Reality and Causality in the Microscopic World: A Discussion from Quantum Transport Theories

Chapter on *"Physics and the Nature of Reality: Essays in Memory of Detlef Dürr".*

Abstract

Paraphrasing Feynmann, perhaps, the main reason why the so-called Copenhagen (or orthodox) quantum theory is so popular among the physicists and engineers is because "I can safely say that nobody understands [it]". Many physicists and engineers take profit of the mathematical machinery of the Copenhagen theory without paying attention to its ontology, which implies that a quantum object has no microscopic properties (unless a property is measured, or the quantum object is an eigenstate of some property). Such an orthodox view of a microscopic world, empty of properties, is specially unsuitable to understand and to develop approaches to predict modern nanoelectronics, as we discuss along this chapter. As an alternative, physicists dealing with the foundations of quantum transport and open systems have developed different approaches in terms of some type of causal motion of electrons. When dealing with nanodevices, the Copenhagen ontology affirms that electrons are nowhere since their positions are undefined until measured, while causal motion approaches say that electrons can be perfectly understood as particles traversing a device with well-defined positions, independently of the measurement. This last view certainly holds true for treatments based on the Bohmian theory. Even for quantum phenomena of light, such as spontaneous emission or photon partition noise, the Bohmian theory allows an explanation from well-defined electromagnetic fields interacting with electrons, which is contrary to the standard Copenhagen approach. The above examples, developed in this chapter, emphasize that the main merit of the Bohmian theory is eliminating the observer/measurement as the "creator" of the microscopic reality, showing that a well-defined description at all times, of the microscopic properties of a quantum system, is available (where particles are particles and fields are fields at all times). Such a microscopic description does not only provide conceptual advantages, but also important numerical ones when electron devices are understood, in general, as non-Markovian open quantum systems.

In front of the incapacity of the standard quantum theory (so called Orthodox interpretation), to answer "when", "for how long" or whether the "are" electrons crossing the transistors of our phones at all, any physicist or engineer that needs to consider that this is indeed the case for the practical development of cutting edge devices, resorts to alternative explanations of quantum phenomena in terms of electrons that actually do cross their transistors [2]. The well known Bohmian explanation is one such alternative to the Orthodox interpretation [3] [4].

As an example of how impractical the interpretative limitation of the Orthodox view can arrive to be: in its terms, it is still obscure something as fundamental for modern electronics as is the current measurement for nano-devices operating at high frequencies. Not for interpretative issues alone, but mainly due to practical issues like the search of an operator for the multi-electron current observable, or the requirement to treat a high frequency measuring apparatus as a non-Markovian environment. As we will discuss along the chapter, satisfying explanations for these, apparently seem to demand the use of modal theories like Bohmian mechanics.

The essential tools for reaching these conclusions are the concepts of conditional wave-function (CWF) and effective wave-function (EWF) introduced by *Dürr et al.* in Ref [5], together with the understanding of the measurement dilemma they enlighten. Thus, we will start the chapter with a small review of these concepts.

The Conditional and Effective Wavefunctions

Given a quantum system of N degrees of freedom described by the real coordinate vector $\vec{X} = (x_1, ..., x_N) \in \Omega_t \subseteq \mathbb{R}^N$, we can describe its evolution in continuous time $t \in \mathbb{R}$, with the use of a complex wavefunction $\Psi(\vec{X}, t) = \rho^{1/2}(\vec{X}, t)e^{iS(\vec{X},t)/\hbar}$ (encoding the two real fields S and ρ), and an associated Bohmian trajectory $\vec{X}^{\vec{\xi}}(t) \equiv \vec{X}(\vec{\xi}, t)$ the initial condition of which is given by the label space vector $\vec{\xi} \in \Omega_0 \subseteq \mathbb{R}^N$ such that $\vec{X}^{\vec{\xi}}(t = 0) = \vec{\xi}$. This trajectory is guided by the wavefunction through the "guidance law", while the wavefunction itself is guided by the Schrödinger Equation. Respectively:

$$\frac{\mathrm{d}x_k^{\xi}(t)}{\mathrm{d}t} = v_k(\vec{X}, t)\Big|_{\vec{X}=\vec{X}^{\xi}(t)} := \frac{1}{m_k}\frac{\partial S(\vec{x}, t)}{\partial x_k}\Big|_{\vec{X}=\vec{X}^{\xi}(t)} \tag{1}$$

$$i\hbar\frac{\partial\Psi(\vec{X}, t)}{\partial t} = \left[\sum_{k=1}^{N}\frac{\hbar^2}{2m_k}\frac{\partial^2}{\partial x_k^2} + U(\vec{X})\right]\Psi(\vec{X}, t) \tag{2}$$

where U denotes the classical potential describing the interaction between the degrees of freedom (since we consider an isolated system we assume no time dependence), m_k is the mass associated with the k-th degree of freedom and v_k is the velocity field piloting the k-th degree of the Bohmian trajectories.

The most general isolated system we could consider is the whole Universe, where $\vec{X}$ would reflect its degrees of freedom or space of **configurations**. We can partition it in a subsystem of interest S, of say, $n < N$ degrees of freedom, labeled by $\vec{x} = (x_1, ..., x_n)$, and its environment E, of configuration coordinates $\vec{y} = (x_{n+1}, ..., x_N)$, such that $\vec{X} = (\vec{x}, \vec{y})$. We could associate one wavefunction to S and one to E, both labeled by the initial global trajectory configuration $\vec{\xi}$, respectively as $\psi^{\xi}(\vec{x}, t) := \Psi(\vec{x}, \vec{y}^{\xi}(t), t)$ and $\phi^{\xi}(\vec{y}, t) := \Psi(\vec{x}^{\xi}(t), \vec{y}, t)$. These are particular cases of the so called **conditional wavefunctions** (CWF-s). In general, a CWF is just evaluating some of the degrees of freedom of a wavefunction along a (Bohmian) trajectory for them, while leaving the rest un-evaluated.

As proved in [6], the full Schrödinger Equation (2), ruling the dynamics of the whole SE system, can be re-written exactly into two coupled dynamical sets of equations ruling the motion of the two presented CWF-s. For S (for E the equations will be the same but changing the indexes and the CWF):

$$i\hbar\frac{\partial\psi^{\xi}(\vec{x}, t)}{\partial t} = \left[\sum_{k=1}^{n}\frac{\hbar^2}{2m_k}\frac{\partial^2}{\partial x_k^2} + U(\vec{x}, \vec{y}^{\xi}(t), t) + G(\vec{x}, \vec{y}^{\xi}(t), t) + i\ J(\vec{x}, \vec{y}^{\xi}(t), t)\right]\psi^{\xi}(\vec{x}, t) \tag{3}$$

with G and J the real and complex parts of the so called **correlation potential**:

$$G(\vec{x},\vec{y}^{\xi}(t),t)=\sum_{j=n+1}^{N}\left\{-\frac{1}{2}m_j\Big(v_j(\vec{x},\vec{y}^{\xi}(t),t)\Big)^2-\frac{\hbar^2}{2m_j\rho^{1/2}(\vec{x},\vec{y}^{\xi}(t),t)}\left(\frac{\partial^2\rho^{1/2}(\vec{x},\vec{y}^{\xi}(t),t)}{\partial x_k^2}\right)\Big|_{\vec{y}=\vec{y}^{\xi}(t)}\right\} \tag{4}$$

$$J(\vec{x},\vec{y}^{\xi}(t),t)=-\frac{\hbar}{2}\sum_{j=n+1}^{N}\frac{\partial}{\partial x_j}v_j(\vec{x},\vec{y},t)\Big|_{\vec{y}^{\xi}(t)} \tag{5}$$

where we recognize as G the difference between the quantum potential and the kinetic energies of E and as J, the spatial variation of the Bohmian velocity of E in its coordinates. All of these are terms that involve a derivative of the phase or the magnitude of the full wavefunction in the E coordinates $\vec{y}$ centered at the trajectory position $\vec{y}^{\xi}(t)$. This means that in order to compute them, we would not have enough with the two CWF-s $\psi^{\xi}(\vec{x},t)$ and $\phi^{\xi}(\vec{y},t)$. We would require the information of the other adjacent CWF-s evaluated in $\vec{y}$ close to but not exactly in $\vec{y}^{\xi}(t)$. This is an explicit manifestation of the so called Quantum Wholeness, by which the dynamics of a single Bohmian trajectory depends on the dynamics of the rest of possible trajectories.[1] Thus, we see that while U introduces all the classical correlation between S and E, the responsible ones for their quantum interaction (and the interaction with the rest of possible Bohmian trajectories) are G and J.

