Linear Algebra

1) Vectors:

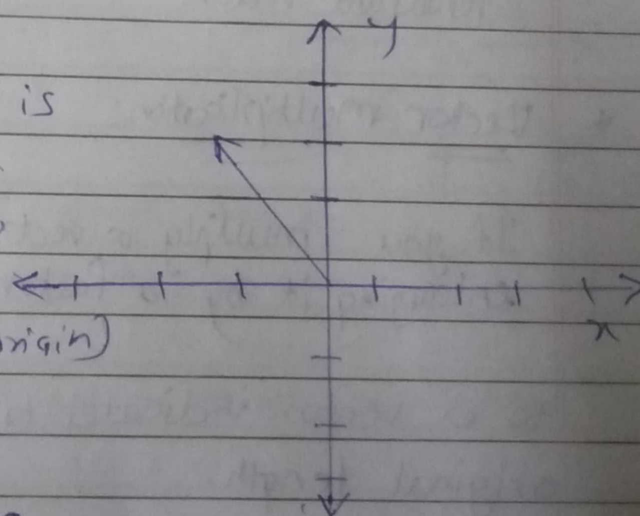
Physics perspective: Vectors are arrows pointing in space defined by its length and direction.

CS perspective: Vectors are ordered list of numbers.

Mathematician perspective: Seeks to generalize both these views. A vector can be anything where there's a sensible notion of adding two vectors and multiply a vector by a number.

In LA, vectors are rooted at origin unlike in physics where they can be rooted anywhere.

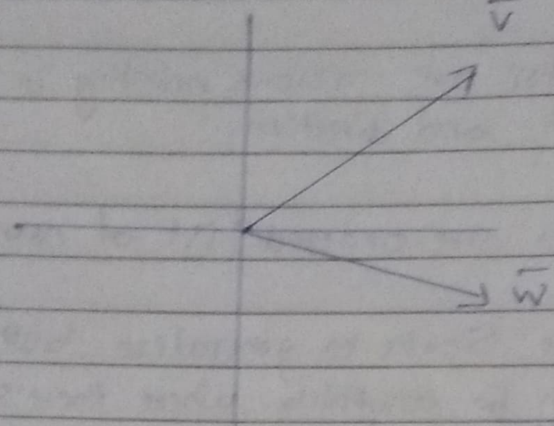
Coordinates of a vector is a pair of numbers that give instructions for how to get from the tail of that vector (at the origin) to its head.

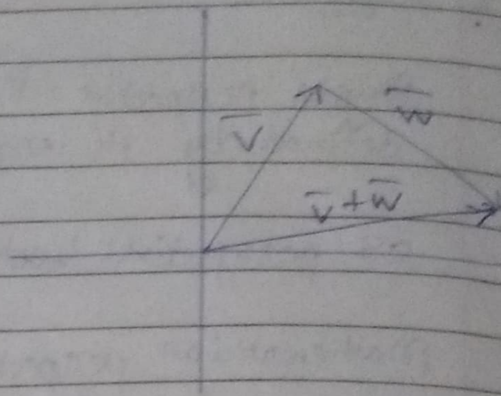


eg: $\begin{bmatrix} a \\ b \end{bmatrix}$ $a \to$ x coordinate, $b \to$ y coordinate

$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $a \to x$, $b \to y$, $c \to$ z coordinate

Every triplet of numbers gives one unique vector in space and every vector in space gives exactly one tr of numbers





Vector addition.

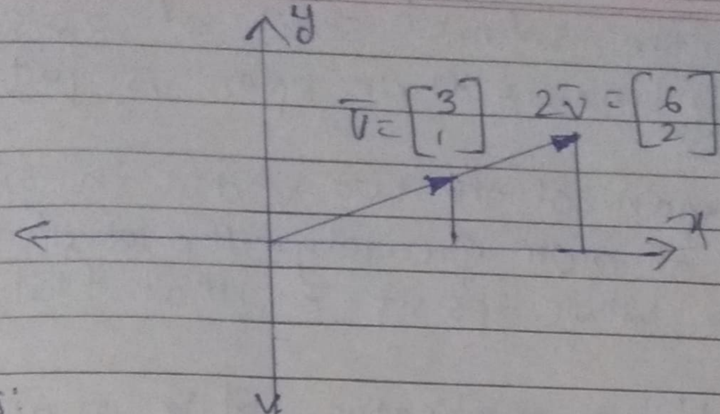
Take the tail of second vector to the head of the first vector. The vector from the tail of first vector to the head of the second vector is the resultant vector.

