Finite Element Method of Analyzing Steady Seepage with a Free Surface

SHLOMO P. NEUMAN AND PAUL A. WITHERSPOON

University of California, Berkeley, California 94720

Abstract. A new iterative approach to steady seepage of ground water with a free surface has been developed using the finite element method. This approach eliminates a number of difficulties that were inherent in the iterative procedures previously used to solve this problem and rapid convergence is now assured in all cases. The method is applicable to heterogeneous porous media with complex geometric boundaries and arbitrary degrees of anisotropy. It can handle problems where the free surface is discontinuous and where portions of the free surface are vertical or near vertical. In addition, infiltration or evapotranspiration at the free surface can be handled with ease. Several examples are included to demonstrate the power of this new approach and to show how it can apply to a wider variety of free surface problems than has been possible before.

INTRODUCTION

In dealing with the seepage of water through earth dams and embankments under steady state conditions where a free surface is present, civil engineers have traditionally relied on the graphical method of flow nets [*Casagrande*, 1940, p. 295, *Cedergren*, 1967]. In analyzing flow to wells in unconfined aquifers, groundwater hydrologists have often based their theory on the Dupuit assumptions. Exact analytical methods of handling such problems have been developed but are often difficult to apply. Extensive treatments of these latter methods are given by *Harr* [1962]; *Polubarinova-Kochina* [1962]; *Aravin and Numerov* [1965]; and *Bear et al.* [1968]. All these methods are limited to flow systems where the porous medium is relatively uniform and the boundary conditions are not too complicated.

In many practical problems, however, the degree of heterogeneity and anisotropy that the engineer encounters in the field may be such that these traditional methods are extremely difficult to apply unless certain simplifying assumptions are made. We need only consider a sequence of homogeneous layers that are nonuniformly anisotropic to realize the limitations of the traditional approach. This approach is further restricted to flow systems with relatively simple boundary configurations.

These difficulties have led to the recent development of numerical methods that enable us to analyze complex systems by using high speed digital computers. *Finnemore and Perry* [1968] have adapted the relaxation technique of *Shaw and Southwell* [1941] to the computer in analyzing seepage through an earth dam. Another finite difference approach to steady state seepage with a free surface has been described by *Jeppson* [1966, 1967, 1968*a*, 1968*b*, 1968*c*, 1968*d*, 1969]. His method requires that the flow region be transformed into another domain that lies in the plane of the velocity potential and stream function. Since he is dealing with nonlinear equations, he obtains a solution using the Gauss-Siedel iterative method of successive over relaxation. His method seems to be limited to systems with simple geometries where the transformation process can be carried out. In systems composed of two irregular layers, his method 'is restricted to homogeneous isotropic mediums or anisotropic mediums, both layers of which have the same horizontal to vertical permeabilities' [*Jeppson*, 1968*c*]. It seems to us that his method is further restricted because the axes of anisotropy must be parallel in all parts of the flow domain.

The problem of seepage with a free surface has also been investigated by *Taylor and Brown* [1967] and *Finn* [1967] using the finite element method. Finn's approach has recently been extended by *Volker* [1969] to include nonlinear flow. The finite element method is based on the calculus of variations and enables us to analyze flow regions having complex geometric boundaries and arbitrary degrees of heterogeneity and anisotropy [*Zienkiewicz and Cheung*, 1965; *Witherspoon et al.*, 1968]. However Taylor and Brown report an ambiguity effect in the vicinity of the seepage face. In reviewing their method in detail, we find that this ambiguity results from a basic lack of convergence in their iterative procedure. The iterative method of Finn is essentially similar to that of Taylor and Brown, and he would have experienced the same difficulties had he not confined himself to those problems where the free surface meets the seepage face tangentially.

The purpose of this paper is to present an improved finite element approach to the problem of steady state seepage with a free surface. We shall first present our method in detail and then use several examples to demonstrate how it can be applied to a wider variety of seepage problems than has been possible before. We will show that the ambiguity effect of Taylor and Brown has been eliminated and that convergence is easily obtained in all cases.

