary constraints for the density $\rho(\vec{x},t_0) = w_0(\vec{x})$ and $\rho(\vec{x},t_f) = w_f(\vec{x})$ and for the Hamilton-Jacobi action of the Universes $S(\vec{x},t_0) = s_0(\vec{x})$ and $S(\vec{x},t_f) = s_f(\vec{x})$, for some fixed times $t_0 < t_f$.

We want to minimize the functional of equation (88) to make sure the density evolves minimizing its Fisher information (the tangent Universes locally repel each other), but we also want the total energy of the fluid of Universes to be constant in time, which was given by the constraint of equation (87). This is then a (quite big) **constrained variational problem**. As explained in detail in Appendix α, this means it is equivalent to the minimization of the following functional with a Lagrange multiplier $\lambda \in \mathbb{R}$:

$$\mathscr{M}[\rho(\vec{x},t), S(\vec{x},t);\ t_0, t_f, w_0(\vec{x}), w_f(\vec{x}), s_0(\vec{x},t), s_f(\vec{x},t)] = \tag{88}$$

$$= \int_{t_0}^{t_f}\iint_{\Omega_t} \mathscr{L}\left(t,\vec{x},S(\vec{x},t),\rho(\vec{x},t),\frac{\partial S}{\partial t},\frac{\partial S}{\partial x_k},\frac{\partial \rho}{\partial t},\frac{\partial \rho}{\partial x_k};\ \lambda\right)dx_1\cdots dx_N\ dt =$$

$$= \int_{t_0}^{t_f} \iint_{\Omega_t} \left\{ \frac{1}{\rho(\vec{x},t)} \sum_{k=1}^{N} \frac{1}{m_k} \left(\frac{\partial \rho(\vec{x},t)}{\partial x_k} \right)^2 + \lambda \rho(\vec{x},t) \left[\sum_{k=1}^{N} \frac{1}{2m_k} \left(\frac{\partial S}{\partial x_k} \right)^2 + V(\vec{x},t) + \frac{\partial S}{\partial t} \right] \right\} dx_1 \cdots dx_N \, dt$$

We will call this "God" functional $\mathscr{M}$, the **action of the fluid of Universes**, while the integrand $\mathscr{L}$ will be called the **Lagrangian density of the fluid of Universes**.

Thus final axiom will be that the density $\rho(\vec{x},t)$ and Hamilton-Jacobi action $S(\vec{x},t)$ defining the time evolution of the whole fluid of Tangent Universe, given the fixed boundaries imposed in the beginning of this subsection, will be given by the critical points of the action of the fluid of Universes.

As explained in Appendix α, if S and ρ extremize the functional $\mathscr{M}$, then they must obey the generalized Euler-Lagrange equations:

$$\frac{\partial \mathscr{L}}{\partial \rho} = \sum_{k=1}^{N} \frac{\partial}{\partial x_k} \frac{\partial \mathscr{L}}{\partial \left(\frac{\partial \rho}{\partial x_k} \right)} + \frac{\partial}{\partial t} \frac{\partial \mathscr{L}}{\partial \left(\frac{\partial \rho}{\partial t} \right)} \tag{89}$$

and

$$\frac{\partial \mathscr{L}}{\partial S} = \sum_{k=1}^{N} \frac{\partial}{\partial x_k} \frac{\partial \mathscr{L}}{\partial \left(\frac{\partial S}{\partial x_k} \right)} + \frac{\partial}{\partial t} \frac{\partial \mathscr{L}}{\partial \left(\frac{\partial S}{\partial t} \right)} \tag{90}$$

A.2.2. The Dynamics of the Hamilton-Jacobi Action of the Universes

Equation (89) evaluated on the Lagrangian density we derived can be easily checked that yields:

$$\lambda \left[\sum_{k=1}^{N} \frac{1}{2m_k} \frac{\partial S}{\partial x_k} + V(\vec{x},t) + \frac{\partial S}{\partial t} \right] + \sum_{k=1}^{N} \frac{1}{m_k} \left[\frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial x_k} \right)^2 - \frac{2}{\rho} \frac{\partial^2 \rho}{\partial x_k^2} \right] = 0 \tag{91}$$

which can be re-written as:

$$-\frac{\partial S}{\partial t} = \sum_{k=1}^{N} \frac{1}{2m_k} \left(\frac{\partial S}{\partial x_k} \right)^2 + V(\vec{x},t) + \sum_{k=1}^{N} \frac{1}{\lambda m_k} \left[\frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial x_k} \right)^2 - \frac{2}{\rho} \frac{\partial^2 \rho}{\partial x_k^2} \right] \tag{92}$$

For historical reasons, we will re-write the constant λ as $\lambda = \frac{8}{\hbar^2}$, where we will call $\hbar$ the **Planck constant**.

Since $\frac{\partial S}{\partial t}$ was the Hamiltonian by equation (55), we see that identifying the momentum using equation (47), we can identify (92) with the Hamilton-Jacobi equation (56). Thus, the required external potential $Q(\vec{x},t)$ that resulted from the imposition of the conservation of total fluid energy and number of Universes and the minimization of Fisher Information is:

$$Q(\vec{x},t) \equiv \sum_{k=1}^{N} \frac{\hbar^2}{8m_k} \left[\frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial x_k} \right)^2 - \frac{2}{\rho} \frac{\partial^2 \rho}{\partial x_k^2} \right] \tag{93}$$

We call this potential, taking into account the interaction between Tangent Universes, the **Quantum Potential**, since it is the only difference we find with classical mechanics (where $Q \equiv 0$), meaning all the "weired" quantum effects will be due to this potential energy term.

Thus, we can write the equation ruling the motion of the Hamilton-Jacobi action for the Universes as:

$$-\frac{\partial S}{\partial t} = \sum_{k=1}^{N} \frac{1}{2m_k} \left(\frac{\partial S}{\partial x_k} \right)^2 + V(\vec{x},t) + \sum_{k=1}^{N} \frac{\hbar^2}{8m_k} \left[\frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial x_k} \right)^2 - \frac{2}{\rho} \frac{\partial^2 \rho}{\partial x_k^2} \right] \tag{94}$$

In the literature, this is called the **Quantum Hamilton-Jacobi** equation.

By knowing $S(\vec{x},t)$, we can get the velocity field for the trajectories of the Universes using equation (57).

A.2.3. The Dynamics of the Density of Universes

On the other hand, if we evaluate our fluid Lagrangian density $\mathscr{L}$ in equation (90), we will easily get the equation of motion for the density of Universes ρ, which is called the **Continuity Equation**:

$$\frac{\partial \rho(\vec{x},t)}{\partial t} = -\sum_{k=1}^{N} \frac{\partial}{\partial x_k}\left(\rho(\vec{x},t)\frac{1}{m_k}\frac{\partial S(\vec{x},t)}{\partial x_k}\right) \tag{95}$$

If we acknowledge that $\frac{1}{m_k}\frac{\partial S(\vec{x},t)}{\partial x_k}$ was the velocity field $\vec{v}(\vec{x},t)$ of the trajectories in equation (57), we see that this equation is just a continuity equation for the density of Universes. It states that the local variation in time for the density of Universes can only be due to the divergence of trajectories. To see this more clearly, we can integrate both sides over an arbitrary bounded volume of configuration space $V \subset \Omega_t \subset \mathbb{R}^N$ with boundary ∂V:

$$\iint_V \frac{\partial \rho(\vec{x},t)}{\partial t} dx_1 \cdots dx_N = -\iint_V \sum_{k=1}^{N} \frac{\partial}{\partial x_k}(\rho(\vec{x},t)\vec{v}(\vec{x},t)) dx_1 \cdots d_N \tag{96}$$

Applying the divergence theorem and having ρ be regular enough, we then get:

$$\frac{\partial}{\partial t}\iint_V \rho(\vec{x},t) dx_1 \cdots dx_N = -\int_{\partial V} \rho(\vec{x},t)\vec{v}(\vec{x},t)\cdot d\vec{S}(\vec{x}) = -\int_{\partial V} \rho(\vec{x},t) v_{normal}(\vec{x},t)\cdot dS \tag{97}$$

with $d\vec{S}(\vec{x})$ the unitary orhtogonal surface differential vector to ∂V in $\vec{x}$ and v_{normal} the component of the velocity field that is normal to the surface of the volume V at $\vec{x}$.

This last equation, which is the integral form of the continuity equation (95), states that the variation in time of the number of Universes inside the volume V (left hand side), must be equal to the total number of Universes getting out of the volume V obeying the velocity field $\vec{v}$. That is: the only way for the number of Universes in a region to change is for them to cross the boundary continously flowing outside or inside the domain. Aka, there is no source or sink of Universes: the number of Universe is conserved locally.

Alternatively, one could see equation (95) as a law stating how to evolve the density field for it to move in accordance with the velocity field. In fact, it is simple to prove (as we do in Appendix β) that this equation can be derived from the axiom summarized by equation (70).

A.2.4. The Wavefuntion and its Dynamics

The Quantum Hamilton-Jacobi equation (94) and the continuity equation (95) form a coupled system of two partial differential equations ruling the motion of the two **coupled** fields $\rho(\vec{x},t)$ and $S(\vec{x},t)$ that define the state of the Fluid of Tangent Universes following all the axioms we imposed.

There is actually a very convenient way to express these two partial differential equations as a single equation. This will be possible thanks to the following two observations:

- The density of Universes is strictly positive by definition $\rho(\vec{x},t) \geq 0 \quad \forall \vec{x},t$.
- If we add a constant to the Hamilton-Jacobi action, the total energy and the velocity field it defines do not change (since they are obtained by partial derivatives). Actually it is as adding a constant to the action $\mathbb{S}$. Since the physical trajectories are obtained by extremizing it, if we extremize $\mathbb{S}+C$ for any real C, the resulting trajectories will be the same, since the whole image of the function is shifted globally.

This means that we could encode these two fields in a same complex field in polar form. We could set the density to be the magnitude of the complex field and the action to be the phase of the complex field. Since the exponent must be unit-less, we will just divide the action S by the constant $\hbar$ which turns out to have its same units. Actually, we will get the most interesting encoding if we set not the density, but its square root as the magnitude of the complex field[12]. We will denote the complex field $\Psi(\vec{x},t)$ encoding the two real fields $\rho(\vec{x},t)$ and $S(\vec{x},t)$ defining the Universe as **the Wavefunction** of the Universe:

$$\Psi(\vec{x},t) \equiv \rho^{1/2}(\vec{x},t)e^{\frac{iS(\vec{x},t)}{\hbar}} \tag{98}$$

where we denote by i the imaginary square root of -1, such that we get the reverse equations:

$$\rho(\vec{x},t) = \Psi(\vec{x},t)^*\Psi(\vec{x},t) = |\Psi(\vec{x},t)|^2 \tag{99}$$

$$S(\vec{x},t) = \hbar \tan^{-1}\left(\frac{\text{Im}\{\Psi\}}{\text{Re}\{\Psi\}}\right) \tag{100}$$

where given a complex number $z \in \mathbb{C}$ with $z = a + ib$ $a, b \in \mathbb{R}$, we denote by z^* the complex conjugate $a - ib$, by $|z|^2$ the magnitude squared $a^2 + b^2$ and by $\text{Re}\{z\}$ and $\text{Im}\{z\}$, the real a and imaginary b parts of z[13].

Note how we can immediately get the velocity field for the trajectories using the complex encoding as:

$$v_k(\vec{x},t) = \frac{\hbar}{m_k}\text{Im}\left\{\Psi(\vec{x},t)^{-1}\frac{\partial}{\partial x_k}\Psi(\vec{x},t)\right\} = \frac{1}{m_k}\frac{\partial S(\vec{x},t)}{\partial x_k} \tag{101}$$

which is computationally more convenient than extracting the phase of the wavefunction and explicitly derivating it.

Now, as a complex number can encode two real numbers and a complex field can encode two real fields, a complex partial differential equation can encode two real partial differential equations! Note first that we can re-write the quantum potential in am ore compact shape as:

$$Q(x,t) = \sum_{k=1}^{N} -\frac{\hbar^2}{2m_k}\frac{1}{\rho^{1/2}}\frac{\partial^2\rho^{1/2}}{\partial x_k^2} \tag{102}$$

Let us now take the Hamilton-Jacobi equation (94) and the continuity equation (95), ruling the coupled motion of ρ and S and write the first as the real part and the second one the imaginary part of an equation:

$$-\frac{\partial S}{\partial t} + i\frac{\partial \rho}{\partial t} = V(\vec{x}) + \sum_{k=1}^{N}\left[\frac{1}{2m_k}\left(\frac{\partial S}{\partial x_k}\right)^2 - \frac{\hbar^2}{2m_k}\frac{1}{\rho^{1/2}}\frac{\partial^2\rho^{1/2}}{\partial x_k^2}\right] - i\sum_{k=1}^{N}\frac{\partial}{\partial x_k}\left(\rho\frac{1}{m_k}\frac{\partial S}{\partial x_k}\right) \tag{103}$$

[12] For the square root function $f(x) = \sqrt{x}$ in the support $[0,\infty)$, there is an inverse function $f(x) = x^2$. Since the density is restricted to be in that support $[0,\infty)$, we can uniquely encode the same information as the density with the squared root of the density.

[13] Note how only taking the complex encoding into account, the action S is not well defined if the density is zero at a certain point.

Then, multiplying the two real parts by $\rho^{1/2}$ and expanding the divergence of the continuity equation using a simple Leibniz rule we get:

$$-\rho^{1/2}\frac{\partial S}{\partial t}+i\frac{\partial\rho}{\partial t}=\rho^{1/2}V(\vec{x})+\sum_{k=1}^{N}\left[\frac{\hbar^2}{2m_k}\left[\frac{\rho^{1/2}}{\hbar^2}\left(\frac{\partial S}{\partial x_k}\right)^2-\frac{\partial^2\rho^{1/2}}{\partial x_k^2}\right]\right]+ \tag{104}$$

$$+i\sum_{k=1}^{N}\frac{-1}{m_k}\left[\frac{\partial S}{\partial x_k}\frac{\partial\rho}{\partial x_k}+\rho\frac{\partial^2 S}{\partial x_k^2}\right]$$

Noting that by the chain rule:

$$\frac{\partial\rho}{\partial x_k}=2\rho^{1/2}\frac{\partial\rho^{1/2}}{\partial x_k} \tag{105}$$

$$\frac{\partial\rho}{\partial t}=2\rho^{1/2}\frac{\partial\rho^{1/2}}{\partial t} \tag{106}$$

We have evaluate them in the imaginary part, and find that a multiplicative factor $\rho^{1/2}$ is canceled out. Then we multiply and divide the imaginary parts by $\hbar$ to get:

$$-\rho^{1/2}\frac{\partial S}{\partial t}+i\frac{\partial\rho^{1/2}}{\partial t}=\rho^{1/2}V(\vec{x})+\sum_{k=1}^{N}\left[\frac{\hbar^2}{2m_k}\left[\frac{\rho^{1/2}}{\hbar^2}\left(\frac{\partial S}{\partial x_k}\right)^2-\frac{\partial^2\rho^{1/2}}{\partial x_k^2}\right]\right]+ \tag{107}$$

$$+i\sum_{k=1}^{N}\frac{-\hbar}{2m_k}\left[2\frac{\partial S/\hbar}{\partial x_k}\frac{\partial\rho^{1/2}}{\partial x_k}+\rho^{1/2}\frac{\partial^2 S/\hbar}{\partial x_k^2}\right]$$

If we take a common factor $i\hbar$ out in the left hand side and group terms in the summation in the right hand side:

$$i\hbar\left(i\rho^{1/2}\frac{\partial S}{\partial t}+\frac{\partial\rho^{1/2}}{\partial t}\right)=\rho^{1/2}V(\vec{x})- \tag{108}$$

$$-\sum_{k=1}^{N}\frac{\hbar^2}{2m_k}\left\{-\rho^{1/2}\left(\frac{\partial S/\hbar}{\partial x_k}\right)^2+\frac{\partial^2\rho^{1/2}}{\partial x_k^2}+i\left[2\frac{\partial S/\hbar}{\partial x_k}\frac{\partial\rho^{1/2}}{\partial x_k}+\rho^{1/2}\frac{\partial^2 S/\hbar}{\partial x_k^2}\right]\right\}$$

If we now multiply both sides by the complex exponential $exp(iS/\hbar)$, we can unwind two Leibniz derivative rules to get:

$$i\hbar\left(i\rho^{1/2}\frac{\partial S}{\partial t}+\frac{\partial\rho^{1/2}}{\partial t}\right)e^{iS/\hbar}=\rho^{1/2}e^{iS/\hbar}V(\vec{x})- \tag{109}$$

$$-\sum_{k=1}^{N}\frac{\hbar^2}{2m_k}\left\{\frac{\partial}{\partial x_k}\left(\frac{\partial\rho^{1/2}}{\partial x_k}e^{iS/\hbar}\right)+\frac{\partial}{\partial x_k}\left(i\rho^{1/2}e^{iS/\hbar}\frac{\partial S/\hbar}{\partial x_k}\right)\right\}$$

where noting that:

$$\frac{\partial e^{iS/\hbar}}{\partial x_k}=ie^{iS/\hbar}\frac{\partial S/\hbar}{\partial x_k} \tag{110}$$

we can have applying it in the left and right hand sides:

$$i\hbar\left(\rho^{1/2}\frac{\partial e^{iS/\hbar}}{\partial t}+\frac{\partial\rho^{1/2}}{\partial t}e^{iS/\hbar}\right)=\rho^{1/2}e^{iS/\hbar}V(\vec{x})-\sum_{k=1}^{N}\frac{\hbar^2}{2m_k}\frac{\partial}{\partial x_k}\left\{\frac{\partial\rho^{1/2}}{\partial x_k}e^{iS/\hbar}+\rho^{1/2}\frac{\partial e^{iS/\hbar}}{\partial x_k})\right\} \tag{111}$$

where unwinding a Leibniz derivative rule in both sides we get:

$$i\hbar\frac{\partial}{\partial t}\left(\rho^{1/2}e^{iS/\hbar}\right)=\rho^{1/2}e^{iS/\hbar}V(\vec{x})-\sum_{k=1}^{N}\frac{\hbar^2}{2m_k}\frac{\partial^2}{\partial x_k^2}\left(\rho^{1/2}e^{iS/\hbar}\right) \tag{112}$$

Finally, identifying $\Psi=\rho^{1/2}e^{iS/\hbar}$, we arrive at the following partial differential equation, which is equivalent to evolving the two equations (94) and (95) in a coupled manner:

$$i\hbar\frac{\partial\Psi(\vec{x},t)}{\partial t}=\left[-\sum_{k=1}^{N}\frac{\hbar^2}{2m_k}\frac{\partial^2}{\partial x_k^2}+V(\vec{x})\right]\Psi(\vec{x},t) \tag{113}$$

This is the famous, so-called, **Schrödinger Equation**.