* Vector multiplication:

If you multiply a vector by 2, we are basically enlarging it by a factor of 2 while $\frac{1}{3}$ multiplied to a vector indicates to squish down to $\frac{1}{3}$rd of its original length.

If a vector is multiplied by a (-ve) number, then it is flipped first and then stretched out or squished.

Scaling: Process of stretching or squishing or sometimes flipping the vector

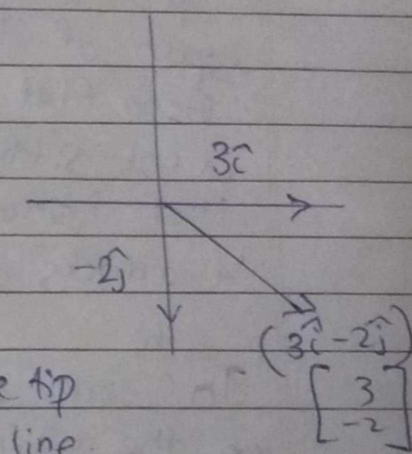


* **Basis Vectors:**
$\hat{i}$ and $\hat{j}$ are the basis vectors of the xy coordinate system. (Set of linearly independent vectors that span the full space)
Think of coordinates as scalars i.e. ~~the~~ number which scales the vector.

* Linear Combination of $\bar{v}$ and $\bar{u}$

$a\bar{v} + b\bar{w}$

Scalars



Why linear?
When you fix one of the vectors, the tip of the other vector draws a straight line.

* **Span:** The span of $\bar{v}$ and $\bar{w}$ is the set of all their linear combinations.

$a\bar{v} + b\bar{w}$

Let a and b vary over all real numbers.
All vectors in 2D space but when they line up their span is all vectors whose ~~to~~ tip sit on a certain line.

Think of individual vectors as arrows while on the other hand think of sets of vectors as points.

The span of most pairs of vectors end up being the entire infinite sheet of 2D space but if they line up, their span is just a line.

The span of the two vectors in 3D is the flat sheet or more precisely the set of all possible vectors whose tips sit on that flat sheet.

3D: Linear combination of $\bar{v}$, $\bar{w}$ and $\bar{u}$:

$a\bar{v} + b\bar{w} + c\bar{u}$

For span, let these constants vary.

If the third vector happens to be sitting on the span of the first two, then the span remains to be a flat sheet, it doesn't change. But if a vector is not sitting on the span of the first two vectors then since it is pointing in a separate direction ~~then~~ it unlocks access to every possible 3D vector.

In the case where the third vector was sitting on the span of the first two vectors or the case where the two vectors line up, at least one of these vectors might be redundant i.e. not adding anything to the span.

Linearly dependent: $\bar{u} = a\bar{v} + b\bar{w}$

If a vector adds another dimension to the span then it is said to be linearly independent.

$\bar{u} \neq a\bar{v} + b\bar{w}$

for all values of a and b

* ② Linear $\underbrace{\text{transformation}}_{\text{function}}$:

A tranformation is linear if it has two properties

-> All lines must remain lines without getting curved.
-> Origin remains fixed in space.

Grid lines remains parallel and evenly spaced

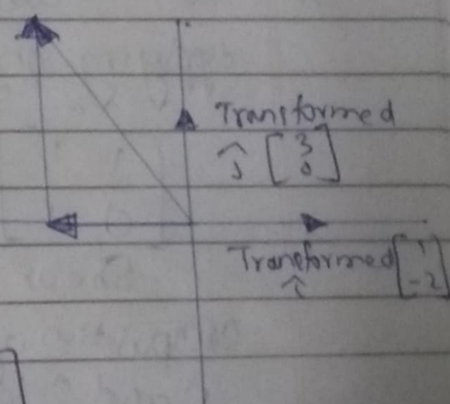
Eg: $\vec{v} = -1\hat{i} + 2\hat{j}$

Transformed $\vec{v} = -1(\text{Tranformed } \hat{i}) + 2(\text{Transformed } \hat{j})$

$$= -1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1(1) + 2(3) \\ -1(-2) + 2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$\hat{i} \to \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $\hat{j} \to \begin{bmatrix} 3 \\ 0 \end{bmatrix}$



$$\begin{bmatrix} x \\ y \end{bmatrix} \to x\begin{bmatrix} 1 \\ -2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1x + 3y \\ -2x + 0y \end{bmatrix}$$