THEORETICAL CONSIDERATIONS

Consider the typical problem of flow through a dam with a toe drain as shown in Figure 1. Let R represent the flow region that generally will have the following four kinds of boundaries: (1) a prescribed head boundary A_1, (2) a prescribed flux boundary A_2, (3) a free surface F, and (4) a seepage face S. This boundary value problem can be described by the following set of equations:

$$\frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial h}{\partial x_j}\right) = 0 \qquad \text{on } R \qquad (1)$$

$$h = H \qquad \text{on } A_1 \qquad (2)$$

$$K_{ij}\frac{\partial h}{\partial x_j} n_i = -V \qquad \text{on } A_2 \qquad (3)$$

$$h = \xi \qquad \text{on } F \qquad (4)$$

$$K_{ij}\frac{\partial h}{\partial x_i} n_i = In_3 = 0 \qquad \text{on } F \qquad (5)$$

$$h = x_3 \qquad \text{on } S \qquad (6)$$

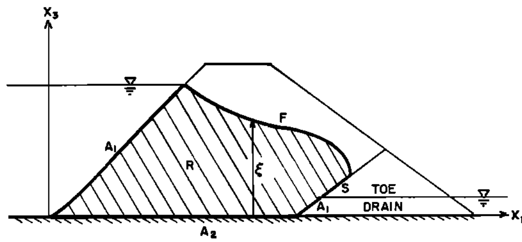

Fig. 1. Cross section of dam showing flow region with a free surface.

Here we introduce ξ in equation 4 to represent the elevation of the free surface above the horizontal datum plane from which head is measured. A generalized variational principle that corresponds to this problem [*Mauersberger*, 1965] may be written

$$\begin{aligned}\Omega(h, \xi) = &\int_R \tfrac{1}{2}K_{ij}\frac{\partial h}{\partial x_j}\frac{\partial h}{\partial x_i}\, dR \\ &- \int_{A_1} (h - H)K_{ij}\frac{\partial h}{\partial x_j} n_i\, dA \\ &- \int_S (h - x_3)K_{ij}\frac{\partial h}{\partial x_j} n_i\, dA \\ &+ \int_{A_2} Vh\, dA \\ &- \int_F (h - \xi)K_{ij}\frac{\partial h}{\partial x_j} n_i\, dA \\ &- \int_F \xi In_3\, dA \end{aligned} \qquad (7)$$

If we could treat both h and ξ as variants at the same time, then a solution could be obtained directly by minimizing the functional (7). In the finite element approach, however, it is necessary that the flow region be fixed in order that the minimization process can be carried out. This has led us to adopt a two step iterative procedure in which the position of the free surface is fixed at the beginning of each iteration, i.e., ξ becomes invariant during each iteration.

Since the true position of F is unknown, only one of the two boundary conditions (4) and (5) can be satisfied at any given time. For the first step of the iteration, it is convenient to adopt boundary conditions (4) and (6) and set $h = \xi$ on F and $h = x_3$ on S. This means that (4) and (6) are now satisfied. Since (2) is easily satisfied, equation 7 reduces to

$$\Omega(h) = \int_R \tfrac{1}{2}K_{ij} \frac{\partial h}{\partial x_j}\frac{\partial h}{\partial x_i}\, dR + \int_{A_2} Vh\, dA \qquad (8)$$

After minimizing (8) using the finite element method described below, we can determine the specific flux V_s on the seepage face.

To satisfy boundary condition (5), we no longer consider h to be fixed on F and S. Instead, for the second step of the iteration we use the prescribed values of I on F and the calculated values of V_s on S and thus treat both F and S as known flux boundaries. This means that equation 7 now reduces to

$$\Omega(h) = \int_R \tfrac{1}{2}K_{ij} \frac{\partial h}{\partial x_j}\frac{\partial h}{\partial x_i}\, dR + \int_S hV_s\, dA + \int_{A2} Vh\, dA - \int_F hIn_3\, dA \qquad (9)$$

After minimizing (9) using the finite element method, the results will satisfy boundary conditions (2), (3), and (5) but not necessarily (4) and (6). If equation 4 is not satisfied with an acceptable error, the final step of the iteration is to shift the position of the free surface in an appropriate manner such that (4) is satisfied as closely as possible. This determines a new position for the free surface and the entire procedure is repeated until $|h - \xi| \leq \epsilon$ everywhere on F, where ϵ is a prescribed error tolerance.

