Solving the Constraint Equation

Puncture Equation

$$\bar{\mathcal{D}}^2 u = -\beta(\alpha + \alpha u + 1)^{-7}$$

Where u is the correction of the Schwarzschild conformal factor for Bowen-York black holes.

and $\frac{1}{\alpha} = \sum_n \frac{M_n}{2 s_n}$

and $\beta = \frac{1}{8} \alpha^7 \bar{A}^L{}_{ij} \bar{A}^{ij}{}_L$

To help with solving this problem we will review the process of linearization of a general non-linear boundary value problems.

Nonlinear Problems

Consider a problem of the form

$$\mathcal{D}^2 f = h(f)$$

, where h is non-linear in f, and $\mathcal{D}$ is the Laplace operator

We then write:

$$f^{[n+1]} = f^{[n]} + \delta f$$

$(n+1)^{th}$ iteration of f

Assuming $\delta f << f^{[n]}$ we can expand the $(n+1)$th iteration of h to the 1st order:

$$\boxed{\begin{aligned} h(f^{[n+1]}) &= h(f^{[n]} + \delta f) \\ &= \underbrace{h(f^{[n]}) + (\delta f)\, h'(f^{[n]})}_{\text{1st order expansion of } (n+1)\text{th iteration of } h \text{ around the } n\text{th iteration}} + \mathcal{O}((\delta f)^2) \end{aligned}}$$

Where $h' = \frac{dh}{df}$

By inserting $f^{[n+1]}$ for f in $\mathcal{D}^2 f = h(f)$:

$$\mathcal{D}^2\left(f^{[n]} + \delta f\right) = h\left(f^{[n]} + \delta f\right)$$
$$= h\left(f^{[n]}\right) + (\delta f)\, h'\left(f^{[n]}\right) + O\left((\delta f)^2\right)$$

We now ignore higher order terms $O\left((\delta f)^2\right)$ and use the linearity of $\mathcal{D}$ to write:

$$\mathcal{D}^2(\delta f) - h'\left(f^{[n]}\right)(\delta f) = -\mathcal{D}^2 f^{[n]} + h\left(f^{[n]}\right)$$

And we know define the <u>residual $R^{[n]}$</u>:

$$R^{[n]} = \mathcal{D}^2 f^{[n]} - h\left(f^{[n]}\right)$$

So we can write:

$$\boxed{\mathcal{D}^2(\delta f) - h'\left(f^{[n]}\right)(\delta f) = -R^{[n]}}$$

Linearized iteration equation.

<u>Steps for computation</u>

- We can solve this equation repeatedly, starting with some initial $f^{[0]}$
- For each iteration we compute $R^{[n]}$ and $h'\left(f^{[n]}\right)$ and then solve the iteration equation for δf
- Use δf to update $f^{[n+1]}$
- Once $R^{[n]}$ drops below a tolerance, terminate iterating the solution.

Note that we identify f with u in the puncture equation, so:

$$h(f) = h(u) = -\beta\left(\alpha + \alpha u + 1\right)^{-7}$$

$u^{[n+1]} = u^{[n]} + \delta u$ and

$$h'(u^{[n]}) = 7\alpha\beta\left(\alpha + \alpha u^{[n]} + 1\right)^{-8}$$

And our specific Linearized iteration becomes:

$$\boxed{\begin{aligned} &\mathcal{D}^2(\delta u) - \left(7\alpha\beta\left(\alpha + \alpha u^{[n]} + 1\right)^{-8}\right)(\delta u) = -R^{[n]} \\ &\text{where } R^{[n]} = \mathcal{D}^2 u^{[n]} + \beta\left(\alpha + \alpha u + 1\right)^{-7} \end{aligned}}$$

which we must iterate using the aforementioned steps for iteration.

Solving in Higher Dimensions

If we identify; for brevity:

$$v = \delta u \quad ; \quad g = -h'(u^{[n]}) \quad \text{and} \quad s = -R^{[n]}$$

We can write:

$$\boxed{\mathcal{D}^2 v + g v = s}$$

We introduce a grid for each dimension. We assume Cartesian coordinates and that the grid spacing, Δ, and grid points, N, are the same in every dimension. So now grid functions have three indices:

$$v_{ijk} = v(x_i, y_i, z_i)$$

Using:

$$\boxed{v_i'' = \frac{v_{i+1} - 2v_i + v_{i-1}}{\Delta^2} + O(\Delta^2)}$$

Finite Difference Equation for second derivative

We can finite difference the Laplace operator ∇^2 for every dimension,

$$(\nabla^2 V)_{ijk} = (\partial_x^2 V)_{ijk} + (\partial_y^2 V)_{ijk} + (\partial_z^2 V)_{ijk}$$

$$= \frac{1}{\Delta^2}\big(V_{i+1,j,k} + V_{i-1,j,k} + V_{i,j+1,k} + V_{i,j-1,k} + V_{i,j,k+1} + V_{i,j,k-1} - 6V_{ijk}\big)$$

In the interior of this grid we have the condition:

$$\boxed{\begin{aligned} &V_{i+1,j,k} + V_{i,j+1,k} + V_{i,j,k+1} + (\Delta^2 g_{ijk} - 6)V_{ijk} \\ &\quad + V_{i-1,j,k} + V_{i,j-1,k} + V_{i,j,k-1} = \Delta^2 S_{ijk} \end{aligned}}$$

Note: valid for $0 < (i,j,k) < N-1$ (interior)

Now we want to set up a matrix equation of the form $A\vec{V} = \vec{S}$ and invert it to solve for $\vec{V}$.

But this is a problem since V_{ijk}, S_{ijk} and g_{ijk} are not one-dimensional. To solve this problem we introduce the super index I:

$$\boxed{I = i + Nj + N^2 k \quad \text{so super index}}$$

Which runs over all combinations of i,j,k so we can write $V_{ijk}, g_{ijk}, S_{ijk}$ into one-dimensional vectors with length N^3.

Now the interior grid expression becomes

$$F_{I+N^2} + F_{I+N} + F_{I+1} + (\Delta^2 G_I - 6)F_I + F_{I-1} + F_{I-N} + F_{I-N^2} = \Delta^2 S_I$$

and on the boundaries of the grid we impose Robin boundary conditions so we are left with:
This means we can write

$$\boxed{A \cdot \vec{v} = \vec{s}}$$

where $\vec{v}$ N^3 dimensional vector indexed with I
$\vec{s}$ N^3 dimensional vector indexed with I
A encodes interior grid relation and boundary conditions

Which we can invert to find v

Note: A is a band diagonal matrix which becomes very large quickly

We identify:

- $\vec{s}$ with $-\vec{r}$, where $\vec{r}$ corresponds to a N^3 dimensional indexed version of R at the nth iteration
- $\vec{v}$ with $\delta\vec{u}$, where $\delta\vec{u}$ is a N^3 dimensional vector indexed with I of the iterative update. This is what we seek