Linear Algebra

Vectors ⇒

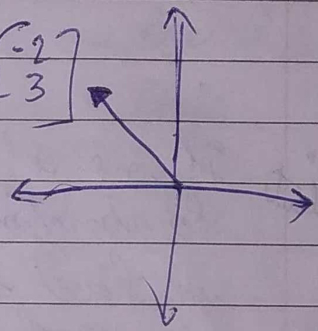
Physics POV - magnitude and direction
CS POV - lists of numbers
like price & area of house $\begin{bmatrix} 60000\$ \\ 1200\,ft \end{bmatrix}$

in 2D

represented by location of tip in coordinate space ~~sol~~ with tail always at origin

where -2 is x coordinate
3 is y

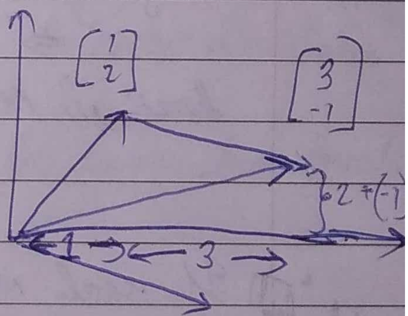
$\begin{array}{c} x \\ y \end{array}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$



and in case of 3D $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ represents a vector

add^n of vectors

we move 2nd vector's tail and put it at 1st vector's head



so their add^n is

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+3 \\ 2-1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}$$

Multiplication of vectors

multiplying a vector by x means new vector's length is x times that of original length.

⇒ The process of stretching or squishing or reversing a vector is called scaling and x is scalar.

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$$2 \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

- vector is also represented in the form $x\hat{i} + y\hat{j}$ and $\hat{i}$ and $\hat{j}$ are basis vectors

- The set of all possible vectors that can be obtained with a linear combination of a given pair of vectors is called span of vectors

$a\vec{v} + b\vec{w}$ where $\vec{v}$ and $\vec{w}$ are pair of vectors.

- dependent If one of the vectors can be expressed as a linear combination of the others or we can say that we have multiple vectors where we can remove one without reducing the span then they are linearly dependent.

$\Rightarrow \vec{u} = a\vec{v} + b\vec{w}$

here $\vec{u}$ ~~come~~ is linear to $a\vec{v} + b\vec{w}$ (not affecting the span)

- independent If each vector adds another dimension to the span they are said to be linearly ~~de~~ independent.

$\vec{u} \neq a\vec{v} + b\vec{w}$

<u>Technical defn of basis</u>

The basis of a vector space is a set of linearly independent vectors that span the full space.

Linear Transformation

It has 2 props-

i) All lines must remain lines, not curved
ii) Origin must remain fixed in posⁿ

Which forces the grid line to remain parallel and evenly spaced

- We can deduce where the vectors land, just by finding where $\hat{i}$ and $\hat{j}$ lands

if $\hat{i} \rightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $\hat{j} \rightarrow \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ where $\hat{i}$ & $\hat{j}$ lands

then $\begin{bmatrix} x \\ y \end{bmatrix}$ lands on $x\begin{bmatrix} 1 \\ -2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} x + 3y \\ -2x + 0y \end{bmatrix}$ General form ~~form~~ of any vector in that particular transformaⁿ

$\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$ in this case

Generally $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is used ~~to~~ for linear transformation ~~where~~

$\hat{i}$ lands $\rightarrow$ $\leftarrow$ $\hat{j}$ lands

~~which~~ this will be used to ~~tell~~ deduce transformation of any vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in the plane

$$\Rightarrow x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

in other words vector $\begin{bmatrix} x \\ y \end{bmatrix}$ lands on x times transformed version of $\hat{i}$ + y times trans version of $\hat{j}$