A.2.5. Summary of the Time Evolution Computational Schemes I

All in all, we can compute the time evolution of an **isolated** quantum system (or a quantum system that allows an effective wavefunction, also known as **closed** quantum system, which we will see in section A.2.7.[14]) using any of the following equivalent frameworks, each having its computational pros and cons.

(I) **The Newton-Bohm approach**. This is the most intuitive set of equations for someone with a "classical" intuition as us humans, who are used to Newton's second law. We are given the initial ($t=t_0$) domain Ω_{t_0} with its boundary conditions, the initial velocity field $\vec{v}(\vec{x},t_0)$ and

[14]In that case, we can have that the potential depends on time $V(\vec{x},t)$. All the approaches would still be the same with this little fix.

the initial density field $\rho(\vec{x}, t_0)$. We consider the ensemble of tangent trajectories such that $\vec{x}(\vec{\xi}, t_0) \equiv \vec{x}^{\xi}(t_0) = \vec{\xi} \in \Omega_{t_0}$. The laws of physics would be for $\vec{\xi} \in \Omega_{t_0}$ and $t \geq t_0$,

$$\begin{cases} Q(\vec{x},t) = -\frac{\hbar^2}{4m_k}\left(\frac{\nabla_{\vec{x}}^2\rho}{\rho} - \frac{1}{2}\frac{||\vec{\nabla}_{\vec{x}}\rho||^2}{\rho^2}\right) \\ m_k\frac{\mathrm{d}^2 x_k^{\xi}(t)}{\mathrm{d}t^2} = -\frac{\partial}{\partial x_k}(V(\vec{x}) + Q(\vec{x},t))\Big|_{\vec{x}=\vec{x}^{\xi}(t)} \qquad \forall k \in \{1,...,n\} \\ \vec{v}(\vec{x}^{\xi}(t),t) = \frac{\mathrm{d}\vec{x}^{\xi}(t)}{\mathrm{d}t} \\ \frac{\partial\rho(\vec{x},t)}{\partial t} = -\vec{\nabla}\cdot(\rho(\vec{x},t)\vec{v}(\vec{x},t)) \end{cases} \tag{114}$$

After computing the pressure potential Q, due to the curvature of the density, the Newton-Bohm equation will tell us the forces on the ensemble of possible trajectories $\vec{x}(\vec{\xi}, t_0) = \vec{\xi} \in \Omega_{t_0}$, letting us know their next positions and which will be their new velocities, which in turn will guide the density along the grid of trajectories through the continuity equation (95). If one wishes to get the associated wavefunction $\psi(\vec{x}, t) = \rho^{1/2}(\vec{x}, t)e^{iS(\vec{x},t)/\hbar}$, one can simply compute $S(\vec{x}, t)$ using that its gradient is the momentum field $(v_1 m_1, ..., v_n m_n)$. Any of the many algorithms to get the potential function from a vector field will suffice.

(II) The Hamilton-Jacobi approach. This second approach is the intermediate between the previous and the next one, such that we need not obtain S by integration of the velocity field. We are given the initial ($t = t_0$) domain Ω_{t_0} with its boundary conditions, the initial Hamilton action field $S(\vec{x}, t)$ and initial density $\rho(\vec{x}, t_0)$. We consider the ensemble of tangent trajectories such that $\vec{x}(\vec{\xi}, t_0) \equiv \vec{x}^{\xi}(t_0) = \vec{\xi} \in \Omega_{t_0}$. The laws of physics would be for $\vec{\xi} \in \Omega_{t_0}$ and $t \geq t_0$,

$$\begin{cases} Q(\vec{x},t) = -\frac{\hbar^2}{4m_k}\left(\frac{\nabla_{\vec{x}}^2\rho}{\rho} - \frac{1}{2}\frac{||\vec{\nabla}_{\vec{x}}\rho||^2}{\rho^2}\right) \\ -\frac{\partial S(\vec{x},t)}{\partial t} = \sum_{k=1}^{N}\frac{1}{2m_k}\left(\frac{\partial S}{\partial x_k}\right)^2 + V(\vec{x}) + Q(\vec{x},t) \\ \frac{\partial\rho(\vec{x},t)}{\partial t} = -\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\rho(\vec{x},t)\frac{1}{m_k}\frac{\partial S(\vec{x},t)}{\partial x_k}\right) \\ m_k\frac{\mathrm{d}\vec{x}_k^{\xi}(t)}{\mathrm{d}t} = \frac{\partial S(\vec{x},t)}{\partial x_k}\Big|_{\vec{x}=\vec{x}^{\xi}(t)} \end{cases} \tag{115}$$

After computing the pressure potential Q due to the curvature of the density, the Hamilton-Jacobi equation will tell us how the Hamilton action per configuration $S(\vec{x}, t)$ will evolve to the next time. With this field, we can obtain the velocity field, which will guide the trajectories to the next positions and the density to its next conformation. This approach has the great advantage that it is not necessary to compute the spatial derivatives of the quantum potential, which can be very noisy to do numerically.

(III) The Schrödinger approach. This is the most common (and computationally most appealing for an isolated system[15]) approach used in the world. Unlike in the other coupled equations, which were real differential equations but non-linear, here, the core equation, evolving the action and density fields is complex but **linear**! We are given the initial ($t = t_0$) domain Ω_{t_0} with its boundary conditions, the initial Hamilton action field $S(\vec{x}, t)$ and initial density $\rho(\vec{x}, t_0)$, or equivalently, the initial wavefunction $\psi(\vec{x}, t) = \rho^{1/2}(\vec{x}, t_0)e^{iS(\vec{x},t_0)/\hbar}$. We consider the ensemble of tangent trajectories such that $\vec{x}(\vec{\xi}, t_0) \equiv \vec{x}^{\xi}(t_0) = \vec{\xi} \in \Omega_{t_0}$. The laws of physics would be for $\vec{\xi} \in \Omega_{t_0}$ and $t \geq t_0$,

$$\begin{cases} i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t} = -\sum_{k=1}^{n}\frac{\hbar^2}{2m_k}\frac{\partial^2}{\partial x_k^2}\psi(\vec{x},t) + V(\vec{x})\psi(\vec{x},t) \\ m_k\frac{\mathrm{d}\vec{x}_k^{\xi}(t)}{\mathrm{d}t} = \frac{\partial S(\vec{x},t)}{\partial x_k}\Big|_{\vec{x}=\vec{x}^{\xi}(t)} \end{cases} \tag{116}$$

Using the Schrödinger Equation (which includes the Hamilton-Jacobi, the definition of quantum potential and continuity equation in a single linear equation), we evolve at the same time the density and action to the next time, and with the action we move or guide the trajectories to

[15]But as we will see, alternative methods will be almost essential for open quantum systems.

their next position. Note that both in the second and this last approach, if we consider all the equations in the Eulerian frame, the trajectories themselves can be obtained just a posteriori. This is why at first nobody realizes about their feasibility. However, we insist that in the Lagrangian or partially Lagrangian frame into which we will need to turn for the description of open quantum systems, the particles will become as a priori as are in the first approach.

Any of these sets of equations, as will the next alternative computational schemes that we will see, may be re-expressed in a fully Eulerian, partially Lagrangian or totally Lagrangian frame. All of this is properly explained and developed in Appendix X.

A.3. Intuition of the Non-Classical thing: the Quantum Potential

We have seen that essentially the only difference between a system of n classical particles and n quantum particles is that the quantum particles behave as if there were an infinite number of alternative n system particles, distributed according to a density ρ in their possible configurations, alternative systems that interact with each other tangentially, never crossing each other in total configuration, through a pressure potential proportional to the curvature of their agglomeration or density. We called this potential the quantum or pressure potential. Then, since we are trapped in a single one of these alternative systems (our brains are part of the n particles), we still see only one of the interacting systems. But, its dynamics encodes, the information of all the rest. Then, if we understand how this potential changes the dynamics one expected classically on the system of particles, we will have mastered quantum mechanics.

We can intuitively understand the effect of the quantum potential (**??**) if we re-express it as:

$$Q(\vec{x},t) = -\frac{\hbar^2}{4m_k}\left(\frac{\nabla_x^2\rho}{\rho} - \frac{1}{2}\frac{||\vec{\nabla}_x\rho||^2}{\rho^2}\right) \tag{117}$$

The first term in Q is the Laplacian ($\nabla_x^2 = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}$) of $\rho(\vec{x},t)$ in each point, normalized by the value of ρ in that point of configuration-space. The Laplacian of a scalar field in a certain point gives the difference between the value of the function in that point and the mean value in its locality. Interpreting this value as a potential, means that the higher the local variation of ρ (the higher the difference between the value in the point and its mean value in the surrounding), the bigger the modulus of the potential will be. In particular, if the Laplacian of ρ is positive, it means that the value in the point is smaller than the mean surrounding density. That is, the density is more convex-like there. This makes the potential at that point be negative -attractive- (noting the minus sign in front of the Laplacian). The fluid element will be more stable there than in points where the variation of the density is more concave-like (where the value of the density is higher than in its local surrounding: $\nabla_x^2\rho < 0$), as these make a positive -repulsive- contribution to the total potential in their locality.

Interpreting ρ as the density of all the possible configurations of the system, this means that the probability of observing each configuration[16] is repelled by the configurations where there is a locally high agglomeration of probability. If we understand ρ as the density of a continuum of possible tangent "Universes", then this simply means that the local agglomeration of possible Universes tends to diverge.

The second term in Q is more straight-forward: it is the modulus of the gradient of the density in each point normalized by the magnitude of the density. This fraction is always a positive value, which means the contribution to the potential will always be positive: it is a destabilizing factor (repels trajectories). That is, the higher the local steepness of the density, the more unstable this zone will be

[16] The density ρ^2 and the Bohmian trajectories of the system are evolved using the same velocity field

for the fluid element. Understanding ρ as the density of a continuum of possible tangent "Universes", this means that the further the locality of a Universe is from the homogeneous density, the more unstable its configuration will be.

Let us now see its effects in some simple scenarios.

A.3.1. Example 1: The Free Particle System

The first example we will visit is the free particle system, where there will be no interaction between the particles within each Universe $V(\vec{x}) \equiv 0\ \forall \vec{x} \in \Omega$. Note that there will still be present the quantum potential interaction between the tangent trajectories due to their local agglomeration in configuration space. This interaction is unavoidable and happens even in the case where within each Universe there is a single particle. It will inevitably interact with its alternative versions in the tangent Universes. Whenever the alternative versions get too concentrated around a similar configuration, they will repel each other. This is essentially the only difference with classical mechanics. Then, let us see which are its consequences in the dynamics of free particles.

We will have that the equation governing each tangent trajectory $\vec{x}(t) = (\vec{q}_1(t), ..., \vec{q}_N(T))$ will be given by the Newton-Bohm equation, Eq. (25), with $V(\vec{x}) = 0$, where we all know that the particle will follow the path towards the minimum of potential energy (the force will always point in the steepest descent direction)

$$m_k \frac{\mathrm{d}\vec{q}_k(t)}{\mathrm{d}t} = -\vec{\nabla}_{\vec{q}_k} Q(\vec{q}_1, ..., \vec{q}_N, t)\Big|_{\vec{q}_j=\vec{q}_j(t)} \quad \forall k \in \{1, ..., n\}. \tag{118}$$

If we get the intuition of what this would (already classically) mean, we will immediately understand what the solution of the Schrödinger Equation encoding it,

$$i\hbar \frac{\partial \Psi(\vec{x}, t)}{\partial t} = \sum_{k=1}^{n} \frac{-\hbar^2}{2m_k} \frac{\partial^2}{\partial x_k^2} \psi(\vec{x}, t), \tag{119}$$

would look like.

In order to get an intuition, consider for example the case in which we have an isolated 1D particle, such that equation (118) would be the 1D second Newton's law with a potential energy field $Q(x, t)$,

$$m_k \frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\frac{\partial Q(x, t)}{\partial x}\Big|_{x=x(t)} \tag{120}$$

where

$$Q(x, t) = \frac{\hbar^2}{4m} \left(\frac{1}{2\rho^2(x, t)} \left(\frac{\partial \rho(x, t)}{\partial x} \right)^2 - \frac{1}{\rho(x, t)} \frac{\partial^2}{\partial x^2} \rho(x, t) \right). \tag{121}$$

Note again that although its scary look, the quantum potential is just the magnitude of the gradient of the density (the maximum variation in that point) normalized by the absolute value of the density there (such that the steeper the density in that locality, the higher the potential will be and the more uncomfortable the trajectory will feel there) plus the negative curvature of the density normalized by the density's value (the laplacian, that will be higher the bigger the value of the density is in that point relative to the average density in the local surrounding). Both terms together mathematically express the mantra "trajectory ensembles that get denser repel each other stronger".

We first note that if the density was such that it was uniform over all the domain Ω (which can happen if $\Omega \subseteq \mathbb{R}$ is bounded for the proper noramlization of the density), say an interval $[a, b] = \Omega$,

$$\rho(\vec{x}, t_0) = \frac{1}{b - a} \tag{122}$$

then the quantum potential would be zero everywhere. In that case, at t_0 Newton's equation (120) would be exactly the same as the free particle in classical mechanics. That is, given an initial velocity field $v(x,t_0) = v_0 \in \mathbb{R}$, since from the continuity equation (95), $\frac{\partial \rho(x,t=t_0)}{\partial t} = 0$, it would have an immediate solution $x(\xi,t) = \xi + v_0 t$ with $v(x,t) = x_0 \ \forall t > t_0$, where $\xi = x(\xi,t_0)$ is the initial position of the ξ-th trajectory. That is, all possible particles would be moving at a constant speed for ever, ignoring the presence of each other. That is, for a wavefunction such that $\psi(x,t_0) = \frac{1}{b-a} e^{\frac{i}{\hbar} m v_0 x}$, the Schrödinger time evolution will be $\psi(x,t) = \frac{1}{b-a} e^{\frac{i}{\hbar} m v_0 x + c(t)}$, with $c(t)$ some constant in space that at most will change in time.