However, if the imaginary potential J vanished for the sub-system, we can see that the CWF for the sub-system would behave as if it was a closed quantum system wavefunction, ruled by the unitary evolution of a Schrödinger Equation like (2), but where the potential energy would now be allowed to be time dependent. The influence of the environment from the point of view of the sub-system, would at most be a time dependent potential energy $U+G$, where the time dependence would be due to the environment's trajectory on U and the immediate CWF-s in the environment axes on G. That is, if J vanished, we see that the CWF alone would serve as a closed qunatum system descriptor of the sub-system. Yet, computationally, we would still require a quantum description for the environment in order to evaluate G. It turns out however that if G was also negligible, then the sub-system would for all purposes be a closed quantum system, only interacting with the environment classically (through U). Whenever this is the case, we say then that the CWF along the Bohmian trajectory of the environment is the **effective wavefunction** (EWF) of the subsystem. The question would then be: when are J and G (the quantum influences of the environment on the subsystem) negligible? It turns out, as qualitatively explained by the influential Ref. [5], when the full wavefunction's variation along the environment's axes $\vec{y}$, in the neighborhood of the environment's trajectory $\vec{y}^{\xi}(t)$, is negligible:

$$\frac{\partial\Psi(\vec{x},\vec{y},t)}{\partial x_j}\Big|_{\vec{y}=\vec{y}^{\xi}(t)}\simeq 0\quad\forall k\in\{1,...,n\},\ \forall j\in\{n+1,...,N\} \tag{6}$$

we will have that the terms involving derivatives of the magnitude $\rho^{1/2}$ or the phase S of the wavefunction Ψ in the environment's axes $\vec{y}$ will vanish.

Graphically, this happens for instance when the full wavefunction Ψ is composed of disjoint and macroscopically separated stackings of almost similar CWF-s in $\vec{x}$ [2] (this will be useful to see why countable spectrum projective measurements effectively generate "collapsed" eigenstates). The same can happen as well if the variation of the full wavefunction in y is arbitrarily slower than the variations in x (which will be useful to see why continous measurement schemes generate very narrow gaussians as "collapsed" wavefunctions).

[1] Note that information about multiple stacked CWF-s for S around $\vec{y}^{\xi}(t)$ is only required in G and J, since the guidance for the trajectory of S $\vec{x}^{\xi}(t)$ is given entirely by the central CWF $\psi^{\xi}(\vec{x},t)=r^{\xi}(\vec{x},t)e^{is^{\xi}(\vec{x},t)/\hbar}$, as $\frac{\mathrm{d}}{\mathrm{d}t}x_k(t)=\frac{1}{m_k}\frac{\partial s^{\xi}(\vec{x},t)}{\partial x_k}\Big|_{\vec{x}=\vec{x}^{\xi}(t)}$ for $k\in\{1,...,n\}$. That is, if it was not for G and J, the CWF would behave as a wavefunction.

[2] More quantitatively, the statement "macroscopically far" means that the variation of the (normalized) conditional wavefunction in the $\vec{y}$ axes happens only if a macroscopically big distance is considered. This implies that for any significant time for the subsystem, the evolution of the environment (the dynamics of the wavefunction in the y axes) will not affect the current shape of the CWF in x.

Measurement understood as unitary evolution using Bohmian CWFs and EWFs

Given a closed quantum system S and its EWF $\psi(\vec{x},t)$, if we wish to do a projective measurement of the property A, as suggested by Von Neumann in [8], we can consider the position of the dial of the measuring apparatus M observed by the experimenter (or the generalized degree of freedom of the measurement result indicator), lets call it $z \equiv x_{n+1}$ (as part of the environment of the sub-system). We prepare the indicator's EWF to be a fiducial state that will make sure its Bohmian position at the first time of the measurement $t=0$, that is $z^{\xi}(t=0)$, is reliably around the rest position of the dial $z=0$ with a high precision and accuracy. For example $\varphi(z,t=0)=\alpha e^{z^2/4\sigma^2}$, or in ket notation $|\varphi(t=0)\rangle_M = \int_{\mathbb{R}} \varphi(z,t=0)\,|z\rangle\,dz$, with α a normalization constant and σ not macroscopically distinguishable. We now let the sub-system's EWF, which in ket notation is $|\psi(t=0)\rangle_S = \int_{\mathbb{R}^N} \psi(\vec{x},t=0)\,|\vec{x}\rangle\,dx$, interact with the measurement ancilla through a Hamiltonian $\hat{H}_{MS}=g(t)\hat{p}_M\otimes\hat{A}_S$, where $\hat{p}_M$ is the momentum operator of the ancilla degree of freedom z and $\hat{A}_S$ is the operator related with the property of the subsystem we wish to measure. $g(t)$ is the interaction strength which has support only in $t\in(0,T)$ (the interaction time), such that $g:=\int_0^T g(t)dt$.

If the observable A has countable spectrum, such that $\hat{A}_S=\sum_k a_k\,|a_k\rangle_S\,\langle a_k|_S$ with $\{|a_k\rangle_S\}_k$ an orthonormal basis of S and $|\psi(0)\rangle_S=\sum_k \alpha_k(0)\,|a_k\rangle_S$, at the last interaction time $t=T$, the unitary Schrödinger evolution $\hat{U}_0^T = exp(-iT\hat{H}_{MS}/\hbar)$ will leave the composed state $|\Phi(0)\rangle_{MS}=|\varphi(0)\rangle_M\otimes|\psi(0)\rangle_S$ as:

$$|\Phi(T)\rangle_{MS}=\sum_k \alpha_k(0)\left(\int e^{-\frac{(z-a_k gT)^2}{4\sigma^2}}\,|z\rangle_M\,dz\right)|a_k\rangle_S \tag{7}$$

This means, as graphically shown in Figure **??**, that if the interaction strength g or time T are big enough, or σ is small enough, the probability density for the Bohmian position of the dial z, will be exclusively concentrated in several roughly disjoint Gaussians of weights $|\alpha_k(0)|^2$, each centered in a different $a_k gT$ position, around which, the CWF for the sub-system (evaluating a particular value for z, say, the observed $z\sim a_k gT$) would be roughly constantly $|a_k\rangle$ for all the wavefunction with significant probability in that region of configuration space. The different CWF-s (the different Gaussians disjointly separating them) have been macroscopically separated in z, since other-wise the dial would not be able to let us know the result of the measurement. Then we see that the CWF obtained by evaluating the observed Bohmian trajectory of z is at $t=T$ an EWF that is the eigenstate of $\hat{A}$ linked with the eigenvalue a_k that was indicated by the observed z, $|a_k\rangle_S$. Not only that, but the observation of z for an ensemble of identical experiments but microscopically different inital Bohmian positions for the dial z, will record a_k with probability $|\alpha_k(0)|^2$, and leave the sub-system in an EWF $|a_k\rangle_S$, just as stated by the measurement postulate in Orthodox quantum mechanics. This is because the weight of the Gaussian enveloping in z the state $|a_k\rangle_S$ has cumulative probability $|\alpha_k|^2$. Now, since the interaction between M and S is off for times $t>T$, the Hamiltonian of the joint system becomes factorisable for $t>T$, meaning we will be able again to evolve for all subsequent times the sub-system as an independent closed quantum system.

