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Analysis of Nonsteady Flow with a Free Surface Using the Finite Element Method

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Abstract. A new iterative, numerical approach to nonsteady flow of groundwater with a free surface using the finite element method has been developed. The method is unconditionally stable and therefore requires only a small number of time steps to reach the steady state. It can handle problems in which the free surface is discontinuous and portions of the free surface are vertical or nearly vertical. Infiltration or evapotranspiration at the free surface is handled with ease, and the effect of the unsaturated zone can be taken into account indirectly by using the concept of delayed yield from storage. In addition to gravity drainage, the method takes into consideration storage due to the elastic properties of the saturated porous medium. In problems involving flow to a well operating at a prescribed rate, both storage in the well and the actual distribution of velocities along the well bore are taken into account. The method can be applied to a wide variety of problems involving complex boundaries and arbitrary degrees of heterogeneity and anisotropy. Several examples are included to demonstrate some of the features of this new approach.

Problems that involve nonsteady flow of groundwater with a free surface in the saturated zone are extremely difficult to solve by exact analytical methods. To simplify the treatment of such problems, groundwater hydrologists have traditionally relied on the well-known Dupuit assumptions. These assumptions lead to the Boussinesq equation, which has been extensively treated by *Polubarinova-Kochina* [1962], *Bear et al.* [1968], and others. Another less common approach to such problems is to solve a linearized version of the basic partial differential equations that govern nonsteady flow with a free surface. Work along these lines has been done by *Belyakova* [1955], *Meyer* [1955], *Galin* [1959], *Polubarinova-Kochina* [1959], and *Dagan* [1966, 1967*a*]. A higher order approximation of these equations has also been considered by *Dagan* [1964, 1967*b*]. All these analytical solutions, however, are limited to flow systems in which the boundaries are simple and geometric, the porous medium is relatively uniform, and the vertical gradients throughout are not too large.

In recent years an increasing number of attempts have been made to simulate nonsteady flow with a free surface by means of electric analogs and high speed digital computers. Much of this work has been based on the Boussinesq equation and is therefore limited to flow regions with small vertical gradients. The most recent example is the work by *Desai and Sherman* [1970], who have used an alternating direction explicit procedure (ADEP) to solve Boussinesq's equation in sloping river banks. A different approach, based on a linearized version of the governing equations, has been adopted by *Stallman* [1965] in developing an electric analog of a water table aquifer.

When the porous medium is assumed to be incompressible, flow everywhere inside the system can be described by a partial differential equation that does not involve derivatives with respect to time (e.g., Laplace's or Poisson's equations). Thus it is possible to replace the original nonsteady state problem with a finite number of equivalent steady state problems. A resistance network analog based on such an approach has been described in detail by *Herbert* [1968], and an essentially similar finite difference model has been developed by *Szabo and McCraig* [1968]. *Parekh* [1967] used the same approach in an attempt to develop a finite element model for transient flow with a free surface. In all these models, however, the position of the free surface at the end of each time step is obtained explicitly from gradients calculated at the beginning of the time step. This approach is equivalent to what is known in numerical analysis as a 'forward difference' scheme, which becomes unstable unless the time intervals are taken to be sufficiently small. To insure stability, the number of time steps required to reach a solution may often be so large as to render this approach inapplicable.

In all of the preceding work, the effect of the unsaturated zone on the transient position of the free surface has been consistently neglected. In recent years, however, there have been a growing number of attempts to consider flow in the saturated and unsaturated zones simultaneously. Finite difference schemes for simulating both of these zones have been employed by *Rubin* [1968], *Taylor and Luthin* [1969], *Freeze* [1969], *Hornberger et al.* [1969], and *Hornberger and Remson* [1970]. An approach developed by petroleum reservoir engineers for two-phase flow has been adapted to this problem by *Green et al.* [1969, 1970] and by *Cooley and Donohue* [1969].

A common difficulty in treating both zones simultaneously is that a great deal more data are required for the unsaturated zone than are required for the saturated zone. In using the petroleum engineering approach to investigate water flow in the unsaturated zone, *Green et al.* [1970] require data on the initial state of saturation, rates of infiltration for both air and water, viscosities and densities of both fluids, capillary pressure characteristics for each part of the flow region, relative permeabilities, and the absolute (or specific) permeability of each part of the system. *Rubin* [1968] and *Taylor and Luthin* [1969] require essentially the same data but only for the water phase. However, the groundwater hydrologist will rarely have such data available in working on specific field problems and will therefore not be able to analyze flow directly in the unsaturated zone. Since he is primarily concerned with the saturated zone, what the groundwater hydrologist really needs to know is the extent to which flow in the unsaturated zone affects his results. It appears that the concept of delayed yield as suggested by *Boulton* [1954] may provide an indirect and more convenient method of handling this problem. According to Boulton's concept, when the water table falls, drainage is not instantaneous and some water is delayed in its downward movement. Boulton has suggested that the delayed yield can be approximated by an exponential function of time.

The question of whether flow in the unsaturated zone must always be taken into account when the saturated zone is studied can also be raised. The available literature does not provide a conclusive answer to this question. *Hornberger et al.* [1969] appear to be the first workers to have investigated this problem using two different models, one that includes the unsaturated zone and one that does not. They found for the particular system studied that, except at a very early time, the effects of the unsaturated zone on the transient position of the water table could be neglected.