For practically any other acceptable initial density however, it will be false that $\frac{\partial \rho(x,t_0)}{\partial x} = 0 \ \forall x \in \Omega$ and thus false that $Q(x,t) = 0$ and from (95), false that $\frac{\partial \rho(x,t_0)}{\partial t} = 0 \ \forall x \in \Omega$. That is, the trajectories will not be straight lines since the trajectories will influence each other, not even when the initial velocities of all the particles are the same $v(x,t_0)$ will they continue evolving in straight lines. The general motion however will still be intuitive because we can understand what the two terms in the quantum potential are.

Imagine the non-trivial case of an initial Gaussian density packet

$$\rho(\vec{x},t_0) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}. \tag{123}$$

Since it reflects an agglomeration of tangent Universes around μ, we can readily imagine that the Quantum potential will be very high in μ and will decrease while we go far from μ symmetrically. Indeed, if we compute the quantum potential we get

$$Q(x,t_0) = -\frac{\hbar^2}{4m\sigma^2}\left[\frac{(x-\mu)^2}{\sigma^2} + 1\right]. \tag{124}$$

This is a concave parabola potential energy, with its maximum at μ and such that the smaller σ (the width of the Gaussian), the steeper it is, reflecting that the trajectories will be separated with a higher force from the center of the Gaussian. So, we can immediately see that even if we had that all the trajectories had a same zero initial velocity $v(x,t_0) = 0$, in the next time step, the trajectories will obtain a non-zero velocity that will make them be more separated relative to the trajectories at μ (in a symmetric fashion). In turn, since the continuity equation causes the density to adapt to the new velocities of the trajectories, the density will broaden its shape to a higher width σ, which will cause a less sharply steep concave parabolic potential, which will add a smaller momentum to the trajectories, but will add a smaller momentum, and so on, towards a constant density distribution. Note that only the trajectory in the origin will remain still, since the quantum potential will be at its peak (thus a zero force) on it.

As we saw in the derivation, the role of the quantum potential is to push a system towards the density profile of maximum entropy, the homogeneous density distribution, through the minimization of Fisher information. We will prove that the time evolution of the Gaussian in free space will still be a Gaussian later on when we see the observable operators, for now just believe it by looking at the simulation frames.

Can we run out of Universes?
If the Gaussian keeps broadening for ever, thus the trajectories keep getting further away from the origin, one could ask whether at some point the only possible Universe around the origin will be the one at $x = 0$, which is still. The answer is no. That is the consequence of having a *continuum* of possible Universes. What will happen of course is, that since the number of Universes relative to the total, in a certain neighborhood $\mathcal{N}$ of the origin is given by $\int_{\mathcal{N}} \rho(x,t)dx$, this number will asymptotically tend towards zero.

It will be interesting as well to see that for an initial condition where all the trajectories have the same non-zero initial velocity $v(x,t_0) = v_0$, they will still evolve towards the broadening of the density profile, such that the peak of the Gaussian will move at a fixed velocity v_0 (as the trajectory that was in the origin in the previous case remained there still, the quantum potential force is zero on it).

The generalization to higher dimensions is straightforward for both the constant density and the Gaussian wavepacket.

All in all, this means that if the fluid of tangent Universes had initially a peaked density, it will evolve towards its flattening. This has the epistemological consequence that although we could assert with great confidence that our Universe was around μ at the initial time, we loose that knowledge very fast, as the wavepacket gets dispersed towards maximum entropy.

A.3.2. Example 2: The Harmonic Oscillator

What if we wanted to avoid this dispersion of tangent trajectories? After all, if the wavepackets get dispersed without bounds, the stability of our observed world would appear impossible! That is precisely the role of the potential energy $V(\vec{x})$. For instance, we have seen that a 1D Gaussian wavepacket centered in the origin with $V(x) \equiv 0$ will broaden unboundedly, because the curvature of the density causes the pressure potential $Q(x,t)$ to be a concave parabola. However, if we had set a potential energy $V(x)$ that was the additive inverse of $Q(x,t_0)$, that is, a convex parabola, or harmonic potential,

$$V(x) = \frac{\hbar^2}{4m\sigma^2}\left[\frac{(x-\mu)^2}{\sigma^2} + 1\right], \tag{125}$$

it would make the total potential be zero, $Q(x,t_0) + V(x) \equiv 0$. Then, if the trajectories were still at t_0, $v(x,t_0) \equiv 0$, they would remain that way for ever. We would have that the Gaussian density continues to be the same function for ever. We would have been able to "confine" the high density of Universes in the parabolic potential.

In fact, this fixed density could be one of the proves that the particle we would observe is not alone. Imagine we observe the particle in $x = 1$, since there is an attractive harmonic potential, with the minimum in $x = 0$, we would expect it to move towards $x = 0$, just as a skateboard does in a U. We would expect to to "fall" to the attraction basin. However it does not. There is something holding it. It is the rest of tangent trajectories, that cannot cross each other and are piled one against the other, that hold it "in mid-air".

If there was a harmonic potential $V(x) = V_0(x-\mu)^2$ centered in the same point as the peak of the Gaussian, but the harmonic potential was not exactly the one that cancels out the pressure potential, then, the total potential would still be quadratic,

$$Q(x,t_0) + V(x) = \left(V_0 - \frac{\hbar^2}{4m\sigma^4}\right)(x-\mu)^2 - \hbar^2/4m\sigma^2 \tag{126}$$

Such that if $V_0 > \frac{\hbar^2}{4m\sigma^4}$ (the potential is stronger than the pressure due to density agglomeration), the parabola would be convex and else concave. Then, in the convex case, the trajectories would feel a net attractive potential towards the origin and while they move they would deform the density increasing it in that region. As we will prove in a coming section, the shape would still be Gaussian, but the width σ would diminish. This would then cause the balance between the potentials to revert at some point, which would start causing a repulsive force in the trajectories relative to the origin, which would widen the Gaussian until the balance reverts back, etc. This would be a "breathing"-like mode, which would eventually thermalize in a static Gaussian if there was something similar to a friction (which we will see that can appear due to the interaction with other particles).

If on the other hand, the Gaussian was displaced, say $\mu > 0$, relative to the harmonic potential $V(x)$ in the origin, and we set the magnitude of $V(x)$ equal to the one for $Q(x,t_0)$, the resulting total potential would be linear, with a positive slope $\mu\frac{\hbar^2}{4m\sigma^4}$,

$$Q(x,t_0) + V(x) = 2\mu\frac{\hbar^2}{4m\sigma^4}x - 4m\sigma^4(\mu^2 + \sigma^2). \tag{127}$$

This would tend to move the trajectories (it can be proven that, preserving the same Gaussian shape, thus same σ) towards the center of the harmonic well, and as the Gaussian approaches the origin, since μ would get smaller, the total potential, which was a line would get its slope reduced until it gets to be a constant in the origin, while then it will become a negative slope line and will push back the Gaussian to the origin. That is, the Gaussian density will oscillate back and forth harmonically around the origin. Finally, if we let the potentials to have different magintudes, it can be proven that the motion will still be harmonic, just that the Gaussian will perform a breathing-like mode while back and forth.

The generalization to higher dimensional systems with harmonic potentials and Gaussians is straightforward. The intuitions would still hold (just that there would now be more directions in which things can move).

A.4. The State of a Partition of the Universe

The Epistemology of the Model

A.4.1. The Conditional Wavefunction

A.5. The Dynamics of a Partition of the Universe

A.5.1. Effective Wavefunctions and Closed Quantum Systems

A.5.2. The Famous Quantum Entanglement

Apendices pendientes: - EL deimos - Jacobian - Putse Hilbert Space and measure and topology - Comparación con las otras interpretaciones - More on discrete Tangent Universes. Explicar tb mi idea de cómo podríamos demonstreitear cuál es el pairwise potential - El asunto sobre la paradoja de que la pdf sea continua y observes algo discreto. Ke en vd deberíamos no abusar del lenguaje y decir un intervalo et al. - La visión de que el Hamiltoniano es dios y no hace falta que haya causalidad material, sino hay una causalidad divina en el movimiento de las cosas. Las dos escuelas azaldu. - Spin introduction appendix. - Maybe photons like modes of an oscillator view. Explore zelan azaltzen dabien fotoie los bohmianos pa poder hacer lo mismo.

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A.6. The Hilbert Space and the Observable Operators

A.6.1. The Hilbert Space

Now that we can define the state of the fluid of Universes with a single (complex) function $\Psi(\vec{x}, t)$, it is interesting to see if we can situate it in any interesting mathematical structure. It turns out that there is one mathematical construction that will be very convenient to manage the wavefunction of the Universe (which will be extensible to the wavefunctions of partitions of the Universe that we will explore in the next section).

Since we defined the density of tangent Universes as a normalized relative density, we imposed that its integral in the whole configuration space was 1 in equation (2). This condition in terms of the wavefunction is:

$$\iint_{\Omega_t} \Psi^*(\vec{x}, t)\Psi(\vec{x}, t)\ dx_1 \cdots dx_N = 1 \quad \forall t \tag{128}$$

we call the left-hand-side the **norm** of the wavefunction, and say that the wavefunction is **normalized** if its norm is equal to 1 (aka (128) is satisfied).[17]

We then imposed a local conservation of the relative number of Universes in time as an axiom, which meant in particular that the total density should also be conserved in time. This meant that if at any time t_0 we knew the wave function was normalized, the dynamics would preserve its norm to unity at all times. This is evident axiomatically, since otherwise we would be saying that there are Universes being created or destroyed. Since this axiom was an imposition in the derivation of the dynamical equations, it is this why we say that the evolution of the wavefunction given by the Schrödinger Equation (113), or equivalently the Hamilton-Jacobi and Continuity Equations (94), (95), is a **unitary time evolution** (they preserve the norm of the wavefunction, of the density at all times).

A.6.2. The Hamiltonian Operator and the Orchestration

Given a quantum system with EWF $\psi(\vec{x}, t)$, let us define the energy of the trajectory that at time t happens to be in $\vec{x}$ (for their non-crossing property, there will only be one, the ξ-th), as

$$\mathscr{H}^{\psi}(\vec{x}^{\xi}(t), t) := H\left(\vec{x}^{\xi}(t), \vec{p}\left(\frac{\mathrm{d}\vec{x}^{\xi}(t)}{\mathrm{d}t}\right), t\right) = \sum_{k=1}^{n} \frac{1}{2} m_k \left(\frac{\mathrm{d}\vec{x}^{\xi}(t)}{\mathrm{d}t}\right)^2 + V(\vec{x}^{\xi}(t)) + Q(\vec{x}^{\xi}(t), t) \tag{129}$$

where H is the Hamiltonian of the system as defined in Eq. (**??**). As we saw previously, this energy depends not only on the instantaneous configuration of a certain trajectory, but also on the velocity field $\vec{v}(\vec{x}^{\xi}(t), t) = \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}^{\xi}(t)$ appearing in the kinetic energy and the density of Universes $\rho(\vec{x}, t)$, through the quantum potential Q. Thus it depends on the whole wavefunction $\psi = \rho^{1/2} e^{iS/\hbar}$ in addition to the position for a trajectory $\vec{x}$.

One might be wondering what is the connection of this trajectory Hamiltonian $\mathscr{H}^{\psi}(\vec{x}, t)$ and the operator we defined in Eq. (**??**) as the Hamiltonian operator, $\hat{H}_{\vec{x}} = \sum_{k=1}^{n} -\hbar^2/2m_k \frac{\partial^2}{\partial x_k^2} + V(\vec{x})$. When we derived the Schrödinger Equation, the operator simply appeared as an actor on the wavefunction, so their connection appears to be little more than their names. Far from being so, as we will see in this section, they are two faces of the same coin.

Their relationship is the following equality

$$\mathscr{H}^{\psi}(\vec{x}, t) = \mathrm{Re}\left\{\frac{\hat{H}_{\vec{x}}\psi(\vec{x}, t)}{\psi(\vec{x}, t)}\right\} \tag{130}$$

[17] Note how for its interpretation as a probability density, this is also a requirement.

where by $\mathrm{Re}\{\cdot\}$, we mean the real part. Let us prove this by developing the expression in the right hand side.

Since $\hat{H}_{\vec{x}} = \sum_{k=1}^{n} -\frac{\hbar^2}{2m_k}\frac{\partial^2}{\partial x_k^2} + V(\vec{x})$, we have in general that,

$$\frac{\hat{H}_{\vec{x}}\psi(\vec{x},t)}{\psi(\vec{x},t)} = \sum_{k=1}^{n} \frac{-\hbar^2}{2m_k}\frac{\frac{\partial^2}{\partial x_k^2}\psi(\vec{x},t)}{\psi(\vec{x},t)} + \frac{V(\vec{x})\psi(\vec{x},t)}{\psi(\vec{x},t)}. \tag{131}$$

Let us develop the first term with the wavefunction in polar form $\psi(\vec{x},t) = \rho^{1/2}(\vec{x},t)e^{iS(\vec{x},t)/\hbar}$

$$\begin{aligned}\frac{\frac{\partial^2}{\partial x_k^2}\psi(\vec{x},t)}{\psi(\vec{x},t)} &= \frac{1}{\rho^{1/2}e^{iS/\hbar}}\left(\frac{\partial^2\rho^{1/2}}{\partial x_k^2}e^{iS/\hbar} - \frac{\rho^{1/2}e^{iS/\hbar}}{\hbar^2}\left(\frac{\partial S}{\partial x_k}\right)^2 + \frac{2ie^{iS/\hbar}}{\hbar}\frac{\partial\rho^{1/2}}{\partial x_k}\frac{\partial S}{\partial x_k} + \frac{i}{\hbar}\rho^{1/2}e^{iS/\hbar}\frac{\partial^2 S}{\partial x_k^2}\right) = \\ &= -\frac{1}{\hbar^2}\left(\frac{\partial S}{\partial x_k}\right)^2 + \frac{1}{2\rho}\frac{\partial^2\rho}{\partial x_k^2} - \frac{1}{4\rho^2}\left(\frac{\partial\rho}{\partial x_k}\right)^2 + \frac{i}{\hbar}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial x_k}\frac{\partial S}{\partial x_k} + \frac{\partial^2 S}{\partial x_k^2}\right)\end{aligned} \tag{132}$$

where we can identify the trajectory momentum $p_k := m_k v_k = \frac{\partial S}{\partial x_k}$ in the k-th degree of freedom and the quantum potential due to the k-th degree of freedom Q_k as seen in Eq. (**??**). For the remaining term in the imaginary part, we can define

$$\wp_k(\vec{x},t) := -\hbar\frac{2}{\rho}\frac{\partial\rho}{\partial x_k} \tag{133}$$

to be the **osmotic momentum**. We will find its relevance in section A.8, for now it will just be a shorthand. With this, we have that

$$\frac{\frac{\partial^2}{\partial x_k^2}\psi(\vec{x},t)}{\psi(\vec{x},t)} = -\frac{1}{\hbar^2}m_k v_k(\vec{x},t)^2 - \frac{2m_k}{\hbar^2}Q_k(\vec{x},t) + \frac{i}{\hbar}\left(-\frac{m_k}{2\hbar}\wp_k(\vec{x},t)v_k(\vec{x},t) + m_k\frac{\partial v_k(\vec{x},t)}{\partial x_k}\right) \tag{134}$$

And therefore,

$$\frac{\hat{H}_{\vec{x}}\psi(\vec{x},t)}{\psi(\vec{x},t)} = \sum_{k=1}^{n}\frac{1}{2}m_k v_k(\vec{x},t)^2 + Q(\vec{x},t) + V(\vec{x},t) - i\frac{\hbar}{2}\left(\frac{1}{2\hbar}\vec{\wp}(\vec{x},t)\cdot\vec{v}(\vec{x},t) + \vec{\nabla}_x\cdot\vec{v}(\vec{x},t)\right) \tag{135}$$

Where we see that the real part is $\mathscr{H}^{\psi}(\vec{x},t)$.

At first this might seem to be a random coincidence. However, it is quite far from being so. Let us diagonalize the Hamiltonian operator to get the eigenstates $\{E_j(x)\}_j$ and eigenvalues $\{\varepsilon_j\}_j$ with $\varepsilon_j \in \mathbb{R}$, such that $\hat{H}_{\vec{x}}E_j(\vec{x}) = \varepsilon_j E_j(\vec{x})$. Then, if the system had a wavefunction matching such an eigenstate, $\psi(\vec{x}) = E_j(\vec{x})$, the energy of the different configurations would be

$$\mathscr{H}^{\psi}(\vec{x},t) = \mathrm{Re}\left\{\frac{\hat{H}_{\vec{x}}E_j(\vec{x})}{E_j(\vec{x})}\right\} = \varepsilon_j \quad \forall\vec{x}\in\Omega_t. \tag{136}$$

That is, for an eigenstate of $\hat{H}_{\vec{x}}$, all the trajectories would have the same energy ε_j! Then, the natural question is the reverse one: are there additional acceptable wavefunctions $\psi(\vec{x},t)$ different from the eigenstates of $\hat{H}_{\vec{x}}$ for which all the possible configurations have the same energy? The answer unfortunately could be affirmative, but not if we add an additional condition. It turns out that the eigenstates $E_j(\vec{x})$ have an additional remarkable property. Given that the system at a certain time $t = t_0$ was an eigenstate $\psi(\vec{x},t_0) = E_j(\vec{x})$, we immediately know it would evolve to the state

$\psi(\vec{x},t) = e^{-\frac{i}{\hbar}\varepsilon(t-t_0)}E_j(\vec{x})$.