The above procedure differs from that of *Taylor and Brown* [1967] and *Finn* [1967] in several ways. They use only one step in each iteration, and during that iteration they treat the free surface as a no-flow boundary and the seepage face as a prescribed head boundary. Since the true length of S is initially unknown, an incorrect prescribed head boundary is imposed on the flow region, and this tends to retard convergence of the solution on F in the vicinity of S. In addition, the method used by Taylor and Brown to shift the position of the free surface at the end of each iteration does not insure that boundary condition (4) is being approached in a consistent manner unless all points on F are being shifted vertically. Thus their results may sometimes diverge instead of converging to a solution as will be demonstrated below. In determining the position of the exit point (i.e., the intersection of F and S), the shifting procedure of Finn relies on the calculated position of the free surface extending beyond the physical limits of the flow region. Such a procedure does not seem applicable to those situations where the free surface always remains within the flow region (e.g., homogeneous dam with a toe drain). In addition, Finn's method requires that the computer program be stopped after each iteration since there is no provision for shifting during execution.

NUMERICAL APPROACH

In solving equations 8 or 9 by the finite element method, the flow region R is subdivided into a network of elements. Within each element, head can be described in terms of the nodal values h_n as

$$h = N_n h_n \qquad (10)$$

where N_n is only a function of the space coordinates. Substituting (10) into (8) and considering a single element e we have

$$\Omega^e(h) = \int_{R^e} \tfrac{1}{2}K_{ij} \frac{\partial N_n}{\partial x_j} h_n \frac{\partial N_m}{\partial x_i} h_m\, dR + \int_{A_2^e} VN_n h_n\, dA \qquad (11)$$

The functional over the entire flow region $\Omega(h)$ is simply the sum of the functionals over all elements. After minimizing $\Omega(h)$ with respect to h_n, we obtain a set of simultaneous linear algebraic equations

$$A_{nm}h_m - Q_n = 0 \qquad n, m = 1, 2, \cdots, N \qquad (12)$$

where

$$A_{nm} = \sum_e \int_{R^e} K_{ij} \frac{\partial N_n}{\partial x_j}\frac{\partial N_m}{\partial x_i}\, dR$$

$$Q_n = -\sum_e \int_{A_2^e} VN_n\, dA$$

and N represents the total number of nodes in the network.

For plane flow it is convenient to adopt a network composed of triangular elements, and for axisymmetric problems we use a network of concentric rings with a constant triangular cross section. For the particular case where K_{ij} is constant and h is linear within each element, A_{nm} and Q_n in (12) have been evaluated and are given by *Wilson and Nickell* [1966] and *Neuman and Witherspoon* [1969, p. 100].

In general Q_n will be zero at all nodal points that do not act as sources or sinks. Moreover, at each nodal point there will be only M values of A_{nm} that are nonzero, where $M - 1$ is the number of elements surrounding the point. This fact enables us to use quadrilateral elements because they can always be subdivided into four triangles and the equation for the central point can be eliminated from (12). As a result, the number of equations and the number of nonzero values of both A_{nm} and Q_n is significantly reduced. The amount of computer storage and time required in obtaining a solution of (12) is further reduced by the fact that the matrix A_{nm} is symmetric. Minimization of equation 9 will also lead to a set of simultaneous equations similar to (12) and can therefore be handled by the same procedure.

In developing the functional in (8) it was assumed that head is known on A_1, F, and S, and therefore boundary conditions (2), (4), and (6) are satisfied. As a result, there will be a total of K nodal points on these three boundaries where head is known, and the number of unknowns in (12) can be reduced from N to $N - K$. Consequently there will be $N - K$ equations of the form

$$A_{nm}h_m = Q_n - \sum_k A_{nk}h_k; \quad n, m \neq k \qquad (13)$$

where k represents those nodal points at which head has been fixed. Equation 13 can be solved for the $N - K$ unknown values of h_m, that can then be substituted in the remaining set of K equations

$$A_{km}h_m = Q_k \qquad (14)$$

to obtain the values of Q_k at each nodal point where head was originally fixed. A similar procedure is used in handling the functional in (9) with regard to the constant head nodal points along A_1.

