Numerical Relativity Theory

Foliations of Spacetime

In numerical work we typically like to solve the initial value or Cauchy problem. We start with a certain state of the fields and matter and then follow how the fields evolve over time.

SPACETIME → SPACE

SPACETIME → TIME

Einstein's Field Equations → CONSTRAINT

Einstein's Field Equations → EVOLUTION

Constraint equations - constrain the fields at each instant of time.

Evolution equations - evolution of the fields

To better understand this process we begin with a simple example, a scalar field.

Foliations of Massless Scalar Field In Flat Spacetime

$$\Box \varphi = \nabla^a \nabla_a \varphi = 4\pi \rho \qquad \text{Field Equation}$$

ρ - mass energy density of matter source term

In flat Minkowski spacetime:

$$g_{ab} = \eta_{ab} \quad \text{(globally)}$$

$$\Box \varphi = g^{ab} \nabla_a \nabla_b \varphi \qquad \eta_{ab} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \eta^{ab} \nabla_a \nabla_b \varphi$$

$$\Box \varphi = (-\partial_t^2 + \mathcal{D}^2)\varphi = 4\pi\rho$$

Note that $\nabla_a = \partial_a$ in Minkowski spacetime and $\mathcal{D}^2$ is the regular Laplacian.

Note that this equation reduces to the Newtonian field equation if we neglect the time derivative.

$$\mathcal{D}^2 \varphi = 4\pi\rho \qquad (G = c = 1)$$

At this point it's worth pointing out the parallel with Einstein's Field Equations (EFEs)

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \qquad \text{Einstein Tensor}$$

$$R_{\mu\nu} = \frac{1}{2} g^{\beta\gamma} \left(\partial_\mu \partial_\gamma g_{\beta\nu} + \partial_\beta \partial_\nu g_{\mu\gamma} - \partial_\mu \partial_\nu g_{\beta\gamma} - \partial_\beta \partial_\gamma g_{\mu\nu} \right) + g^{\beta\gamma} \left(\Gamma^\sigma_{\mu\nu} \Gamma_{\sigma\beta\nu} - \Gamma^\sigma_{\mu\nu} \Gamma_{\sigma\beta\gamma} \right) \qquad \text{Ricci Tensor}$$

This is a contraction of the Riemann tensor written in terms of the metric tensor.
The reason why we have done this is to point out that the RHS of the EFEs are dependent on

second derivatives of the metric tensor in an analogous manner to which

$$(-\partial_t^2 + D^2)\varphi = 4\pi\rho$$

is dependent on the second derivative of φ.
If we define a new variable

$$K = -\partial_t \varphi$$

We can rewrite the above as two equations

$$\left.\begin{aligned} \partial_t \varphi &= -K \\ \partial_t K &= -D^2\varphi + 4\pi\rho \end{aligned}\right] \text{Evolution Equations}$$

In this case we note there are no constraint equations. In this case, getting the wave equation in this form was easy, since we were dealing with a scalar field

A good warmup for dealing with general relativity decompositions is electrodynamics which we will now take a look at it.

Electrodynamics

Maxwell's equations for $\vec{E}$ and $\vec{B}$ are written:

$$D \cdot \vec{E} = 4\pi\rho \qquad \partial_t \vec{E} = D \times \vec{B} - 4\pi\vec{j}$$

$$D \cdot \vec{B} = 0 \qquad \partial_t \vec{B} = -D \times \vec{E}$$

D is the spatial gradient operator
ρ is the charge density
$\vec{j}$ is the charge current density

Useful $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

We can write

$\vec{B} = \nabla\times\vec{A}$ $\qquad$ $\vec{A}$ is the vector potential

Because

$$\nabla\cdot\vec{B} = \nabla\cdot(\nabla\times\vec{A}) = 0$$

Now $\nabla\times\vec{B}$ becomes

$$\nabla\times\nabla\times\vec{A} = -\nabla^2 A + \nabla(\nabla\cdot\vec{A})$$

Now turning to $\partial_t B = -\nabla\times E$ we get

$$\partial_t(\nabla\times\vec{A}) = \nabla\times\partial_t\vec{A} = -\nabla\times\vec{E}$$

Since $\nabla\times\nabla\varphi = 0$ we can write

$$\partial_t\vec{A} = -\vec{E} - \nabla\varphi$$

(φ arbitrary gauge variable)

With all of these insights
Maxwell's equations reduce to:

$$\partial_t\vec{A} = -\vec{E} - \nabla\varphi$$

$$\partial_t\vec{E} = -\nabla^2\vec{A} + \nabla(\nabla\cdot\vec{A}) - 4\pi\vec{j}$$

In index notation (in Minkowski spacetime):

$$\partial_t A_i = -E_i - \nabla_i\varphi$$

$$\partial_t E_i = -\nabla^2 A_i + \nabla_i\nabla^j A_j - 4\pi j_i$$

Notice the parallels with the scalar wave eqn:

$$\partial_t A_i = -E_i - D_i \varphi \quad \Big| \quad \partial_t \varphi = K$$

$$\partial_t E_i = -D^2 A_i + D_i D^j A_j - 4\pi j_i \quad \Big| \quad \partial_t K = -D^2 \varphi + 4\pi \rho$$

- We include : $D_i D^j A_j$, gradient of divergence
  $D_i \varphi$, gradient of gauge variable.
- Another difference is that we must include the satisfaction of gauss' law.

$$D_i E^i = 4\pi \rho \quad \text{constraint equation}$$

So we achieve the decomposition of Maxwelles equations:

$$\left.\begin{aligned} \partial_t A_i &= -E_i - D_i \varphi \\ \partial_t E_i &= -D^2 A_i + D_i D^j A_j - 4\pi j_i \end{aligned}\right] \text{(evolution)}$$

$$D_i E^i = 4\pi \rho \quad \text{(constraint)}$$

So you see that we must solve the constraint equation for valid initial data.

This provides the framework for decomposing the gravitational field equations.

However this example does not show how to split a spacetime tensor into space and time parts,

Faraday Tensor

The electric and magnetic fields form the relativistic anti-symmetric rank 2 tensor,

$$F^{ab} = \begin{bmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B_x & 0 \end{bmatrix}$$

Note: $F^{ab} = -F^{ba}$

E^a = electric field four vector

B^a = magnetic field four vector

For an observer with a four-velocity u^a we demand

$$u_a E^a = 0 \quad \text{and} \quad u_a B^a = 0$$

So that

$$F^{ab} = u^a E^b - u^b E^a + u_d \epsilon^{dabc} B_c$$

ϵ^{abcd} = Levi-Civita tensor

= +1 even permutations

= -1 odd permutations

= 0 otherwise

Given the transformation law for tensors

$$F^{ab'} = \Lambda^{a'}_{\ c} \Lambda^{b'}_{\ d} F^{cd} \qquad \Lambda^{a'}_{\ c} = \frac{\partial x^{a'}}{\partial x^c}$$

We can get a glimpse of how the fields transmute.
In terms of this tensor Maxwell's equations become

$$\nabla_b F^{ab} = 4\pi j^a$$

$$\nabla_{[a} F_{bc]} = \nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab}$$
$$= 0$$

And $j^a = (\rho, j^i)$ where $i = 1, 2, 3$
We make note of a subtle connection with general relativity:

$$\nabla_a \nabla_b F^{ab} = 0$$ (implies conservation law)

Plays the same role as the Bianchi Identity in general relativity.

$$\nabla_a \nabla_b F^{ab} = \nabla_a J^a$$
$$= 0$$

Which leads to the continuity equation

$$\partial_t \rho = -D_i j^i$$

We observed that $D_i B^i = 0$ and wrote

$$B^i = \varepsilon_{ijk} D^j A^k$$ ie. A^k vector potential

We now express F^{ab} in terms of A^a:

$$F^{ab} = \nabla^a A^b - \nabla^b A^a$$

So

$$\boxed{\nabla_b \nabla^a A^b - \nabla_b \nabla^b A^a = 4\pi j^a}$$ Field Equation

Now A^a takes the same role as ϕ in Newtonian gravity.

Theory	Fundamental Quantity	Rank
Newtonian Mechanics	ϕ	0
Electrodynamics	A^a	1
General Relativity	g_{ab}	2

There is another consequence of the "electromagnetic Bianchi identity":

- Because $\nabla_a \nabla_b F^{ab} = 0$ we get

$$\nabla_a\left(\nabla_b \nabla^a A^b - \nabla_b \nabla^b A^a\right) = \nabla_a 4\pi j^a$$
$$= 0$$

- So we only get 3 independent wave equations
- The redundancy results in the <u>constraint equation</u>.
- This means one component of A^a is left undetermined. This redundancy is the origin of the gauge freedom in electromagnetism.

Degrees gauge freedom = # constraint equations

3+1 decomposition of tensors

$$j^a = (\rho, j^i) \qquad A^a = (\varphi, A^i)$$

3+1 decomposition of equations

First we consider the time component and set $a = T$;

$$\nabla_b \nabla^T A^b - \nabla_b \nabla^b A^T = 4\pi j^T$$

$$\nabla_j \nabla^T A^j - \nabla_j \nabla^j \varphi = 4\pi \rho$$

crucial step

We are able to do this because

$\nabla_T \nabla^T A^T - \nabla_T \nabla^T A^T = 0$ so we can restrict $b = j$ (ie. only spatial)

In special relativity we identify:

$\nabla_j = D_j$, $\nabla_T = \partial_T$ and

$$\nabla^T = \eta^{Ta} \nabla_a = -\partial_T$$

So

$$D_j \left(-\partial_T A^j - D^j \varphi \right) = 4\pi \rho$$

Furthermore we make the identification

$$E^i = F^{Ti} = -\partial_T A^i - D^i \varphi$$

So $D_j E^j = 4\pi \rho$ (ie. constraint equation)

Next considering the spatial components we set $a=i$ and find

$$\nabla_T(\nabla^i A^T - \nabla^T A^i) = -\nabla_j \nabla^i A^j + \nabla_j \nabla^j A^i + 4\pi j^i$$

Above we have grouped $b=T$ on LHS and $b=j$ RHS.