For the present work we have adopted the approach that there are probably many situations in which the effects of the unsaturated zone can either be neglected completely or be approximated satisfactorily by using the concept of delayed yield. We have therefore developed a numerical method for analyzing nonsteady flow with a free surface that permits either of these options. The method is an extension of a finite element technique that was recently applied to steady state problems [*Neuman and Witherspoon,* 1970*a*]. The method is based on the original form of the nonlinear governing equations and therefore enables one to investigate nonsteady free surface problems for a wide variety of conditions. Large vertical gradients are easily handled in systems with complex boundaries and arbitrary degrees of anistropy and heterogeneity. When dealing with radial flow to a well operating at a prescribed rate, one must take into account both storage in the well and the actual distribution of velocities along the well bore. The method can also handle the effects of elastic storage, an important capability in analyzing multiple aquifer systems [*Neuman and Witherspoon,* 1969*a*]. In addition, water can be added to or taken away from the free surface at prescribed rates to simulate infiltration and evapotranspiration. To obtain variations with time, the method uses an implicit, time centered scheme that is accurate and unconditionally stable. As a result only a small number of time steps are required to reach the final steady state.

THEORETICAL CONSIDERATIONS

Consider the typical problem of flow through a dam with a toe drain (Figure 1). Initially water in the reservoir is maintained at a constant level L_0, and the free surface has developed a steady position, as indicated by the dashed line in Figure 1. If the water level in the reservoir drops instantaneously or at some prescribed rate to a new elevation L_1, the free surface will start falling until a new state of equilibrium is established. Let R represent the flow region that in general will have the following four kinds of boundaries: (1) prescribed head boundaries A_1, (2) prescribed flux boundaries A_2, (3) a free surface F, and (4) seepage faces S. This initial boundary value problem can be described [*Neuman and Witherspoon*, 1970*b*] by the following set of equations:

$$\frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial h}{\partial x_j}\right) = S_s\frac{\partial h}{\partial t} \tag{1}$$

$$h(x_i, 0) = h_0(x_i) \tag{2}$$

$$\xi(x_1, x_2, 0) = \xi_0(x_1, x_2) \tag{3}$$

$$h(x_i, t) = H(x_i, t) \qquad \text{on} \quad A_1 \tag{4}$$

$$K_{ij}\frac{\partial h}{\partial x_j}n_i = -V(x_i, t) \qquad \text{on} \quad A_2 \tag{5}$$

$$\xi(x_1, x_2, t) = h(x_1, x_2, \xi, t) \quad \text{on} \quad F \tag{6}$$

$$K_{ij}\frac{\partial h}{\partial x_j}n_i = \left(I - S_y\frac{\partial \xi}{\partial t}\right)n_3 \quad \text{on} \quad F \tag{7}$$

$$h(x_i, t) = x_3 \qquad \text{on} \quad S \tag{8}$$

We have introduced ξ in equations 3, 6, and 7 to represent the elevation of the free surface above the horizontal datum plane from which head is measured. The elastic specific storage S_s is often set equal to zero when one deals with unconfined flow in a homogeneous porous medium. However there are many situations in which an unconfined aquifer is being supplied with significant amounts of leakage from below. In these cases the effects of compressibility on the nonsteady behavior of the entire system may be quite important. If we want to treat such systems by the finite element method, we must retain S_s in formulating the governing equations.

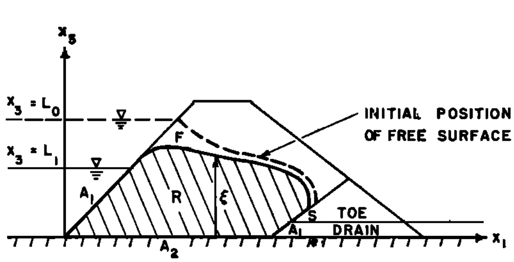

Fig. 1. Cross section of a dam showing nonsteady flow region with a free surface.

A generalized variational principle that corresponds to equations 1–8 was developed by *Neuman and Witherspoon* [1970*b*]. It has the form

$$\begin{aligned}\Omega(h, \xi) = &\int_R \left(\tfrac{1}{2}K_{ij}\frac{\partial h}{\partial x_j}\frac{\partial h}{\partial x_i} + S_s h\frac{\partial h}{\partial t}\right) dR \\ &- \int_{A_1}(h - H)K_{ij}\frac{\partial h}{\partial x_j}n_i\, dA \\ &+ \int_S (h - x_3)K_{ij}\frac{\partial h}{\partial x_j}n_i\, dA \\ &+ \int_{A_2} Vh\, dA - \int_F (h - \xi)K_{ij}\frac{\partial h}{\partial x_j}n_i\, dA \\ &- \int_F \xi\left(I - S_y\frac{\partial \xi}{\partial t}\right)n_3\, dA \end{aligned} \tag{9}$$

where the time derivatives of h and ξ are considered to be unvaried. If one could vary h and ξ at the same time, then a solution could be obtained directly by minimizing the functional 9. In the finite element approach, however, it is necessary to fix the flow region so that the minimization process can be carried out. Thus in the iterative process to be described below, ξ must remain unvaried during each iteration.

To meet this requirement we assume that the prescribed head boundary conditions 4, 6, and 8 can be satisfied independently of the minimization process. This assumption leads to a simplified variational principle

$$\begin{aligned}\Omega(h) = &\int_R \left(\tfrac{1}{2}K_{ij}\frac{\partial h}{\partial x_j}\frac{\partial h}{\partial x_i} + S_s h\frac{\partial h}{\partial t}\right) dR \\ &+ \int_{A_2} Vh\, dA - \int_F h\left(I - S_y\frac{\partial h}{\partial t}\right)n_3\, dA\end{aligned} \tag{10}$$

which is less general than (9) because it does not guarantee that the minimizing function will satisfy boundary conditions 4, 6, and 8.