In summary, as seen from the subsystem alone, it will have looked like the unitarily evolved state $|\psi\rangle_S$ "collapsed" into one of the eigenstates of $\hat{A}_S$, with a probability equal to the magnitude squared of the projections to those eigenstates $|\alpha_k(0)|^2$, and will now continue to be unitarily evolvable (for it is an EWF with no interaction with M). The randomness arises due to the fact that we cannot know $z^{\xi}(0)$ with an arbitrary precision, unless we made a projective measurement on it before the experiment. But to measure a continous observable, we need yet another ancilla coupling and measurement, as explained in what follows. The thing is that this chain of couplings allows us to decide where arbitrarily between the observer's mind and the subsystem we place the "collapse", just as required by Von Neumann himself in [8][3], without the need of a physical collapse, just with a unitary evolution coupling bigger partitions of the Universe, that we can reasonably decide at which point to consider as EWF-s.

[3]Von Neumann did not believe in a physical collapse as explained by [**?**], he instead believed an explanation of the measurement should be possible setting an apparent collapse at an arbitrary point of the measurement Neumann chain.

If the observable A had a continuous spectrum, such that $\hat{A}_S = \int a \left|a\right\rangle_S \left\langle a\right|_S da$ with $\{\left|a\right\rangle_S\}_a$ an improper state Rigged Hilbert space orthonormal basis associated with S, with $\left|\psi(0)\right\rangle_S = \int \psi(a,0) \left|a\right\rangle da$, then the unitary coupling evolution will yield a composed state at time $t = T$:

$$\left|\Phi(T)\right\rangle_{MS} = \int \left(\int \psi(a,0) e^{-\frac{(a-\frac{z}{gT})^2}{4\Delta^2}} \left|a\right\rangle_S da \right) \left|z\right\rangle_M dz \quad \text{with} \quad \Delta := \frac{\sigma}{gT} \tag{8}$$

where, as graphically shown in Figure **??**, if g or T are big enough, or σ is small enough, the resulting wavefunction is stretched in z such that the variation is so slow in z, that if the interaction between the S and M is stopped for $t > T$, the subsystem will be in an EWF equal to the CWF sliced by evaluating z in the observed Bohmian position $z^\xi(T)$. This EWF will be a very narrow Gaussian (tending towards a Dirac delta), and will be the EWF of the subsystem with a probability density roughly equal to $|\psi(a,0)|^2$, as stated by the "collapse" postulate of Orthodox quantum mechanics. We know the state will need to be macroscopically stretched for our measuring dial to show significantly different positions, which is the reason why the measurement seems to result in a "collapse" as seen from the subsystem.

Now, either the assumption that for time $t > T$, M does not interact anymore with S, or that the environment entanglement with the subsystem is lost by some sort of thermalization, mean that the information of the subsystem that was "leaked" to the environment (the so called "empty waves", which are the rest of CWF-s that are not sliced at $z^\xi(T)$[4]), do not interact back with the EWF of the subsystem. Any of these two assumptions then mean that the environment effectively forgets the entanglement achieved with the subsystem. This is an environment behavior we could call Markovian or memory-less.

Then, since the environment description will only be useful for the measurement time, and we can then discard it (as we consider it for an ideal measurement to be Markovian), we can explain the projection of the state of S to a subspace of the Hilbert space of S directly with a set of effective-"collapse" projectors $\{\hat{\Pi}_k\}_k$ without the need to formalize M, as is typically done, just without forgetting that this is a short-cut in the explanation. In quantum mechanics we "measure" the a posteriori state of the system with probabilities due to the a priori state.

If we were now interested on the post-measurement description of the sub-system alone, but irrespective of the measured result, we could independently unitarily evolve each of the possible projected states (EWF-s), rememebering which was the probability for each of them. Then if a second measurement is performed at a later point, we could simply treat a measurement on each of the independent states and weight them with the joint probability of the previous and the current measurements. Perhaps more compactly, we could instead build a "matrix" where each "column" (each slot that will save a state-vector independently of the rest under linear operations) would be a possible effective state-vector, where we could also save the probabilities as the coefficient of the state-vector. If we now apply linear operations we wish to apply (say the unitary evolution) in both sides of the "matrix" (with an Herminitan conjugate in the right), we will be applying the operation to each state-vector (each "column") independently of the rest. That is, we could define some operators on state-vectors, that will serve as state containers like:

$$\hat{\rho}_S(T) = \sum_k |\alpha_k(0)|^2 \left|a_k\right\rangle_S \left\langle a_k\right|_S \quad \text{or} \quad \hat{\rho}_S(T) = \int |\psi(a,0)|^2 \left|a\right\rangle_S \left\langle a\right|_S da \tag{9}$$

This is the so called **density matrix** of the subsystem S. Notice that for subsequent unconditional measurements, the density matrix will get more and more mixed (the squared trace will diminish).[5]

[4] Empty waves that technically could interact back with the CWF selected by $z^\xi(T)$ if their macroscopic separation was made microscopic again.

[5] Also, note that these are "diagonal" representations of the density matrix but we could equivalently express it in other orthonormal bases, which means we can loose the microscopic deterministic detail of what is happening if we only specify a density matrix. For a probabilistic operational description of S (an epistemological description), this will suffice, but we have just seen that an ontologically meaningful view of the matrix is always available.

A Bohmian Narrative for General Quantum Operations

At this point, notice that the MS coupling and subsystem EWF branching due to a macroscopic separation in configuration space for different CWF-s, is not necessary to be part of a measurement by an observer. Such an apparent collapse on the subsystem could also happen as the effect of a more general environment for the system. From the perspective of the subsystem alone, such an interaction with the environment could be seen as a non-unitary evolution that makes the density matrix of the system get more mixed. Yet, a requirement for this environment, that acts as an ideal projective measurement, is that the portion of the enviornment that got entangled with the subsystem and caused its effective collapse rapidly thermalizes or never again interacts with the subsystem. This is a possible narrative for a general Markovian environment.

Given a density matrix $\hat{\rho}_{MS}$ for a composed MS system (which can be pure if say $\hat{\rho}_{MS} = |\Psi(T)\rangle_{MS} \langle\Psi(T)|_{MS}$), its partial trace over M is, given an arbitrary orthonormal base of M $\{|b\rangle\}_b$, defined as:

$$tr_M[\hat{\rho}_{MS}] = \sum_k \langle b| \otimes \hat{I}_S(\hat{\rho}_{MS}) \, |b\rangle \otimes \hat{I}_S \tag{10}$$

where it would be an integral if the base is made of improper states. It can be proven that the resulting S density matrix, so called **reduced density matrix**, is the same irrespective of the employed basis. Now, note that the partial trace of M on the pure state $|\Psi(T)\rangle_{MS} \langle\Psi(T)|_{MS}$ under the effective "collapse" conditions for g, T, σ, precisely yields the unconditional post-measaurement density matrices of equation (9).

In general, this indicates that the partial trace of a partition A of a composite Hilbert space AS can always be interpreted as how S would be left if an unconditional ideal projective measurement was performed on A. That is, if we coupled an ancilla M to A to ideally measure an observable of A and performed the effective collapse of A. The effective collapse of A would cause the effective collapse of the CWF-s of S entangled with each collapsed orthonormal state of A. The resulting density operator of S would exactly be the partial trace of A. Note very importantly that if the traced out partition is not really projectively measured (coupling a measurement ancilla to it and evolving until macroscopic distinguish-ability is achieved) and the interaction between A and S is not thermalized or does not cease indeterminately, then the obtained reduced density matrix for S will not evolve unitarily: each CWF of the subsystem will still interact with the different adjacent slices of the full wavefunction. That is, these CWF-s will not be EWF-s. Yet, for statistical predictions about S at that time, the information in the reduced density matrix will be enough. Thus, the reduced density matrix is just a structure that allows punctual measurement statistics on S to be predicted, but in general the environment's effect will need to be taken into account in its time evolution. Unless of course, a real unconditional measurement of A is performed (by an outer environment or by an observer) and the A-S interaction ceases. Interesting enough, the observable measured on the traced out partition is irrelevant for the resulting reduced density matrix, which is what allows the versatility of the so called pure unravellings.