Assuming Minkowski spacetime:

$$\partial_T E_i = -D_j D^j A_i + D_i D^j A_j - 4\pi j_i$$

Note we have used the fact that in flat spacetime we may commute second derivatives

$$D_j D_i A^j = D_i D_j A^j = D_i D^j A_j$$

If $\eta_{\mu\nu}$ is mostly +ve then:

$$A^1 = \eta^{1a} A_a = \eta^{10} A_0 + \eta^{11} A_1 \ldots$$
$$= A_1$$

- $A^j = A_j$
- $A^0 = \eta^{0a} A_a$
$$= -A_0$$
- $\partial_T = -\partial^T$

Finally we have factored out a -ve sign on the LHS.

Notice $\nabla_T(\nabla^i A^T - \nabla^T A^i) = -\partial_T(-\partial_T A^i - \partial^i \varphi)$

Summary

- So we find the time component of the field equation resulted in the constraint eqn.

- The spatial components result in the evolution equations.

Time = constraint

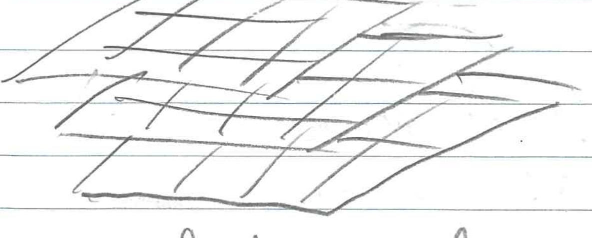
$$\nabla_b \nabla^T A^b - \nabla_b \nabla^b A^T = 4\pi j^T$$

Space = evolution

$$\nabla_T (\nabla^i A^T - \nabla^T A^i) = - \nabla_j \nabla^i A^j + \nabla_j \nabla^j A^i + 4\pi j^i$$

Spacetime Kinematics : 3+1 Split of Spacetime

So far we have assumed that spacial slices of our decomposition have to line up with a given coordinate time T. Now we are going to relax this assumption. This general coordinate freedom is essential in general relativity.
We assume we are given $t = t(x^a)$ that depends, in some way, on our original coordinates.



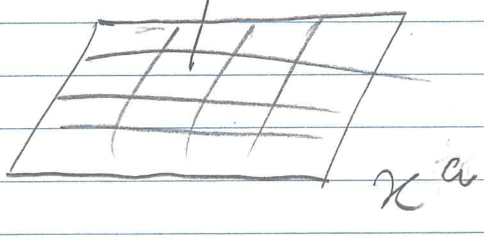
Stack of hypersurfaces

We can think of the level surfaces of constant t as 3D "contour surfaces" that stack up to cover 4D spacetime

Note:

- The hyper surfaces are spacelike ie $ds^2 < 0$ (with $-+++$ metric signature)
- We refer to them as "time slices"

$$n_a = -\alpha \nabla_a t$$



n_a = normal to the time slices

Note that $g^{ab} n_a n_b = -1$

We also should point out that α measures the ratio between the advance of proper time and coordinate time.

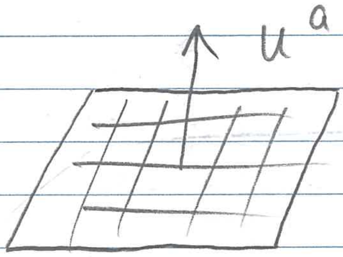
In general

$\alpha = \alpha(x^a)$ ie. it is a function of the coordinates

Finally the choice of the -ve sign makes the contravariant time component of n^a +ve and future orientated.

- We think of $n^a = u^a$ as an observer who moves normal to the time slices

"Normal Observer"



Moves normal to time slices.

Now consider two events seperated by $d\tau$ (infinitesimal proper time interval) along a "normal observer's" worldline,
So the vector pointing from one event the other is given by $n^a d\tau$ which let's us write:

$$dt = \left(\frac{\partial t}{\partial x^a}\right) n^a d\tau$$

$$= \nabla_a t \left(n^a d\tau\right)$$

$$= -\frac{1}{\alpha} n_a n^a d\tau$$

$$= \frac{d\tau}{\alpha} \quad \text{or} \quad d\tau = \alpha\, dt$$

We made use of the results:

$$\nabla_a t = \partial_a t \quad (t \text{ is a scalar})$$

$$n_a = \frac{-1}{\alpha}(1, 0, 0, 0)$$

So we interpret α as the lapse function:

- A measure of the change in proper time $d\tau$ for a change in coordinate time dt.

We should note that we also have freedom to choose our spatial coordinates which affects the contravariant components of n^a:

$$n^a = \frac{1}{\alpha}(1, -\beta^i)$$

Where β^i is any spatial vector.
This normal vector still satisfies $g_{ab}n^a n^b = -1$ as required.

We consider a few situations to help interpret this vector β^i:

- $\beta^i = 0$ spatial coordinates of β^i don't change ie. we think of the spatial coordinates being attached to the normal observer
- $\beta^i \neq 0$ "normal observer" moves wrt to the spatial coordinates. ie. the labels move away from the "normal observer" with β^i as the observer moves through spacetime.

Given we have discovered $dt = \frac{d\tau}{\alpha}$ we now look to find dx^i:

$$dx^i = \left(\frac{\partial x^i}{\partial x^a}\right)(n^a d\tau)$$

$$= \delta^i_a (n^a d\tau)$$

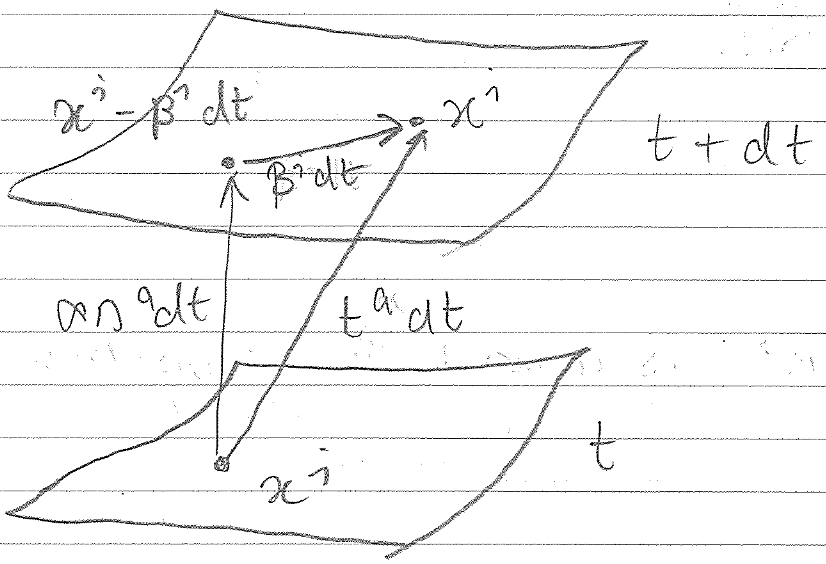
$$= n^i d\tau$$

$$= -\beta^i dt$$

$n^i = \frac{-1}{\alpha}\beta^i$ and $d\tau = \alpha dt$

Note $i = 1, 2, 3$ are the spatial indices

We interpret β^i as the "shift vector" as during an advance $d\tau$ in proper time; an observer attached to the spatial coordinates appears to be shifted $\beta^i dt$ wrt to the normal observer



t^a = tangent vector in spacetime
β^i = shift vector
n^a = normal observer 4-velocity
α = lapse function

We see the coordinate observer with constant x^i moves on a shifted tangent vector; by an amount $\beta^i dt$ wrt the observer.

$t^a = \alpha n^a + \beta^a = (1, 0, 0, 0)$ time vector

Where $\beta^t = 0$ by definition

- This vector is tangent to the coordinate observer's worldline.

3+1 Split of Electrodynamics

To begin with we need the projections of tensors either onto spatial slices or along the normal vector.

$$A^a = A^a_{\parallel} + A^a_{\perp}$$

$A^a_{\parallel}$ = Part of A^a parallel to n^a

$A^a_{\perp}$ = Part of A^a perpendicular to n^a.

The magnitude of $A^a_{\parallel}$ is given by the projection of A^a along n^a

$\varphi = -n_b A^b$ (dot product projection)

$$A^a_{\parallel} = \varphi n^a = -n^a n_b A^b$$

Remember $A^a = (\varphi, A^i)$

This implies

$$A_{\perp}{}^a = A^a - A^a_{\parallel}$$
$$= (g^a{}_b + n^a n_b) A^b$$

We used $A^a = g^a{}_b A^b$ $\qquad \delta^a_b$ = kronecker delta

$$= \delta^a_b A^b$$
$$= A^a$$

$\gamma^a_b = (g^a_b + n^a n_b)$ Projection Operator

It projects A^a into a slice perpendicular to n^a. (into a spatial slice)
So in summary we have:

$$A^a = \varphi n^a + A^a_\perp$$ Spacetime decomposition of rank 1 tensor

We will now consider projections of a few other tensors to help us when we come to the decomposing general relativistic objects.
First we define:

- K_{ab} = extrinsic curvature

  This measures how a spatial slice is warped in a embedded higher dimensional manifold.