NUMERICAL APPROACH

In using the finite element method to solve for the minimizing function of (10), the flow region R is subdivided into a network of elements. It is convenient to adopt a network composed of triangular elements for plane flow and a network composed of concentric rings of constant triangular cross section for axisymmetric problems. We shall assume that K_{ij} and S_s are constant within each element and that V, I, and S_y are constant on each segment of the boundary at any given time t. Within each element head can be described in terms of the nodal values h_n as

$$h = N_n h_n \tag{11}$$

where N_n are linear functions of the coordinates, and the repeated indices represent summation over all n. If we now consider a single element e and substitute (11) into (10), we obtain

$$\begin{aligned}\Omega^e(h) = \int_{R^e} \biggl(\tfrac{1}{2}K_{ij} \frac{\partial N_n}{\partial x_j} h_n \frac{\partial N_m}{\partial x_i} h_m \\ + S_s N_n h_n \frac{\partial h}{\partial t} dR \\ + \int_{A_2^e} V N_n h_n \, dA \\ - \int_{F^e} N_n h_n \left(I - S_y \frac{\partial h}{\partial t}\right) n_3 \, dA \end{aligned} \tag{12}$$

Since $\partial h/\partial t$ is unvaried, it will be treated below in a slightly different manner.

The functional over the entire flow region $\Omega(h)$ is simply the sum of the functionals over all elements. After minimizing $\Omega(h)$ with respect to h_n, one obtains

$$\begin{aligned}A_{nm}h_n + \sum_e \int_{R^e} S_s N_n \frac{\partial h}{\partial t} dR - Q_n - C_n \\ + \sum_e \int_{F^e} N_n S_y \frac{\partial h}{\partial t} n_3 \, dA = 0 \end{aligned} \tag{13}$$

$$n, m = 1, 2, \cdots, N$$

where N represents the total number of nodes in the network, and

$$A_{nm} = \sum_e \int_{R^e} K_{ij} \frac{\partial N_n}{\partial x_j} \frac{\partial N_m}{\partial x_i} dR$$

$$Q_n = -\sum_e \int_{A_2^e} V N_n \, dA$$

$$C_n = \sum_e \int_{F^e} N_n I n_3 \, dA$$

In the preceding equations, as well as throughout the rest of our discussion, the symbol Σ_e indicates summation over all elements adjacent to nodal point n. The term N_n and the terms A_{nm} and Q_n have been evaluated and are given by *Wilson and Nickell* [1966] and *Neuman and Witherspoon* [1969*b*, p. 100].

The partial derivatives of head with respect ment of the free surface L^e, as shown in Figure 2. Since I and n_3 are constant on each segment, C_n is easily evaluated and is given by

$$C_n = \tfrac{1}{2} \sum_e \alpha I \, \Delta x_1^e \tag{14}$$

where $\alpha = 1$ for plane flow and $\alpha \approx \pi[(x_1)_n + (x_1)_m]$ for axisymmetric flow. Thus C_n is simply the average net vertical infiltration across the two free surface segments adjacent to nodal point n.

The partial derivatives of head with respect to time that appear in (13) represent the change in head at a fixed point in space. Our method, however, requires that the network be allowed to contract and expand to accommodate the movement of the free surface. Thus at least some of the nodal points in the network are movable and change their positions with time. Moreover when the free surface changes its position over some time interval Δt as indicated in Figure 2, it is often necessary to shift nodal points along directions other than the vertical. For example, nodal point m in Figure

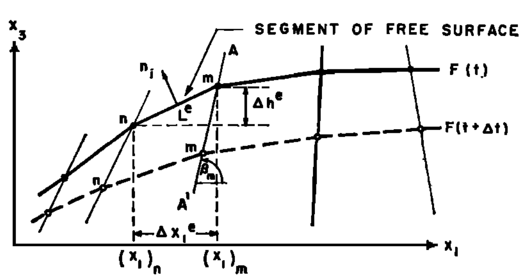

Fig. 2. Scheme for a shifting free surface.

2 is to be shifted along the line AA', which makes an angle β_m with the horizontal. In the process of determining the new position of m on the line AA' at $t + \Delta t$, it is necessary to replace $\partial h/\partial t$ with a total derivative dh/dt representing the rate at which the point m moves along the prescribed direction AA'. The same steps must be followed for other movable nodal points lying on the line AA'.

In general, the total derivative of head is given by

$$\frac{dh}{dt} = \frac{\partial h}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial h}{\partial t} \quad \text{in} \quad R \tag{15}$$

Since the sides of the elements do not curve, the rate dx_i/dt at which the network expands or contracts must vary linearly with the space coordinates within each element. The same condition holds true for dh/dt because, within each element, h is a linear function of x_i. Thus equation 15 can be written

$$\frac{\partial h}{\partial t} = N_m\left(\frac{dh}{dt}\right)_m - \frac{\partial N_m}{\partial x_i} h_m N_p \left(\frac{dx_i}{dt}\right)_p \quad \text{in} \quad R \tag{16}$$

On the free surface head is a function only of x_1 and t. Note that in Figure 2

$$\frac{\partial h}{\partial x_1} = \frac{\Delta h^e}{\Delta x_1^e} \quad \text{on} \quad F^e \tag{17}$$

and

$$\left(\frac{dx_1}{dt}\right)_m = \left(\frac{dx_1}{dh}\frac{dh}{dt}\right)_m = \cot\beta_m\left(\frac{dh}{dt}\right)_m \quad \text{on} \quad F^e \tag{18}$$