A special problem arises during the second step of the iterative procedure at the particular nodal point where F and S intersect. During the first step of the iteration, the Q_k calculated from (14) for this point represents the flux across both F and S in the neighborhood of the point. For the second part of the iteration, however, we only need to know that portion of Q_k that flows across S. It is possible to calculate this portion of Q_k exactly, but if the elements in the vicinity of the intersection of F and S are sufficiently small, it is more convenient to approximate this portion by setting it equal to one-half the value of Q_k that was obtained at the adjacent nodal point on S.

PROCEDURE FOR SHIFTING FREE SURFACE

It will be recalled that at the end of the second step of each iteration, it is necessary to shift the position of the free surface such that boundary condition (4) is satisfied as closely as possible. Consider a portion of the free surface at the start of the ith iteration F_i as shown on Figure 2. A typical nodal point m initially has coordinates $(r_m, z_m)_i$, but at the end of the second step of the iteration, the value of head that has been calculated for this point is found to be $(h_m)_i$. If boundary condition (4) is to be satisfied, the position of F for the $i + 1$ iteration should pass through the point whose coordinates are $(r_m, h_m)_i$. Similarly F must also pass through other calculated positions $(r_n, h_n)_i$ and $(r_p, h_p)_i$, etc., as indicated on Figure 2 by the dashed line F'_{i+1}.

In changing the position of the free surface, it is often necessary to shift nodal points along directions other than the vertical. For example, the point m is to be shifted along the line AA', which makes an angle γ with the vertical on Figure 2. In this case the new position of m is taken as the intersection of AA' with F'_{i+1} at $(r_m,$

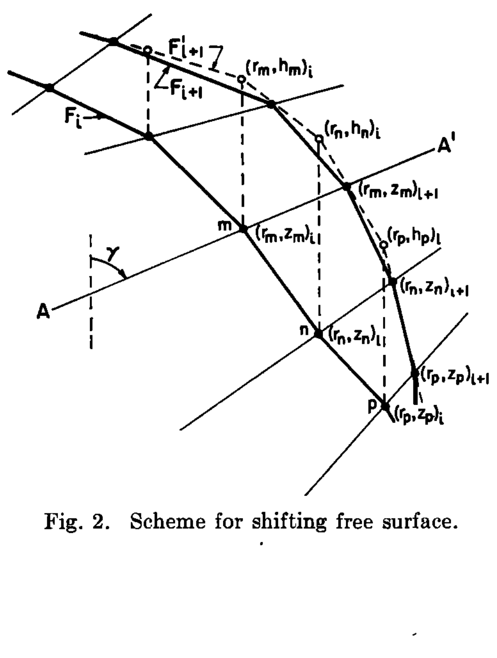

Fig. 2. Scheme for shifting free surface.

$z_m)_{i+1}$. Similarly the points n and p are shifted to new locations whose coordinates are $(r_n, z_n)_{i+1}$ and $(r_p, z_p)_{i+1}$, respectively. The line F_{i+1} that passes through these new points is adopted as the position of the free surface for the $i + 1$ iteration.

When the direction of shifting of any point crosses an interface between two materials or crosses a physical boundary of the system, it will usually be necessary to shift F by a lesser amount than would result from the above procedure. To insure convergence, this shifting must be done in such a way that the rate at which boundary condition (4) is approached is the same everywhere on F. This is accomplished by introducing a correction factor $\alpha \leq 1$, that changes the vertical coordinate of F'_{i+1} for any given node m from $(h_m)_i$ to $(z_m)_i + \alpha[(h_m)_i - (z_m)_i]$. In the program, α is initially set equal to unity and is changed during each iteration to its maximum allowable value such that F'_{i+1} always remains within the prescribed boundaries.