Given all this narrative, we could now easily obtain a generalized measurement scheme were we could determine after the measurement in which state the system is, without such an state necessarily being part of the eigenbasis of a Hermitian operator. For this, we could arrive to couple S states (not necessarily orthogonal to each other) in which we could decompose a state of S, each with a state of an orthonormal basis $\{|m\rangle_A\}_m$ of an ancilla A, employing a suitable coupling unitary evolution $\hat{U}_{AS}$. Such that beginning with the product of a fiducial ancilla state and a certain subsystem state $|\theta\rangle_A \otimes |\psi\rangle_S$, we get the entangled state $\hat{U}_{AS} \, |\theta\rangle_A \otimes |\psi\rangle_S = \sum_m |m\rangle_A \otimes |\psi_m\rangle_S$ where $|\psi_m\rangle_S := \langle m|_A \otimes \hat{I}d_S\Big(\hat{U}_{AS} \, |\theta\rangle_A \otimes |\psi\rangle_S\Big)$ is an unnormalized subsystem state that we will call the **general conditional state** (GCS) for the m-th observation of the environment A. Then an ideal projective measurement on A for the $|m\rangle_A$ basis (coupling a dial to A and dividing A in EWF-s consisting of the measured eigenstates), would generate ancilla-subsystem unnormalized EFW-s given by the Bohmian position of the dial equal to $|m\rangle_A \otimes |\psi_m\rangle_S$, each with a probability equal to its norm squared, which is equal

to the norm of the GCS of S conditioned on m, $N^2 := |\langle\psi_m|\psi_m\rangle|^2$. These AS states are a product of the measured state for A and the (unnormalized) state entangled with it for S. If we then set off the interaction between A and S, the S will be in the EWF $|\psi_m\rangle_S / N$ with probability N^2. If we wish we could now shortcut the formalization of the ancilla and its projective measurement, by just considering the general measurement operators (called POVM-s) $\{\Omega_m := |m\rangle \otimes \hat{Id}\hat{U}_{AS} |\theta\rangle_A \otimes\}_m$ such that the only requirement for them is that since $\Omega_m |\psi\rangle_S = |\psi_m\rangle_S$ will be the (unnormalized) post-measurement state and its squared norm will be the probability to observe m: $\sum_m \Omega_m^\dagger \Omega_m = \hat{Id}_S$ (so that probabilities add up to one). The reason why we can choose any linear operator in Ω_m will be seen in the next paragraph. The treatment of such an operation on the subsystem for a density matrix can be straightforwardly derived.

In order to finish integrating the density matrix formalism and any general quantum operation with this Bohmian view (even the generality of the previous derivation of a POVM), we can invoke the Gelfand-Naimark-Segal theorem, following which, for any most general operation we can perform on a density matrix $\hat{\rho}_S$ of a system S (any complete-positive, convex linear and not trace increasing superoperator), say, for the operation $\mathfrak{S}$, there exists at least an ancilla system A with a pure state $|\theta\rangle_A$ and a coupling unitary evolution $\hat{U}_{AS}$ such that:

$$\mathfrak{S}[\hat{\rho}_S] = tr_A\Big[(\hat{\Pi}_A \otimes \hat{Id}_S)\hat{U}_{AS}(|\theta\rangle_A \langle\theta|_A \otimes \hat{\rho}_S)\hat{U}_{AS}^\dagger\Big] \tag{11}$$

which can be interpreted as a unitary coupling of the initially independent system and an ancilla, and posterior partially selective ideal projective measurement of A (where only the eigenstates of non-null eigenvalue of $\hat{\Pi}_A$ are left and the rest are discarded)[6]. In particular, if the coupling of S and A perfectly entangles the eigenstates of $\hat{\Pi}_A$ with some orthonormal basis of S, this will be visualizable as a projective measurement of S. Else, it will be visualizable as, a so called, generalized measurement (POVM) of S. In the trivial case where $\hat{U}_{AS} = \hat{U}_A \otimes \hat{U}_S$, this will just be the Schrödinger unitary evolution.

Markovian to Non-Markovian measurement contexts. THz electronics demands.

We saw in the previous pragraph that a portion A of the environment could be called Markovian, if it interacted with the subsystem S such that for the reduced density it could be viewed as if every time step Δt an ideal projective measurement was made on A (thus effectively collapsing S to the states entangled with the measured basis of A), and such that A never again interacts with S. A possible realization of this is when a different portion of the environment (a different ancilla) interacts with the subsystem at each Δt and is then ideally measured in a way that this ancilla never again interacts with the system (or the SA entanglement is somehow thermalized before their next interaction). From the perspective of S, this is to perform a POVM on S every Δt. Among other formalizations, the "Past-Future Independence" definition of Markovianity by Wiseman et al.[17], perfectly matches this view.

In fact, as shown by Ref [16], such a continuous monitorization of different ancillas coupled to the system at each time, can be used to derive one of the most general dynamical equations for the reduced density matrix of a subsytem in a Markovian environment. A kind of so called Lindblad Master Equations. Generalizing them to arbitrary Markovian environments, then requires considering several continuous measurements of different properties of the bath, but the same idea still holds.

The fact that the dynamics of the reduced density matrix of a subsystem can be understood in these terms means that instead of solving directly the non-linear Markovian Master Equation, we could do the following. Find an observable W for some (fictitious or not) environment ancillas, ancillas that get entangled with S and are then projectively measured producing the same average effect on the reduced

[6]Note that for the projective measurement of A, we will need to include a measurement ancilla M and do all the coupling and macroscopic determination of the Bohmian position of M.

density of S as the predicted one by the Master equation. Then, we could evolve a pure state-vector of the subsystem choosing at each bath projective measurement per Δt, each POVM for the subsystem one of the possible stochastic results. This would generate a linked in time pure state $\left|\psi_{w(t)}(t)\right\rangle$, associated to a certain measured continous monitorization (or unravelling) of the bath $w(t)$[7]. Such a pure state is called a **quantum trajectory**, linked to a particular, so called, "noise realization" $w(t)$ for its environment. As we saw previously that the reduced density matrix of a system is how it would be left if an unconditional ideal measurement was performed on the environment, this tells us that we should be able to obtain the reduced density for the subsystem by averaging the ensemble of all possible quantum trajectories for the unraveling of the W observable of the bath:

$$\hat{\rho}_S(t) := tr_{ES}[\hat{\rho}_{ES}(t)] = \mathbb{E}_{w(t)}\left[\left|\psi_{w(t)}(t)\right\rangle\left\langle\psi_{w(t)}(t)\right|\right] \tag{12}$$

Computationally, this means that if we got an equation ruling the stochastic time evolution of the pure quantum trajectory $\left|\psi_{w(t)}\right\rangle$ and its noise realization $w(t)$, we would be able to massively parallelize the computation of the reduced density matrix. This is profitable because a density matrix is a computationally more complex structure than a state vector. Additionally, the obtained reduced density matrix is necessarily positive definite by construction, which is not the case for other methods to evolve Markovian master equations. Equations of these kind are so called, Stochastic Schrödinger Equations (SSE-s). Such a pure state quantum trajectory can always be physically interpreted by the Orthodox interpretation as a so called pure unravelling (where one would invoke the collapse at each Δt).

From our Bohmian perspective, this works because at each time t, the ideal measurement of the environment portion A entangled with the system S, makes the A-S CWF obtained by conditioning the A-S-measurement-apparatus wavefunction to the dial position $w(t)$ be converted into an EWF, just as explained when describing the narrative for POVMs. As was clear for POVMs in general, notice again that the measured property of the bath, indicated by $w(t)$, does not need to be its position (even in Bohmian mechanics). It is the position of the dial measuring the property of the bath, that must be a position (which is actually what $w(t)$ is in each time, assuming the Bohmian postulate that a measurement is always a position measurement in the end). Thus, we conclude that a quantum trajectory is exactly a normalized subsystem CWF (in ket notation), which every significant Δt is converted into an EWF (thus the normalization).