Thus (16) becomes

$$\frac{\partial h}{\partial t} = N_m\left(\frac{dh}{dt}\right)_m\left(1 - \frac{\Delta h^e}{\Delta x_1^e}\cot\beta_m\right) \quad \text{on} \quad F^e \tag{19}$$

Substituting (16) and (19) into (13) and recognizing from Figure 2 that

$$\begin{aligned} \int_{F^e} N_n N_m n_3 \, dA &= \frac{\alpha}{3}\Delta x_1^e \qquad n = m \\ \int_{F^e} N_n N_m n_3 \, dA &= \frac{\alpha}{6}\Delta x_1^e \qquad n \neq m \end{aligned} \tag{20}$$

where α is given in equation 14, one finally obtains

$$\begin{aligned} A_{nm}h_m + B_{nm}\left(\frac{dh}{dt}\right)_m & - B_{npim}'\left(\frac{dx_i}{dt}\right)_p h_m - Q_n \\ & - C_n + D_{nm}\left(\frac{dh}{dt}\right)_m = 0 \end{aligned} \tag{21}$$

$$n, m = 1, 2, \cdots, N$$

where

$$B_{nm} = \sum_e \int_{R^e} S_s N_n N_m \, dR$$

$$B_{npim}' = \sum_e \frac{\partial N_m}{\partial x_i}\int_{R^e} S_s N_n N_p \, dR$$

$$D_{nm} = \sum_e \frac{\alpha}{3} S_y(\Delta x_1^e - \Delta h^e \cot\beta_m) \qquad n = m$$

$$D_{nm} = \sum_e \frac{\alpha}{6} S_y(\Delta x_1^e - \Delta h^e \cot\beta_m) \qquad n \neq m$$

The integrals associated with the terms B_{nm} and B'_{npim} have been evaluated and are given by *Wilson and Nickell* [1966] and *Neuman and Witherspoon* [1969*b*, p. 100].

Obviously C_n and D_{nm} are zero everywhere except at the free surface. Q is always zero at the free surface, because C takes care of an infiltration or loss of water at this boundary. Q will also be zero at all other nodal points that do not act as sources or sinks. If one adopts the usual assumption that the elastic storage can be neglected (i.e., $S_s = 0$), then B and B' vanish from (21), and the equation is greatly simplified.

To save computer time, it is usually convenient to hold part of the network fixed as we have done previously for the steady state [*Neuman and Witherspoon,* 1970*a*]. In this fixed part of the network, the coefficients will be constant and need be determined only once at the beginning of a problem. The term dx_i/dt will obviously vanish, and thus computations will be greatly reduced. If one is able to arrange the network such that S_s can be neglected in the variable part (but not necessarily in the fixed part), then the term B'_{npim} $(dx_i/dt)_p$ vanishes completely from (21).

EFFECT OF THE UNSATURATED ZONE

The problem of delayed yield from storage in the unsaturated zone can be handled in the following manner. First, different values of specific yield must be prescribed at different values of time at each point on the free surface. Since the manner in which specific yield varies with time is not yet clearly understood, one must rely on the few data available in the current literature. Some data have recently been published by *dos Santos and Youngs* [1969].

Second, a mass balance must be maintained so that the amount of water remaining in the unsaturated zone is always known. At each time t a prescribed amount of the remaining water can be added to the free surface by using the term C in equation 21. If one accepts *Boulton's* [1954, 1963] assumption that delayed yield is an exponential function of time and is proportional to drawdown at the water table, one can easily incorporate that assumption into the problem at hand. Other rates of delayed yield can also be prescribed, depending on the circumstances. Additional field and laboratory research is needed to develop a better understanding of how S_y and delayed yield vary in space and time.

INTEGRATION WITH RESPECT TO TIME

Note that (21) is a set of nonlinear ordinary differential equations that must be integrated with respect to time. In performing this integration, it is convenient to divide the time domain into a discrete number of time steps $\Delta t = t^{k+1} - t^k$, where t^k represents a time level at which the position of the free surface is known. To obtain a solution at the new time level t^{k+1}, the simplest approach is to replace the time derivatives in (21) with finite differences.

In the so-called forward difference scheme, equation 21 is expressed at the old time level t^k as

$$A_{nm}{}^k h_m{}^k + (B_{nm}{}^k + D_{nm}{}^k) \frac{h_m{}^{k+1} - h_m{}^k}{\Delta t} - \left[B_{npim}{}'^k \frac{(x_i)_p{}^{k+1} - (x_i)_p{}^k}{\Delta t} \right] h_m{}^k = Q_n{}^k + C_n{}^k \tag{22}$$

If the coefficient of the third term is zero, (22) becomes a set of simultaneous linear algebraic equations that can be solved directly for $h_m{}^{k+1}$. If the coefficient of the third term is not zero, the equations are nonlinear and must be solved by iteration. In either case this scheme becomes unstable unless Δt is kept sufficiently small, and therefore an extremely large number of time steps may often be required to obtain a satisfactory solution.