Another problem arises when part of the free surface becomes essentially vertical, as in the case of seepage from a pond or canal (Figure 3). In such cases, the horizontal coordinates of any two adjacent nodal points on F may be so close together that the shifting scheme described above fails to adequately adjust the horizontal coordinates of these points. We overcame this problem by introducing an additional correction factor β which, for any nodal point m, is used by the program whenever

$$|(z_m)_{i+1} - (z_m)_i| < \beta\alpha|(h_m)_i - (z_m)_i| \tag{15}$$

When (15) holds, nodal point m is shifted arbitrarily along AA' such that

$$(r_m)_{i+1} = (r_m)_i + \beta\alpha[(h_m)_i - (z_m)_i] \tag{16}$$

Convergence is further assured by adjusting the magnitude of β during execution using

$$\beta_{i+1} = \beta_i \frac{E_{i+1}}{E_i} \tag{17}$$

where E_i is the maximum difference between elevation and the computed head on F at the end of the ith iteration and is given by

$$E_i = \max_F |(h_m)_i - (z_m)_i| \tag{18}$$

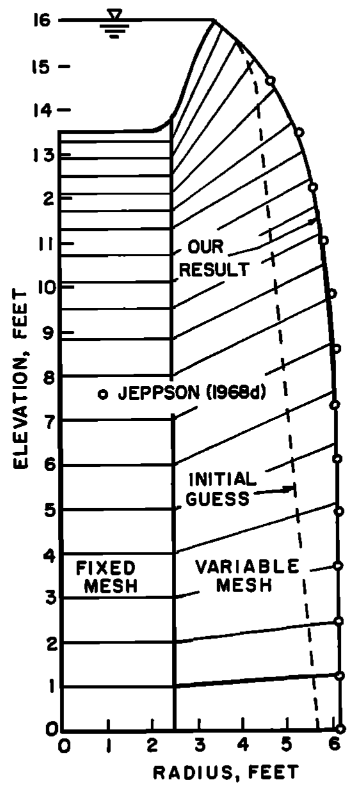

Fig. 3. Seepage from a circular pond to a horizontal drain.

The initial value of β will depend on the direction along which the nodal points are being shifted. Our experience indicates that a β of 0.2 to 0.3 is desirable when the direction is close to horizontal. Larger values of β (0.6 to 0.8) should be used when $\gamma < 45°$. To permit us to examine whether or not the initial value of β has been properly chosen so as to achieve rapid convergence, the program can be stopped after any iteration. An examination of the results will usually reveal whether a change in β is indicated. The problem can then be restarted at the same iteration with a new β and a solution obtained.

APPLICATION TO VARIOUS PROBLEMS

All the above features have been incorporated in a computer program that is able to handle both plane and axisymmetric flow. To demonstrate the versatility of this program, we present below solutions for several different kinds of problems.

1. We shall first consider the axisymmetric problem of flow from a circular pond, where the porous medium is isotropic and homogeneous. Problems of this kind have previously been handled by *Jeppson* [1968*d*] using a finite difference approach. We chose one of his problems and a comparison of our solution with his for the position of the free surface is shown in Figure 3. In addition a set of lines from our network is included to indicate the directions along which shifting of the free surface took place. It is not necessary that the shifting process be carried out over the entire network. We therefore divide the network into one portion where the mesh can conveniently be held fixed and another portion where the mesh is allowed to expand or contract. As a result the matrix A_{nm} for the fixed mesh need only be computed once at the beginning of the problem. The initial guess for the free surface is also indicated on Figure 3.

Figure 4 shows how convergence to a solution with an E_i (equation 18) of about 0.4% of maximum available head was reached. The flattening of this curve indicates that from the practical standpoint, a satisfactory solution has been obtained. The deviations of the plotted points from the average curve are the effect of the arbitrary manner in which horizontal coordinates are shifted using equation 16.