2x2 matrix: $\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}$ $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$

Where $\hat{i}$ lands Where $\hat{j}$ lands

$$5\begin{bmatrix} 3 \\ -2 \end{bmatrix} + 7\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{array}{cc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \begin{bmatrix} x \\ y \end{bmatrix} \end{array}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \text{First basis vector} & \text{Second basis vector} \end{array}$$

$$x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

Shear transformation:

$\hat{i}$ remains fixed while $\hat{j}$ moves over the coordinate $(1,1)$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Multiplying a matrix to a vector means computationally to transform the given vector.

* Matrix multiplication: It has a geometric meaning of applying one transformation then another.

$\xleftarrow{\text{Read Right to Left}}$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{Shear}}\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Rotation}} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Composition}}$$

Composition of a rotation and a shear. Record where $\hat{i}$ and $\hat{j}$ land.

Consider an example:

first $\hat{j}$ lands here

$$\underbrace{\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}}_{M_2}\underbrace{\begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}}_{M_1} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

first $\hat{i}$ lands here

$\hat{i}$ lands after second transformation:

$$\therefore \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\hat{j}$ lands after second transformation:

$$\therefore \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix} = -2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$$

$$M_1 M_2 \neq M_2 M_1$$

Associativity: $(AB)C = A(BC)$

(3) Three dimensional transformation:

$$\underbrace{\begin{bmatrix} 0 & -2 & 2 \\ 5 & 1 & 5 \\ 1 & 4 & -1 \end{bmatrix}}_{\text{second transformation}} \underbrace{\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}}_{\text{first transformation}}$$

Where $\hat{i}$ lands after first transformation:

$$\begin{bmatrix} 0 & -2 & 2 \\ 5 & 1 & 5 \\ 1 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$$

$$= 0 + \begin{bmatrix} -6 \\ 3 \\ 12 \end{bmatrix} + \begin{bmatrix} \cancel{8}12 \\ 30 \\ -6 \end{bmatrix} = \begin{bmatrix} 6 \\ 33 \\ 6 \end{bmatrix}$$

Where $\hat{j}$ lands after first transformation:

$$\begin{bmatrix} 0 & -2 & 2 \\ 5 & 1 & 5 \\ 1 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

$$= 1\begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} + 4\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} + 7\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} + \begin{bmatrix} -8 \\ 4 \\ 16 \end{bmatrix} + \begin{bmatrix} 14 \\ 35 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 44 \\ 10 \end{bmatrix}$$

Where $\hat{k}$ lands after first transformation:

$$\begin{bmatrix} 0 & -2 & 2 \\ 5 & 1 & 5 \\ 1 & 4 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$= 2\begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} + 5\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} + 8\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 10 \\ 2 \end{bmatrix} + \begin{bmatrix} -10 \\ 5 \\ 20 \end{bmatrix} + \begin{bmatrix} 16 \\ 40 \\ -8 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 55 \\ 14 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 0 & -2 & 2 \\ 5 & 1 & 5 \\ 1 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 6 \\ 33 & 44 & 55 \\ 6 & 10 & 14 \end{bmatrix}$$

(4) **Determinant**: Scaling factor, the factor by which a linear tranformation changes any area is called the determinant of that transformation.

Determinant of a tranformation would be 3 if that tranformation increases the area of region by a factor of 3.

Determinant of a 2D tranformation would be 0 if it squishes all of space onto a line or even ~~no~~ onto a single point since then the area of any region would become 0.

~~the ha~~ Whenever the orientation of space ~~get~~ is reversed, the determinant will be (-ve). Absolute value of determinant tells the factor by which the areas have been scaled.