Using the propagator definition (**??**) and the definition of the exponential of an operator (**??**) together with the complex exponential

$$\psi(\vec{x},t) = \hat{U}_{\vec{x}}\psi(\vec{x},t_0) = e^{-\frac{i}{\hbar}\hat{H}_{\vec{x}}(t-t_0)}\psi(\vec{x},t_0) = \left(1 - \frac{i}{\hbar}(t-t_0)\hat{H}_{\vec{x}} + \frac{i^2}{\hbar^2}(t-t_0)\hat{H}_{\vec{x}}\hat{H}_{\vec{x}} + ...\right)\psi(\vec{x},t_0) =$$
$$= \left(1 - \frac{i}{\hbar}(t-t_0)\varepsilon_j + \frac{i^2}{\hbar^2}(t-t_0)\varepsilon_j^2 + ...\right)E_j(\vec{x}) = e^{-\frac{i}{\hbar}\varepsilon_j(t-t_0)}E_j(\vec{x}) \tag{137}$$

Then, because $e^{-\frac{i}{\hbar}\varepsilon_j(t-t_0)}$ is a global phase that only depends on t, both the velocity field and the density of trajectories will be the same at all times. To see why, note that if $E_j(\vec{x}) = \rho^{1/2}(\vec{x})e^{iS(\vec{x})/\hbar}$,

$$\psi(\vec{x},t) = \rho^{1/2}(\vec{x})e^{i\frac{S(\vec{x})-\varepsilon_j(t-t_0)}{\hbar}} \tag{138}$$

and therefore, for all $t \geq t_0$, $|\psi(\vec{x},t)|^2 = \rho(\vec{x}) = |E(\vec{x})|^2$ and $\vec{v}(\vec{x},t) = \frac{1}{m_k}\vec{\nabla}(S(\vec{x}) - \varepsilon_j(t-t_0)) = \frac{1}{m_k}\vec{\nabla}S(\vec{x}) = \vec{v}(\vec{x},t_0)$. As a consequence, if we have that the wavefunction was an eigenstate of the Hamiltonian operator, not only the energy of all the trajectories will be the same everywhere in $\vec{x}$, but also at all time t!

Now yes, let us proof why, a state that fulfills both conditions, of constancy in time and configuration, must be an eigenstate of the Hamiltonian operator $\hat{H}_{\vec{x}}$. That is, let us prove that these two conditions uniquely characterize the Hamiltonian operator $\hat{H}_{\vec{x}}$, since by knowing the eigenstates and eigenvalues of $\hat{H}_{\vec{x}}$, we can fully determine it!

We first need to prove that denoting by $\text{Im}\{\cdot\}$ the imaginary part,

$$\text{Im}\left\{\frac{\hat{H}_{\vec{x}}\psi(\vec{x},t)}{\psi(\vec{x},t)}\right\} = \frac{\hbar}{2\rho(\vec{x},t)}\frac{\partial\rho(\vec{x},t)}{\partial t} = \hbar\frac{\partial}{\partial t}log(\rho(\vec{x},t)) \tag{139}$$

If we take the imaginary part that we already computed in (135) and write it all as a function of ρ and S, we can undo a chain rule to get

$$\text{Im}\left\{\frac{\hat{H}_{\vec{x}}\psi(\vec{x},t)}{\psi(\vec{x},t)}\right\} = -\frac{\hbar}{2}\sum_{k=1}^{n}\frac{1}{m_k}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial x_k}\frac{\partial S}{\partial x_k} + \frac{\partial^2 S}{\partial x_k^2}\right) = -\frac{\hbar}{2}\frac{1}{\rho}\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\rho\frac{1}{m_k}\frac{\partial S}{\partial x_k}\right) \tag{140}$$

If you take a look at to the continuity equation for the density to follow the velocity field, Eq. (95), you will immediately get the desired Eq. (**??**).

Then, by defining

$$\mathscr{H}_{\mathbb{C}}^{\psi}(\vec{x},t) := \frac{\hat{H}_{\vec{x}}\psi(\vec{x},t)}{\psi(\vec{x},t)} \quad \text{such that } \mathscr{H}_{\mathbb{C}}^{\psi}(\vec{x},t) = \mathscr{H}^{\psi}(\vec{x},t) + i\hbar\frac{\partial}{\partial t}log(\rho(\vec{x},t)) \tag{141}$$

we realize that to look for states that make the imaginary part of $\mathscr{H}_{\mathbb{C}}^{\psi}$ zero is equal to looking for states that have $\frac{\partial\rho(\vec{x},t)}{\partial t} = 0\ \forall t$[18], meaning that we would be demanding that the density $\rho(\vec{x},t)$ is constant in time. Therefore, imposing that $\mathscr{H}_{\mathbb{C}}^{\psi}$ yields the same real value $E \in \mathbb{R}$ for all $\vec{x}$, means that at the same time we impose the imaginary part to be zero (the Eulerian density is constant in time) and

[18] One could also consider the case in which $\rho(\vec{x},t)$ spikes to infinity as well, but for vectors $\psi \in L^2(\Omega_t)$ this is not a possibility, although we will see one example that appears in a limiting case.

that all configurations have the same energy E. That is, we have written the two conditions in one as

$$\frac{\hat{H}_{\vec{x}}\psi(\vec{x},t)}{\psi(\vec{x},t)} = E \quad \forall \vec{x}, t \Longleftrightarrow \hat{H}_{\vec{x}}\psi(\vec{x},t) = E\psi(\vec{x},t), \tag{142}$$

which means that $\psi(\vec{x},t)$ must be an eigenstate of eigenvalue E of the operator $\hat{H}_{\vec{x}}$ for both conditions on $\mathscr{H}^{\psi}_{\mathbb{C}}$ to be satisfied. Finally, since the eigenstates also fulfill that the velocity field is stationary, we even win that "for free".

In summary, a state $\psi(\vec{x},t)$ is such that all the tangent trajectories have the same energy $\mathscr{H}^{\psi}(\vec{x},t) = E$ in all configurations $\vec{x}$ at all times t, if and only if, $\psi(\vec{x},t)$ is an eigenstate of the Hamiltonian operator, such that $\hat{H}_{\vec{x}}\psi(\vec{x},t) = E\psi(\vec{x},t)$.

Because we saw that a Hermitian operator is always diagonalizable over real eigevalues, such that its eigenstates and eigenvectors uniquely determine it, we have that if we are able to find the states for which all trajectories have the same energy at all times, we will diagonalize $\hat{H}_{\vec{x}}$. Thus, this is an alternative method (quite unorthodox) for diagonalization. The most interesting thing will be the reverse however, that by abstractly diagonalizing a Hamiltonian, we will get very interesting particular states.

But, why are these states really interesting?

- No matter which Universe is ours, if one prepares an EWF that happens to be an eigenstate of its Hamiltonian, then, with probability one, one will know that the energy of the system is the eigenvalue, and that it will be so, while the potential energy $V(\vec{x})$ remains the same. We will see that this is the key misconception about measurements in quantum mechanics. That what we typically look for, is this kinds of states, because not only they are ontologically determined as all states are within our theory, but in particular these ones are also epistemologically determined (regarding the energy), and thus, will be phenomenologically appealing for many things.

- Since $\hat{H}_{\vec{x}}$ is a Hermitian operator, it diagonalizes over a complete orthonormal Hilbert set $\{E_j(\vec{x})\}_j$, such that

$$\iint_{\Omega} E_j^*(\vec{x})E_k(\vec{x})dx = \delta_{jk}. \tag{143}$$

As we saw, this means that an arbitrary state $\psi(\vec{x},t)$ can always be written as (the limit of) an "infinite linear combination" of the eigenstates of $\hat{H}_{\vec{x}}$ as

$$\psi(\vec{x},t_0) = \sum_{k=0}^{\infty} \alpha_k E_j(\vec{x}) \tag{144}$$

for certain coordinates $\alpha_k \in \mathbb{C}$ obtainable as

$$\alpha_k = \iint E_k^* \psi(\vec{x},t_0)dx \tag{145}$$

But of course, this can be done with any complete orthonormal Hilbert set. What makes this one special? The key is that once we know the projection coordinates α_k of an arbitrary state $\psi(\vec{x},t_0)$ on the Hamiltonian eigenvector set, even if it is for a single time t_0, its whole time evolution will be completely determined. This is because of the superposition principle or linearity of the Schrödinger Equation. Indeed

$$\hat{U}^t_{\vec{x}}\psi(\vec{x},t_0) = \hat{U}^t_{\vec{x}}\sum_{k=0}^{\infty}\alpha_k E_k(\vec{x}) = \sum_{k=0}^{\infty}\alpha_k \hat{U}^t_{\vec{x}} E_k(\vec{x}) = \sum_{k=0}^{\infty}\alpha_k e^{-\frac{i}{\hbar}\varepsilon_k(t-t_0)}E_k(\vec{x}). \tag{146}$$

That is, the time evolution only sets a **relative** phase shift on each coefficient, $\alpha_k \xmapsto{\hat{U}} \alpha_k e^{-\frac{i}{\hbar}\varepsilon_k(t-t_0)}$, or in general, if we write the coefficients as functions of time $\alpha_k(t) = \alpha_k(t_0)e^{-\frac{i}{\hbar}\varepsilon_k(t-t_0)}$. Note how this does change the total density and velocity fields of the arbitrary state obtained with the sum, unlike what happened when the state was already an eigenstate: in that case the phase was a **global** one! EL PAN BIMBO!!!

We can prove this in a more formal way, by taking the Schrödinger Equation and developing it with the linear combination at the initial time $\psi(\vec{x}, t_0) = \sum_{k=0}^{\infty} c_k(t_0) E_k(\vec{x})$,

$$i\hbar \frac{\partial \psi(\vec{x}, t)}{\partial t} = \hat{H}_{\vec{x}} \psi(\vec{x}, t) \Rightarrow \sum_{k=0}^{\infty} i\hbar \frac{\partial c_k(t)}{\partial t} = \sum_{k=0}^{\infty} c_k(t) \hat{H}_{\vec{x}} E_k(\vec{x}) = \sum_{k=0}^{\infty} c_k(t) \varepsilon_k E_k(\vec{x}). \tag{147}$$

then, applying the operation $\int_\Omega E_j(\vec{x})^* \cdot dx$ in both sides of the equation

$$\sum_k i\hbar \frac{\partial c_k(t)}{\partial t} \int_\Omega E_j(\vec{x})^* E_k(\vec{x}) dx = \sum_k c_j(t) \varepsilon_j \int_\Omega E_j(\vec{x})^* E_k(\vec{x}) dx \Rightarrow i\hbar \frac{\partial c_j(t)}{\partial t} = c_j(t) \varepsilon_j \tag{148}$$

where we used the orthonormality of the eigenstates. The last differential equation can be directly integrated to get

$$c_k(t) = c_k(t_0) e^{-\frac{i}{\hbar} \varepsilon (t - t_0)}. \tag{149}$$

which gives what we wanted.

- A remarkable property of the expansion of an arbitrary state $\psi(\vec{x}, t)$ in this basis, $\psi(\vec{x}, t) = \sum_k \alpha_k(t) E_k(\vec{x})$, is that by knowing the coefficients $\alpha_k(t)$ and the eigen-energies ε_k alone, we immediately know the ontological average energy of the possible trajectories, or equivalently, since the total amount of trajectories is normalized to 1, the total energy of the fluid of tangent trajectories, as

$$\mathbb{E}[\mathscr{H}; \psi](t) = \int_\Omega \mathscr{H}^\psi(\vec{x}, t) \rho(\vec{x}, t) dx = \sum_k |\alpha_k(t)|^2 \varepsilon_k \tag{150}$$

Given that $\forall z \in \mathbb{C}$, $\frac{1}{2}(z + z^*) = \text{Re}\{z\}$ and meaning by $c.c.$ the complex conjugate of the previous part of the same expression,

$$\int_\Omega \mathscr{H}^\psi(\vec{x}, t) \rho(\vec{x}, t) dx = \int_\Omega \text{Re}\left\{ \frac{\hat{H}_{\vec{x}} \psi(\vec{x}, t)}{\psi(\vec{x}, t)} \right\} \psi^* \psi dx = \frac{1}{2} \int_\Omega \left(\psi^* \hat{H}_{\vec{x}} \psi + \psi \hat{H}_{\vec{x}} \psi^* \right) dx = \tag{151}$$

$$= \frac{1}{2} \int_\Omega \sum_j \left(\alpha_j^*(t) E_j^*(\vec{x}) \right) \sum_k \left(\alpha_k(t) \varepsilon_k E_k(\vec{x}) \right) dx + c.c. = \frac{1}{2} \sum_k \sum_j \alpha_j^*(t) \alpha_k(t) \varepsilon_k \int_\Omega E_j^*(\vec{x}) E_k(\vec{x}) dx + c.c.$$

Using that by orthonormality $\int_\Omega E_j^*(\vec{x}) E_k(\vec{x}) dx = \delta_{jk}$,

$$\int_\Omega \mathscr{H}^\psi(\vec{x}, t) \rho(\vec{x}, t) dx = \frac{1}{2} \sum_j |\alpha_k(t)|^2 \varepsilon_k + c.c. = \sum_j |\alpha_k(t)|^2 \varepsilon_k. \tag{152}$$

This feature will be crucial in the chapter about measurement.

We can now almost see why we call it *Quantum*!
Because in general, the cardinality of the orthonormal Hilbert basis in L^2 must be countable, the set of eigenstates of the Hamiltonian operator are in general countable. That is, it is not possible to have for any energy a fluid of tangent Universes with a velocity field and density that precisely give all the trajectories the same energy (and thus make the energy epistemologically deterministic). This delicate orchestration that gives all the trajectories the same energy can only be made for certain particular energies and not the other ones, only for a discretized grid or quantized grid of energies is this orchestration possible. If one wants to go from such a state to another one, only crossing states for which the epistemological energy is deterministic, it is not possible (as it was in classical mechanics)! You need to introduce discrete or quantized chunks of energy into the system to make it go from such a state to another one, and only that exact amount of energy may produce that exact transition. That is why we call it quantum! Because as we will see in what follows, the fact that the energy in this sense, considering the entire fluid, is quantized, will also cause that other observables like the angular momentum are be quantized. But notice, that this does not at all mean that we can have a Universe with an arbitrary (non-quantized) energy, just that the special arrangement where all Universes have the same energy are limited in a sense.

As we anticipated in this gray box, what we have found is way more than a Hamiltonian diagonalization technique, since it will now allow us to meaningfully define what we will call in general, observable operators! We will first walk through some very important examples, and we will state a general result afterwards.

NOT ONLY THE TOTAL ENERGY OF THE FLUID BE CONSERVED BUT ALSO THE ENERGY OF EACH ONE OF THE TRAJECTORIES! EN GENERAL SI OMEGA NOT BOUNDED LA UNICA CONDICION DE CONTORNO QUE TENDRA SENTIDO PARA QUE ESTE NORMALIZADA LA FK DE ONDA ES QUE SE HAGA CERO AL TENDER AL INFINITO.

A.6.3. Example 1: The Free Particles - the Momentum Operator

Let us retake the free particle system example where there is no potential V imposing hindrance on the motion of the fluid towards maximum diffusion or minimum Fisher information, $V(\vec{x}) \equiv 0$. If so, the Hamiltonian operator would be left as $\hat{H}_{\vec{x}} = \sum_{k=1}^{n} \frac{-\hbar^2}{2m_k}\frac{\partial^2}{\partial x_k^2}$, which would make the energy of each configuration $\mathscr{H}^\psi$

$$\mathscr{H}^\psi(\vec{x},t) = \sum_{k=1}^{n} \frac{1}{2} m_k v_k(\vec{x},t)^2 + Q(\vec{x},t). \tag{153}$$

Then we could ask which are the states for which the energy of every trajectory is the same and is a constant in time. By the previous result we know that must be those states that diagonalize the Hamiltonian. To see the equivalence, let us consider the diagonalization from the two directions.