$90° \# \rightarrow$

Matrix multiplication as composition

composition is the ~~tran~~ rotation of plane by 90° then applying shear

and doing this in single step and not successively ~~st~~

- and in doing so $\hat{i}$ lands on 1, 1 and $\hat{j}$ on -1, 0 and it's 2×2 matrix is known as ~~shear~~ composition matrix.

$$\therefore \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{then shear}} \left(\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{first rotation}} \begin{bmatrix} x \\ y \end{bmatrix} \right) = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{composition}} \begin{bmatrix} x \\ y \end{bmatrix}$$

This new matrix captures the overall effect of rotation + shear

* ~~So~~ when we apply 2 transformations to a plane then instead of using 2 diff matrices which gives location of vector $\hat{i}$ & $\hat{j}$, we use composition matrix which combines effect of both transformation

It is a result of multiplication of the two matrices of $\hat{i}$ & $\hat{j}$

eg if first transformation gives $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ as location of $\hat{i}$ & $\hat{j}$

and 2^{nd} ~~one~~ trans $\begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}$ " "

then composi^n matrix is

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$$

Three dimensional Linear matrix :

same as 2D , but in 3×3 matrix

example $\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ are locations where $\hat{i}$ $\hat{j}$ $\hat{k}$ lands

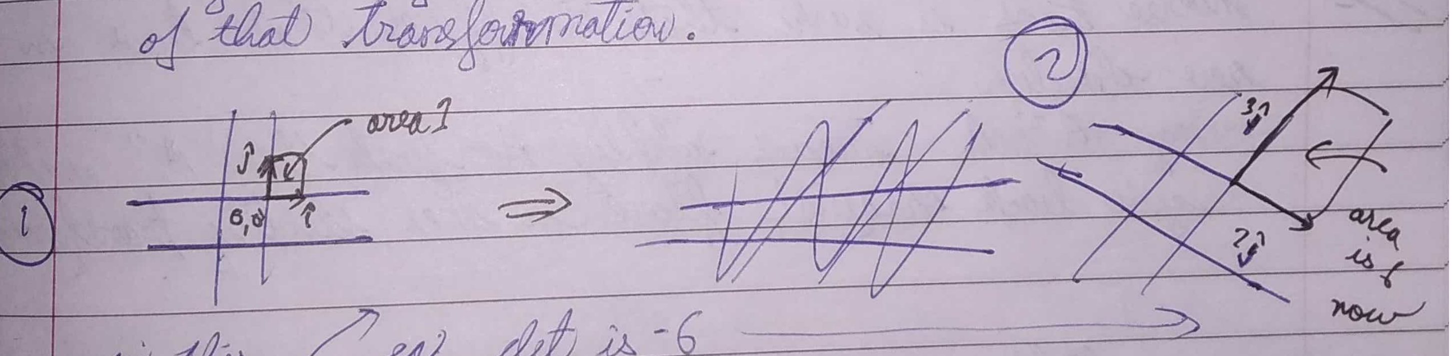
then $\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} + y\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + z\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$

gives location of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Determinant :

when we apply transformation then area enclosed by $\hat{i}$ & $\hat{j}$ changes to something else

- The factor by which the area changes is the det of that transformation.



in this ↗ eg det is -6

and if orientation of space is inverted

i.e. in (1) $\hat{j}$ is to left of $\hat{i}$ and in (2) $\hat{i}$ is left of $\hat{j}$

determinant is -ve.

What about negative det in 3D

Use Right hand rule , middle fing points in $\hat{j}$ index in $\hat{i}$ and thumb in $\hat{k}$ and if after transf it is possible to do so using left hand only , then orientation is flipped

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Linear system of equations

eg
$$\begin{aligned}2x + 5y + 3z &= -3\\ 4x + 0y + 8z &= 0\\ 1x + 3y + 0z &= 2\end{aligned} \longrightarrow \underbrace{\begin{bmatrix}2 & 5 & 3\\ 4 & 0 & 8\\ 1 & 3 & 0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\ y\\ z\end{bmatrix}}_{\vec{X}} = \underbrace{\begin{bmatrix}-3\\ 0\\ 2\end{bmatrix}}_{\vec{V}}$$

can be written as $A\vec{x} = \vec{v}$ — (1)