- D_a = spatial covariant derivative

  Spacial analogue of 4D covariant derivative.

- $\mathcal{L}_n$ = Lie derivative

  Measures those changes in a tensor field that are not the result of a coordinate transform

$$\boxed{\gamma^a{}_b = g^a{}_b + n^a n_b}$$ Projection Operator

We apply it to some objects from covariant electrodynamics.

$$j^a = g^a{}_b j^b = (\gamma^a_b - n^a n_b) j^b$$
$$= n^a(-n_b j^b) + \gamma^a{}_b j^b$$

$$j^a = \rho n^a + j^a_\perp$$
$$\rho = -n_b j^b$$ (charge density observed by normal observer)
$$j^a_\perp = \gamma^a_b j^b$$ (charge - current density)

For more general tensors we project each index individually. This means we can have:

- completely normal
- Mixed Projections
- completely spatial

$\nabla_a n_b$ Projections

An important example is $\nabla_a n_b$

$$\nabla_a n_b = g^c{}_a g^d{}_b \nabla_c n_d$$

$$= (\gamma^c{}_a - n^c n_a)(\gamma^d{}_b - n^d n_b)\nabla_c n_d$$
$$= \gamma^c{}_a \gamma^d{}_b \nabla_c n_d - \gamma^d{}_b n_a n^c \nabla_c n_d$$
$$- \gamma^c_a n_b n^d \nabla_c n_d + n_a n_b n^c n^d \nabla_c n_d$$

Making a note that $[\nabla, g] = 0$ (ie. they commute)

$\gamma^c{}_a \gamma^d{}_b \nabla_c n_d$ (complete spatial projection of $\nabla_a n_b$)

- It measures how the normal vector changes in one hyper surface. This quantity plays a crucial role in general relativity.

$$\boxed{K_{ab} = -\gamma^c{}_a \gamma^d{}_b \nabla_c n_d}$$ (Extrinsic Curvature)

The term $a_a \equiv n^b \nabla_b n_a$ is the acceleration of a "normal observer."

Given the geodesic equation $a^a = \dfrac{Du^a}{D\tau} = u^b \nabla_b u^a = 0$

We deduce $a_a = 0$ as expected since the normal observer is travelling along a geodesic:

$$n^a \nabla_b n_a = 0 \qquad (a_a = 0)$$

So the last two terms in $\nabla_b n_a$ dissappear. The decomposition then reduces to:

$$\boxed{\nabla_a n_b = -K_{ab} - n_a a_b}$$

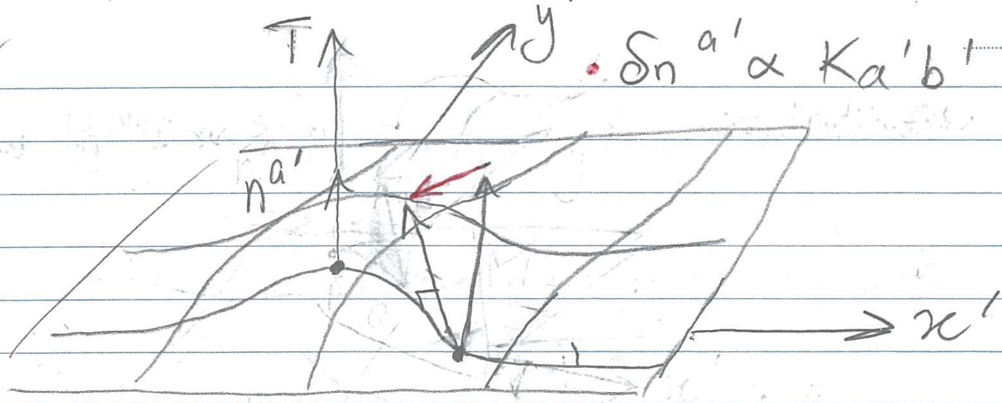
Because $\gamma^d{}_b n_a n^c \nabla_c n_d$

$= \gamma^d{}_b n_a a_d$

$= (g^d{}_b + n^d n_b) n_a a_d$

$= n_a a_b + \cancel{n^d n_b n_a a_d}^{\,0} \qquad (n^d a_d = 0)$

• $\delta n^{a'} \propto K_{a'b'}$



K_{ab} measures how much a hyper surface is warped by computing differences between normal vectors

- The covariant derivative of the normal field $n^{a'}$ along the curve between P and Q measures departure from parallel transport
- $\delta n^{a'} \neq 0$ indicates the surface is warped.

Projections of F^{ab}

$$F^{ab} = g^a{}_c\, g^b{}_d\, F^{cd}$$

$$= (\gamma^a{}_c - n^a n_c)(\gamma^b{}_d - n^b n_d)\, F^{cd}$$

$$= \gamma^a{}_c \gamma^b{}_d F^{cd} - \gamma^b{}_d n^a n_c F^{cd}$$

$$- \gamma^a{}_c n^b n_d F^{cd} + \cancel{n^a n^b n_c n_d F^{cd}}^{\,0}$$

Because F^{ab} is anti-symmetric $n_c n_d F^{cd} = 0$

We can simplify the middle two terms if we evaluate the tensor in the "normal observer's" frame. ($u^a = n^a$) Doing this we find:

$$E^a = n_b F^{ab}$$

$$B^a = \frac{\epsilon^{abcd}}{2} u_b F_{dc}$$

The first term of the projected Faraday tensor also encodes the spatial covariant derivative.

$$\gamma^a{}_c \gamma^b{}_d F^{cd} = \gamma^a{}_c \gamma^b{}_d \left(\nabla^c A^d - \nabla^d A^c\right)$$

$$= \gamma^a{}_c \gamma^b{}_d \left(\nabla^c (\varphi n^d + A^d_\perp) - \nabla^d(\varphi n^c + A^c_\perp)\right)$$

$$= \gamma^a{}_c \gamma^b{}_d \left(n^d \nabla^c \varphi - n^c \nabla^d \varphi + \varphi \nabla^c n^d - \varphi \nabla^d n^c + \nabla^c A^d_\perp - \nabla^d A^c_\perp\right)$$

$$\gamma^a{}_c \gamma^b{}_d \left(n^d \nabla^c \varphi - n^c \nabla^d \varphi\right) = 0 \quad \text{(identically)}$$

$$\gamma^a{}_c \gamma^b{}_d \left(\varphi \nabla^c n^d - \varphi \nabla^d n^c\right) = 0 \quad (K_{ab} = -K_{ba})$$

The last terms leave us with the commutation of the spatial derivative. The spatial derivative is given by:

$$\boxed{D_a A^b_\perp \equiv \gamma^c{}_a \gamma^b{}_d \nabla_c A^d_\perp}$$

Now using this we can express

$$\gamma^a{}_c \gamma^b{}_d F^{cd} = D^a A^b_\perp - D^b A^a_\perp$$

This leaves the Faraday tensor as:

$$\boxed{F^{ab} = D^a A^b_\perp - D^b A^a_\perp + n^a E^b - n^b E^a}$$

Now we use A^a to express the normal projection of F^{ab},

$$n^c F_{cd} = n^c (\nabla_c A_d - \nabla_d A_c)$$
$$= n^c \left(\nabla_c (\varphi n_d + A_d^\perp) - \nabla_d (\varphi n_c + A_c^\perp) \right)$$
$$= n^c (\varphi \nabla_c n_d + n_d \nabla_c \varphi + \nabla_c A_d^\perp - \varphi \underbrace{\nabla_d n_c}_{0} - n_c \nabla_d \varphi - \nabla_d A_c^\perp)$$
$$= \varphi a_d + n^c n_d \nabla_c \varphi + \nabla_d \varphi + n^c \nabla_c A_d^\perp + A_c^\perp \nabla_d n^c$$

$a_d = n^c \nabla_c n_d$, $n^a \nabla_b n_a = 0$, $-n_c \nabla_d A_c^\perp = A_c^\perp \nabla_d n^c$

Using the relation $a_a = D_a \ln(\alpha)$
(the acceleration of a "normal observer" is related to the lapse):

$$\varphi a_d + n^c n_d \nabla_c \varphi + \nabla_d \varphi$$
$$= \varphi D_d \ln(\alpha) + (g^c{}_d + n^c n_d) \nabla_c \varphi$$
$$= \varphi D_d \ln(\alpha) + \gamma^c{}_d \nabla_c \varphi$$
$$= \frac{1}{\alpha} (\varphi D_d \alpha + \alpha D_d \varphi)$$
$$= \frac{1}{\alpha} D_d (\alpha \varphi)$$

We used $\gamma^c{}_d \nabla_c \varphi = D_d \varphi$

i.e. covariant derivative projection

It should be noted that, given A_d is a covector field:

$$\boxed{\mathcal{L}_n A_d^{\perp} = n^c \nabla_c A_d^{\perp} + A_c^{\perp} \nabla_d n^c}$$

Lie Derivative of $A_d^{\perp}$ along n^a vector field

MTW pg. 517 Lie Derivative Examples

$$\mathcal{L}_n T = \left(T_{\alpha\beta,\mu} n^{\mu} + T_{\mu\beta} n^{\mu}{}_{,\alpha} + T_{\alpha\mu} n^{\mu}{}_{,\beta}\right) dx^{\alpha} \otimes dx^{\beta}$$

$$\mathcal{L}_n \sigma = \left(\sigma_{\alpha,\beta} n^{\beta} + \sigma_{\beta} n^{\beta}{}_{,\alpha}\right) dx^{\alpha}$$

Subtle Difference between Covariant Derivative and Lie Derivative

- Covariant derivative vanishes if A^a is parallel transported along n^a.
- Lie derivative along n^a vanishes if the changes in A^a result merely from a infinitesimal coordinate transform generated by n^a.

ie. Lie derivative measures changes in the tensor field that are not produced by a coordinate transformation generated by n^a.