However, what if we had an environment that gets entangled with S, but which never really allows us to consider a branching in different EWF-s (effective collapse). What if the CWF-s of S, conditioned on different bath observable values were allowed to interact in any future time, and were not converted into EFW-s? That is, what if the quantum trajectories could interact between them, such that the evolution of each of them depended on the rest? Then "the information leaked" onto the environment from S (the "empty waves"), would be able to affect back S in any significant future time for S. Such an environment with "memory" of the entanglement achieved with S would be called a non-Markovian environment. Then, it turns out that from a Bohmian interpretation, we could still continue talking about "pure state quantum trajectories", which would be the CWF-s for S (in any desired representation), conditioned on any position for the environment interacting with S (or if one wished not to use the position, then conditioned on the position of a dial coupled with an observable of the bath interacting with S). Since in Bohmian mechanics, measurement and collapse is just described as another unitary evolution of the whole, while the positions and derived properties are ontologically real at all times, then no problem at all.

Contarily, in Orthodox mechanics, a CWF (normalized or not, in an arbitrary representation basis), does not have a physical interpretation, unless it is an EWF, that is, unless the conditining variable is projectivelly measured, in which case the CWF is an (unnormalized) post-measurement state. As a

[7]Remember that at each time a different generalized measurement is performed on S, meaning this trajectory $w(t)$ reflects the Bohmian positions of different measurement dials at each Δt step. Thus, its non-differentiable nature is not a problem at all.

consequence, if a SSE is found for a non-Markovian dynamical equation ruling a reduced density matrix (a master equation), the conditional pure state evolved by the SSE in the Orthodox interpretation can only be understood as the state in which S would be left on, if the environment was measured...but its not! If it was, the evolution of the state would be pretty different (we would neglect the interaction between the CWF-s in the enviornment axes). Thus, the linking of such states in time, can only be understood if we get out from the Orthodox interpretation and use concepts like the CWF of Bohmian mechanics. Of course, mathematically, one could derive such non-Markovian SSEs as pragmatical computational tools to reconstruct the reduced density matrix, but one would need to avoid any additional consideration for the quantum trajectory unless accepting some sort of ontological reality (independent of measurement) for the conditioning property of the environment.

From this Bohmian perspective it is easy to notice why SSE-s for non-Markovian environments will never be exact for a general case. One of the main properties a SSE needs to have is that it should allow the time evolution of a single conditional state independently of the rest of possible conditional states, which is precisely asking that there is no quantum influence by adjacent CWF-s $\frac{\partial \Psi}{\partial y} \simeq 0$, influence which is the main characteristic of quantum mechanics in comparison with classical mechanics (the so called quantum wholeness). In fact, this is asking for these CWF-s to be EWF-s as we saw in the beginning of the chapter, which would then allow a Markovian interpretation for the SSE, thus the contradiction. Yet, it is indeed possible to find approximate SSEs also in non-Markovian environments. This is because, a whole ensemble of CWF-s does not need to be an ensemble of EWF-s to allow the computation of the reduced density matrix at each time via an ensemble average![8]

To see that this is so independently of the nature of the environment, we can see an example for an arbitrary composed pure state (then the generalization to mixed states would be trivial). Given the arbitrary state $|\Psi\rangle_{ES}$ for the environment E and system S, with position observables $\vec{y}$ and $\vec{x}$ respectively, just as introduced in the beginning, in ket notation we would have that at each time[9]:

$$|\Psi(t)\rangle_{AS} = \int \left|\vec{y}^{\xi}(t)\right\rangle_A \otimes \left|\psi^{\xi}(t)\right\rangle_S d\xi = \int |\vec{y}\rangle_A \otimes \left|\psi^{\xi(y,t)}\right\rangle_S dy \tag{13}$$

Then tracing out A in $\hat{\rho}_{AS}(t) = |\Psi(t)\rangle_{AS} \langle\Psi(t)|_{AS}$, we would get the reduced density for S:

$$tr_E[\hat{\rho}_{AS}(t)] = \int \langle\vec{y}| \hat{\rho}_{AS}(t) |\vec{y}\rangle \, dy = \int \left|\psi^{\xi(y,t)}(t)\right\rangle \left\langle\psi^{\xi(y,t)}(t)\right| dy = \mathbb{E}_{\xi(y,t)}\left[\left|\psi^{\xi(y,t)}(t)\right\rangle \left\langle\psi^{\xi(y,t)}(t)\right|\right] \tag{14}$$

where we have used the most general arbitrary expression for an AS state.

Having this clear narrative in terms of Bohmian CWF-s for non-Markovian open quantum systems, in which SSE-s need to be derived as ad-hoc approximations for particular settings, is not only theoretically insightful, but it can be a powerful practical tool to look for reasonable SSE-s. As an example we have developed two frameworks.

Non-Markovian SSE for many-electron two-terminal devices operating at THz frequencies

In the pragmatical view we have mentioned about Markovian open quantum systems, we said the dynamics should be interpretable as if every Δt a POVM (instantaneously for S) took place. This means that in such a picture, the entanglement and interaction bewteen the subsystem and the enviornment should decay in a time scale τ_{decay} much smaller than any characteristic time scale for

[8] In fact, a whole set of CWF-s, if the system state was not mixed, would also allow the reconstruction of the full wavefunction! In which case a reduced density matrix would not be necessary.

[9] Note that since Bohmian trajectories do not cross each other in configuration space, if we sampled "all" Bohmian trajectories for which $\vec{x}(t_0)$ is a fixed position, at each time, we would have a CWF per each position in $\vec{y}$, thus, we have that the states $\left\{\left|\psi^{\xi(y,t)}(t)\right\rangle_S := \langle y(\xi,t)|_A |\Psi(t)\rangle_{AS}\right\}_y$ are all the possible slices of the $\vec{y}$ axis. This avoids considering all ξ, since we would introduce redundant CWF-s.

the subsystem τ_S (related with Δt), $\tau_S >> \tau_{decay}$. However, for nanoscale electronic devices operating at very high frequencies (order of THz), the relevant dynamics and "measurement" time intervals are both below picoseconds time-scales, meaning $\tau_S \sim \tau_{decay}$, leading to a necessary practical consideration of a non-Markovian scenario.

We observe that following Bohmian mechanics, we have already shown a way to obtain SSE-s for arbitrary settings in equation (3). In principle, this equation is coupled to the dynamical equation of the CWF of the environment and actually to all the other subsystem CWF-s in the vicinity of the Bohmian trajectory (which is why it is non-Markovian). However, for specific scenarios, we can make educated guesses for the quantum correlation terms G and J, and the classical potential U to render a SSE for individual CWF-s. As an example, in the BITLLES simulator described in Refs. [**?**],[], the classical potential is evaluated through the solution of the Poisson equation[], while G and J are modeled by a proper injection model [] as well as proper boundary conditions [] that include the correlations between the active region and reservoirs. Even electron-phonon decoherence effects can be included effectively as shown in Ref. [].

In an electron device, the number of electrons contributing to the electrical current are mainlu those in the active region. This number fluctuates as there are electrons entering and leaving the active region. This creation and destruction of electrons leads to an abrupt change in the degrees of freedom of the subsystem CWF. This problem can be circumvented in Bohmian mechanics by decomposing the system CWF $\psi^{y^\xi(t)}(\vec{x},t)$ into a set of CWF-s for each electron. That is, for each of the $n/3$ electrons $\vec{x}_k := (x_{3k+1}, x_{3k+2}, x_{3k+3})$ with $k \in \{0,..,n/3-1\}$, we define a single particle CWF $\phi^\xi_k(\vec{x}_k,t) := \psi^{y^\xi(t)}(\vec{x}_k, \vec{x}_{\neg k} = \vec{x}^\xi_{\neg k}(t), t)$ with $\vec{x}_{\neg k} = (\vec{x}_1,..,\vec{x}_{k-1},\vec{x}_{k+1},...,\vec{x}_{n/3})$ the position of the electrons in the active region except for the k-th one. Then, we can consider a set of $n(t)/3$ equations of motion now having as sub-subsystems each of the electrons in the active regions, toe volve each CWF ϕ^ξ_k. These equations will have the shape of (3) just with a small change of indices and letters. Note that what we have just done is to consider the subsystem of the non-Markovian open quantum system of interest as itself composed of several open quantum systems that will interact with each other non-Markovian-ly.