2. We next consider the axisymmetric problem of flow to a well in an unconfined aquifer that has been investigated experimentally by *Hall* [1955] using a sand box model. A numerical solution for this problem that considers flow

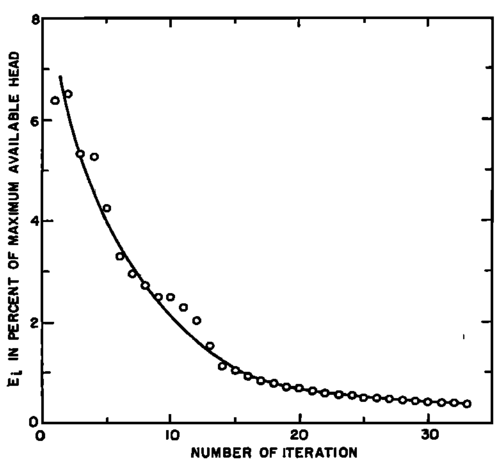

Fig. 4. Convergence for circular pond problem.

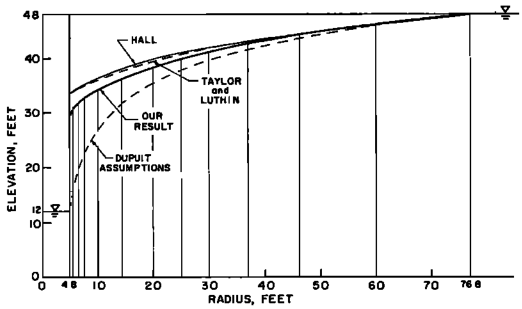

Fig. 5. Seepage toward a well.

in the unsaturated region has also been published by *Taylor and Luthin* [1969]. A comparison of our solution with both of their results and with a solution obtained using the Dupuit assumptions is given in Figure 5. In general, our solution lies slightly below that of Hall and Taylor and Luthin. Our free surface approaches the well tangentially, as is suggested by theory. Again a set of lines is included to indicate the vertical direction along which shifting took place in obtaining a solution. Our initial guess was taken to be the solution given by Hall.

3. We next consider the problem of plane flow through a homogeneous dam to demonstrate how a solution is obtained when the directions of shifting intersect a physical boundary. The results are shown in Figure 6*A*, and the exit point compares favorably with the results calculated by *Casagrande*'s [1940, p. 304] method.

In our first attempt to solve this problem, we used the seepage face as one of the directions along which shifting took place. This proved unsatisfactory because the elements near the exit point became too elongated in the direction of maximum gradient. However, when the directions of shifting were changed to those indicated in Figure 6*A*, a satisfactory solution was obtained after only seven iterations (Figure 6*B*.

4. We shall now consider the problem of a homogeneous dam with a toe drain similar to that investigated by *Taylor and Brown* [1967]. The positions of the free surface as obtained by our method after the fifth and tenth iterations are given in Figure 7*A*. The result for the tenth iteration is essentially the same as that obtained by flow net analysis and was adopted as our solution. The position of our

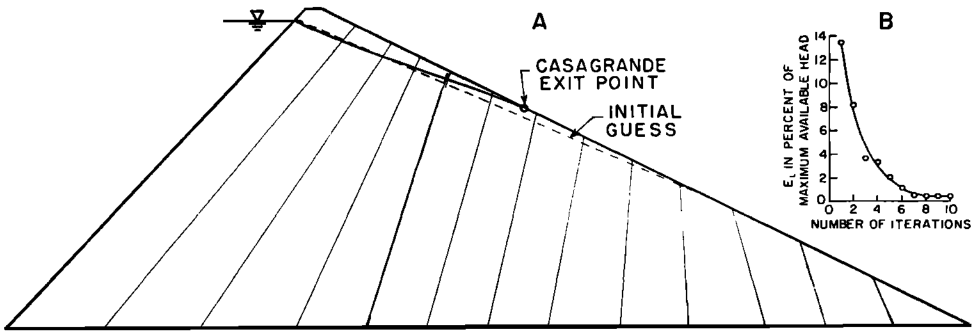

Fig. 6. Seepage through a homogeneous dam.

initial guess is indicated, and the directions of shifting were taken parallel to the upstream face of the dam.

We also solved this problem for the same network configuration and same initial guess using a program provided by Professor R. E. Taylor and based on the method of Taylor and Brown. The results after the tenth and twentieth iterations are shown in Figure 7*B*. It may be seen that their solution for the free surface agrees with ours everywhere except in the vicinity of the seepage face where a lack of convergence is apparent.