Takeaway

The Lie derivative plays an important role in formulating field equations in 3+1 decompositions as it measures changes in the tensor field that are not attributed to a coordinate transformation

Lie Derivative Rules

$$\mathcal{L}_\omega v^a = [\omega^b, v^a] = \omega^b \nabla_b v^a - v^b \nabla_b \omega^a$$

$$\mathcal{L}_\omega v_a = \omega^b \nabla_b v_a + v_b \nabla_a \omega^b \quad \text{(covector field)}$$

Summary of $n^c F_{cd}$

$$n^c F_{cd} = \varphi a_d + n^c n_d \nabla_c \varphi + \nabla_d \varphi + n^c \nabla_c A_d^\perp + A_c^\perp \nabla_d n^c$$

$$= \frac{1}{\alpha}\left(\varphi D_d \alpha + \alpha D_d \varphi\right) + \mathcal{L}_n A_d^\perp$$

$$= \frac{1}{\alpha} D_d(\alpha\varphi) + \mathcal{L}_n A_d^\perp$$

$$\alpha \mathcal{L}_n A_d^\perp = \mathcal{L}_t A_d^\perp - \mathcal{L}_\beta A_d^\perp$$

$$= \partial_t A_d^\perp - \mathcal{L}_\beta A_d^\perp$$

$$n^c F_{cd} = \frac{1}{\alpha}\left(D_d(\alpha\varphi) + \partial_t A_d^\perp - \mathcal{L}_\beta A_d^\perp\right)$$

Note $\quad n^c F_{cd} = -E_d$

An important property of the Lie derivative is:

$$\mathcal{L}_n A_d^{\perp} = n^c \nabla_c A_d^{\perp} + A_c^{\perp} \nabla_d n^c$$

$$= n^c \partial_c A_d^{\perp} + A_c^{\perp} \partial_d n^c$$

ie. the Christoffel symbol terms cancel eachother.

- Taking the Lie derivative along αn^a, of any spatial tensor results in a tensor that is spatial aswell.

- We therefore want to take Lie derivatives along αn^a to ensure E^a and $A_a^{\perp}$ remain spatial

We therefore write:

$$\alpha n^a = t^a - \beta^a$$

Where $\beta^a = (0, \beta^i)$, $t^a = (1, 0, 0, 0)$

We therefore can write

$$\alpha \mathcal{L}_n A_d^{\perp} = \mathcal{L}_{\alpha n} A_d^{\perp} = \mathcal{L}_t A_d^{\perp} - \mathcal{L}_\beta A_d^{\perp}$$

We then see

$$\mathcal{L}_t A_d^{\perp} = \partial_t A_d^{\perp}$$

Now, we write the normal projection of F^{ab} as

$$n^c F_{cd} = \frac{1}{\alpha}\left(D_d(\alpha \varphi) + \partial_t A_d - \mathcal{L}_\beta A_d^{\perp} \right)$$

Now we rearrange the previous eqn to:

$$\partial_t A_a = -\alpha E_a - D_a(\alpha\varphi) + \mathcal{L}_\beta A_a \quad \left(\begin{array}{c}\text{arbitrary}\\ \text{slices}\end{array}\right)$$

Which comparing to

$$\partial_t A_i = -E_i - D_i\varphi \quad \left(\begin{array}{c}\text{constant } t\\ \text{time slices}\end{array}\right)$$

We see the former is a generalization for arbitrary time slices

Decomposing $\nabla_a F^{ab}$

As we have seen the divergence of the Faraday tensor yields the evolution equation for the spatial components of A^a.

$$\boxed{\begin{aligned}\nabla_a F^{ab} = \gamma^b{}_c\left(\mathcal{L}_n E^c - E^c K + \nabla_a\left(D^a A^c_\perp - D^c A^a_\perp\right)\right)\\ - n^b D_a E^a\end{aligned}}$$

Where $\mathcal{L}_n E^c = n^a \nabla_a E^c - E^a \nabla_a n^c$

$$K = g^{ab} K_{ab} = -\nabla_a n^a$$

= trace of extrinsic curvature, aka mean curvature

We made use of

$$\nabla_a F^{ab} = g^b{}_c \nabla_a F^{ac}$$

$$= (\gamma^b{}_c - n^b n_c)\nabla_a F^{ac}$$

$$E^a = n_c F^{ac}, \qquad \nabla_a n_c = -K_{ac} - n_a a_c$$

Proof for $\nabla_a F^{ab}$

$$\nabla_a F^{ab} = g^b{}_c \nabla_a F^{ac}$$

$$= \gamma^b{}_c \left[\nabla_a\left(D^a A^c_\perp - D^c A^a_\perp\right) + \nabla_a\left(n^a E^c - n^c E^a\right)\right] - n^b n_c \nabla_a F^{ac}$$

$$= \gamma^b{}_c \left[\nabla_a\left(D^a A^c_\perp - D^c A^a_\perp\right) + \mathcal{L}_n E^c - E^c K - n^c \nabla_a E^a\right] - n^b n_c \nabla_a F^{ac}$$

$$= \gamma^b{}_c \left[\mathcal{L}_n E^c - E^c K + \nabla_a\left(D^a A^c_\perp - D^c A^a_\perp\right)\right] - n^b D_a E^a$$

Finally in deriving $\nabla_a F^{ab}$ we should be reminded

$$F^{ab} = D^a A^b_\perp - D^b A^a_\perp + n^a E^b - n^b E^a$$

Preceeding onwards we will investigate the term $\gamma^b{}_c \nabla_a S^{ac}$ where we have made the substitution

$$S^{ac} = \left(D^a A^c_\perp - D^c A^a_\perp\right)$$

For clarity this is the third term in $\nabla_b F^{ab}$.

This term can be written

$$\gamma^b{}_c \nabla_a S^{ac} = \gamma^b{}_c g^a{}_d g^e{}_a \nabla_e S^{dc}$$

$$= \gamma^b{}_c \gamma^a{}_d \gamma^e{}_a \nabla_e S^{dc} + \gamma^b{}_c n^a n_d n^e n_a \nabla_e S^{dc}$$

Since S^{dc} is purely spatial the first term is the covariant spatial derivative, $n_d \nabla_e S^{dc} = -S^{dc} \nabla_e n_d$, $\gamma^a{}_d n_a = 0$ and $n_d S^{dc} = 0$. So

$$\gamma^b{}_c \nabla_a S^{ab} = D_a S^{ab} + \gamma^b{}_c S^{dc} n^e \nabla_e n_d$$

By making use of $\gamma^b{}_c S^{dc} = S^{db}$, the definition of the "normal observers" acceleration, namely $n^c \nabla_c n_a \equiv a_a$, and reinserting our definition of S^{ac} we find:

$$\gamma^b{}_c \nabla_a \left(D^a A^c{}_\perp - D^c A^a{}_\perp\right)$$

$$= \alpha^{-1} D_a \left(\alpha D^a A^b{}_\perp - \alpha D^b A^a{}_\perp\right)$$

Since D^a, $A^c{}_\perp$ are purely spatial $\nabla_a \to D_a$ and we should note that we have made use of $a_a = D_a \ln \alpha$. With this new expression we write $\nabla_a F^{ab}$:

$$\boxed{\nabla_a F^{ab} = \alpha^{-1} D_a \left(\alpha D^a A^b{}_\perp - \alpha D^b A^a\right) + \gamma^b{}_a \mathcal{L}_n E^a - E^b K - n^b D_a E^a}$$

3+1 Decomp $\nabla_a F^{ab}$

- First 3 terms are spatial
- Last term is normal

By inserting this new expression for $\nabla_b F^{ab}$ into the field equation given by:

$$\nabla_b F^{ab} = 4\pi j^a$$

We can obtain Maxwell's equation in a 3+1 decomposition. By considering the spatial and normal parts seperately we can obtain the constraint and evolution equations in 3+1 decomposition form. First we see the spatial projection yields the evolution equation:

$$\partial_t E^b = D_a\left(\alpha D^b A^a_\perp - \alpha D^a A^b_\perp\right) + \mathcal{L}_\beta E^b + \alpha E^b K - 4\pi\alpha j^b_\perp$$

Note: this is for generally orientated hypersurfaces

The normal projection of $\nabla_a F^{ab}$ yields the constraint equation:

$$D_a E^a = 4\pi\rho \quad \left(\begin{array}{l}3+1\\ \text{constraint eqn}\end{array}\right)$$

Collecting our results so far we present the evolution equations for a 3+1 decomposition of Maxwell's equations:

$$\boxed{\begin{aligned} \partial_t A_a^\perp &= -\alpha E_a - D_a(\alpha\varphi) + \mathcal{L}_\beta A_a^\perp \\ \partial_t E^a &= D_b\left(\alpha D^a A^b_\perp - \alpha D^b A^a_\perp\right) + \alpha E^a K \\ &\quad - 4\pi\alpha j^a_\perp + \mathcal{L}_\beta E^a \end{aligned}}$$

(3+1 evolution eqns)

This pair of equations now generalizes

$$\partial_t A_i = -E_i - D_i \varphi$$

$$\partial_t E_i = -D^2 A_i + D_i D^j A_j - 4\pi j_i$$

The above equations are only valid for a very specific set of time slices.