Now, the active region of an electron device (the subsystem S of interest) is connected to the ammeter (acting as a measuring apparatus M) by a macroscopic cable (representing the portion of the environment A that gets entangled with the subsystem). The electrical current read by position of the dial in the ammeter (correlated with the Bohmian trajectory of the charge carriers in the cable, which are correlated with the electrons in the active region) is the relevant observable we are interested to predict. Thus, in principle the evaluation of the electrical current should require keeping track of the environment's degrees of freedom as well. However, at THz frequencies, the electrical current is not only the particle current but also the displacement current. It is known that the total current defined as the particle plus displacement currents, is a divergenceless vector [**?**]. In consequence, the total current evaluated at the end of the active region is equal to the total current evaluated at the cables, so the variable of the environment associated to the total current $z(t) \equiv I(t)$ can be equivalently computed at the borders of the open system. This current in turn (its expectation), can be computed from the Bohmian trajectories of the electrons in the active region with the two-terminal device current given in equation (**??**). Let us note that although computed inside the active region, $I(t)$ is the electrical current given by the ammeter. Since the cable has macroscopic dimensions, it can be shown that the measured current at the cables is just equal to the "unmeasured current" by the active region electrons (plus a source of nearly white noise which is only relevant at high frequencies), as shown in Ref. [**?**]. Essentially the argument is that the electrons in the metallic cables have a very short screening time, meaning the electric field generated by an electron in the cable spatially decreases rapidly due to the presence of many other mobile charge carriers in the cable that screen it out. Thus, the contribution of this outer electron to the displacement current at hte border of the active region is negligible.

In Ref. [**?**] we provide some numerical results demonstrating the ability of the presented method, simulating a two-terminal electron device whose active region is a graphene sheet contacted to the

outer by two (ohmic) contacts. To take into account the electromagnetic environment of the electron device, we model the interaction between the graphene device with the environment through a resistor and a capacitor connected in series through ideal cables. For more details the reader is referred to the indicated article.

Towards a general framework to look for SSEs

Equation (3) as stated is a good point to start to look for SSE-s in any environment setting, where the interaction with the environment can be approximated ad-hoc for particular systems through educated guesses over G and J. Yet we have developed a second framework, now at the cost of having a suitable guess for the conditional eigenstates of the relevant parts of the enviornment, to look for SSEs (equations to evolve CWF-s independently), based on the Born-Huang ansatz of the full wavefunction. For this, given the full, subsystem-environment Hamiltonian (for the evironment portion quantically interacting with the system) $\hat{H}(\vec{x},\vec{y},t)=\sum_{k=1}^{N}\frac{-\hbar^2}{2m_k}\frac{\partial^2}{\partial x_k^2}+U(\vec{x},\vec{y},t)+V(\vec{x},t)$, [10] we can define the transversal section Hamiltonian as $\hat{H}_x(\vec{y},t):=\sum_{k=m+1}^{N}\frac{-\hbar^2}{2m_k}\frac{\partial^2}{\partial x_k^2}+U(\vec{x},\vec{y},t)$. Then, we define the set of eigenstates $\{\Phi_x^j(\vec{y},t)\}_j$ with eigenvalues $\{\varepsilon_x^j(t)\}_j$, parametrized by the chosen section $\vec{x}$, to be the solution of: $\hat{H}_x(\vec{y},t)\Phi_x^j(\vec{y},t)=\varepsilon_x^j(t)\Phi_x^j(\vec{y},t)$. We could call these $\Phi_x^j(\vec{y},t)$, transversal section eigenstates (TSE). Since the hermiticity of the operator $\hat{H}_x(\vec{y},t)$ implies the TSE-s form an orthonormal basis for the space $\vec{y}$ of $\vec{x}$, we could expand the ansatz:

$$\Psi(\vec{x},\vec{y},t)=\sum_j\Lambda^j(\vec{x},t)\Phi_x^j(\vec{y},t)=\sum_j\varphi_j(\vec{x},\vec{y},t) \tag{15}$$

with $\Lambda^j(\vec{x},t):=\int\Phi_x^j(\vec{y},t)\Psi(\vec{x},\vec{y},t)dy$ the projection coefficients and $\varphi_j(\vec{x},\vec{y},t):=\Lambda^j(\vec{x},t)\Phi_x^j(\vec{y},t)$.

Now, using this expansion in the Schrödinger Equation, after a manipulation, and evaluating the full wavefunction along the trajectory for the environment $\vec{y}=\vec{y}^{\xi}(t)$[11], we get by denoting $\varphi_\xi^k(\vec{x},t):=\varphi^k(\vec{x},\vec{y}^{\xi}(t),t)$ that:

$$i\hbar\frac{\partial}{\partial t}\varphi_\xi^k(\vec{x},t)=\left[-\sum_{s=1}^{m}\frac{\hbar^2}{2m_s}\frac{\partial^2}{\partial x_s^2}+\varepsilon^k(\vec{x},t)+V(\vec{x},t)+i\hbar\frac{\mathrm{d}}{\mathrm{d}t}log(\Phi_x^k(\vec{y}^{\xi}(t),t))\right]\varphi_\xi^k(\vec{x},t)+ \tag{16}$$

$$+\sum_{j=0}^{\infty}\sum_{s=1}^{m}\frac{-\hbar}{2m_s}\left(\frac{1}{\Phi_x^j(\vec{y}^{\xi}(t),t)}\frac{\partial^2\Phi_x^j(\vec{y}^{\xi}(t),t)}{\partial x_s^2}+2\frac{\partial}{\partial x_s}log(\Phi_x^j(\vec{y}^{\xi}(t),t))\left[\frac{\partial}{\partial x_s}-\frac{\partial}{\partial x_s}log(\Phi_x^j(\vec{y}^{\xi}(t),t))\right]\right)\varphi_\xi^j(\vec{x},t)$$

Which means that since the CWF for the subsystem can be recovered from $\psi^{\xi}(\vec{x},t)=\sum_j\varphi_\xi^j(\vec{x},t)$, we have obtained a set of exact linear equations involving only n dimensional states that allow the evolution of single CWF-s. In principle, since a general Hilbert space will need to have a countably infinite number of orthonormal states in a basis, they would be an inifnite number of equations. However, we shall reasonably truncate the series at a relatively fixed finite point (at which for example the norm of the CWF surpasses a certain threshold). Even allowing it to vary in time is a possibility.

Now, the difficulty of these equations would relay on the knowledge of the TSE-s $\Phi_x^j(\vec{y},t)$. However, providing educated guesses for them seems to be more reasonable than guessing for G and J, and might be a starting point for the derivation of ad hoc SSE-s for particular environments.

[10] The classical potentials are allowed to have a time dependence to account for classical interaction with the rest of the environment.

[11] In reality, we could chose any trajectory we like for the environment here if we are not interested in computing Bohmian trajectories. Thus, it could be the result of a measurement of the bath, or just a set of trajectories that leads us to the construction of the reduced density with the least number of them.

Properties of the Pilot Wave vs Properties of the Trajectory: Two approaches to predict observables.

From the above explanation of the ideal projective measurement, we can see that a measurement operator is just a way to gather an orthonormal basis of the subsystem $\{|m\rangle\}_m$, with the corresponding physical observable value m corresponding to each orthonormal state $|m\rangle$. Thus, we see that what we are really measuring is a property of the wavefunction. In particular we are measuring how the system state-vector will be left after the measurement, with a probability due to the state before the measurement, among several possible states in whcih the a priori state can be linearly decomposed. This is in contrast with a property of the underlying Bohmian trajectory of the subsytem $\vec{x}^{\xi}(t)$, which unlike the wavefunction, has a definite value at all times. This was not interesting until, very recently the community started to acknowledge that properties predicted for Bohmian trajectories can indeed be experimentally measured, when Wiseman proposed his protocol to operationally measure the velocity of a quantum particle [12].

More recently, we derived a very insightful observation that extends this concept, allowing, if wished, to measure any property of a Bohmian trajectory, and in fact, to attribute for any quantum observable, a determined value to each Bohmian trajectory. This led to several practical advantages, as we will explain now, following [13], [14].