In the backward difference scheme equation 21 is expressed at the new time level t^{k+1} as

$$A_{nm}{}^{k+1} h_m{}^{k+1} + (B_{nm}{}^{k+1} + D_{nm}{}^{k+1}) \frac{h_m{}^{k+1} - h_m{}^k}{\Delta t} - \left[B_{npim}{}'^{(k+1)} \frac{(x_i)_p{}^{k+1} - (x_i)_p{}^k}{\Delta t} \right] h_m{}^{k+1} = Q_n{}^{k+1} + C_n{}^{k+1} \tag{23}$$

Theoretical considerations, as well as our experience, indicate that the backward difference scheme is unconditionally stable. However, more accurate results can often be obtained by taking the arithmetic average of (22) and (23). This operation leads to the unconditionally stable and highly accurate time centered (or Crank-Nicholson) scheme

$$\left\{ A_{nm}{}^{k+1} + \frac{1}{\Delta t} [B_{nm}{}^k + B_{nm}{}^{k+1} - B_{npim}{}'^{(k+1)}((x_i)_p{}^{k+1} - (x_i)_p{}^k) + D_{nm}{}^k + D_{nm}{}^{k+1}] \right\} h_m{}^{k+1}$$
$$= Q_n{}^k + Q_n{}^{k+1} + C_n{}^k + C_n{}^{k+1} - \left\{ A_{nm}{}^k - \frac{1}{\Delta t} [B_{nm}{}^k + B_{nm}{}^{k+1} + B_{npim}{}'^k((x_i)_p{}^{k+1} - (x_i)_p{}^k) + D_{nm}{}^k + D_{nm}{}^{k+1}] \right\} h_m{}^k \tag{24}$$

The time centered scheme was adopted in our investigation of nonsteady flow with a free surface.

SOLUTION BY ITERATION

Expression 24 is a set of nonlinear algebraic equations that can be solved by iteration. Within each time step the coefficients in (24) are relaxed through an iterative procedure until convergence is achieved. To demonstrate this process, it is helpful to drop the superscript k from all known terms and replace $k + 1$ by another superscript j to represent iterations. Thus within each time step, equation 24 takes the form

$$\begin{aligned}\Big\{A_{nm}{}^{i} + \frac{1}{\Delta t}[B_{nm} + B_{nm}{}^{i} \\ - B_{npim}{}'^{i}((x_i)_p{}^{i} - (x_i)_p) \\ + D_{nm} + D_{nm}{}^{i}]\Big\} h_m{}^{j+1} \\ = Q_n + Q_n{}^{i} + C_n + C_n{}^{i} \\ - \Big\{A_{nm} - \frac{1}{\Delta t}[B_{nm} + B_{nm}{}^{i} \\ + B_{npim}{}'((x_i)_p{}^{i} - (x_i)_p) \\ + D_{nm} + D_{nm}{}^{i}]\Big\} h_m \qquad (25)\end{aligned}$$

When there is no seepage face (see problem 2 in the last section of the paper), equation 25 can be solved directly for $h_m{}^{j+1}$. However in the more common case, in which a seepage face is present, the fact that the length of the seepage face at the end of the time step is unknown presents a problem. As has been demonstrated in connection with steady seepage problems [*Neuman and Witherspoon*, 1970*a*], this difficulty can be overcome by dividing each iteration into two steps, $j + (1/2)$ and $j + 1$. This idea is explained in detail in *Neuman and Witherspoon* [1970*a*] and will be described again below.

For the first step it is convenient to adopt boundary conditions (6) and (8) and treat both the free surface and the seepage face as prescribed head boundaries. Thus if we set $h_m{}^{j+(1/2)} = \xi_m{}^{j}$ on F and $h_m{}^{j+(1/2)} = (x_3)_m{}^{j}$ on S, we need only to solve for those nodal points that do not lie on F and S. As a result D and C do not appear, and (25) becomes

$$\begin{aligned}\Big\{A_{nm}{}^{i} + \frac{1}{\Delta t}[B_{nm} + B_{nm}{}^{i} - B_{npim}{}'^{i}((x_i)_p{}^{i} \\ - (x_i)_p)]\Big\} h_m{}^{j+(1/2)} \\ = Q_n + Q_n{}^{i} - \Big\{A_{nm} - \frac{1}{\Delta t}[B_{nm} \\ + B_{nm}{}^{i} + B_{npim}{}'[((x_i)_p{}^{i} - (x_i)_p)]\Big\} h_m \\ (26)\end{aligned}$$

It should be obvious that one does not have to solve (26) for values of h_m on A_1 because here the head is always equal to the prescribed value H (i.e., boundary condition 4 is easily satisfied). Equation 26 can be solved directly for $h_m{}^{j+(1/2)}$, and the results can then be used to obtain fluxes $Q_s{}^{j+(1/2)}$ at the seepage face explicitly from

$$\begin{aligned}\Big\{A_{sm}{}^{i} + \frac{1}{\Delta t}[B_{sm} + B_{sm}{}^{i} - B_{spim}{}'^{i} \\ \cdot((x_i)_p{}^{i} - (x_i)_p)]\Big\} h_m{}^{j+(1/2)} \\ = Q_s + Q_s{}^{i+(1/2)} \\ - \Big\{A_{sm} - \frac{1}{\Delta t}[B_{sm} + B_{sm}{}^{i} \\ + B_{spim}{}'((x_i)_p{}^{i} - (x_i)_p)]\Big\} h_m \qquad (27)\end{aligned}$$

where the subscript s refers to nodal points on S.

In the usual case in which elastic storage S_s is neglected, B and B' vanish from (26) and (27). Since the right-hand side of equation 1 is now zero, values of h_m at the old time level must satisfy

$$A_{nm}h_m = Q_n \qquad (28)$$

for all nodal points n that are not on the free surface. This case, of course, is the same as the steady state case for seepage with a free surface [*Neuman and Witherspoon*, 1970*a*]. As a result, equations 26 and 27 can be replaced by the much shorter expressions

$$A_{nm}{}^{i} h_m{}^{i+(1/2)} = Q_n{}^{i} \qquad (29)$$

and

$$A_{sm}{}^{i} h_m{}^{i+(1/2)} = Q_s{}^{i+(1/2)} \qquad (30)$$

respectively.