Summary of solving constraint-evolution eqns
Now that we have the generalized decomposition of Maxwell's equations we could proceed and solve the Cauchy problem as follows:

1. Choose E^a initial data that satisfies:

$$D_a E^a = 4\pi \rho$$

2. Choose φ which determines:
   - the gauge of the electromagnetic fields
   - the lapse, α
   - the shift, β^i

3. We then integrate the generalized evolution equations obtaining solutions for E^a and $A_a^\perp$ for all times
   We can combine $A_a^\perp$ with $A_{\parallel}^a = \varphi n^a$

$$A^a = A_{\parallel}^a + A_{\perp}^a$$

from which we constanct the Faraday tensor F^{ab} by using:

$$F^{ab} = \nabla^a A^b - \nabla^b A^a$$

or $B^i = \epsilon^{ijk} D_j A_k$.

N: we also require the spatial metric on each hyper surface

Takeaways

$$\partial_t A_a^{\perp} = -\alpha E_a - D_a(\alpha\varphi) + \mathcal{L}_\beta A_a^{\perp}$$

$$\partial_t E^a = D_b(\alpha D^a A^b_{\perp} - \alpha D^b A^a_{\perp}) + \alpha E^a K - 4\pi j^a_{\perp}\alpha + \mathcal{L}_\beta E^a$$

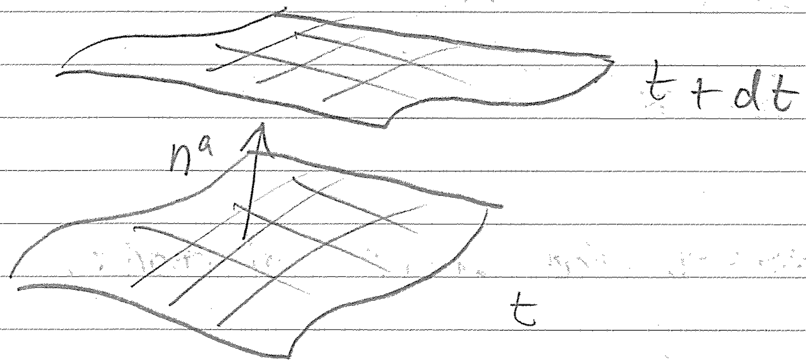
EVOLUTION

$$D_a E^a = 4\pi\rho$$

CONSTRAINT

We have expressed Maxwell's equations in a decomposed form whereby β^i and α are free parameters that specify our time and space coordinates

In other words the intuition behind α and β^i is shown below:



We set arbitrary spatial coordinates on t. Now when the "normal observer" moves to the new hypersurface $t+dt$ they find:

- Their watches shift by a proper time $d\tau = \alpha\, dt$
- Their spatial coordinate labels have shifted by

$$(x^i - \beta^i)\, dt$$

NOTE: determining proper distances requires the spatial metric: the projection of the spacetime metric onto spatial hypersurfaces.

3+1 Decomposition of General Relativity (GR)

Note: This task is much more difficult if we don't understand how to decompose Maxwell's equations.

Einstein's equation

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}}$$

(in units where $G = c = 1$; natural units)

Where $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor,

$R_{\mu\nu} = R^{\beta}{}_{\mu\beta\nu}$ is the Ricci tensor

$R = R_{\mu\nu} g^{\mu\nu}$ Ricci Scalar

$T_{\mu\nu}$ is the energy momentum tensor

Note : we have omitted $\Lambda g_{\mu\nu}$ as we are not interested in cases where it is important.

For reference we give the Riemann tensor components:

$$\boxed{\begin{aligned} R^{\beta}{}_{\lambda\nu\mu} &= \Gamma^{\beta}_{\lambda\mu,\nu} + \Gamma^{\sigma}_{\lambda\mu}\Gamma^{\beta}_{\sigma\nu} - \Gamma^{\beta}_{\lambda\nu,\mu} - \Gamma^{\sigma}_{\lambda\nu}\Gamma^{\beta}_{\sigma\mu} \\ &= \left\langle \omega^{\beta}, \left\{ [\nabla_\nu, \nabla_\mu] - \nabla_{[\partial_\nu, \partial_\mu]} \right\} \partial_\lambda \right\rangle \end{aligned}}$$

where $\partial_\nu \equiv \frac{\partial}{\partial x^\nu}$, $\nabla_\nu \equiv \nabla_{\partial_\nu}$

and ω^β is a differential 1-form

and $\beta = 0, 1, 2, 3$ (ie, 4D spacetime)

Einstein's equation state that given

$$\nabla_\mu G^{\mu\nu} = 0 \qquad \text{(contracted Bianchi identity)}$$

by the well known contracted Bianchi identity

$$\nabla_\mu T^{\mu\nu} = 0$$

which encodes the local conservation of energy and momentum.

Another consequence of this divergenceless is that the equations are not independent.
Given the symmetries of $g_{\mu\nu}$ and $R_{\mu\nu}$ via the no torsion condition:

$$T(\partial_\mu, \partial_\nu) = \nabla_\mu \partial_\nu - \nabla_\nu \partial_\mu - \cancel{[\partial_\mu, \partial_\nu]} = 0$$

$$= 0$$

$$\nabla_\mu \partial_\nu = \nabla_\nu \partial_\mu \qquad (\nabla_\mu \partial_\nu = \Gamma^\beta_{\nu\mu} \partial_\beta)$$

$$\Gamma^\beta_{\mu\nu} = \Gamma^\beta_{\nu\mu}$$

Einstein's equations reduce to 10 equations. With the divergenceless property we further infer that 4 of the equations are not independent.

Decomposing the metric

We first will decompose the metric g_{ab} as before:

$$g_{ab} = g^c{}_a g^d{}_b g_{cd} = (\gamma^c{}_a - n^c n_a)(\gamma^d{}_b - n^d n_b) g_{cd}$$

$$= \gamma^c{}_a \gamma^d{}_b g_{cd} - n_a n_b$$

Notice

$$\gamma^c{}_a \gamma^d{}_b g_{cd} = g_{ab} + n_a n_b$$

is the projection operator, with both indices downstairs

$$\gamma_{ab} \equiv g_{ab} + n_a n_b \qquad \text{spatial metric}$$

We should note that $\gamma^a{}_a = N-1$ where $N=4$ in GR. i.e. the trace of this operator is 3. That is the hypersurfaces will have $d=3$ dimensions as expected.

- The spatial metric plays the same role g_{ab} does for spacetime tensors.

For example:

$$\beta_a = \gamma_{ab} \beta^b \qquad (\beta^b \text{ is a spatial rank 1 tensor})$$

We will use the convention:

$a, b, c \ldots$ spacetime indices

$i, j, k \ldots$ spatial indices

Example of spatial metric

Since the contravariant time components of a spatial tensor must vanish ($n_b \beta^b = 0$) we can write:

$$g^{ab} = \gamma^{ab} - n^a n^b = \begin{bmatrix} -\alpha^{-2} & \alpha^{-2}\beta^i \\ \alpha^{-2}\beta^j & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{bmatrix}$$

(Remember $n^a = (-\alpha, 0, 0, 0)$)

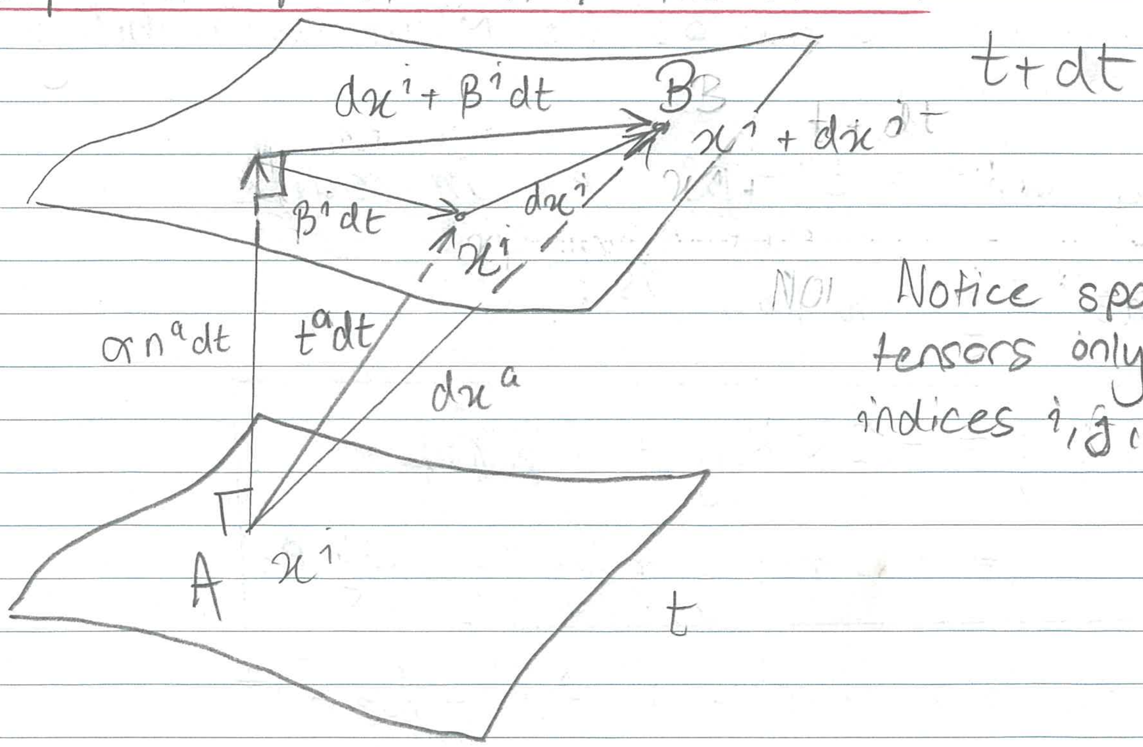
And

$$g_{ab} = \begin{bmatrix} -\alpha^2 + \beta_l \beta^l & \beta_i \\ \beta_j & \gamma_{ij} \end{bmatrix}$$