To see the power of the result, we will first compute the states that make all the trajectories have the same energy at all times by diagonalizing the Hamiltonian and then we will try to obtain them as if we did not have the equivalence.

The states $\psi(\vec{x})$ that diagonalize the Hamiltonian must fulfill

$$\hat{H}_{\vec{x}}\psi(\vec{x}) = E\psi(\vec{x}) \iff \sum_{k=1}^{n} \frac{-\hbar^2}{2m_k}\frac{\partial^2}{\partial x_k^2}\psi(\vec{x}) = E\psi(\vec{x}). \tag{154}$$

If we define a change of variables such that $\tilde{x}_k = \frac{x_k}{m_k}$ and define $\tilde{\psi}(\tilde{\vec{x}}) = \psi(\vec{x}(\tilde{\vec{x}}))$, the eigenstate equation is left as if all masses were the same

$$\sum_{k=1}^{n} \frac{-\hbar^2}{2} \frac{\partial^2}{\partial \tilde{x}_k^2} \tilde{\psi}(\tilde{\vec{x}}) = E\tilde{\psi}(\tilde{\vec{x}}) \Longleftrightarrow \sum_{k=1}^{n} \frac{\partial^2}{\partial \tilde{x}_k^2} \tilde{\psi}(\tilde{\vec{x}}) = \frac{-2E}{\hbar^2} \tilde{\psi}(\tilde{\vec{x}}). \tag{155}$$

For reasons that will become clear soon, let us choose a plane wave solution proposal $\tilde{\psi}(\tilde{\vec{x}}) = \alpha e^{i\vec{k}\cdot\tilde{\vec{x}}}$, where $\alpha \in \mathbb{C}$ and $\vec{k} = (k_1, ..., k_n) \in \mathbb{R}^n$.[a] Given $E \geq$,[b] Introducing it in the eigenvalue problem, we get such a solution is an eigenstate if

$$k_1^2 + \cdots + k_n^2 = \frac{2E}{\hbar^2} \Rightarrow ||\vec{k}||^2 = \frac{\sqrt{2E}}{\hbar}, \tag{156}$$

where we discard the negative sign case (for $||\vec{k}|| > 0$). This condition means that given a set $(k_1, ..., k_n)$ for which $||\vec{k}||^2 = 2E/\hbar$, the new $\vec{k}'$ obtained by multiplying any number of the k_j by -1, will also be an eigenstate of the same energy E. What is more, due to the linearity of the differential equation (by the superposition principle), any linear combination of them will also be an eigenstate of the same energy. Thus, for now we know that the linear combinations of the functions

$$\tilde{\psi}(\tilde{\vec{x}}) = \alpha e^{i\vec{k}\cdot\tilde{\vec{x}}} \qquad \text{for any } \vec{k} \in \mathbb{R}^n \text{ s.t.} ||\vec{k}|| = \frac{\sqrt{2E}}{\hbar} \tag{157}$$

will be eigenstates of the same energy E, or equivalently, reverting the change of variables, those of

$$\psi_{\vec{k}}(\vec{x}) = \alpha e^{i(k_1 m_1 + \cdots + k_n m_n)\cdot\vec{x}} \qquad \text{for any } \vec{k} \in \mathbb{R}^n \text{ s.t.} ||\vec{k}|| = \frac{\sqrt{2E}}{\hbar}. \tag{158}$$

These are what are called, **degenerate** states in energy. In particular, because the gradient of the phase divided by $\hbar$ is the momentum field, we would have for the j-th degree of freedom

$$p_j(\vec{x}) = \frac{\partial}{\partial x_k}(\hbar \; phase\{\psi\}(\vec{x}))) = \hbar m_j k_j \tag{159}$$

which is constant in $\vec{x}$. In fact, this means that E is equal to the kinetic energy, which is physically appealing. Thus, if we renamed the constant $k_j = \frac{p_j}{\hbar m_j}$ for some other constant p_j, which will turn out to be equal to the momentum of all the trajectories associated to that wavefunction (if it is not superposed with other wavefunctions), the general result could be re-written as

$$\psi_{\vec{p}}(\vec{x}) = \alpha e^{i\frac{\vec{p}\cdot\vec{x}}{\hbar}} \qquad \text{for any } \vec{p} \in \mathbb{R}^n \text{ s.t.} \sum_{k=1}^{n} \frac{p_k^2}{2m_k} = E. \tag{160}$$

A first necessary constraint to make each an acceptable state would be that α is such that the norm of the wavefunction is unity $\int_\Omega \psi^*(\vec{x})\psi(\vec{x})dx = 1$.
One might ask whether there are additional eigenstates that yield an energy E. Without imposing anything else about Ω, there might. However, we will prove that these are enough for all the relevant cases, even when we take the limit to an unbounded domain Ω.

[a] For maximum generality, we should let the constants $\vec{k} = (k_1, ..., k_n) \in \mathbb{C}^n$. However, if $k_j \in \mathbb{C}$, this introduces an exponentially decaying or increasing factor multiplying the state as $e^{-\operatorname{Im}\{k_j\}x_j}$. If the domain Ω was unbounded, this would be directly unacceptable for the density would grow with no bounds. But, even if the domain Ω is bounded, for the two relevant cases of wavefunction zero in the boundary and continous periodic function, the state will still be unacceptable. Thus, we will only consider the case in which $k_j \in \mathbb{R}$.

[b] If $E < 0$, we get that some of the k_j must be complex. So even if formally there is a state that is even normalizable if the domain Ω is bounded, exactly for the same reason mentioned in the previous footnote, such states will be unacceptable.

Thus, as proven in the gray-box, if no additional constraint is placed, the plane waves, $\{\alpha e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}\}_{\vec{p}\in\mathbb{R}^n}$, with $\alpha \in \mathbb{C}$ a normalization constant are eigenstates of the Hamiltonian with respective eigenvalues $\{E_{\vec{p}}\}_{\vec{p}\in\mathbb{R}^n} = \{\sum_{j=1}^n \frac{p_j^2}{2m}\}_{\vec{p}\in\mathbb{R}^n}$. For now the energy eigenvalues appear **not** to be "quantized", because we could find such a vector $\vec{p}$ for any desired $E \geq 0$. Moreover, each possible eigenvalue $E_{\vec{p}}$ is multiply degenerated. At least 2^n times degenerate, since if we find one such $\vec{p}$, we can obtain 2^n other $\vec{p}'$ by setting a -1 in any of the entries of $\vec{p}$.

Bounded Domain

In order to give physical meaning to this, we must define a particular domain Ω. Let us assume first $\Omega \subseteq \mathbb{R}^n$ is bounded. For simplicity it will be a parallelotope $\Omega = \prod_{j=1}^n (0, L_j)$. Without still saying what happens in its boundary, this readily imposes the normalization condition for the eigenstates such that, up to a global phase, we can fix $\alpha = \frac{1}{\sqrt{V}}$, for

$$\int_\Omega \alpha^* e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{x}} \alpha e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}} = 1 \Longrightarrow |\alpha|^2 = \frac{1}{\prod_{j=1}^n L_j}. \tag{161}$$

where $V := \prod_{j=1}^n L_j$ is the n-volume of the parallelotope Ω. Such a bounded domain must necessarily be considered in a numerical simulation for example, so it is not at all as artificial as it may sound. What is more, the domain of an effective wave-function of a subsystem, which we will study in the next sections may naturally be bounded.

Now, let us try two important different behaviours for the fluid in its boundary, which we will denote by $\partial\Omega$.

(a) Confined Universe

If we define the boundary of the domain Ω to be a confinement wall (as the one appearing in open-world videogames when you arrive to the edge of the map), it will be a boundary where there can be no Universe, $\rho(\vec{x})\Big|_{\vec{x}\in\partial\Omega} = 0$, and no Universe can get out of it, $\vec{v}(\vec{x})\Big|_{\vec{x}\in\partial\Omega} = 0$. By the continuity equation this will make sure that $\rho(\vec{x},t)\Big|_{\vec{x}\in\partial\Omega} = 0\ \forall t$ (some call these ends of the fluid as "infinite potential energy walls"). These two conditions together, immediately imply that

$$\psi(\vec{x})\Big|_{\vec{x}\in\partial\Omega} = 0 \tag{162}$$

Since a general state $\psi(\vec{x}) = \frac{1}{\sqrt{V}} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}$ does not fulfill it, we might think that there is no energy eigenstate in this case. However, we remember that the plane wave with momentum $\vec{p} = (p_1, ..., p_n)$ and all its 2^n variants $\vec{p}' = (-p_1, p_2, ..., p_n), \vec{p}'' = (-p_1, -p_2, ..., p_n), \vec{p}''' = (p_1, -p_2, p_3, ..., p_n)$ etc. had the same energy. Thus, we still have the chance that a superposition of them is acceptable, since such a superposition would still be an eigenstate of energy E.

Let us index by j the j-th of the 2^n variants of $\vec{p} := (p_1, ..., p_n)$ with $j \in \{1, ..., 2^n\}$, such that $j = 0$ is $\vec{p}$ itself. Consider the j-th variant of $\vec{p}$ is given by $\vec{p}'(j)$ and that the number of negative signs it has relative to $\vec{p}$ is $\sigma(j)$. Then, we claim the equality

$$\sum_{j=1}^{2^n} (-1)^{\sigma(j)}\ e^{\frac{i}{\hbar}\vec{p}'(j)\cdot\vec{x}} \;=\; (2i)^n \sin\Big(\frac{p_1x_1}{\hbar}\Big)\sin\Big(\frac{p_2x_2}{\hbar}\Big)\cdots\sin\Big(\frac{p_nx_n}{\hbar}\Big). \tag{163}$$

To see why this is true, let us take the result and develop it using the definition of the sine in terms of plane waves $sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$,

$$(2i)^n \sin\Big(\frac{p_1x_1}{\hbar}\Big)\sin\Big(\frac{p_2x_2}{\hbar}\Big)\cdots\sin\Big(\frac{p_nx_n}{\hbar}\Big) = \tag{164}$$

$$= (2i)^n \frac{\Big(e^{\frac{i}{\hbar}p_1x_1} - e^{-\frac{i}{\hbar}p_1x_1}\Big)}{2i}\cdot\frac{\Big(e^{\frac{i}{\hbar}p_2x_2} - e^{-\frac{i}{\hbar}p_2x_2}\Big)}{2i}\cdots\frac{\Big(e^{\frac{i}{\hbar}p_nx_n} - e^{-\frac{i}{\hbar}p_nx_n}\Big)}{2i} =$$

$$= \Big(e^{\frac{i}{\hbar}(p_1x_1+p_2x_2)} - e^{\frac{i}{\hbar}(-p_1x_1+p_2x_2)} - e^{\frac{i}{\hbar}(p_1x_1-p_2x_2)} + e^{\frac{i}{\hbar}(-p_1x_1-p_1x_1)}\Big)\cdot\Big(e^{\frac{i}{\hbar}p_3x_3} - e^{-\frac{i}{\hbar}p_3x_3}\Big)\cdots =$$

$$= e^{\frac{i}{\hbar}(p_1x_1+\cdots+p_nx_n)} - e^{\frac{i}{\hbar}(-p_1x_1+p_2x_2+\cdots+p_nx_n)} - e^{\frac{i}{\hbar}(-p_1x_1-p_2x_2+p_3x_3+\cdots+p_nx_n)} + \cdots + e^{\frac{i}{\hbar}(-p_1x_1-\cdots-p_nx_n)}.$$

Much in a similar way, we can prove the equality

$$\sum_{j=1}^{2^n} e^{\frac{i}{\hbar}\vec{p}'(j)\cdot\vec{x}} \;=\; 2^n \cos\Big(\frac{p_1x_1}{\hbar}\Big)\cos\Big(\frac{p_2x_2}{\hbar}\Big)\cdots\cos\Big(\frac{p_nx_n}{\hbar}\Big). \tag{165}$$

Therefore the following linear combination

$$\phi_{\vec{p}}(\vec{x}) = \frac{A}{2^n}\sum_{j=1}^{2^n} e^{\frac{i}{\hbar}\vec{p}'(j)\cdot\vec{x}} + \frac{B}{(2i)^n}\sum_{j=1}^{2^n}(-1)^{\sigma(j)}\ e^{\frac{i}{\hbar}\vec{p}'(j)\cdot\vec{x}}, \tag{166}$$

is equal to

$$\phi_{\vec{p}}(\vec{x}) = A\cos\Big(\frac{p_1x_1}{\hbar}\Big)\cdots\cos\Big(\frac{p_nx_n}{\hbar}\Big) + B\sin\Big(\frac{p_1x_1}{\hbar}\Big)\cdots\sin\Big(\frac{p_nx_n}{\hbar}\Big). \tag{167}$$

As proven in the gray-box, the state

$$\phi_{\vec{p}}(\vec{x}) = A\cos\Big(\frac{p_1x_1}{\hbar}\Big)\cdots\cos\Big(\frac{p_nx_n}{\hbar}\Big) + B\sin\Big(\frac{p_1x_1}{\hbar}\Big)\cdots\sin\Big(\frac{p_nx_n}{\hbar}\Big) \tag{168}$$

is a linear combination which has an associated eigenstate E. Now, this state can be made to satisfy the boundary condition, but this will restrict the allowed $\vec{p}$.

Any point $\vec{d} \in \partial\Omega$ can be written as $\vec{d} = (a_1L_1, ..., a_nL_n)$ with $a_j \in [0,1]$ and at least one of the $a_j = 0$ or $a_j = L_j$ (the faces of the box that touch the origin and the ones touching the point $(L_1, ..., L_n)$ respectively). Then if $\vec{d} = \vec{0}$

$$\psi(\vec{d})\Big|_{\vec{d}\in\partial\Omega} = 0 \Longrightarrow\; A = 0 \tag{169}$$

wiping out the cosines. Then, for any of the faces of the box that cross the origin, $\phi_{\vec{p}} = 0$ immediately (since at least one of the $x_j = 0$ there). Finally, if we evaluate $\vec{d} = (L_1, ..., L_n)$

$$\psi(\vec{x})\Big|_{\vec{x}\in\partial\Omega} = 0 \Longrightarrow\; \sin\Big(\frac{p_1L_1}{\hbar}\Big)\cdots\sin\Big(\frac{p_nL_n}{\hbar}\Big) = 0, \tag{170}$$

taking into account that even if only one of the $a_j = L_j$ and for the rest $a_k \in (0,1)$ it must happen that

$$\psi(\vec{x})\Big|_{\vec{x}\in\partial\Omega} = 0 \Longrightarrow \sin\left(\frac{p_1 a_1 L_1}{\hbar}\right)\cdots\sin\left(\frac{p_j L_j}{\hbar}\right)\cdots\sin\left(\frac{p_n a_n L_n}{\hbar}\right) = 0, \tag{171}$$

this implies that for all the $j \in \{1,...,n\}$

$$\sin\left(\frac{p_j L_j}{\hbar}\right) = 0 \implies p_j = \frac{\hbar\pi}{L_j}s_j \quad \text{with } s_j \in \mathbb{Z}. \tag{172}$$

That is, only a countable number of momenta $\vec{p}$ have an eigenstate! Therefore, we will only be able to orchestrate a wavefunction were all the trajectories have the same energy for a quantized number of energies, namely, if we denote by $\vec{s} = (s_1, ..., s_n)$ the vector of quantum numbers, for the energies

$$E_{\vec{s}} = \sum_{k=1}^{n} \frac{1}{2m_k}\left(\frac{s_j\hbar\pi}{L_j}\right)^2 \quad \text{with } \vec{s} \in \mathbb{Z}^n \tag{173}$$

where the corresponding eigenstates would be

$$\phi_{\vec{s}}(\vec{x}) = \sqrt{\frac{2^n}{V}}\sin\left(\frac{s_1\pi x_1}{L_1}\right)\cdots\sin\left(\frac{s_n\pi x_1}{L_n}\right) \tag{174}$$

where we found B by requiring the norm to be 1. Moreover, we can check that these functions are orthonormal, such that

$$\int_{\Omega} \phi^*_{\vec{s}}(\vec{x})\phi_{\vec{s}'}(\vec{x})dx = \delta_{\vec{s}\vec{s}'}. \tag{175}$$

There is one last thing to prove in order to really state this conclusion however. When we derived the plane wave eigenstates, two gray-boxes above, we did not prove whether there were other functions that could be potential solutions to the Hamiltonian eigenstate problem. We can now confidently say that for the chosen domain and boundary conditions there are no more.