Vector $\vec{x}$ is such that after applying $\underset{A}{\text{trans}}$ lands on $\vec{v}$

Diff cases with det

i) If det $\neq 0$, means space does not squishes into zero ~~as~~ area

means there is only one vector that lands on $\vec{v}$

Inverse -

Inverse trans is such that it applies that trans in rev direction

eg $A^{-1}.A$ means apply A first then A^{-1} will give back original plane or does identity transformation

on multiplying (1) by A^{-1}

$\vec{x} = A^{-1}\vec{v}$

Rank means number of dimensions in the output of transformation

Rank of matrix

rank = 1: On applying transformation, if all vectors lie on a line ~~this~~.
The trans^n is said to have rank 1

rank = 2: if all vectors lie on a plane, rank = 2

both rank 1 & 2 refer to D = 0

if det ≠ 0, means trans is a plane, so rank = 3

Column space: The set of all possible outputs of $A\vec{v}$ is the column space of the matrix

any trans^n

the column of matrix tells us where basis vec lands and the span of columns give all possible locations of basis vectors.

def of rank: It is the no. of dimensions in column space

Null space

If a 2d trans ~~converts~~ squishes space onto a line then there is line that squishes onto the origin.
or if a 3D transf squishes onto a plane then there is a full plane that squishes on the origin

- This set of vectors that lands on the origin is called null space of kernel.
- It is the space of all vectors that become null.

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- In terms of linear system of equations, when v is a zero vector
  Then null space gives ~~gy~~ all possible soln to the eq

Conclusion
i) Column space lets us understand when a soln exists
ii) Null space tells us what set of all poss soln looks like

For Non square matrices
if $\hat{i}$ & $\hat{j}$ after transfor lands on $\begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}$ slicing in 3D plane

here the column space of the matrix, the place where all the vectors lands is a 2D plane but matrix is full rank since no. of dimension in output space = no. of dimension in input space

so a ~~k~~ 3x2 matrix $\begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}$ tells that ~~k~~ 2 basis vectors land on 3D space coordinates

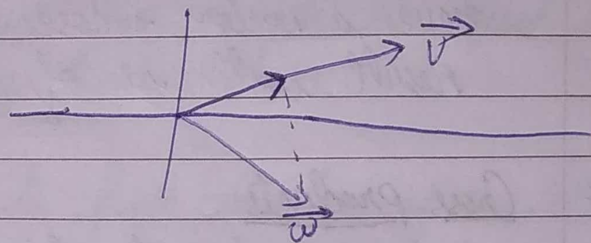
and 2x3 $\begin{bmatrix} 2 & 0 & 1 \\ 2 & -1 & 1 \end{bmatrix}$ tells that 3 basis vectors lands on 2D space

Dot Products

in matrix form $\begin{bmatrix}4\\1\end{bmatrix} \cdot \begin{bmatrix}2\\-1\end{bmatrix} = 4\times 2 + (1\times -1)$

in vector form:

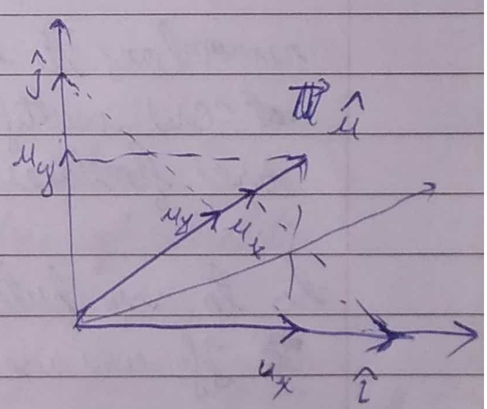
eg $\underset{\vec{v}}{\begin{bmatrix}4\\1\end{bmatrix}} \cdot \underset{\vec{w}}{\begin{bmatrix}2\\-1\end{bmatrix}}$ = (Length of projected $\vec{w}$) (Length of v)