The line element becomes

$$ds^2 = g_{ab}\, dx^a dx^b$$
$$= -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

Graphical Representation of Line Element



Notice spatial tensors only have indices i, j, k

The graphical representation gives us the intuition:

- $-\alpha^2 dt^2$ measures proper time along normal vector from t to $t+dt$
- $\gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$ measures the proper length from x^i to $x^i + dx^i$

Extrinsic Curvature and the Spatial Metric

First we state the important property:

$$\mathcal{L}_n \gamma_{ab} = n^c \nabla_c \gamma_{ab} + \gamma_{cb} \nabla_a n^c + \gamma_{ac} \nabla_b n^c$$

Which leads to the result

$$\boxed{K_{ab} = -\frac{1}{2} \mathcal{L}_n \gamma_{ab}}$$

where we have used $a_a = n^b \nabla_b n_a$,

$$\nabla_a n_b = -K_{ab} - n_a n_b \quad \text{and} \quad \nabla_a g_{bc} = 0$$

(or otherwise known as the metric compatibility condition)

Using $\alpha n^a = t^a - \beta^a$ we can rewrite $K_{ab} = -\frac{1}{2} \mathcal{L}_n \gamma_{ab}$ as:

$$\boxed{\begin{aligned} \partial_t \gamma_{ij} &= -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij} \\ &= -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \end{aligned}}$$

Because $\alpha \mathcal{L}_n A_d^{\perp} = \mathcal{L}_t A_d^{\perp} - \mathcal{L}_\beta A_d^{\perp}$

So we think of the extrinsic curvature as measuring the time derivative of the spatial metric.

Covariant Derivatives and Christoffel Symbols

In GR the spacetime metric g_{ab} is dynamical as implied by the field equations and is famously in full limelight in Wheeler's expression:

"Spacetime tells matter how to move,
Matter tells spacetime how to curve"

- John Wheeler 2000

By the same reasoning K_{ij} and γ_{ij} are also dynamical in a 3+1 decomposition of GR.

We need to update our definitions of the covariant derivative and christoffel symbols to reflect the dynamic nature of γ_{ij}.

Given a spatial vector V^j we define:

$$\boxed{D_i V^j = \partial_i V^j + V^k \Gamma^j_{ki}}$$

Spatial Covariant derivative

$$\boxed{\Gamma^i_{jk} = \frac{1}{2}\gamma^{il}\left(\partial_k \gamma_{jl} + \partial_j \gamma_{lk} - \partial_l \gamma_{jk}\right)}$$

Remember $i = 1, 2, 3$ (ie spatial indices)
$a = 0, 1, 2, 3$ (spacetime indices)

Γ^i_{jk} is known as the spatial christoffel symbol of the second kind.

We should make note of the property:

$$\boxed{\gamma^a{}_i \gamma^j{}_b \nabla_a V^b = \partial_i V^j + V^k \Gamma^j_{ki}}$$

Where V^b is a spatial vector.

Next on the list is the Riemann tensor and it's contraction the Ricci tensor:

$$\boxed{R^i{}_{jkl} = \Gamma^i_{jl,k} + \Gamma^m_{jl}\Gamma^i_{km} - \Gamma^i_{jk,l} - \Gamma^m_{jk}\Gamma^i_{lm}}$$

Spatial Riemann Tensor

And we give the spatial Ricci tensor in terms of γ_{ij}, which is a little verbose:

$$\boxed{\begin{aligned} R_{ij} = \frac{1}{2}\gamma^{kl}(\partial_k\partial_l\gamma_{ij} + \partial_i\partial_j\gamma_{kl} - \partial_i\partial_l\gamma_{kj} \\ - \partial_k\partial_j\gamma_{il}) + \gamma^{kl}(\Gamma^m_{il}\Gamma_{mkj} - \Gamma^m_{ij}\Gamma_{mkl}) \end{aligned}}$$

Spatial Ricci Tensor

Where $\Gamma_{mkl} = \Gamma^n_{kl}\gamma_{nm}$ is the spatial christoffel symbol of the 1st kind.

Intuition of Spatial Ricci Tensor

- The spatial Riemann tensor is not the spatial projection of the spacetime Riemann tensor. Niether is the spatial Ricci tensor the spatial projection of it's cousin.
- If we parallel transport a vector around an infinitesimal loop on the hypersurface and after the round trip it is different, $R_{ij} \neq 0$.

Infact the Spacetime Riemann tensor is related to its spatial counterpart by:

$$\boxed{\gamma^{p}{}_{i}\,\gamma^{q}{}_{j}\,\gamma^{r}{}_{k}\,\gamma^{s}{}_{l}\,R_{pqrs} = R_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk}}$$

Where $\gamma^{p}{}_{i}$ is the projection operator,

R_{pqrs} is the spacetime fully covariant Riemann tensor

K_{ik} is the extrinsic curvature of the hypersurface

R_{ijkl} is the fully covariant spatial Riemann tensor

And to derive our 3+1 decomposition of $G^{\mu\nu} = 8\pi T^{\mu\nu}$ we need the codazzi equation:

$$\boxed{\gamma^{p}{}_{i}\,\gamma^{q}{}_{j}\,\gamma^{r}{}_{k}\,n^{s}\,R_{pqrs} = D_j K_{ik} - D_i K_{jk}}$$

Which relates spatial covariant derivatives of the extrinsic curvature to 3 spatial and 1 normal projection of the spacetime Riemann tensor.

We can now make projections of G_{ab} and T_{ab} and equate them to find the fully decomposed field equations

ADM Equations

Projecting along n^a we find:

$$\boxed{R + K^2 - K_{ij}K^{ij} = 16\pi\rho}$$ Hamiltonian constraint

Where $\rho \equiv n^a n^b T_{ab}$ total mass energy density

$K = \gamma^{ij} K_{ij}$ is the trace of the extrinsic curvature (ie. mean curvature)

Using a mixed projection we find:

$$\boxed{D_j\left(K^{ij} - \gamma^{ij}K\right) = 8\pi j^i}$$ Momentum Constraint

Where $j^i = -\gamma^{ia} n^b T_{ab}$ is the mass energy flux

We think of these constraints as ensuring the hypersurface fits into the curvature of the surrounding spacetime.

Finally using a complete spatial projection we get the evolution equations:

$$\boxed{\begin{aligned} \partial_t \gamma_{ij} &= -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij} \\ \partial_t K_{ij} &= \alpha\left(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij}\right) \\ &\quad - D_i D_j \alpha - 4\pi\alpha M_{ij} + \mathcal{L}_\beta K_{ij} \end{aligned}}$$

EVOLUTION EQUATIONS

Where $M_{ij} = 2S_{ij} - \gamma_{ij}(S - \rho)$

$S_{ij} = \gamma^a{}_i \gamma^b{}_j T_{ab}$

$S \equiv \gamma^{ij} S_{ij}$ is the trace of the stress according to a normal observer

ADM Equations

Evolution Equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}$$

$$\partial_t K_{ij} = \alpha\left(R_{ij} - 2K_{ik}K^k{}_j + K K_{ij}\right) - D_i D_j \alpha + 4\pi\alpha M_{ij} + \mathcal{L}_\beta K_{ij}$$

Constraint Equations

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho \quad \text{(Hamiltonian)}$$

$$D_j\left(K^{ij} - \gamma^{ij}K\right) = 8\pi j^i \quad \text{(momentum)}$$

We next move to solving the constraint equations by making use of conformal decompositions which leads to the puncture method for finding black hole initial data.

In the boxed ADM equations above we should note:

Dynamical variables: γ_{ij}, K_{ij}

Gauge variables: α, β^i

Matter variables: ρ, j^i, M_{ij}

Number of evolution eqns: 12

Number of constraint eqns / number of gauge choices: 4

Dynamical degrees of freedom: 2

Solving the Hamiltonian Constraint

Solving the initial-value or Cauchy problem in GR entails finding solutions for γ_{ij} and K_{ij}.

The Hamiltonian constraint:

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho$$

and the momentum constraint

$$D_j\left(K^{ij} - \gamma^{ij}K\right) = 8\pi j^i$$

By the use of a conformal transformation:

$$\gamma_{ij} = \varphi^4 \bar{\gamma}_{ij}$$

$\bar{\gamma}_{ij}$ = conformally related metric

φ = conformal factor

γ_{ij} = physical metric

$\bar{\gamma}_{ij}$ also defines a conformal geometry which we can explore by slicing etc.