Given any arbitrary (Hermitian) operator $\hat{G}$, describing the observable property G for the subsystem S, with normalized EWF $|\psi(t)\rangle$, let us define the function $G^{\psi}(x,t) := \frac{\langle\vec{x}|\hat{G}|\psi(t)\rangle}{\langle\vec{x}|\psi(t)\rangle}$, which is the weak value of $\hat{G}$ for the state $|\psi(t)\rangle$ at time t, post-selected at $\vec{x}$ (we will discuss its meaning later on). Then, we could define a real function $G^{\psi}_B(\vec{x},t) = \mathbb{R}e\{G^{\psi}(x,t)\}$, which we could say is the property G of the Bohmian trajectory passing from $\vec{x}$ at time t. Until here it seems just a cumbersome definition. Now, let us compute the expected value for $\hat{G}$ and try to write it as a function of $G^{\psi}(x,t)$:

$$\langle\hat{G}\rangle(t) = \langle\psi(t)|\,\hat{G}\,|\psi(t)\rangle = \int \langle\psi(t)|\,|\vec{x}\rangle\,\langle\vec{x}|\,\hat{G}\,|\psi(t)\rangle\,dx = \int \langle\psi(t)|\vec{x}\rangle\,\langle\vec{x}|\psi(t)\rangle\,\frac{\langle\vec{x}|\,\hat{G}\,|\psi(t)\rangle}{\langle\vec{x}|\psi(t)\rangle}dx \tag{17}$$

$$\langle\hat{G}\rangle(t) = \int |\psi(\vec{x},t)|^2 G^{\psi}(\vec{x},t)dx \tag{18}$$

which means that the ensemble average of the (possibly complex) $G(x,t)$ of the Bohmian trajectories (following quantum equilibrium [5]), gives the same expected value for the observable as using the operator. But here comes the most interesting point: since $\hat{G}$ is an observable, its expected value will be a real number, meaning that $\langle\hat{G}\rangle = \mathbb{R}e\{\langle\hat{G}\rangle\}$, which means that:

$$\langle\hat{G}\rangle(t) = \int |\psi(\vec{x},t)|^2 G^{\psi}_B(\vec{x},t)dx \tag{19}$$

Such that the real property $G^{\psi}_B(\vec{x}(\vec{\xi},t),t)$ of the $\vec{\xi}$-th Bohmian trajectory, seems to mean something relevant about G, since its ensemble average gives the same expected value as the operator expected value. What is even more: what if $|\psi\rangle$ was an eigenstate of $\hat{G}$ with eigenvalue g? What would the suggested Bohmian property related with G be in that case?

$$G^{\psi}_B(x) = \mathbb{R}e\left\{\frac{\langle\vec{x}|\,\hat{G}\,|\psi\rangle}{\langle\vec{x}|\psi\rangle}\right\} = \mathbb{R}e\left\{\frac{\langle\vec{x}|\psi\rangle\,g}{\langle\vec{x}|\psi\rangle}\right\} = g \tag{20}$$

which means that an eigenstate of $\hat{G}$ must be a state for which every Bohmian trajectory has the same value for the property G. So not only this identification serves to compute the expected value, but it is even a tool to construct the whole $\hat{G}$ operator itself! We could look for eigenstates of the observable G by looking at states that yield Bohmian trajectories with the same value g for G^{ψ}_B. This will be useful, because there are observables, like the many electron current in a nano-electronic device, for which there is no clear measurement operator, but there is a Bohmian trajectory related current. In fact,

this would explain why for an eigenstate, there is a 100% probability that we observe the eigenvalue of the property also in the Bohmian measurement scheme.

What is even more: since we placed no restriction on $\hat{G}$, this means that for **any** observable of a quantum system, we are mathematically safe to assume each Bohmian trajectory has an ontologically determined value for all of them simultaneously.

As the icing of the cake, it turns out, as we proved in [13], that if we set as $\hat{G}$, the momentum operator $\hat{p_k}$ of the k-th degree of freedom, the Bohmian trajectory property $G_B^{\psi}(\vec{x},t)$ is exactly equal to the Bohmian velocity field $v_k(\vec{x},t)$. If we set as $\hat{G}$, the Hamiltonian operator $\hat{H}$, the property $G_B^{\psi}(\vec{x},t)$ turns out to be exactly equal to the Bohmian energy (kinetic plus classical and quantum potentials) of the Bohmian trajectory. And the list goes on. Even, if wished, for the three components of spin.

As if all this was not already hard to digest, there is even an additional theoretical point to be remarked. It is about the definition we have made for the Bohmian property G_B^{ψ} associated with G, as the real part of the so called weak value. It turns out, as is so well known [9], the real part of our weak value can be experimentally measured as follows. First couple an ancilla with the subsystem of EWF $|\psi\rangle$, through the Von Neumann Hamiltonian explained in the beginning of the chapter, which if coupled with a big enough interaction strength λ, produces the separation in macroscopically separated eigenstates of $\hat{G}$. Yet, let the interaction strength be very small, such that the system state is only slightly perturbed (just a little amount of information is leaked to the environment). Then, the ancilla position is strongly measured (through a strong coupling of a second ancilla, macroscopic separation etc., which causes also the effective collapse of the system a qué). The interesting part of the Neumann interaction is that the position of the ancilla will always have the same expected value as the coupled observable (in our case $\hat{G}$). But there is still a step more. Right after the weak measurement of $\hat{G}$ has been performed, a third ancilla is rapidly coupled to the (only slightly perturbed) system, with a Neumann interaction to projectively measure the position of the system (which will be the Bohmian trajectory position of the system). Then, instead of computing the expected observed G by averaging the weak measurements, we could only average the weak measurements that after-wards led to a strong measurement (Bohmian position for the system) at a certain $\vec{x}$. This procedure then, exactly gives the number $G_B^{\psi}(x)$, and is called a psot-selected weak measurement. One could say, this is just juggling with numbers due to several observations, but, one could also perfectly legitimately assume (especially after what we have revealed) that this is the measured expected value for G, because all the times that our Bohmian trajectory was at $\vec{x}$, the system had indeed the property $G_B^{\psi}(\vec{x})$. Of course, phenomenologically this does not "prove" Bohmian trajectories are ontologically real, since that is unprovable to begin with. Yet, this leaves a yet more appealing interpretation, in the opinion of the authors.

This Bohmian discovery, as already anticipated has several practical applications.

On the one hand, this means that we can numerically predict the expected value for an observable without the need to have explicitly defined its formal operator. Instead, one can derive the observable in the language of Bohmian mechanics, which is very akin to classical mechanics (this would be $G_B^{\psi}(x,t)$) and compute the ensemble average of the property for the trajectories. This is how we achieved for example the capability to predict the total electrical current crossing the active region of a two-terminal nano-device operating at high frequencies (THz). Combining such time and length scales implies a full quantum description of the system must be made. Most importantly, the total current at such frequencies is the sum of the conduction (particle flux) and displacement (time-derivative of the electric field) components. This makes it hard to even ask what the operator should look like. However, we can define the total current for the Bohmian trajectory of a k-th electron $\vec{x}(\vec{\xi_k},t)$ of charge q through a surface σ as:

$$I_k^{\xi}(t) = \int_{\sigma} \vec{J}^{\xi}(\vec{r},t)\cdot d\vec{s} + \int_{\sigma} \varepsilon(\vec{r},t)\frac{\partial \vec{E}^{\xi}(\vec{r},t)}{\partial t}\cdot d\vec{s} \tag{21}$$

where $\varepsilon(\vec{r},t)$ is the electric permittivity , $\vec{J^{\xi}}(\vec{r},t) = q\frac{\mathrm{d}\vec{x}_k^{\xi}(t)}{\mathrm{d}t}\delta(\vec{r}-\vec{x}_k^{\xi}(t))$ the particle current density, and $\vec{E}^{\xi}(\vec{r},t)$ is the electric field generated by the electron, as a solution to the Gauss equation. Then, as proven in [], after our group develped a quantum Ramo-Pelegrini theorem, that for two teerminal devices of distance L with metalic contact surface of width and height $w,h >> L$, the total current contribution by this electron is:

$$I_k^{\xi}(t) = \frac{q}{L}v_x^{\xi}(\vec{x}=\vec{x}_k(t),t) \tag{22}$$

where v_x is the longitudinal direction Bohmian velocity of the electron. Then the total current at the surface σ, the sum of these contributions, will be the same as the one measured in the ammeter. This is because when an electron gets out of the active region and into the macroscopic cable connected to the ammeter, it will be screened as explained in. Then by the conservation of the total current, the expectation of the current measured in the ammeter will match with the expected current in the ammeter.