- dot product will be negative if $\vec{v}$ & $\vec{w}$ face in opp direc
$\vec{v}.\vec{w} = 0$, if $\vec{v} \perp \vec{w}$

Note:

- use a 2D vector on a line
- here projecting $\hat{i}$ on $\hat{u}$ and vice versa is symmetric so whatever $\hat{i}$ lands on, is where $\hat{u}$ lands on



So projection of $\hat{i}$ on $\hat{u}$ is same a
" of $\hat{u}$ on $\hat{i}$

So for a 2D vector on 1D line
the matrix of where $\hat{i}$ and $\hat{j}$ land is
$[\hat{i} \quad \hat{j}]$
which is also $[u_x \quad u_y]$
So we can use $[u_x \quad u_y]$ to find pos^n of random vectors in that space.

$$[u_x \quad u_y]\begin{bmatrix}x\\y\end{bmatrix} = u_x . x + u_y . y$$

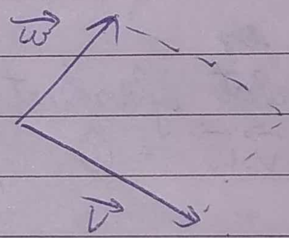
So $\begin{bmatrix} u_x & u_y \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = u_x \cdot x + u_y \cdot y$ — (I)

is same as dot product of em $\begin{bmatrix} u_x \\ u_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = u_x \cdot x + u_y \cdot y$ — (II)

"This is duality"
The transformations of same vector in space to 1D will have a vector whose ~~dot product~~ is ~~same as that of~~ result will as of (I) & (II)

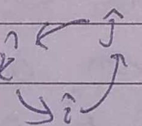
* **Cross products**

gives area of certain figure
$\vec{v} \times \vec{w}$ = Area of $\parallel^{m}$



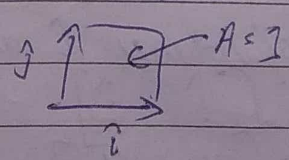
here $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

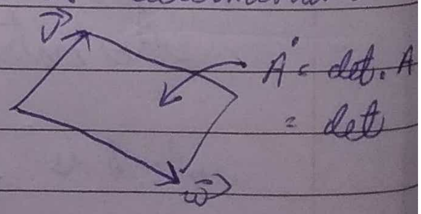
remember it like
~~a~~ CCW multiplications are positive



So to compute this area, we use determinant
~~To~~ If we use $\vec{v}$ at $(3,1)$ and $\vec{w}$ at $(2,-1)$

Then assume $\vec{v}$ & $\vec{w}$ as posⁿ where $\hat{i}$ & $\hat{j}$ land upon transformation & since area of $\hat{i} \times \hat{j} = 1$ then after transformation their ~~determnat~~ determinant will give area

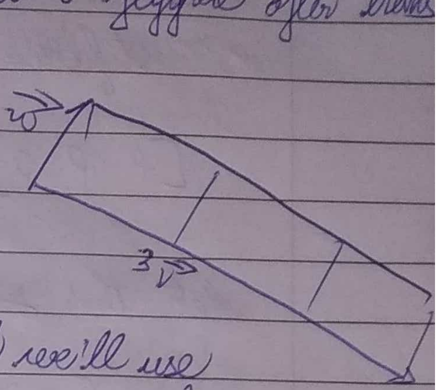




$\therefore$ Area $= \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix}$

$\therefore \vec{v} \times \vec{w}$ is ~~+~~ -ve as its orientation flipped after trans

also scaling a vector $\vec{v}$ by 3
means scaling $\|^{m}$ area by 3
$\therefore (3\vec{v}) \times \vec{w} = 3(\vec{v} \times \vec{w})$