For the second step of the iteration, we no longer treat F and S as prescribed head boundaries. Instead, since Q is zero at the free surface and has just been calculated on the seepage face, we can now write (25) as

$$\begin{aligned} \Big\{A_{nm}{}^{i} &+ \frac{1}{\Delta t}\,[B_{nm} + B_{nm}{}^{i} \\ &- B_{npjm}{}'^{i}((x_j)_p{}^{i} - (x_j)_p) \\ &+ D_{nm} + D_{nm}{}^{i}]\Big\}\, h_m{}^{i+1} \\ &= Q_n + Q_n{}^{i+(1/2)} + C_n + C_n{}^{i} \\ &- \Big\{A_{nm} - \frac{1}{\Delta t}\,[B_{nm} + B_{nm}{}^{i} \\ &+ B_{npjm}{}'((x_j)_p{}^{i} - (x_j)_p) \\ &+ D_{nm} + D_{nm}{}^{i}]\Big\}\, h_m \qquad (31) \end{aligned}$$

and solve for $h_m{}^{i+1}$ everywhere including F and S. Again B and B' vanish if $S_s = 0$.

To determine whether these new values are acceptable, we define the maximum relative error on F as

$$E = \max_{F} \left| \frac{h_m{}^{i+1} - h_m{}^{i}}{h_m{}^{i+1} - h_m} \right| \qquad (32)$$

If E is found to be sufficiently small, the iteration is completed, and one can proceed to the next time step.

If E is larger than a prescribed error tolerance, the final step of the iteration is to shift the position of the free surface. As we mentioned earlier, each nodal point m on F must be shifted along a prescribed direction until its elevation is equal to the most recently calculated value of $h_m{}^{i+1}$. Next the variable portion of the finite element network must be expanded or contracted to accommodate the new position of the free surface. Finally new values of all coefficients in (25) that depend on geometry must be calculated. The two-step iteration is then repeated until E in (32) is sufficiently small.

A special problem arises during the second step of each iteration at the particular nodal point where F and S intersect. The value of $Q_s{}^{i+(1/2)}$ calculated for this point during the first step of the iteration from equation 27 or 30 represents the flux across both F and S in the neighborhood of the point. In using equation 31, however, we should have used only that portion of $Q_s{}^{i+(1/2)}$ that flows across S. The use of more than that portion introduces a slight error that can be corrected arbitrarily by placing the node along a straight line that passes through the two nearest nodal points on F.

WELL WITH A PRESCRIBED DISCHARGE

When a well has been completed in an unconfined aquifer and discharges at some prescribed rate, flow into the well bore is not uniform along its length. As shown in Figure 3, the internal boundaries along the well bore consist of a seepage face S and a boundary A_1, along which at any instant of time the head is equal to the elevation of the water level $L(t)$. The variation of flux across these two boundaries in a problem previously solved for the steady state [*Neuman and Witherspoon*, 1970a] is shown in Figure 4. Note that flux increases more or less linearly along the seepage face and becomes essentially constant below the water level in the well.

From this example it is apparent that to treat this variable flux boundary appropriately, one must consider the actual distribution of fluxes along the well bore. Since the variation of flux depends on the length of the seepage face, one must take into account the effect of storage in the well.

The total discharge from the pump $Q_P(t)$ can be divided into two components, the discharge from the aquifer into the well $Q_A(t)$ and the amount of discharge contributed from well storage $Q_W(t)$ (Figure 3). In general, one always has

$$Q_P(t) = Q_A(t) + Q_W(t) \qquad (33)$$

where $Q_P(t)$ is a prescribed function of time.

If we assume that all three quantities vary linearly during each time step Δt, then the total volume of water produced at the surface is simply

$$\Delta V = \frac{\Delta t}{2}[Q_P^{\,k} + Q_P^{\,k+1}]$$
$$= \frac{\Delta t}{2}[(Q_A^{\,k} + Q_A^{\,k+1}) + (Q_W^{\,k} + Q_W^{\,k+1})] \quad (34)$$

where k and $k+1$ represent the old and the new time levels, respectively. The volume of water removed from well storage is given by

$$\Delta V_W = \frac{\Delta t}{2}[(Q_P^{\,k} + Q_P^{\,k+1}) - (Q_A^{\,k} + Q_A^{\,k+1})] \quad (35)$$

and can also be expressed as

$$\Delta V_W = \Delta L \pi (r_W^{\,2} - r_t^{\,2}) \quad (36)$$

where ΔL is the change in the height of the water level in the well, r_w is the effective radius of the well, and r_t is the outside radius of the production pipe. Combining equations 35 and 36, one obtains the following expression for ΔL:

$$\Delta L = \frac{\Delta t}{2\pi(r_w^{\,2} - r_t^{\,2})}[(Q_P^{\,k} + Q_P^{\,k+1}) - (Q_A^{\,k} + Q_A^{\,k+1})] \quad (37)$$

Each side of equation 37 contains one unknown, ΔL on the left side and Q_A^{k+1} on the right side. To incorporate (37) into the two-step iterative procedure adopted in the present work, we shall again drop the superscript k and replace $k+1$ by another superscript j, to indicate iterations within a time step:

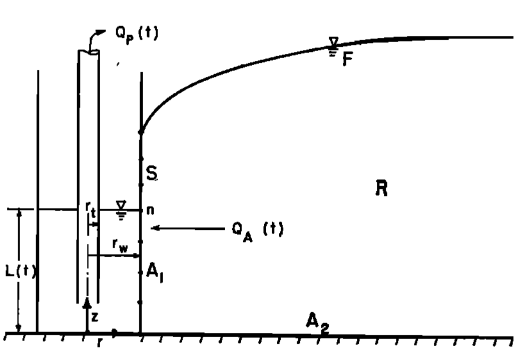

Fig. 3. Flow through an unconfined aquifer to a well with storage.