We can also introduce the inverse conformal metric:

$$\gamma^{ij} = \varphi^{-4}\bar{\gamma}^{ij}$$

And the conformal covariant derivative:

$$\bar{D}_i V^j = \partial_i V^j + V^k \bar{\Gamma}^j_{ki}$$

Where;

$$\bar{\Gamma}^i_{jk} = \frac{1}{2}\bar{\gamma}^{il}\left(\partial_k \bar{\gamma}_{jl} + \partial_j \bar{\gamma}_{lk} - \partial_l \bar{\gamma}_{jk}\right)$$

In fact there is a relationship between $\bar{\Gamma}^i_{jk}$ and Γ^i_{jk}

$$\Gamma^i_{jk} = \bar{\Gamma}^i_{jk} + 2\left(\delta^i_j \partial_k(\ln \psi) + \delta^i_k \partial_j \ln \psi - \bar{\gamma}_{jk}\bar{\gamma}^{il}\partial_l \ln \psi\right)$$

Then we express all the other fundamental tensor objects in this space:

$$\boxed{R = \psi^{-4}\bar{R} - 8\psi^{-5}\bar{D}^2\psi}$$ Conformal Ricci Scalar

Where

$$\bar{D}^2\psi = \bar{\gamma}^{ij}\bar{D}_i\bar{D}_j\psi$$ Conformal Laplacian

Inserting these terms into the Hamiltonian constraint we find:

$$\boxed{\bar{D}^2\psi + \frac{\psi}{8}\bar{R} + \frac{\psi^5}{8}\left(K_{ij}K^{ij} - K^2\right) = -2\pi\psi^5\rho}$$

Conformal Hamiltonian Constraint

- This can be solved for ψ given K_{ij} and ρ with standard numerical techniques

Schwarzschild and Brill-Lindquist Solutions

In order to obtain analytical solutions to the Hamiltonian constraint we have to make simple choices for matter sources ρ, the extrinsic curvature K_{ij}, and the conformally related metric $\bar{\gamma}_{ij}$.

Assumptions

- We will focus on vacuum spacetimes so that $\rho = 0$, $j^i = 0$ in the momentum constraint.
- We also assume time symmetry, whereby all derivatives of γ_{ij} are zero and the line interval is invariant under time reversal. This implies $\beta^i = 0$ and hence the evolution equation

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}$$

implies $K_{ij} = 0 = K$ and the momentum eqn is satisfied identically.

- Finally we choose conformal flatness;

$$\bar{\gamma}_{ij} = \eta_{ij}$$

and with this choice $\bar{R} = 0$ so the conformal Hamiltonian reduces to;

$$\bar{D}^2 \psi = 0$$

In this case since $\bar{R} = 0$ $\bar{D}^2$ is the ordinary laplacian. For spherical symmetry;

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\psi}{dr}\right) = 0$$

Which is solved by

$$\psi = A + \frac{B}{r}$$

We choose $A = 1$ and $B = \frac{M}{2}$ so the conformal factor approaches unity as $r \to \infty$.

$$\boxed{\psi = 1 + \frac{M}{2r}}$$

Conformal Factor Solution

The spatial metric $\gamma_{ij} = \psi^4 \eta_{ij}$ is the metric of a Schwarzschild black hole on a slice of constant Schwarzschild time. M corresponds to the mass that an observer very far away would measure.

Given the fact that

$$\bar{D}^2 \psi = 0$$

is a linear equation we can construct initial data describing multiple black holes. This is encapsulated in the Brill-Lindquist formula:

$$\boxed{\psi = 1 + \sum_{n=1}^{N} \frac{M_n}{2r_n}}$$

Multiple Black Holes Initial Data

Where $r_n \equiv |x^i - x^i_n|$ measures the distance at a coordinate location x^i from the nth black hole

Note:

- This solution holds only at the initial moment of time symmetry
- We still need to use the evolution equations which are non-linear.

• To construct initial data describing black holes in binary orbit we need to abandon the assumption of time symmetry meaning that there is non-zero extrinsic curvature and we must solve both constraint equations together

The Conformal Transverse-Traceless Decomposition

We start with the Helmholtz theorem which says we can write any vector as the sum of two parts

$$E^i = E^i_L + E^i_T = -D^i\varphi + E^i_T$$

E^i_L = Longitudinal zero curl part

E^i_T = Transverse zero divergence part
$(D_i E^i_T = 0)$

Inserting the decomposition into $D_i E^i = 4\pi\rho$ we find:

$$D_i E^i = -D_i D^i \varphi + D_i E^i_T$$

$$= -D^2\varphi = 4\pi\rho$$

This says:

- E^i_T is freely specifiable
- E^i_L is constrained

Taking note of this insight from Maxwell's equations we apply this logic to the GR momentum constraint equation!

GR Conformal - Traceless Decomposition

Next we consider the momentum constraint

$$D_j\left(K^{ij} - \gamma^{ij} K\right) = 8\pi j^i$$

We split K_{ij} into it's trace K and it's traceless part A_{ij}:

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij} K$$

and now we perform a conformal decomposition for A_{ij} and K. There is no advantage in rescaling K since it's a scalar. The traceless part A^{ij} will appear in $D_i K^{ij}$. When we transform the divergence conformally the derivative terms involving φ cancel leaving us with:

$$D_j A^{ij} = \varphi^{-10} \bar{D}_j \bar{A}^{ij}$$

provided we transform A_{ij} like

$$A^{ij} = \varphi^{-10} \bar{A}^{ij} \quad \text{and} \quad A_{ij} = \varphi^{-2} \bar{A}_{ij}$$

We made use of $\bar{\gamma}_{ij}$ to lower the indices.

As a generalization of the Helmholtz theorem we can write a <u>traceless rank-2 tensor</u> as a sum of:

- Longitudinal part

$$\bar{A}^{ij}_L = (\bar{L}W)^{ij} = \bar{D}^i W^j + \bar{D}^j W^i - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k W^k$$

- Transverse-traceless part whose divergence vanishes

$$\bar{D}_j A^{ij}_{TT} = 0$$

The divergence of $\bar{A}^{ij}$ becomes

$$\bar{D}_j \bar{A}^{ij} = \bar{D}_j \bar{A}^{ij}_L = \bar{D}_j (\bar{L}W)^{ij}$$

$$= \bar{D}_j \bar{D}^j W^i + \frac{1}{3}\bar{D}^i(\bar{D}_j W^j) + \bar{R}^i{}_j W^j$$

$$\equiv (\bar{\Delta}_L W)^i$$

Where $\bar{\Delta}_L$ is the vector Laplacian

Now by inserting $\bar{A}^{ij} = \bar{A}^{ij}_L + \bar{A}^{ij}_{TT} = \varphi^{10} A^{ij}$;

$$K_{ij} = \varphi^{-10} \bar{A}^{ij}_L + \varphi^{-10} \bar{A}^{ij}_{TT} + \frac{1}{3}\gamma^{ij} K$$

$$= \varphi^{-10} (\bar{L}W)^{ij} + \varphi^{-10} \bar{A}^{ij}_{TT} + \frac{1}{3}\gamma^{ij} K$$

by the Helmholtz theorem, we can obtain an equation for the vector potential W^i; by inserting K_{ij} into $D_j(K^{ij} - \gamma^{ij} K) = 8\pi j^i$

$$\boxed{(\bar{\Delta}_L W^i) - \frac{2}{3}\varphi^6 \bar{\gamma}^{ij} \bar{D}_j K = 8\pi \varphi^{10} j^i}$$

CONFORMAL TRANSVERSE-TRACELESS MOMENTUM CONSTRAINT (CTTMC)

We used:

$$\bar{D}_j \bar{A}^{ij} = (\bar{\Delta}_L W)^i \quad \text{and} \quad D_j A^{ij} = \varphi^{-10} \bar{D}_j \bar{A}^{ij}$$

- We freely choose the components $\bar{A}^{ij}_{TT}$ and K
- The momentum constraint then fixes the components of $\bar{A}^{ij}_L$.

Bowen - York Solutions

In order to construct black-hole solutions, we focus on the vacuum case, which implies $j^i = 0$:

$$(\bar{\Delta}_L W)^i - \frac{2}{3}\psi^6 \bar{\gamma}^{ij} \bar{D}_j K = 0 \qquad \text{MOMENTUM CONSTRAINT}$$

We also choose conformal flatness ($\bar{\gamma}_{ij} = \eta_{ij}$) so $\bar{R}^i{}_j = 0$ and all covariant derivatives become partial derivatives, in cartesian coordinates.

Next we choose the freely specifiable parts of the extrinsic curvature.
We choose:

- $\bar{A}^{ij}_{TT} = 0$ which we will make use of later
- $K = 0$, maximal slicing. The slicing maximizes the volume of the slice as measured by "normal observes".

Remember the spatial slices are 3D and embedded in a 4D Lorentzian geometry. So setting $K=0$ corresponds to maximizing the volume,

When we choose $K=0$ all terms involving ψ vanish. That means the momentum constraint decouples from the Hamiltonian constraint.
So we can solve the momentum constraint on it's own. In cartesian coordinates the momentum constraint is:

$$(\Delta_L W^i) = \boxed{\partial^j \partial_j W^i + \frac{1}{3}\partial^i \partial_j W^j = 0}$$

BOWEN-YORK EQUATION

We used $\bar{R} = 0$, $\bar{D}^i = \partial^i$ and $\bar{D}_j = \partial_j$

Intuition of CTTMC

- K_{ij} is a rank-2 symmetric tensor in 3 dimensions so it has 6 degrees of freedom
- $\bar{A}^{ij}_{TT}$ has 2 degrees of freedom

$$DOF(\bar{A}^{ij}_{TT}) = 6 - 1 - 3 = 2$$

  - Lose one DOF because $\bar{A}^{ij}_{TT}$ is traceless
  - Lose three DOF because the divergence vanishes

Summary

- If we combine the transverse-traceless decomposition of K_{ij} with the conformal decomposition of the metric we complete one possible decomposition of the initial-value problem.