* But this is not the cross product that we'll use
Cross product gives us another vector normal to both the vectors and it's length is equal to area of $\|^{m}$ enclosed by two vectors

Use right ~~hand~~ thumb rule to find direction of new vector

- Mathematically cross product of $\vec{v}(v_1, v_2, v_3)$ and $\vec{w}(w_1, w_2, w_3)$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{vmatrix} \hat{i} & v_1 & w_1 \\ \hat{j} & v_2 & w_2 \\ \hat{k} & v_3 & w_3 \end{vmatrix}$$

but why do we use basis vectors here
This ~~is~~ is explained with the idea of duality

How this is done is -
1) Define a 3D → 1D linear trans in terms of $\vec{v}$ and $\vec{w}$
2) Find its dual vector
3) Show that this dual is $\vec{v} \times \vec{w}$

let there be a variable vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with $\vec{v}$ & $\vec{w}$ to be fixed
so it's like a func^n

$$f\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det \begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix} \quad -(1)$$

and this func^n can be defined as a 1×3 matrix multiplied with $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to give 1×1 matrix

$$[?\ \ ?\ \ ?] \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det \begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}$$

let this ~~row~~ unknown matrix be P and by duality we get

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{vmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{vmatrix} \rightarrow (II)$$

This gives

$$P_1 x + P_2 y + P_3 z = x(v_2 w_3 - v_3 w_2) + y(v_1 w_3 - v_3 w_1) + z(v_1 w_2 - v_2 w_1)$$

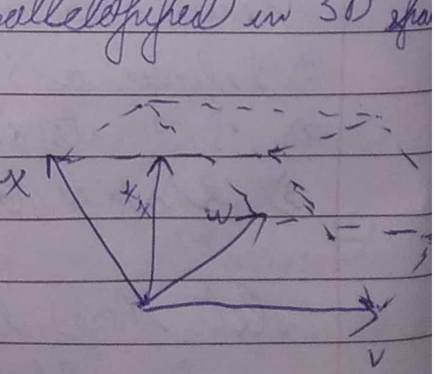
comparing LHS and RHS, we can see that

$$P_1 = v_2 w_3 - v_3 w_2 \qquad P_2 = v_1 w_3 - v_3 w_1 \qquad P_3 = v_1 w_2 - v_2 w_1$$

So we can see that it is similar to just plugging in $\hat{i}\ \hat{j}\ \hat{k}$ in place of $x\ y\ z$

also eq (1) gives volume of parallelopiped in 3D space

~~one~~ We can think of it as taking projection of x on $\perp^r$ to both v & w (so it kind of becomes the height)



then volume = $\underbrace{\text{length of projec}^n}_{P} \times \underbrace{(v \times w)}_{\text{area of } \parallel^m \text{ on base}}$

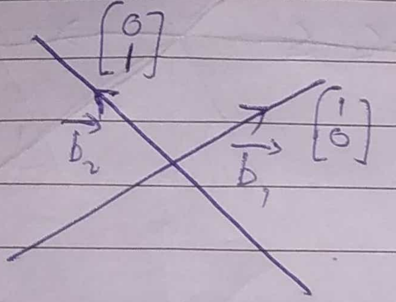
It is same as if we assume to be $\perp r$ to $\vec{v}$ & $\vec{w}$.
where P is projec^n of $x\ y\ z$ as per (II)

So that is why we use basis vectors in 1st column as both these vectors give same p on projection

Change of basis

~~If we take~~ a

eg :-



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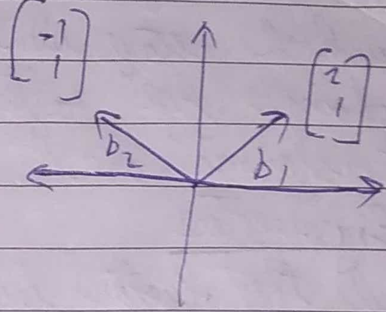