5. Finally we consider an example of the complex type of problem that is easily handled by our program (Figure 8). In this case, a dam with a sloping core and horizontal drain rests on a slightly permeable foundation whose bedding

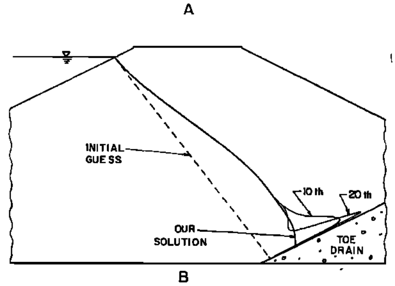

Fig. 7. Seepage through a dam with toe drain.

planes are inclined to the horizontal. Both sections of the dam are assumed to have an isotropic permeability as indicated, whereas the foundation is anisotropic with its principal permeability K_1 parallel to the assumed bedding.

Because of the permeability contrast within the dam, an internal seepage face develops along the interface between the sloping core and the rest of the dam. The possibility of such a situation has previously been pointed out by *Casagrande* [1940, p. 303]. Because of this, we can anticipate the development of a partially saturated region beneath the overhanging slope. Since our program is not designed to treat unsaturated flow, the problem arises as to how much of this flow moves vertically downward to the free surface and how much moves laterally above the saturated zone to the horizontal drain and is thus permanently lost from the system.

We solved this problem by considering two limiting cases. The free surface marked *A* on Figure 8 represents the lower limit for this free boundary when all water that flows across the overhanging seepage face is lost. A few isopotentials for this case have also been included on the figure to indicate general directions of flow The free surface marked *B* represents the upper limit if all water moving across the overhanging slope flows vertically downward through the unsaturated region so as to join the free surface. The actual free surface should therefore lie between these two limits. It should be noted that in obtaining these solutions, the directions of shifting vary from nearly horizontal along the drain to a direction that is parallel to the interface between the two sections of the dam. Shifting of the free surface is done independently in each section of the dam.

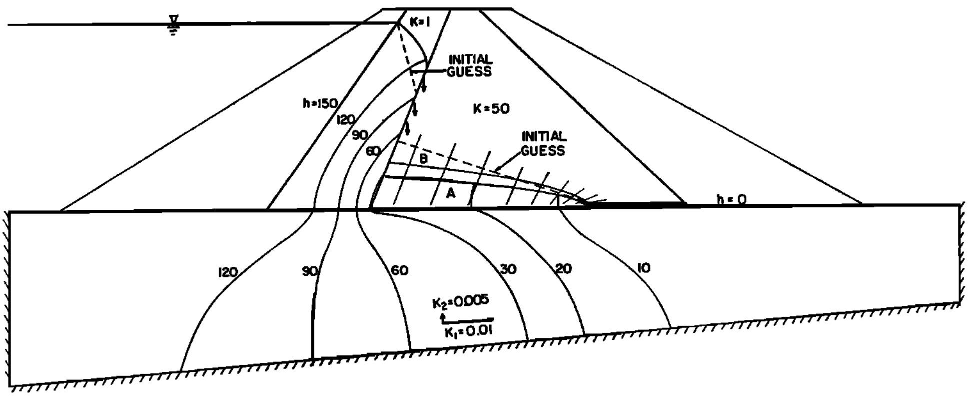

Fig. 8. Seepage through a dam with sloping core and horizontal drain on a slightly permeable foundation.

NOTATION

- h, hydraulic head, L;
- H, hydraulic head on prescribed head boundary, L;
- I, net vertical specific rate of infiltration at free surface, L/T;
- K_{ij}, permeability tensor, L/T;
- n_i, unit outer normal vector, L;
- r, radial coordinate, L;
- V, specific flux on prescribed flux boundary, L/T;
- x_i, coordinate vector, L;
- z, vertical coordinate, L;
- ξ, elevation of free surface above horizontal datum plane from which h is measured, L.

REFERENCES

Aravin, V. I., and S. N. Numerov, *Theory of Fluid Flow in Undeformable Porous Media,* 511 pp., Israel Program for Scientific Translations, Jerusalem, 1965.