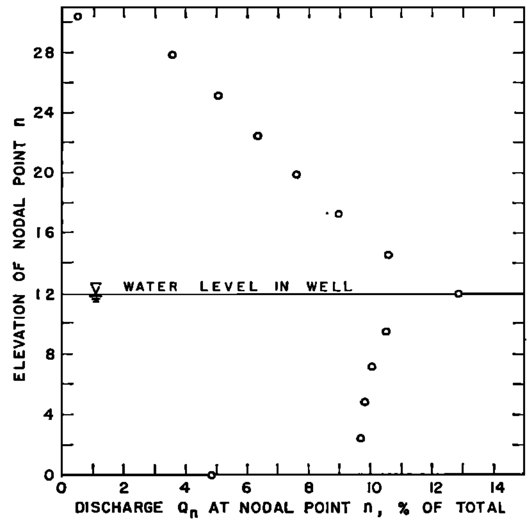

Fig. 4. Variation of steady state discharge at nodal points along a well in an unconfined aquifer.

$$\Delta L^j = \frac{\Delta t}{2\pi(r_w^{\,2} - r_t^{\,2})}[Q_P' - Q_A - Q_A^{\,j}] \quad (38)$$

Here $Q'_P = Q_P^k + Q_P^{k+1}$, and ΔL has been superscripted to indicate that it can vary from iteration to iteration. Prior to the first step of the iteration, a new elevation for the water level in the well is determined by using equation 38 together with

$$L^j = L - \Delta L^j \quad (39)$$

where L is the elevation at the old time level. Next the particular nodal point along the well bore with the elevation nearest to L^j is shifted to the new elevation (nodal point n in Figure 3). The coordinates of all the other nodes along the well bore are readjusted to be equally spaced along each segment S and A_1. All other nodes lying in the immediate vicinity of the well must also be rearranged so that the internal angles of the elements near the well do not change significantly. Finally new values for the coefficients A^j, B^j, B'^j, and D^j are determined.

During the first step of the iteration the well is treated as a prescribed head boundary, whereas during the second step both segments are treated as prescribed flux boundaries in the manner discussed above for the seepage face. One obtains Q_A^{j+1} simply by adding all the values of Q_n^{j+1} along the well bore.

The procedure for the first iteration of the initial time step differs slightly from the procedure described earlier. At $t = 0$ no gradients have yet been generated in the aquifer, and therefore $Q_A^k = 0$ when $k = 0$. Our present experience indicates that to use (38) it is helpful to set the initial value of Q_A^j at 1/2 Q'^P and at the same time to set the initial value of ΔL^j at the distance between the top two nodes on the well. Equation 38 is then used to calculate the initial value of Δt which is adopted for the first time step.

APPLICATION TO VARIOUS PROBLEMS

All the features described above have been incorporated into a computer program that is able to handle both plane and axisymmetric flows. The program uses a one-step iterative procedure when there is no seepage face and a two-step procedure when such a boundary is present. To demonstrate the versatility of the program, we present below solutions for several different kinds of problems. In all the problems $S_s = 0$.

1. We first consider the problem of axisymmetric flow to a well of large diameter that is being pumped at a constant rate. *Taylor and Luthin* [1969] have developed a finite difference method of solving transient flow to a well with a free surface that also considers flow in the unsaturated region. They have published only their final, steady state solution for a special case, and we have recently solved the same problem by finite element methods [*Neuman and Witherspoon,* 1970*a*].

In our finite element approach, we used consistent units and arbitrarily adopted a permeability of $K = 0.1$ and a storage of $S_y = 0.3$. We determined that for the geometry used by Luthin and Taylor, we had a steady state flow rate of 39.045, which was used to obtain a nonsteady solution, storage in the well being considered as shown in Figure 5. Figure 5 shows the positions of the water level in the well and the water table in the aquifer at various values of time.

The unconditional stability of our method is evident in Figure 5 because we were able to increase Δt logarithmically, and the same steady state was reached after only 10 time steps. The iterations required to obtain a solution for a