As a second sounded example of the application of this discovery, it provides an operational answer to the search of non-contextuality in the definitions of measurements involving two different times. Saying that a property of a system is contextual means that its value depends on the environment employed to convey that information to the observer. It turns out that for a single-time (static) measurement of the expectation of a property G of a subsystem, which-ever fiducial state we employ in the measurement protocols described in this and the previous sections (either strong or weak), the measured value will be the same, and will be the same as the value we can calculate only knowing the pre-measurement EWF of the subsystem (thus it is a non-contextual property). However, if we want to know the correlation information (say the expectation of the product of observables) between an observable G at time t_1 and an observable F at time t_2, attempting to avoid any measurement apparatus to enter in the picture, we see that to begin with, if their operators do not commute, $\langle\hat{G}(t_2)\hat{F}(t_1)\rangle$ in the Heisenberg formalism will in general give a complex number. Then, we could correlate the result of a strong measurement of G at time t_1 and a strong measurement of F at time t_2, but since the measurements will project the state to EWF eigenstates, the backaction of the measuring device will be present. Apparently, the best we could do is to look for $\mathbb{R}e\Big\{\langle\hat{G}(t_2)\hat{F}(t_1)\rangle\Big\}$, which is the correlation of a weak measurement of G at time t_1 and a strong measurement of F at time t_2, which seems to provide a more "unperturbing" view of a correlation of the dynamic variables. Yet, as it is shown in [13], even an ideal weak measurement does perturb the system in a way that is dependent for example, on the particular fiducial state employed for it (among the acceptable ones).

This has led for example in quantum thermodynamics to a "no-go" theorem [**?**] stating that there cannot exist a work superoperator that simultaneously satisfies all the physical properties required from it. Other thermodynamic properties that are history dependent suffer from the same issue: any definition of them involving weak, strong or even collective measurements are all dependent on the particular measuring scheme employed, which causes to be as many quantum work definitions as measuring schemes exist.

A similar issue raises when trying to measure the maximum working frequency of state-of-the-art transistors and hence the performance of modern computers. For this, the time spent by electrons in nano-scale transistors, the dwell time in the active region, must be measured. One could place position detectors in the two ends $\vec{a}$ and $\vec{b}$ of the active region, but this would result in a highly contextual measurement, when in operation no computer has such detectors in the ends of its transistors.

All this calls for a way to paradoxically measure "unmeasured" dynamical properties of a quantum subsystem. Dynamical properties that could be predicted in a simulation without any explicit introduction of a particular measuring ancilla. It turns out, such properties about the unmeasured wavefunction are for instance the Bohmian property $G_N^{\psi}(\vec{x},t)$ previously mentioned, which "paradoxically" can indeed be measured, but are a property strictly belonging to the unmeasured system wavefunction.

The problem for non-contextual two-time correlations of dynamical variables for a system wavefunction

$|\psi(t)\rangle$, can be simply circumvented by computing the expectation as would be operationally done by definition: for a system with homogeneously sampled trajectories $\{\vec{x}^{\xi}(t)\}_{\xi\in\Sigma}$ and ontologically deterministic observables $G_B^{\psi}(\vec{x},t)$ and $F_B^{\psi}(\vec{x},t)$, the frequentist definition would be:

$$\langle G(t_2)F(t_1)\rangle = \lim_{|\Sigma|\to\infty}\frac{1}{|\Sigma|}\sum_{\xi\in\Sigma} G_B^{\psi}(\vec{x}^{\xi}(t_2),t_2)F_B^{\psi}(\vec{x}^{\xi}(t_1),t_1) = \tag{23}$$

$$= \int |\psi(\vec{\xi},0)|^2\ \mathbb{R}e\left[\frac{\langle\vec{x}^{\xi}(t_2)|\,\hat{G}\,|\psi(t_2)\rangle}{\langle\vec{x}^{\xi}(t_2)|\psi(t_2)\rangle}\right]\mathbb{R}e\left[\frac{\langle\vec{x}^{\xi}(t_1)|\,\hat{G}\,|\psi(t_1)\rangle}{\langle\vec{x}^{\xi}(t_1)|\psi(t_1)\rangle}\right]d\xi$$

This correlation value not only is context-free, which means is an "unmeasured" property that gives information about the system alone, but it can also be experimentally computed employing (quite a lot of) weak measurements.

In a similar way, we can solve the problems concerning a quantum work definition, by first noting that given a general system Hamiltonian $\hat{H} = \sum_k \frac{-\hbar^2}{2m_k}\frac{\partial^2}{\partial x_k^2} + V(\vec{x},t)$, it is immediate to obtain using the wave-function in polar form (and the Hamilton-Jacobi equation encoded in the Schrödinger equation) that:

$$E_B^{\psi}(\vec{x}^{\xi}(t),t) = \mathbb{R}e\left[\frac{\langle\vec{x}^{\xi}(t)|\,\hat{H}\,|\psi(t)\rangle}{\langle\vec{x}^{\xi}(t)|\psi(t)\rangle}\right] = \sum_{k=1}^{n}\frac{1}{2}m_k v_k(\vec{x}^{\xi}(t),t)^2 + V(\vec{x}^{\xi}(t),t) + Q(\vec{x}^{\xi}(t),t) \tag{24}$$

with Q the well known Bohmian quantum potential. Thus, $E_B^{\psi}(\vec{x}^{\xi}(t),t)$ is as anticipated, the Bohmian total energy of the trajectory at time t. Then, following classical mechanics, we can compute its associated Bohmian work as the energy difference: $W^{\xi}(t_1,t_2) = E_B^{\psi}(\vec{x}^{\xi}(t_2),t_2) - E_B^{\psi}(\vec{x}^{\xi}(t_1),t_1)$, which is non-contextual. Finally, the definition of the non-contextual quantum work would be the ensemble average of this trajectory work (also definible in the limit as an integral following the quantum equilibrium hypothesis once again [5]):

$$\langle W(t_1,t_2)\rangle = \lim_{|\Sigma|\to\infty}\frac{1}{|\Sigma|}\sum_{\xi\in\Sigma}\Big(E_B^{\psi}(\vec{x}^{\xi}(t_2),t_2) - E_B^{\psi}(\vec{x}^{\xi}(t_1),t_1)\Big) \tag{25}$$

This proves once again that the Bohmian formalism is not only theoretically helpful for the community, but represents a practical advantage, both numerical and operationally.

Finally, we could give a reasonable Bohmian answer to the search of an "unmeasured" dwell time, as the expected time spent by the Bohmian trajectory of the electron within the active region $\Gamma \subset \mathbb{R}^3$. Mathematically, the dwell time τ for the ξ-th trajectory of the k-th electron with EWF $\psi^{\xi}(\vec{x}_k,t)$ is by definition given by the Lebesgue integral:

$$\tau_B^{\xi} = \int_0^{\infty} dt \int_{\Gamma}\delta(\vec{r} - \vec{x}_k^{\xi}(t))dr \tag{26}$$

which makes the expected time $\langle\tau\rangle$ be, by the quantum equilibrium hypothesis:

$$\langle\tau\rangle = \lim_{|\Sigma|\to\infty}\frac{1}{|\Sigma|}\sum_{\xi\in\Sigma}\tau_B^{\xi} = \int_0^{\infty} dt\int_{\Gamma}|\psi^{\xi}(\vec{r},t)|^2 dr \tag{27}$$

which turns out to be a well-known expression employed to predict the dwell time [**?**].

Could we understand light matter interaction through well defined "Bohmian-electromagnetic fields"?

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