Bear, J., D. Zaslavsky, and S. Irmay, *Physical Principles of Water Percolation and Seepage,* 465 pp., UNESCO, Paris, 1968.

Casagrande, A., Seepage through dams, in *Contributions to Soil Mechanics 1925–1940,* Boston Society of Civil Engineers, 1940.

Cedergren, H. R., *Seepage, Drainage, and Flow Nets,* 489 pp., John Wiley, New York, 1967.

Finn, W. D. Liam, Finite-element analysis of seepage through dams, *J. Soil Mech. Found. Div., Amer. Soc. Civil Eng., 93*(SM 6), 41, 1967.

Finnemore, E. J., and B. Perry, Seepage through an earth dam computed by the relaxation technique, *Water Resour. Res., 4*(5), 1059, 1968.

Hall, H. P., An investigation of steady flow toward a gravity well, *La Houille Blanche, 10,* 8, 1955.

Harr, M. E., *Groundwater and Seepage,* 315 pp., McGraw-Hill, New York, 1962.

Jeppson, R. W., Techniques for solving free-streamline, cavity, jet and seepage problems by finite differences, *Dep. Civil Eng., Tech. Rep. 6ᵋ* Stanford University, Stanford, California, 196(

Jeppson, R. W., Finite difference solutions to free surface flow through nonhomogeneous porou media, *Utah Water Res. Lab. Tech. Rep. WG 52-1,* Utah State University, Logan, Utah, 196'

Jeppson, R. W., Seepage from ditches—Solutio by finite differences, *J. Hydraul. Div., Ame Soc. Civil Eng., 94*(HY 1), 259, 1968*a*.

Jeppson, R. W., Seepage through dams in the com plex potential plane, *J. Irrig. Drainage, Ame Soc. Civil Eng., 94*(IR 1), 23, 1968*b*.

Jeppson, R. W., Seepage from channels throug layered porous mediums, *Water Resour. Res 4*(2), 435, 1968*c*.

Jeppson, R. W., Axisymmetric seepage throug homogeneous and nonhomogeneous porous m diums, *Water Resour. Res., 4*(6), 1277, 1968*d*.

Jeppson, R. W., Free surface flow through heter geneous porous media, *J. Hydraul. Div., Ame Soc. Civil Eng., 95*(HY 1), 363, 1969.

Mauersberger, P., A variations principle for stead state groundwater flow with a free surface, *Pu Appl. Geophys., 60,* 101, 1965.

Neuman, S. P., and P. A. Witherspoon, Transier Flow of Groundwater to Wells in Multipl Aquifer Systems, *Geotech. Eng. Rep. 69-1,* Un versity of California, Berkeley, 1969.

Polubarinova-Kochina, P. Ya., *The Theory (Groundwater Movement,* 613 pp., Princetc University Press, Princeton, New Jersey, 196

Shaw, F. S., and R. V. Southwell, Relaxatio methods applied to engineering problems, Problems relating to the percolation of flui through porous material, *Proc. Roy. Soc. Lo don, A, 178,* 1, 1941.

Taylor, R. L., and C. B. Brown, Darcy flow sol tions with a free surface, *J. Hydraul. Di Amer. Soc. Civil Eng., 93*(HY 2), 25, 1967.

Taylor, G. S., and J. N. Luthin, Computer met ods for transient analysis of water-table aqu fers, *Water Resour. Res., 5*(1), 144, 1969.

Volker, R. E., Nonlinear flow in porous media b finite elements, *Proc. Amer. Soc. Civil Eng.*, *95*(HY 6), 2093, 1969.

Wilson, E. L., and R. E. Nickell, Application of finite element method to heat conduction analysis, *Nuclear Engineering and Design*, p. 276, North Holland Publishing, Amsterdam, 1966.

Witherspoon, P. A., I. Javandel, and S. P. Neuman, Use of the finite element method in solving transient flow problems in aquifer systems, in *The Use of Analog and Digital Computers in Hydrology, AIHS Publ. 81*, vol. 2, p. 687, 1968

Zienkiewicz, O. C., and Y. K. Cheung, Finite elements in the solution of field problems, *The Engineer*, p. 220, September 24, 1965.

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