DECOMP. $$K_{ij} = \psi^{-10}(\bar{L}W)^{ij} + \psi^{-10}\bar{A}^{ij}_{TT} + \frac{1}{3}\gamma^{ij}K$$

MOMENTUM $$(\bar{\Delta}_L W)^i - \frac{2}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_j K = 8\pi\psi^{10} j^i$$

HAMILTONIAN $$\bar{D}^2\psi + \frac{\psi}{8}\bar{R} + \frac{\psi^5}{8}(K_{ij}K^{ij} - K^2) = -2\pi\psi^5\rho$$

- We specify $\bar{\gamma}_{ij}$, K and $\bar{A}^{ij}_{TT}$ and solve the Hamiltonian and Momentum constraints for ψ and W^i respectively.

- The transverse-traceless decomposition has analytical solutions for extrinsic curvatures that describes black holes with linear or angular momentum

Note:

- Since $\partial^j \partial_j W^i + \frac{1}{3}\partial^i \partial_j W^j = 0$ is linear we can use it to construct solutions for multiple black holes.

Bowen-York Solutions

Consider a black hole centred at x^i_{BH}. At a coordinate location x^i we define the (coordinate) unit vector pointing away from the black hole as:

$$L^i = \frac{x^i - x^i_{BH}}{s}$$

Where $s = ((x_i - x_i^{BH})(x^i - x^i_{BH}))^{\frac{1}{2}}$ is the coordinate distance. We should note:

$$\partial_i s = L_i \quad \text{and} \quad \partial_i L_j = \frac{1}{3}(\delta_{ij} - L_i L_j)$$

Using these expressions we can write the solution to the Bowen-York equation:

$$\boxed{W^i = \frac{\bar{\epsilon}^{ilk} L_l J_k}{s^2} \qquad \text{Bowen-York Solution Form I}}$$

Here $\bar{\epsilon}^{ijk}$ is the 3D and conformally related version of the Levi-Civita tensor.

J^k is not to be confused with j^i, it is the vector whose components are constant in cartesian coordinates.

We now compute $\partial_j W^j$ and $\partial^m \partial_j W^j$ to verify the solution.

$$\partial_j W^i = \frac{\bar{\epsilon}^{ilk} J_k}{s^3} \left(\delta_{jl} - 3L_j L_l\right)$$

and

$$\partial^m \partial_j W^i = \frac{3\bar{\epsilon}^{ilk} J_k}{s^4} \left(5L^m L_j L_l - L_l \delta^m{}_j - L^m \delta_{jl} - L_j \delta^m{}_l\right)$$

Setting $i=j$ in the first term results in the contraction of $\bar{\epsilon}^{ilk}$ which is 0, so $\partial_j W^j = 0$ and $\partial^i \partial_j W^j = 0$

Setting $m=j$ in the second term we get

$$\partial^j \partial_j W^i = \frac{3\bar{\epsilon}^{ilk} J_k}{s^4} \left(5L^j L_j L_l - L_l \delta^j{}_j - L^j \delta_{jl} - L_j \delta^j{}_l\right)$$

$$0 = \left(5L_l - 3L_l - L^l - L_l\right)$$

Note: $L^l = L_l$ (via conformal flatness)

We used $\delta^j{}_j = 3$ (ie. trace of metric = # dimensions)

and $L^j L_j = 1$

So the W^i given is indeed a solution.

For conformal flatness and Cartesian coordinates:

$$\bar{A}_L^{ij} = \bar{D}^i W^j + \bar{D}^j W^i - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k W^k$$
$$= \partial^i W^j + \partial^j W^i - \frac{2}{3}\eta^{ij}\partial_k W^k$$

Remember $\bar{\gamma}^{ij} = \eta^{ij}$ $\bar{D}^j = \partial^j$ for conformal flatness

If we insert our definition for $\partial_j W^i$ we find:

$$\bar{A}_L^{ij} = \frac{6}{s^3} L^{(i}\epsilon^{j)kl} J_k L_l$$

Where ( ) is the symmetrization operator

ie. Take a rank 2 tensor T_{ab} then;

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba})$$

$$T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ab})$$

Now we consider a vector potential:

$$W^i = -\frac{1}{4s}(7P^i + L^i L_k P^k)$$

Where P^i are constant when expressed in Cartesian coordinates.

This vector potential also solves $\partial^j\partial_j W^i + \frac{1}{3}\partial^i\partial_j W^j = 0$.

We use $\partial_i W^i = \frac{3L_i P^i}{2s^2}$.

Finally we claim that the extrinsic curvature corresponding to W^i is:

$$\bar{A}_L^{ij} = \frac{3}{2s^2}\left(P^i L^j + P^j L^i - (\eta^{ij} - L^i L^j)L_k P^k\right)$$

Remember $K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K$ (ie. A_{ij} is the traceless part)

Bowen-York Solutions Form II

$$\bar{A}^{ij}_{L} = \frac{6}{s^3} L^{(i} \bar{\epsilon}^{j)kl} J_k L_l$$

$$\bar{A}^{ij}_{L} = \frac{3}{2s^2}\left(p^i L^j + p^j L^i - (\eta^{ij} - L^i L^j) L_k p^k\right)$$

And note we can obtain K^{ij} via:

$$K^{ij} = \varphi^{-10} \bar{A}^{ij}_{L}$$

Where $L^i = (x^i - x^i_{BH}) \frac{1}{s}$

$$s = \left((x_i - x_i^{BH})(x^i - x^i_{BH})\right)^{\frac{1}{2}}$$

p^i = linear momentum of black hole
J^i = angular momentum of black hole

Since the Bowen-York solution Form I is linear:

- We can add together several Bowen-York solutions
- We can use superposition to construct multiple black holes with linear and angular momentum.
- We can add Bowen-York solutions centred on two different coordinate locations.

Note: we still need to find φ and we do this by revisiting the Hamiltonian constraint.

Puncture Initial Data for Black Holes

Under the previous assumptions of the Bowen-York solutions the Hamiltonian constraint

$$\bar{D}^2\psi - \frac{\psi}{8}\bar{R} + \frac{\psi^5}{8}\left(K_{ij}K^{ij} - K^2\right) = -2\pi\psi^5\rho$$

Becomes:

$$\bar{D}^2\psi + \frac{1}{8}\psi^{-7}\bar{A}^L_{ij}\bar{A}^{ij}_L = 0$$

Where A^{ij}_L is given by the Bowen-York solutions. Remember $\bar{A}^{ij} = \bar{A}^{ij}_L + \bar{A}^{ij}_{TT}$ and $A^{ij} = \psi^{-10}\bar{A}^{ij}$.

Now given $\psi = 1 + \frac{M}{2r}$ we expect the Bowen-York solutions to diverge at the centers of black holes.

A method for dealing with the divergence of the conformal factor is the puncture method.

Puncture Method

- Close to each black hole's center, which we refer to as the puncture, the conformal factor is dominated by $\psi \sim \frac{\mathcal{M}}{s}$

  Where $\mathcal{M}$ is the puncture mass.

Note in the reformulated Hamiltonian constraint the term $\bar{A}^L_{ij}\bar{A}^{ij}_L$ diverges as s^{-6}.

This is suppressed by $\psi^{-7} \sim s^7$.

- Close to the puncture the Laplace operator acting on φ must vanish, which leads to solutions that diverge as M/s.
  (ie. $\bar{D}^2\varphi = 0$, close to the puncture)

This observation leads to the approach below. We write

$$\varphi = 1 + \frac{1}{\alpha} + u$$

where $\frac{1}{\alpha} = \sum_n{}' \frac{M_n}{2s_n}$

$\frac{M_n}{s_n}$ describes the singular behaviour for each black hole.

u is the correction term.

Inserting $\varphi = 1 + \frac{1}{\alpha} + u$ into

$\bar{D}^2\varphi + \frac{1}{8}\varphi^{-7}\bar{A}^L{}_{ij}\bar{A}^{ij}{}_L = 0$ and

using $\bar{D}^2\left(\frac{1}{s_n}\right) = 0$ for $s_n > 0$, we obtain an equation for u,

$$\boxed{\bar{D}^2 u = -\beta\,(\alpha + \alpha u + 1)^{-7}}$$

Puncture Equation

where $\beta = \frac{1}{8}\alpha^7\bar{A}^L{}_{ij}\bar{A}^{ij}{}_L$

Intuition of the Puncture Equation

- Given the choices for the Bowen-York solutions and the $\mathcal{M}_n$ puncture masses which specify the black holes' locations and momentum β becomes an analytical function
- The equation is a non-linear elliptic equation that can be solved with standard numerical techniques.

Summary

$$\varphi = 1 + \frac{1}{\alpha} + u \qquad \text{Conformal Factor}$$

$$\sum_n \frac{\mathcal{M}_n}{2 s_n} = \frac{1}{\alpha}$$

$$\bar{D}^2 u = -\beta(\alpha + \alpha u + 1)^{-7} \qquad \text{Reduced Hamiltonian}$$

The key insight to the puncture method is:

$$\bar{D}^2 \varphi = 0 \quad \text{close to the puncture}$$

This is because in the original Hamiltonian:

$$\bar{D}^2 \varphi + \frac{1}{8} \varphi^{-7} \bar{A}^L_{ij} \bar{A}^{ij}_L = 0$$

$$\varphi^{-7} \bar{A}^L_{ij} \bar{A}^{ij}_L \rightarrow \underline{s} \quad \text{close to the puncture}$$

References

[1] E. [illegible], [illegible] [illegible]
[illegible] . He'