This is because we can realize that by the discrete Fourier series theorem, we know that for any function $\varphi(\vec{x})$ confined in Ω (in reality, that is periodic in the unit cell Ω), there exist complex coefficients $\{A_{\vec{s}}, B_{\vec{s}}\}_{\vec{s}\in\mathbb{Z}^n}$ such that

$$\varphi(\vec{x}) = \sum_{\vec{s}\in\mathbb{Z}^n}\left(A_{\vec{s}}\cos\left(\frac{s_1\pi x_1}{L_1}\right)\cdots\cos\left(\frac{s_n\pi x_1}{L_n}\right) + B_{\vec{s}}\sin\left(\frac{s_1\pi x_1}{L_1}\right)\cdots\sin\left(\frac{s_n\pi x_1}{L_n}\right)\right). \tag{176}$$

In particular, any function $\varphi(\vec{x})$ that fulfills the boundary condition of being zero in $\partial\Omega$ will have that all coefficents $A_{\vec{s}} = 0$, such that any function satisfying the conditions will be a superposition of the energy eigenstates we have found. Since this means that the eigenstates we have found form a complete orthonormal basis, they are "all" the eigenstates of the Hamiltonian operator (taking into account the boundary). If there was another one, we could write it as a combination of the ones we have found, so it would not be a "linearly independent" eigenstate.

Note that, as the time evolution of an eigenstate only introduced a global phase, the states we have found have a null velocity field everywhere in the domain (since they have no spatially varying complex phase) at all times,

$$v_k^{(\vec{s})}(\vec{x},t) = \frac{1}{m_k}\frac{\partial \text{phase}(\phi_{\vec{s}}(\vec{x}))}{\partial x_k} = 0 \quad \forall\vec{x}\in\Omega \;\; \forall t \geq t_0, \tag{177}$$

meaning the kinetic energy of each configuration must be zero at all times. This leaves alone the quantum potential in the trajectory energy expression, $\mathscr{H}^\psi = Q$. Since the time evolution of an eigenstate leaves an initial density $\rho_{\vec{s}}(\vec{x}) = |\phi_{\vec{s}}|^2(\vec{x})$ constant in time, the energy $\mathscr{H}^\psi$ will be constant in time. Finally, we can see it will also be constant in position by explicitly computing the derivatives of the density.

Result Interpretation

We can interpret all this from the point of view of the trajectories. If the trajectories were for instance, in constant motion, they would try to surpass the boundary where their density must be zero. So at least we know that a constant velocity field is not possible. Let us imagine we set it to zero at a certain time. In order for the tangent trajectories to continue at rest in the next time, there should be no force acting on them. Since the only force that can appear is the one due to the variation of the quantum potential, we need it to be constant in space. So, we look for a function that has a curvature that allows this. And the sine is such a function. As you see, it seems that with similar argumentation we could have found the same orchestrations without knowing the Schrödinger Equation and thus that they are to be eigenstates of the Hamiltonian operator. Indeed, we can diagonalize the Hamiltonian operator using the tangent Universe interpretation as we prove in the next gray-box. Yet, we will see that without the eigenstate trick, the problem is fairly more complicated to be resolved. However, it might add light to why the eigenstates happened to be in particular the ones we found.[19]

In order to make the notation less cumbersome, consider the 1D case and that the domain is bounded as $\Omega_t = (0, L) \subset \mathbb{R}$. We want to have a constant $\mathscr{H}^\psi$ in x, such that $\frac{\partial \rho(x,t)}{\partial t} = 0$. From the continuity equation, we know that the second condition implies

$$\frac{\partial}{\partial x}(v(x,t)\rho(x,t)) = 0 \Rightarrow v(x,t)\rho(x,t) = c(t) \quad \forall x \in [a,b], \forall t > t_0 \tag{178}$$

with $c(t) \in \mathbb{R}$ constant in space. This implies that rather

- Both v and ρ are (non-zero) constant in space
- ρ is zero.
- They are not constant in space but are everywhere the multiplicative inverse of each other.
- v is zero.

Let us first consider the **confined system boundary condition (a)**, such that for all $\vec{x} \in \delta\Omega$, $\rho(\vec{x}, t) = 0$.
We note that for this system, the first of the possible implications we just obtained, is not possible, since if the ρ is to be continuous and at the edges of the domain it must be zero, for it to be constant, it should be zero everywhere, which leaves us an unacceptable state. This also implies that the second case is discarded. The third case could be possible but we will rule it out because with only states for which the last condition holds, we will be able to find a set of solutions the linear combinations of which can give any possible state (even the ones that are not eigenvalues).[a]

[a] There could be an eigenstate that has indeed v and ρ as multiplicative inverses of each other. However, by linearity of the Hamiltonian operator and the completeness of the basis of eigenstates we will find, it will be necessarily a combination of such solutions with the same energy eigenvalue, and all of these solutions will just fulfill the fourth condition.

[19] In addition in some cases we will have no other choice, since, as we will see in part C, there are scenarios where the full Hamiltonian diagonalization is inaccessible in practice, but there are phenomenological results of interest that depend on it.

Let us then consider that $v(x,t) = 0\ \forall x \in \Omega$ and $\forall t \geq t_0$. Now, we should try to look for $\rho(x,t_0)$ such that this is possible and such that $\mathscr{H}^\psi = E$ is preserved at all times. Since the velocity field is constantly zero, we have that the kinetic energy of the states we are looking for should be zero, but in $\mathscr{H}^\psi(x,t)$ (135) we still have the quantum potential Q, depending on ρ. It should be constant. Indeed, this would also allow that given $v(x,t_0) = 0\ \forall x \in \Omega$, we have $v(x,t) = 0\ \forall t > t_0$, because the force on the trajectories is the derivative of the quantum force (which would be zero). Therefore, we have enough with looking for the densities that cause $\hbar^\psi(x,t) = Q(x,t) = E \in \mathbb{R}$. By defining a change of variable function $R(x,t)$ such that $R^2 = \rho$, the expression for Q becomes

$$Q(x,t) = -\frac{\hbar^2}{2m}\frac{1}{\rho^{1/2}(x,t)}\frac{\partial^2 \rho^{1/2}(x,t)}{\partial x^2} = -\frac{\hbar^2}{2m}\frac{1}{R(x,t)}\frac{\partial^2 R(x,t)}{\partial x^2} \tag{179}$$

meaning we are looking for $R(x,t)$ such that

$$Q(x,t) = E \Longrightarrow R(x,t) = R(x) \text{ and } -\frac{\partial^2 R(x)}{\partial x^2} = E\,R(x). \tag{180}$$

This partial differential equation is similar to the one we solved for the wavefunction in the diagonalization of the Hamiltonian, but now the solutions we are looking for are strictly real. In order to find a general solution, note first that we are looking for a function that is its own second derivative. We could suggest a sine or a cosine. Since it is a linear differential equation we can try both at once as $R(x) = A\sin kx + B\cos kx$, to we get the condition

$$k^2(A\sin kx + B\cos x) = E(A\sin kx + B\cos x) \Rightarrow k^2 = E \Rightarrow k = \pm\sqrt{E} \tag{181}$$

thus, since it is a linear differential equation, we would have that one set of solutions would be $R(x) = A\sin(\sqrt{E}x) + B\cos(\sqrt{E}x)$ for any $A, B \in \mathbb{R}$. Note that we consider $\sin(\sqrt{E}x)$ and $\sin(-\sqrt{E}) = -\sin(\sqrt{E})$ as a same solution, since the negative sign would be absorbed in a new constant. Yet this is not really a solution allowing any E, since it should also satisfy the boundary conditions, only the $R(x)$ such that

$$R(0) = 0 \Rightarrow B = 0 \text{ and then } R(L) = 0 \Rightarrow \sin(\sqrt{E}L) = 0 \Rightarrow \sqrt{E} = \frac{\pi\ r}{L}\text{with } r \in \mathbb{Z} \tag{182}$$

are solutions. And then $R(x) = \sqrt{\frac{2}{L}}\sin\frac{\pi r}{L}x$ (note the normalization condition) now yes, would be a possible solution for any $r \in \mathbb{Z}$, yielding a constant energy $E_r = \pi^2 r^2/L^2$. Let us see that there are no more solutions.

By the Fourier theorem, any real (periodic) function in $(0, L)$ can be written as a linear combination of sines and cosines of frequencies that are commensurate with the interval, but if the solution must be zero at $x = 0$, for any combination $c_1 sin(kx) + c_2 cos(kx)$, we will have $c_2 = 0$ and the series will only be made out of sines. The commensurate sines are those $sin(\kappa x)$ such that

$$sin(\kappa x)\Big|_{x=L} = 0 \Rightarrow sin(\kappa L) = 0 \Rightarrow \kappa x = \pi r \text{ for } r \in \mathbb{Z}. \tag{183}$$

where r and $-r$ are essentially the same sine. Thus in general we have that any function $f(x)$ in this domain can be written for some coefficients $\{A_r\}_{r\in\{1,2,\ldots\}} \subset \mathbb{R}$ as

$$f(x) = \sum_{r=1}^{\infty} A_r \sin\left(\frac{x\pi r}{L}\right). \tag{184}$$

But as we proved in the previous paragraph, this is nothing but a linear combination of functions that each is a solution for the differential equation (180) with a different E. Then this proves by the linearity of the equation (180) the only solutions are the found ones. Therefore, they should be an eigenbasis for the Hamiltonian of the system. Indeed, they match with the ones we found outside the gray-box by direct diagonalization.

KOEFIZIENTIEK ZELAN ATARA EIN HILBERT SPACEN INTRODUKZIÑOAN, ETA AZALDU HAN IGUAL ADIBIDE MODURE FOURIER DISKRETOA!

EIN UNA DE LET US CALL THIS “THE ORCHESTRATION” ta holan hortik aurrera eztozu hainbeste errepikatu biher mantra guztixe.

(b) Pac-Man Universe

Let us now assume a bounded domain case where particles do not need to avoid “touching” the boundaries. That is, that they can go with a certain velocity against the boundary. If we assume that any particle entering from one face of the parallelotope will get out from the other face (this is a “Pac-Man Universe”, or if you prefer a Universe that is periodic over and over again in unit cells Ω), this mathematically means that for any $l_1, ..., l_n \in \mathbb{Z}$

$$\psi(\vec{x}) = \psi(\vec{x} + (l_1 L_1, \cdots, l_n L_n)) \tag{185}$$

which will mean the same conditions must be held for the velocity field and the density.

For this to be satisfied we cannot accept all the plane waves that formally appeared to be eigenstates at the beginning of the section, $\phi(\vec{x}) = \frac{1}{\sqrt{V}} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}$. Only those plane waves fulfilling the mathematical statement of periodicity in the pac-man box are acceptable. To know which ones in particular we apply the condition to the generic eigenstate $\alpha e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}$, to get that

$$e^{\frac{i}{\hbar}\vec{p}\cdot(l_1 L_1, \cdots, l_n L_n)} = 1 \tag{186}$$

which can happen only if

$$\vec{p} \cdot (l_1 L_1, \cdots, l_n L_n) = 2\pi\hbar r \quad \text{ for } r \in \mathbb{Z}. \tag{187}$$

This is the condition for the so-called “reciprocal lattice vectors”, relative to the “direct lattice” defined by the n parallelotope edges $\{(L_1, 0, ..., 0), ..., (0, ..., 0, L_n)\}$.

If we define an $\mathbb{R}^n$ basis as $\{\vec{P}_j\}_{j=1}^n = \{(2\pi\hbar/L_1, 0, ..., 0), ..., (0, ..., 0, 2\pi\hbar/L_n)\}$, since any vector $\vec{p} \in \mathbb{R}^n$ can be written in this basis, such that for some $s_1, .., s_n \in \mathbb{R}$,

$$\vec{p} = s_1 \vec{P}_1 + \cdots + s_n \vec{P}_n, \tag{188}$$

evaluating equation (187) we get the condition for acceptable plane-wave

$$s_1 l_1 + \cdots + s_n l_n = r \quad \text{ for } r \in \mathbb{Z} \tag{189}$$

Then, since $l_j \in \mathbb{Z}$ and $r \in \mathbb{Z}$, the equation can only be satisfied if $s_1, ..., s_n \in \mathbb{Z}$. This means that not all the momenta $\vec{p} \in \mathbb{R}^n$ are allowed, but only a countable number of them. Namely

$$\vec{p}_{\vec{s}} = \left(s_1 \frac{2\pi\hbar}{L_1}, ..., s_n \frac{2\pi\hbar}{L_n} \right) \quad \text{ with } \vec{s} = (s_1, ..., s_n) \in \mathbb{Z}^n. \tag{190}$$

Thus, we can only find states with constant energy for all trajectories for certain “quantized” energies

$$E_{\vec{s}} = \sum_{j=1}^{n} \frac{1}{2m_j} \left(s_j \frac{2\pi\hbar}{L_j} \right) \tag{191}$$

of associated states

$$\phi_{\vec{s}}(\vec{x}) = \frac{1}{\sqrt{V}} e^{\frac{i}{\hbar}\left(s_1 \frac{2\pi\hbar}{L_1} + \cdots + s_n \frac{2\pi\hbar}{L_n} \right)} \quad \text{ with } \vec{s} \in \mathbb{Z}^n. \tag{192}$$

Notice that these states are orthonormal to each other

$$\int_{\Omega} \phi^*_{\vec{s}}(\vec{x})\phi_{\vec{s}'}(\vec{x})dx = \delta_{\vec{s}\vec{s}'} \,. \tag{193}$$

Then, by noting that by the Fourier series theorem (in its plane-wave version), for any function $\varphi(\vec{x})$ that is periodic in the cell Ω there exist coefficients $\{C_{\vec{s}}\}_{\vec{s}\in\mathbb{Z}^n}$ such that it can be written as a superposition of precisely the orthonormal states we found

$$\varphi(\vec{x}) = \sum_{\vec{s}\in\mathbb{Z}^n} C_{\vec{s}} e^{\frac{i}{\hbar}\vec{p}_{\vec{s}}\cdot\vec{x}}, \tag{194}$$

we realize that the eigenstates of the Hamiltonian operator we have found form a complete orthonormal basis for our conditions, meaning there are no additional functions that give the energy orchestration that are not among the energies we listed before.

From the point of view of the trajectories, this time we have found a "reverse" case of stationary states. Now the density is constant in space, thus the quantum potential is what is zero everywhere, while the trajectories are not necessarily stationary and can move constantly.

With a similar development as the one made in case (a), we could have obtained the eigenstaets of this case from trajectory based arguments alone without the usage of the Schrödinger equation and the wavefunction.

Unbounded Domain

Let us now consider what happens when we take the domain Ω to be the whole $\mathbb{R}^n$. We found in the beginning of the example that for the free particle Hamiltonian $\hat{H}_{\vec{x}} = \sum_{j=1}^n \frac{-\hbar^2}{2m_j}\frac{\partial^2}{\partial x_j^2}$, all the plane-waves $\phi_{\vec{p}}(\vec{x}) = \alpha e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}$ with $\vec{p}\in\mathbb{R}^n$ satisfied the eigenvalue equation with eigenvalue $\varepsilon_{\vec{p}} = \sum_{j=1}^n \frac{p_j^2}{2m_k}$ (the kinetic energy),

$$\hat{H}_{\vec{x}}\phi_{\vec{p}}(\vec{x}) = \left(\sum_{j=1}^n \frac{p_j^2}{2m_k}\right)\phi_{\vec{p}}(\vec{x}). \tag{195}$$

This made them the candidates for energy eigenstates of the free system, and yet we found that when constraining the domain, only some of them or combinations of degenerate of them were acceptable wavefunctions. In this case we are placing no restriction at all, so it seems there is nothing against all of them being valid eigenstates. This would mean that a "really free" (in the sense that the space is not constrained) particle system would not have its orchestrated energies "quantized". However, it turns out that none of our plane-waves is normalizable if $\vec{x}\in\mathbb{R}^n$, not even by considering linear combinations of the degenerate ones. Therefore none of them is an state in L^2. The reason is quite clear: they do not even fulfill that $lim_{|\vec{x}|\to\infty}\phi_{\vec{p}}(\vec{x}) = 0$. In fact, it can be proven that $\hat{H}_{\vec{x}}$ cannot be diagonalized over a set of acceptable quantum states. We can qualitatively understand this, because if a state must fulfill that $lim_{|\vec{x}|\to\infty}\phi_{\vec{p}}(\vec{x}) = 0$, but $\rho \neq 0$ for some $\vec{x}$, then the quantum potential will take at least two different values and its derivative (thus the force the tangent trajectories exert on each other) will be non-null and a stationary state like the one we looked for will not be possible. Or if you prefer, since the effect of the quantum potential is to evolve the states towards the minimum Fisher information (which happens at a constant density, such as the plane-waves), we will not be able to stop the dynamics of the density.