In original plane 2

How to translate b/w coordinate systems

a point $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ in plane 1 is in the form $-1\vec{b_1} + 2\vec{b_2}$

So to find location in plane 1

multiply coordinates with location of new basis vector as per plane 2

$$\Rightarrow \quad -1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$ ← are coordinates in plane 2

So here let, $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ is change of basis matrix

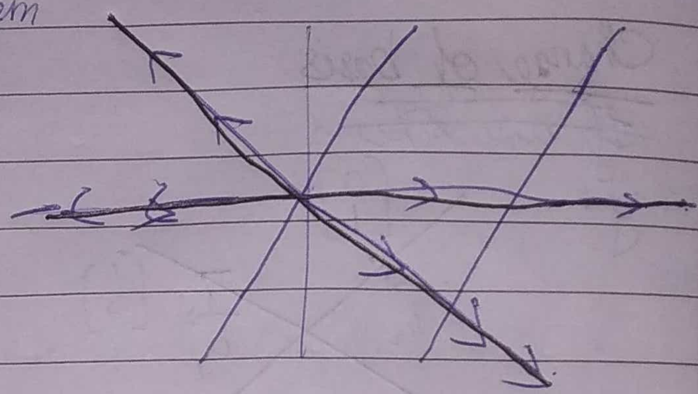
$$A \underbrace{\begin{bmatrix} x_i \\ y_j \end{bmatrix}}_{\text{new grid}} = \underbrace{\begin{bmatrix} x_o \\ y_o \end{bmatrix}}_{\text{our grid}}$$

(new ~~basis~~ grid, our ~~basis~~ grid)

and A^{-1} gives coordinates of new grid using our grid

Eigenvalues & Eigenvectors

When a transformation is applied to a plane
for example : shear transformⁿ



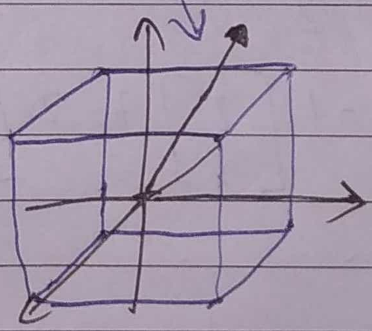
- The line of vectors which stays in its original posⁿ after transformaⁿ are **Eigenvectors**

and the factor by which these vectors are stretched (squished) are eigenvalues

-ve Eigenvalues means that the vector has been flipped

Why eigenvectors?

They help tell us the axis of rotation if u assume a 3D scenario



Result

$A\vec{v} = \lambda\vec{v}$ where A is transf matrix means the transforⁿ that u app

$\vec{v}$ is Eigenvector λ is Eigenvalue

$A\vec{v}$ is Matrix - vector multiplication

$\lambda\vec{v}$ " Scalar - " "

~~To~~ m

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To make RHS matrix vector multiplication as well we can assume $\lambda = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \lambda I$

so now

$$A\vec{v} = (\lambda I)\vec{v}$$

$$\therefore (A - \lambda I)\vec{v} = 0 \quad \text{where } \vec{v} \text{ is non zero vector}$$

for this to be true, $|(A - \lambda I)| = 0$
which ultimately depends on value of λ.

eg: if $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$
then $|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) = 0$

$\therefore$ for $\lambda = 3, 2$ the relation is zero

Note: Transformation matrix A is the matrix which contains the final location of $\hat{i}$ and $\hat{j}$ after transform^n

→ What if both basis vectors are eigen vectors
eg: when flipped plane is, the basis vectors stay in place

then the transformation matrix in such a case has the eigen values (which are the non zero values in diagonals

This case is known as eigenbasis//

If we want to compute the for eg 100^{th} then convert the transformation matrix to eigenbasis, where ~~ass~~ take any two eigen vectors and make them the basis of new matrix.