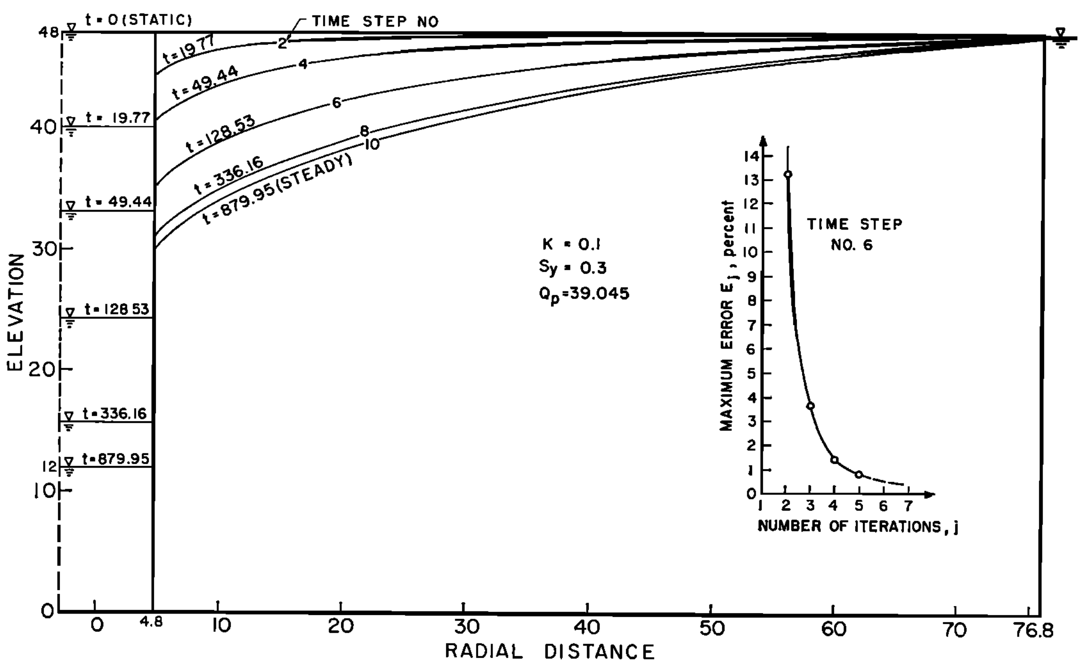

Fig. 5. Nonsteady solution for flow through an unconfined aquifer to a well with storage.

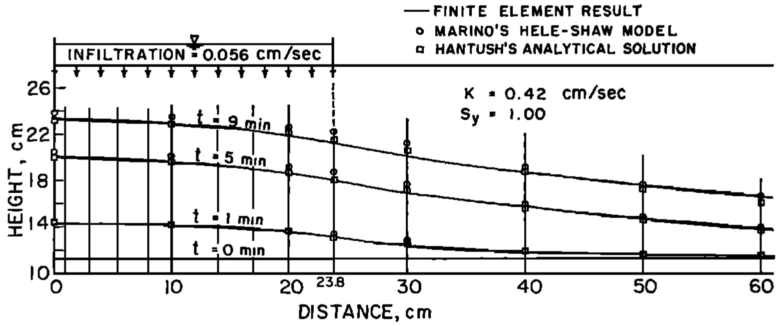

Fig. 6. Nonsteady solution for buildup of a mound due to infiltration.

typical time step are also included in figure 5. Note that after five two-step iterations, the maximum error E_f was less than 1%.

2. We next consider the problem of the growth of a groundwater ridge in response to uniform recharge from a strip of finite width and infinite length (Figure 6). An analytical solution to this problem has been developed by *Hantush* [1963] and has been verified experimentally with a Hele-Shaw model by *Marino* [1967]. One can see from Figure 6 that our finite element solution compares favorably with both the analytical and experimental results. A set of lines is included to indicate the vertical direction along which shifting took place in obtaining a solution. Since no seepage face is involved, a typical time step required three one-step iterations to reduce E_f from more than 20% to less than 0.1%.

It should be mentioned that since our program is capable of handling axisymmetric flow, one could easily investigate the problem of the growth and decay of groundwater mounds under circular or nearly circular spreading basins.

3. Finally we consider the more difficult problem of how the free surface within a dam adjusts itself when the reservoir level is suddenly lowered. We shall start with the steady state solution that we previously obtained for a homogeneous dam with a toe drain [*Neuman and Witherspoon,* 1970*a,* Figure 7]. At time $t = 0$ the water level in the reservoir drops instantaneously from an elevation of 100 to an elevation of 40. Figure 7 shows how our method gives the transient position of the free surface as it readjusts to a new steady state position. Note that the directions of shifting are varied so as to be approximately orthogonal to the free surface.

In obtaining the present solution we have arbitrarily restricted the number of iterations for each time step to three. For this reason the results are probably less accurate than those presented in the previous examples.

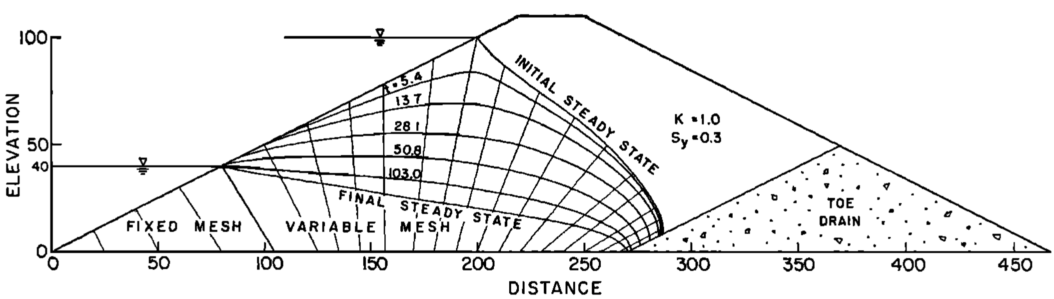

Fig. 7. Nonsteady solution for seepage through a dam showing effect of a sudden lowering in reservoir level.

NOTATION

- h, hydraulic head, L;
- H, hydraulic head on prescribed head boundary, L;
- I, net vertical specific rate of infiltration at free surface, L/T;
- K_{ij}, permeability tensor, L/T;
- n_i, unit outer normal vector, L;
- Q, discharge, L^3/T;
- r, radial coordinate, L;
- S_s, elastic specific storage, L^{-1};
- S_y, specific yield;
- t, time, T;
- V, specific flux on prescribed flux boundary, L/T;
- x_i, coordinate vector, L;
- z, vertical coordinate, L;
- ξ, elevation of free surface above horizontal datum plane from which h is measured, L.

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