Nevertheless, because the plane-waves do fulfill formally both the domain constraints and the eigenfunction equation, and thus in a formal sense they do determine the Hamiltonian, it is common to still call them "improper" quantum states. Improper in the sense that they can still be thought of as the limit of an acceptable quantum state, say, when the periodic box we considered in the bounded domain case is made big enough (in fact, since in numerical simulations an unbounded domain cannot be considered, over the simulation grids the approximations are quite acceptable). And in this sense, by lifting the normalization condition for the density, plane waves do describe an state for which all the trajectories have the same energy and actually a same momentum $\vec{p}$ at all times (they are minimal Fisher information states).

It turns out that this formal consideration can be taken seriously because even if they are not acceptable states, apart from the analogy, they do fulfill very interesting properties that Hamiltonian eigenstates possessed. To see the analogy, let us consider that $\{\varphi_j(\vec{x})\}_{j\in\mathbb{Z}^n}$ is an acceptable wavefunction orthonormal basis (indexed by a countable set, say $\mathbb{Z}^n$).

- They satisfy a "continous version" of the orthonormality condition

$$\int_{\mathbb{R}^n} \phi^*_{\vec{p}\,'}(\vec{x})\phi_{\vec{p}}(\vec{x})dx = \delta(\vec{p}\,' - \vec{p}) \quad \text{just as} \int_{\mathbb{R}^n} \varphi^*_{j\,'}(\vec{x})\varphi_j(\vec{x})dx = \delta_{j\,'j} \tag{196}$$

  where $\delta(\vec{y} - \vec{y}_0)$ is the Dirac delta, the continous index analogous of the Kronecker delta $\delta_k k_0$, or its continous limit

$$\delta(\vec{y} - \vec{y}_0) = \begin{cases} 0 \text{ if } \vec{y} \neq \vec{y}_0 \\ \infty \text{ if } \vec{y} = \vec{y}_0 \end{cases} \qquad \text{just as } \delta_{kk_0} = \begin{cases} 0 \text{ if } k \neq k_0 \\ 1 \text{ if } k = k_0 \end{cases} \tag{197}$$

  This expression is to be taken seriously only when it is inside an integral, for $\delta(\vec{y} - \vec{y}_0)$ is a **distribution**, such that, it just "selects" one of the indexes of a smooth function $f(\vec{y})$ with which it is integrated (just as the Kronecker delta used to do),

$$\int_{\mathbb{R}^n} f(\vec{y})\delta(\vec{y} - \vec{y}_0)dy = f(\vec{y}_0) \quad \text{just as} \sum_{k\in\mathbb{Z}^n} f_k\delta_{kk_0} = f_{k_0} \tag{198}$$

  Ein analogixe tio EXPLIKE SARTZEN DOZUNIEN EL HILBERT SPACE LO QUE ES EL PAN BIMBO Y CÓMO LAS FUNCIONES EN REALIDAD SIEMPRE HAN SIDO VECTORES! NIKE SEA EN UN GRAYBOX. Klaro eske Hilbert space is just an intermediate thing entre finita dimensional y improper de estas. En la del pan bimbo ein analogixe explicitamente esaten lo poco que le gustará a un matemático pero lo cierto que es que no es más que una analogía. As it happened in the kronecker delta case, this is a useful feature whenever we already have an integral over plane waves (what previously was a sum on the eigenstates), since integrating over we just select one of the eigenstates. An example of its usefulness will be given in the two proves of the end of the list.

- Any state can be expressed using combinations of these states. Since they are an uncountably many number of states we consider a "continuous linear combination", such that for any state $\psi(\vec{x})$, there exist coefficients $\tilde{\psi}(\vec{p})$ such that

$$\psi(\vec{x}\,) = \int_{\mathbb{R}^n} \tilde{\psi}(\vec{p}\,)\ \phi_{\vec{p}}(\vec{x})dp \quad \text{just as } \psi(\vec{x}\,) = \sum_{j\in\mathbb{Z}^n} c_j\varphi_j(\vec{x}), \tag{199}$$

  where the coeffients are also given by the "projections"

$$\tilde{\psi}(\vec{p}\,) = \int_{\mathbb{R}^n} \phi^*_{\vec{p}}(\vec{x})\ \psi(\vec{x}\,)dx \quad \text{just as } c_j = \int_{\mathbb{R}^n} \varphi_j(\vec{x})^*\ \psi(\vec{x})dx. \tag{200}$$

  The reason why this is true, is that since $\phi_{\vec{p}}(\vec{x}\,) = e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}$, equation (200) is just the Fourier transform and equation (199) the inverse Fourier transform

$$\tilde{f}(\vec{p}) = \int_{\mathbb{R}^n} f(\vec{x})e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}dx \quad \text{and} \quad f(\vec{x}) = \int_{\mathbb{R}^n} \tilde{f}(\vec{p})e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{x}}dx. \tag{201}$$

- The expectation value of the energy for an arbitrary state can be computed using "only" the projection coefficient magnitudes,

$$\mathbb{E}[\mathscr{H};\psi](t) = \int_{\mathbb{R}^n} \mathscr{H}^\psi(\vec{x},t)\rho(\vec{x},t)dx = \int_{\mathbb{R}^n} \varepsilon_{\vec{p}}\ |\tilde{\psi}(\vec{p},t)|^2 dp \quad \text{just as } \mathbb{E}[\mathscr{H};\psi](t) = \sum_{j\in\mathbb{Z}^n} \varepsilon_j |c_j(t)|^2. \tag{202}$$

Given that $\forall z \in \mathbb{C}$, $\frac{1}{2}(z+z^*) = \text{Re}\{z\}$ and meaning by *c.c.* the complex conjugate of the previous part of the same expression (assuming the Hamiltonian operator derivatives in $\vec{x}$ and the integrals over $\vec{p}$ commute),

$$\int_\Omega \mathscr{H}^\psi(\vec{x},t)\rho(\vec{x},t)dx = \int_\Omega \text{Re}\left\{\frac{\hat{H}_{\vec{x}}\psi(\vec{x},t)}{\psi(\vec{x},t)}\right\}\psi^*\psi dx = \frac{1}{2}\int_\Omega \left(\psi^*\hat{H}_{\vec{x}}\psi + \psi\hat{H}_{\vec{x}}\psi^*\right)dx = \tag{203}$$

$$= \frac{1}{2}\int_\Omega \int_{\mathbb{R}^n} \left(\tilde{\psi}^*(\vec{p}\,',t)\phi^*_{\vec{p}\,'}(\vec{x})\right)dp' \int_{\mathbb{R}^n} \left(\tilde{\psi}(\vec{p},t)\varepsilon_{\vec{p}}\phi_{\vec{p}}(\vec{x})\right)dx + c.c. =$$

$$= \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tilde{\psi}^*(\vec{p}\,',t)\tilde{\psi}(\vec{p},t)\varepsilon_{\vec{p}}\int_\Omega \phi^*_{\vec{p}}(\vec{x})\phi_{\vec{p}}(\vec{x})dxdpdp' + c.c. =$$

Using that by orthonormality condition we obtain a delta, which selects one of the p as

$$= \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tilde{\psi}^*(\vec{p}\,',t)\tilde{\psi}(\vec{p},t)\varepsilon_{\vec{p}}\delta(\vec{p}-\vec{p}\,')dpdp' + c.c. = \frac{1}{2}\int_{\mathbb{R}^n} \tilde{\psi}^*(\vec{p},t)\tilde{\psi}(\vec{p},t)\varepsilon_{\vec{p}}dp + c.c. \tag{204}$$

And thus,

$$\int_\Omega \mathscr{H}^\psi(\vec{x},t)\rho(\vec{x},t)dx = \frac{1}{2}\int_{\mathbb{R}^n} |\tilde{\psi}(\vec{p},t)|^2\varepsilon_{\vec{p}}dp + c.c. = \int_{\mathbb{R}^n} \varepsilon_{\vec{p}}|\tilde{\psi}(\vec{p},t)|^2 dp$$

- Given we know the projection coefficients of an initial quantum state $\psi(\vec{x},t_0)$ are $\tilde{\psi}(\vec{p},t_0)$, the time evolution of an state for the free particle Hamiltonian can be computed just adding relative phases

$$\psi(\vec{x},t) = \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}\varepsilon_{\vec{p}}(t-t_0)}\tilde{\psi}(\vec{p},t_0)\phi_{\vec{p}}(\vec{x})dp \quad \text{just as} \quad \psi(\vec{x},t) = \sum_{j\in\mathbb{Z}^n} e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{x}}c_j\varphi(\vec{x}) \tag{205}$$

Given we know the initial condition of a certain state and we know its projection coefficients $\psi(\vec{x},t_0)=\int_{\mathbb{R}^n}\tilde{\psi}(\vec{p},t_0)\phi_{\vec{p}}(\vec{x})dp$, by the Schrödinger Equation (113), and assuming that the integral over $\vec{p}$ and the derivatives in $\vec{x}$ in the Hamiltonian operator commute,

$$i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t}=\hat{H}_{\vec{x}}\psi(\vec{x},t)\Rightarrow\int_{\mathbb{R}^n}i\hbar\frac{\partial\tilde{\psi}(\vec{p},t)}{\partial t}\phi_{\vec{p}}(\vec{x})dp=\int_{\mathbb{R}^n}\tilde{\psi}(\vec{p},t)\hat{H}_{\vec{x}}\phi_{\vec{p}}(\vec{x})dp=\int_{\mathbb{R}^n}\varepsilon_{\vec{p}}\tilde{\psi}(\vec{p},t)\phi_{\vec{p}}(\vec{x})dp \tag{206}$$

by applying the operation $\int_{\mathbb{R}^n}\phi^*_{\vec{p}'}(\vec{x})\cdot dx$ on both sides and assuming both integrals commute

$$\int_{\mathbb{R}^n}i\hbar\frac{\partial\tilde{\psi}(\vec{p},t)}{\partial t}\left(\int_{\mathbb{R}^n}\phi^*_{\vec{p}'}(\vec{x})\phi_{\vec{p}}(\vec{x})dx\right)dp=\int_{\mathbb{R}^n}\varepsilon_{\vec{p}}\tilde{\psi}(\vec{p})\left(\int_{\mathbb{R}^n}\phi^*_{\vec{p}'}(\vec{x})\phi_{\vec{p}}(\vec{x})dx\right)dp \tag{207}$$

$$\int_{\mathbb{R}^n}i\hbar\frac{\partial\tilde{\psi}(\vec{p},t)}{\partial t}\delta(\vec{p}-\vec{p}')dp=\int_{\mathbb{R}^n}\varepsilon_{\vec{p}}\tilde{\psi}(\vec{p})\delta(\vec{p}-\vec{p}')dp\Rightarrow i\hbar\frac{\partial\tilde{\psi}(\vec{p}',t)}{\partial t}=\varepsilon_{\vec{p}}\tilde{\psi}(\vec{p}',t).$$

Finally, this equation has an immediate solution for each $\vec{p}$,

$$\tilde{\psi}(\vec{p},t)=\tilde{\psi}(\vec{p},t_0)e^{-\frac{i}{\hbar}\varepsilon_{\vec{p}}(t-t_0)} \tag{208}$$

which is what we wanted to prove.

There are actually several mathematically rigurous ways to treat continious spectrum "formal eigenfunctions" as the ones we have encountered, rather through Rigged Hilbert Spaces or through the von Neumann development CITEU. However, we will not consider them here for ease of reading.

Note that with the last of the properties we can now prove what we said a pair of sections ago about the boradening of an initially Gaussian wavepacket

DEMOSTREU!

The Ket Notation

Klaro has de introducir position eigenstates, verás que son tal que tampoco son estados cuánticos, pero que permiten una notación muy interesante y útil, idatzi un operator como en su diagonal shape quedaría y tal.

Haz incapié en que no es más que notación y que como bien decía un profesor, no heu de ser esclaus de la notació.

A.6.6. A General Observable Operator

What if we found a complete set of quantum states for which not the energy but some other property like the momentum, the angular momentum etc., call it property B, is the same for all trajectories at all times? That is, imagine we define a trajectory property B, as given by a particular function $\mathscr{B}^{\psi}(\vec{x},t)$, and we find a complete orthonormal set of functions $\{B_k(\vec{x})\}_{k\in\sigma}$ for which $\mathscr{B}^{B_k}(\vec{x},t)=b_k\in\mathbb{R}$ for all $\vec{x},t$. Then, not only the set of orthonormal functions would be insightful orchestrations for which no matter which would our trajectory be, with a probability 1, we would know the property B's value "forever", but, in addition, if we knew the expansion coefficients of an arbitrary function $\psi(\vec{x})$ in that base, such that $\psi(\vec{x},t)=\sum_{k\in\sigma}c_k(t)B_k(\vec{x})$, we would immediately know the expected value of the property B for that state as

$$\mathbb{E}[\mathscr{B};\psi] = \int_\Omega \rho(\vec{x},t)\mathscr{B}^\psi(\vec{x},t)dx = \sum_{k\in\sigma} b_k|c_k(t)|^2. \tag{209}$$

Now, there is a very convenient way to gather these states and values in a same mathematical structure, such that we can share them or refer to them altogether naming a single thing. We can build with them an operator that diagonalizes over them and has as eigenvalues the corresponding constant property B. More importantly, this would gives us a very convenient

But not only that. It turns out that this will give us the hint onto how to measure stuff in a very convenient way in part B and will very naturally allow us to see what the famous Heisenberg uncertainty formula means. And yet, not only that, but we will obtain a means to get more such bases for derived properties other than B together with a lifehack to build Hamiltonian operators, just getting inspired by classical mechanics.

But not only that! Este set nos permitiría saber la bohmian property como tal también para cualquier estado! Cómo? Pues allí es donde entra la necesidad última de definir un operador para nosotros. Ezta bape erreza demostretie........

If we ever find such a complete orhtonormal set of functions with associated real numbers giving the constant property B for each of them,

Ein ounbounded arinau expliketan de manera formal ein dogune dala, en el que sin poner ninguna restricción danak ziela eigenstatek. Pb? que ninguno es un estado de L2 valido, ezinzu normalizeu. Eta esan edmostreu ahal dala eztauela normalizeu ahal dozun eigenstateik, eta arrazoie dala edozein estado zabaldu eingo dala hacia el minimo fisher information. Baaaina, esan de manera formal que funciona como los otros por ejemplo para la evolución temporal tal. Orduen oin sartun momentum operatorrera, azaldu zein izengo zan oemga boundeden eta zein dan free spacen. orduen definidu momentum operator del free space zein dan eta idatzi finalmente hamiltonianoa ondo.

The Momentum Operator

Notice that interestingly, all the trajectories in the eigenstates of the free Hamiltonians we have considered, not only had the energy, but also the velocity equal for all trajectories at all times.

On this line, we might be interested in having a

Itxi klaru baten p momentume zala ta bestien ez!

On this line, we might want to know which are the states (density and velocity fields, or equivalently, the wavefunctions) such that the velocity is the same constant for all the trajectories at all times. Then, if we want the velocity of the state to be constant at all times

ta gero ein adibidie del potenical central eta definidu momentu angularra.

We could want to look for a complete orthonormal basis made out exclusively of states for which tal es constante. En cuyo caso para lo de la esperanza para esa base expansion de tal es sumar los coeficientes al cuadrado, y tal que podrás hacer un truco en la part B tanto lo de naive measurement como el von neumann normal. Si es función de rho y de v, que lo será, sabes que si impones rho cte y vel cte sale eigenstate izen bidala de hamiltoniano. Pues podemos buscar un Hamiltoniano para el que obtengas una base de orthornomals con no todos los tal para ellos igual a cero, porque si no el operador que te saldrá es el operador cero. Y entonces con esa base formas un operador hermítico nuevo, tal que sabes que siempre que conmute con el hamiltoniano como diagonalizarán en la misma base ortonormal ba ttak constante en el tiempo.