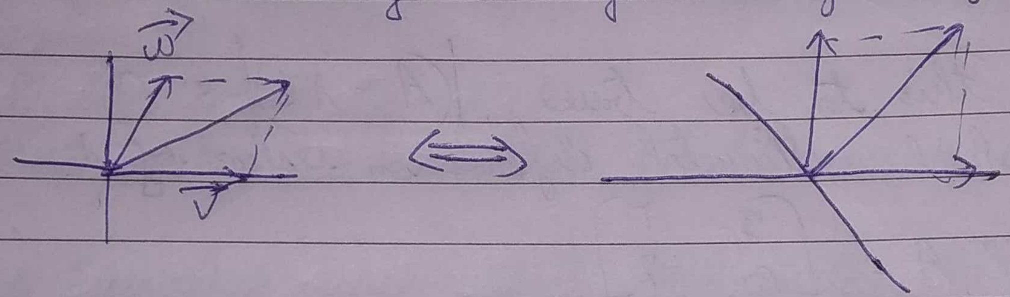
so if new ~~eigenmatr~~ transfoⁿ matrix is $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$
then calculating 100 th power of such a transⁿ is ~~so~~ easy //

Formal defⁿ of linearity

- Additivity :

$$L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w})$$

⇒ This result stays true for both pre & post transforma



- Scaling: $L(c\vec{v}) = c\,L(\vec{v})$

* Derivative $\left(\frac{d}{dx}\right)$ is linear

as $\frac{d}{dx}(ax^3 + bx) = \frac{d}{dx}(ax^3) + \frac{d}{dx}(bx)$ [additive]

and $\frac{d}{dx}(3x) = 3\frac{d}{dx}(x)$ [Scaling]

Polynomials are expressed in the form of matrix as ∞ matrix
with coeff of increasing power of x in sequence

eg $3x^2 + 5x + 8 = \begin{bmatrix} 8 \\ 5 \\ 3 \\ \vdots \end{bmatrix}$

$4x^{10} - 5x^7 + 2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -5 \\ 0 \\ 0 \\ 4 \end{bmatrix}$

~~func in another way act as vectors~~ only
Func is kind of like a vector

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and $\frac{d}{dx}$ is represented as
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Eg So $\frac{d}{dx}(1x^3 + 5x^2 + 4x + 5)$ in matrix form

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 5 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 4\cdot 1 \\ 2\cdot 5 \\ 3\cdot 1 \\ 0 \\ \vdots \end{bmatrix} = 3x^2 + 10x + 4$$

This is possible only cuz derivative is LINEAR

LA concepts	Alt names when applied to functions
Linear transformations	Linear operators
Dot products	Inner products
Eigen vectors	Eigen functions

What are vector spaces ?

- The vector ish things like arrows, ~~number spaces~~ lists of numbers or functions are called vector spaces
- and the whole concept of linear algebra applies to these vector spaces.
- The rules that vector addition & scaling have to abide by are called axioms

In modern theory of linear algebra there are 8 axioms

1) $\vec{u} + (\bar{v} + \bar{w}) = (\bar{u} + \bar{v}) + \bar{w}$
2) $\bar{v} + \bar{w} = \bar{w} + \bar{v}$
3) There is a vector 0 such that $0 + \bar{v} = \bar{v}$ ~~for~~ $\forall\ \bar{v}$
4) For every vector $\bar{v}$ there is a vector $-\bar{v}$ so that $\bar{v} + (-\bar{v}) = 0$

5) $a(b\bar{v}) = (ab)\bar{v}$
6) $1\bar{v} = \bar{v}$
7) $a(\bar{v} + \bar{w}) = a\bar{v} + a\bar{w}$
8) $(a+b)\bar{v} = a\bar{v} + b\bar{v}$

* If someone wants to create any new vector space ~~then their vectors space must above be true~~ then these axioms must be verified with those vectors b4 applying the results of linear algebra.

* Vectors can be anything as long as the scaling & adding vectors follow these axioms //