Orduen te fijas en que este operador su parte real operada en un estado es literalmente la versión bohmiana. Orduen, oin una vez tienes un par cualquier producto de ellos os suma o tal en espacio de operadores, sabrás que el operador que te sale es la cosa versión bohmiana, tal que sus eigenstates son tal ke para le bohmiano es fim y tal que para

It is now time to see the main perk that the operator eigenstate view of the trajectory orchestration provides us. As we introduced in 6.1., any pair of (Hermitian) operators $\hat{A}_{\vec{x}}, \hat{B}_{\vec{x}}$ that commute with each other will share a common eigenstate set (each with its own eigenvalues and degeneracies). From the trajectory point of view, this means that if the Hamiltonian $\hat{H}_{\vec{x}}$ commutes with some other Hermitian operator on the wavefunctions $\hat{G}_{\vec{x}}$ (that is, $\hat{H}\hat{G}\psi(\vec{x}) = \hat{G}\hat{H}\psi(\vec{x})$ for any acceptable $\psi(\vec{x})$), then, each state $E_j(\vec{x})$ of the orthonormal eigenbasis of $\hat{H}_{\vec{x}}$ will fulfill that $\hat{G}E_j(\vec{x}) = gE_j(\vec{x})$ for some real g. Thus, not only will we have a constant energy for any trajectory and any time, but will also have another number

Oin the reverse way to find eigenstates. Ta gero el unbound domain ta gero explike zelan oin dinamika temporala super domineta dekozun. Quizás hasta merezca la pena hemen sartzie position representation eta zelan ez izen arren operadore balidoak en Hilbert space, son muy utilizados ni que sea de form asimbólica (aunque existen formalizaciones matemáticas eta redirigidu irakurlie beste batzutara gure badeuen irakurri), ta orduen behin honegaz oin idatzi beste base ortonormala del harmonix oscilator en este sentido cómo se escribe gure badozun.

Ze koefiziente????

We can now evolve an arbitrary initial condition!

As we explained in 6.2., one of the main reasons why computing the eigenstates is useful is that we are now able to obtain the time evolution of an arbitrary initial condition $\psi(\vec{x}, t_0) = \phi(\vec{x})$. As an example, imagine we have a Gaussian

Bale, hau dana va en part A, ze hasta contiene los ejemplos y la conexión con la ket notation de forma superb natural.

- Orduen si buscas los estados para los que una cantidad es la misma para toda trayectoria y encima es una cantidad conservada en el tiempo (si conmuta con el hamiltoniano)-z te sale el espectro de un operador hermítico (en concreto del hamiltoniano que tocaría en ese momento). Ejemplos, ein para el harmónico (conmuta ocn nada, ejemplo de hamiltoniano en general) y pare el central (conmuta con el momento angular-¿ operador momento angular), los eigenstates de ese hamiltoniano conservan la energía y tods las trajs con la misma energía, y en particular en ese sistema vemos que se conserva el momento angular tb, es más, si defines el operador momento angular, ves que conmuta con H, la base de estados del momento angular te la da este hamiltoniano. En free partcile, sacas los estados que cumplen eso, aka los eigenstates del hamiltoniano, entonces si defines un operador momento2, que por tanto conmuta con un operador momento, te da estos eigenstates! Como consideras omega_t benetan son numerables, consideralo cúbico y quietp por ejemplo (que en verdad es como si fuese un potencial infinito cage), ta gero ein la aprox de que sería si hubiese infinito espacio. Conmuta con el momento lineal. Ba algo así debe poder hacerse con la posición no? Igual aproximando desde el potencial harmonico o algo? Tipo en el limite de que la masa sea enorme comparada con omega del potencial harmonico Oin haces con la posición.

En la de harmonic hacer lo de operadores de creacion y anikilasion pero desde nuestro punto de vista, sin mas, en plan preparasao para cuando hable de fotones.

Azaldu que momentum op coefficients es Fourier transform de la de la posición y de alli tb s epuede motivar bien una delta en el espacio. En general observable bat zelan definiduten dozu en función de las x y p?

COMMUTATION RELATIONS AZALDU! Ze naturalak diren. AZALDU HEISENBERG'S UN-

CERTAINTY

Adibide bakoitza dibujoakaz gudot!

Entonces qué ventaja tiene saberse los eigenstaes del H, bueno, que son una base ortonormal del espacio de Hilbert fijo! Orduen aparte de ser estados para los que fijo te sabes sin randomness cual es su energia etc. ba tal.-¿ measurement dilemma hortik etorriko da. Komente base honetan erreza dala esperantza ontologikoa kalkuletie, baia bueno ahal zenuzela erabili trayektoriak bebai.

Jarri apartado bat nun komentas los axiomas de von Neumann, ta ikusi berdiñe diñoiela baia purely phenomenologicamente, lo que nos lleva a preguntas turbias y paradojas turbisimas, ta kritikeu zer gure dabien esan. Hasi measurementen txapterra horren komentario bategaz? Ez, hobeto bukaeran komente zer diñoien.

Apartado bat komentetan el entanglement!! Lo natural que es, sin más, como el measurement no crea nada, pues ya está. Konekte con la no localidad.

Apartado bat con charged particle hamiltonian y coulomb interaction y tal.

Apartadoan en el del spinor, incluir spin operator, qiue es el que todo el mundo usa para explicarte todas las paradojas...en fin. Intente meikietan sense ia. baia posiblemente super ad-hoca izengo da.

Klaro, guzti hau ez doa en la de measurement, ze es ontologico todo esto, pero si quieres probear something ya es part B.

En section B Habría de ponerse ni que sea un apartado comentando la no localidad, como conseqcia natural de que el potencial cuantico tal, que es la unica paradoja que no es paradoja.

A.6.5. Example 3: The Central Potential - the Angular Momentum Operator

A.6.4. Example 2: The Harmonic Oscillator - the Position Operator

Ein hamiltonianoan eigenstaetk topeu gero interpreteu ein dinamika temporalana

A.6.6. A General Observable Operator

ARGITZAPENA, JAINKOAREN GRAZIAZ!

Klaro ezta inungo misterioa! Zerba lo de kasualidad expresiño random honen parte reala da hau ta imagianrioa da hori? Bueno, ze $H\psi$ justo SEan eskumetara dauena da ta badakizu forma polarrien wf sartzen badozun urtetan dana diela la ecuacione de HJ con la energía total igual a la variación de la acción S (bider la densidad), eta la continuity equation. Asike zati densidadie edo zati WFa eitzen badozun (bixek ondo supongo baia hobeto zati wf ze holan weak value protocolak exaktamente hau emotetzu con una medición naive) urtetan dana ba literalmente da continuity eqt zati rho, por eso la derivada temporal de la desnidad y la energía total o elk Hamiltoniano. Orduen por qué casualidad da la energía bohmiana, pues por esto! Y porque si justo pongo un hamiltonian operator que conmute con un obsrevable por ejemplo el momento angular o uno que me invente resulta que saldrá la momentum bohmiana? La cosa es que en verdad es alrevés! Tú primero en base bohmiana sabes lo que se conserva o tal y escribes el hamiltoniano en su función, inicluso en el caso de una partícula cargada que de hecho será el siguiente ejemplo-¿ Cómo conserguir el hamiltonian operator de una partícula cargada? Lo suelen hacefr por operadores, pero no, nosotros haremos classical mechanics y luego el truco es ponerlo como parte real de una continuity equation con una velocity field asociada a la momentum lineal, asike generalized momentume ezta emoten deuena el gradiente de S, sino la lineal (creo...y si es alrevés pues tb bien, pero uno de los dos), y entonces al llegar a la eq de Schrödinger, la hamiltonian operator que te queda, eso es! Y ese hamiltonian operator pues tendrá la propiedad de que conmute la property esa con el hamiltonian, que si conserva la energía conserva esa cantidad. Por tanto podemos querer describir estados para los que la cantidad conservada sea X, pues la pones en la parte imaginaria de una ecuación con la continuity equation y ya está?

Demostreu zeba horregaitzik funzionetan deuen eso de de repente ponerle operadores x y p que no conmuten. Osea que cuando cogían por ejemplo lo de el hamiltoniano clásico de la partícula cargada y le ponían de pronto que p y x eran operadores, era literalmente eso!!!! Asike llámale a esto QUANTIZATION!!!

Orduen siempre que hagas que dos cosas conjugadas derepente sean operadores que no conmutan quizá, y sólo quizá, si escribes el hamiltoniano de esa primera cosa que te da una Hamilton-Jacobi equation, si luego haces una continuity equation aparte y consideras que esa cosa (aquí era la posición, pero podría ser la amplitud de un campo por ejemplo o más bien quizá su expansión en fourier, las k sobre las que tiene soporte), sean trayectorias con una densidad asociada que se mueve según el campo de velocidades de esas trayectorias, obtines la ecuación de la mecánica cuántica que toque!!!!!!

Osea que literalmente si es un field(x) lo que cuantizas, las trayectorias serían $field(x,t)^{\xi}$, como cuando tenías $x(j)$ y ahora las trayectorias son $x(j,t)^{\xi}$. Asike tendrías por ejemplo la time evolution de posibles campos electricos en el espacio, y todos los posibles time evolutions de posibles campos eléctricos interaccionarían de forma tangente entre sí!

Asike billatzen bazun la HJ equation para partículas relativistas y haces esto quizá, y sólo quizá te salga la KG o Dirac o algo. Aunque en relativista problemita bat agertzen da eta da qué tiempo? claro. Proper time supongo... Aunque si el campo electromagnético lo puedes cuantizar de esta forma y es lo más relativista que hay...

Y así saldría que la razón por la que no conmuten esos dos operadores es que haces el caso de free system y haces bounded y luego al límite defines el momentum operator y luego el position operator como su conjugado en cierto sentido.

A.6.6. The Uncertainty Principle

A.6.7. The Ket Notation

A.7. The Axioms in Short

We can summarize the postulates we have assumed to arrive to non-relativistic Quantum Mechanics as follows:

- **CM 1:** The Universe is made of a countable number of particles, each with three degrees of freedom.
- **QM 1:** Our observed Universe is one of a continuity of possible Universes, all with the same exact number of degrees of freedom. Even if they all exist contingently, we only perceive one. We can say so, because we always observe the fundamental constituents of reality to be point-like, but they behave as if they were a continuity of them.
- **QM 2:** The state of the fluid of Universes is defined by the relative density of possible Universes in each configuration $\rho(\vec{x}, t)$ and their velocity field $\vec{v}(\vec{x}, t)$. Thus, our best a priori guess about which of them is our Universe is using the relative density of Universes as probability density. The randomness observed in quantum mechanics is due to our lack of knowledge of the disposition of all the degrees of freedom of the Universe.
- **QM 3:** The velocity field is Lipschitz continuous making the ensemble of trajectories for the possible Universes an homeomorphism. In particular, we consider it to be differentiable also with respect to the initial conditions. Thus, trajectories of Universes never cross each other.
- **QM 4:** There is an interaction between the possible Universes, making them cause a mutual influence in their dynamics. We say they are Tangent Universes.
- **CM 2:** Within each Universe, the stationary action axiom holds ruling the trajectory of the Universe. The Lagrangian and its symmetries will yield the dynamic equations for the trajectory of the Universe.
- **CM 3:** Physical space within each Universe is homogeneous and isotropic.
- **CM 4:** Within each Universe, Galilean Relativity holds.
- **CM/QM 5:** There is a global time axis for all the Tangent Universes, which is homogeneous.
- **CM 6:** The particles within each Universe have a possible pairwise interaction.
- **QM 5:** There is a stationary action axiom ruling the evolution of the density and the velocity field.
- **QM 6:** The density will need to evolve in a way that minimizes its total Fisher information weigthed by the mass of each degree of freedom. This implies the Universal time axis has a preferred direction of flow.
- **QM 7:** The number of Universes in any locality is conserved.
- **QM 8:** The total energy of the fluid of Universes is conserved.

If we only assumed the postulates with a **CM** tag, we would arrive at classical particle mechanics. It is the postulates with tags **QM** that end up making non-relativistic quantum mechanics.

A.7.1. The “Official” Postulates of Quantum Mechanics

A.8. Spin in Non-Relativistic Quantum Mechanics

A.9. This was just a Medley of so Many Other Narratives!

References

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Part B

The Measurement

" Measuring a Quantum System means knowing the state of the system after perturbing it, with probabilities due to the state before the measurement."

B.1. The Von Neumann Chain and Perturbing the System

B.1. State Preparation

Imagine we isolate a subsystem of the Universe, say, the quantum system we are studying in our lab, so as to have an EWF $|\psi\rangle$ [20].

Now, in principle, since we do not know exactly the wavefunction $|\psi\rangle$ in which the system is, we would like to prepare it to a known wavefunction before any experiment is done. Not only that, but we would like to prepare it into a wavefunction for which for all the tangent subsystems, the value of some control variable B (e.g. the energy, the angular momentum etc.) are the same, such that without needing to know about the rest of Universes we can be confident that ours has a certain reference value. Then, we would like to have an experimental protocol to convert the subsystem wavefunction $|\psi\rangle$ to such fiducial states where all tangent Universes agree in the value of some observable B.

B.2. The Property of one or all the Universes?

The answer is seemingly no, since that would imply that $\mathscr{H}^{\psi}(\vec{x},t) = E$ for all $\vec{x}$. YO ESPERO QUE SÍ QUE SEA CIERTO, ELSE LA DEFINICION DE LAS BOHMIAN EN FUNCION Yet that is still to be proven. What is true is that the eigenstates of the Hamiltonian are wavefunctions where all the trajectories have the same energy.

In general, it turns out that given a property of the trajectories $\mathscr{B}^{\psi}(\vec{x},t)$ (in the Eulerian frame), we can define a

Consider that we are interested in knowing the property B (it could be the position, the velocity, the energy, the angular momentum etc.) of a certain partition of the whole Universe. For example of the quantum particle system we have prepared into a known EWF $|\psi\rangle$ in the lab. You could imagine for instance a molecule that we have isolated in a cryostat, which we isolate from potential interactions with the rest of the world (its environment), including us, the measurement apparatus etc. We wait enough time for it to achieve an EWF (what one can informally say, "until its entanglement with the environment is lost"). Let us denote by B a general observable property of our subsystem of interest S. For example B might be the position, the velocity, the energy or the angular momentum of some subsystem we are studying

[20] We will later see how we can achieve this. For now just assume we have done so, for instance by putting it inside a cryostat at near 0K, with almost zero vibration and energy transmission from the environment, that is, with negligible interaction with the environment.

B.3. The Apparently Collapsing Measurement

B.2.1. Discrete Spectrum Measurement

B.2.2. Continuous Spectrum Measurement

B.4. Non-contextual Characterization of the Quantum System

B.3. The Generalized Measurement

B.5. Link with Other Narratives

Part C
The Density Matrix

" A wavefunction keeps track of
Tangent Universes while a Density Matrix
keeps track of Parallel Wavefunctions."

C.1. The Way to Keep Track of Parallel Realities

ATAL BAT AZALTZEN MEASUREMENT EN DENSITY MATRIX FORMALISM ZEAN IZENGO ZAN TA ZE ZENTZU EUKIKO BAN HORREK. TA BUENO DANA DE UNITRAY TIME EVOLUTION ETC AZALDU PARA DENSITIES ZELAN IZENGO ZAN.

C.2. The Reduced Density Matrix

C.3. The Unconditional Measurement and the Choice of Basis

C.4. Pure Unravellings

C.5. Complete Positive Maps: Any Quantum Operation is a Measurement

C.6. Noise, Decoherence and the Environment

Part D

Markovianity and Master Equations

" The Quantum Measurement and the Decohering Noise have exactly the same mathematical formulation. Are they the same thing?"

D.1. Some Possible Quantum Markovianity Definitions

D.1.1. Past-Future Independence

D.1.2. Etc.

D.2. Continuous Measurements: Introduction to Master and Stochastic Schrödinger Equations

D.3. The Most General Markovian Master Equations: The Lindblad Equations

D.4. Markovian Stochastic Schrödinger Equations: Pure Unravellings

D.5. The Most General non-Markovian Master Equation: The Nakajima-Zwanzig Equation

D.6. Non-Markovian Stochastic Schrödinger Equations: the Conditional Wavefunction

D.6.1 Wiseman's

D.6.2 Ours

D.7. What if there was No Macro-scale?

[11]

Part E

Even Quantum Fields and Relativistic Equations Fit Well

" Eee menuda flipada"

E.1. Photons

E.2. Phonons and other quasi-particles

E.3. Then, what is really Quantization? Is this a new Quantization Method?

E.4. Relativistic Tangent Universes: the Klein-Gordon Equation

Part X
Appendices

"There is always more to say"

X.1. Basic Probability Concepts and Lebesgue Integration

X.2. Basic Concepts about the Trajectory Ensemble $\vec{x}(\vec{\xi}, t)$

X.3. Extremizing Multivariate Functionals with Constraints

X.4. The Hilbert Space of Functions

X.5. Stochastic Differential Equations

References

[11] Vladimir Igorevich Arnol'd. *Mathematical methods of classical mechanics*, volume 60. Springer Science & Business Media, 2013.