Determinant of 3D tranformation tells how much volume gets scaled.

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Gaussian elimination
Row echelon form

* Usefulness of matrices

(1) Solving linear equations

$$\begin{matrix} 2x + 5y + 3z = -3 \\ 4x + 0y + 8z = 0 \\ 1x + 3y + 0z = 2 \end{matrix} \longrightarrow \underbrace{\begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\bar{x}} = \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}}_{\bar{V}}$$

$\boxed{A\bar{x} = \bar{V}}$ $\Rightarrow$ Act of ~~multiplying~~ transforming vector $\bar{x}$ so that it gets aligned with $\bar{V}$

A ↓ Transformation

* Inverse matrix:

1) If A was a counterclockwise rotation by 90° then inverse of A would be a clockwise rotation by 90°.

2) If A was a rightward shear that pushes $\hat{j}$ one unit to the right then inverse of A would be a leftward shear that pushes $\hat{j}$ one unit to the left.

$A^{-1}A = I$ (The transformation that does nothing is an identity transformation)

$A\bar{x} = A^{-1}\bar{V}$ $\Rightarrow$ Performing the transformation in reverse and following $\bar{V}$.

~~As~~ # of equations = # of unknowns $\Rightarrow$ Unique solution.

* But when $\det(A) = 0$ i.e. the transformation squishes space into a smaller dimension, there is no inverse

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* This is because if we reverse the transformation i.e. from a squished state to initial state it would require transforming a vector into a whole line full of vectors which a function can't do since it can take a single input ~~and~~ and give ~~out~~ a single output

* det (A) = 0 : for 3D space the transformation squishes the space into a plane, line or in an extreme case to a single point

* Even if det (A) = 0 $\Rightarrow$ Solution exist $\Rightarrow$ Vector ($\vec{v}$) should lie on that squished plane/line.

* Rank: Number of dimensions in the output

If the transformation squishes the space to a one dimensional line then the rank is one.
If the transformation squishes the space to a ~~two~~ two dimensional plane then the rank is two (i.e. all the vectors lie on a plane) (for 3D)

2x2 matrix:
Rank 2 is the best case when the two basis vectors ~~fill~~ spans all of 2D space. [det (A) ≠ 0]
Rank 1 is the case when the two basis vectors squishes into a line [det (A) = 0]

3x3 matrix:
Rank 2 is the case when the three basis vectors ~~span~~ lies on a plane. [det (A) ≠ 0]
Rank 3 $\Rightarrow$ det (A) ≠ 0)

* Set of all possible outputs for a matrix whether it's a line, a plane, 3D space, etc is called the column space of the matrix

In other words, column space ⇒ Span of columns of the matrix.

Rank: # of dimensions in the column space.

__Null space__: Set of vectors that land on the origin is the null space or kernel. It is the space of all vectors that become null i.e. they land on the zero vector

In case of linear system of equations when v happens to a zero vector the null space gives you all the possible solution to the equations..

* Non square matrix:

3x2 matrix : $\begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 9 \end{bmatrix}$

mapping 2 dimensions to 3 dimensions
- Two columns indicate that the input space has two basis vectors.
- Three rows indicate that the landing spots for each of these basis vectors is described with three separate coordinates.

3 basis vectors

$$\overbrace{\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}}^{\text{3 basis vectors}} \Big\} \text{ 2 coordinates for each landing spots}$$

(2x3)

3D → 2D

Dot Product:

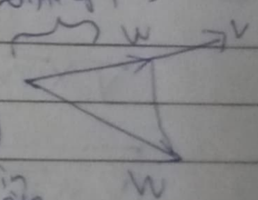
Two vectors of the same dimension:

$$\begin{bmatrix} 6 \\ 2 \\ 8 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 8 \\ 5 \\ 3 \end{bmatrix} = 6\cdot1 + 2\cdot8 + 8\cdot5 + 3\cdot3$$

Imagine projecting w onto the line that passes through the origin and tip of v.

pointing in the same direction

length of projected $\bar{w}$



$\bar{v}\cdot\bar{w}$ = (Length of projected $\bar{w}$)(Length of $\bar{v}$)

= −(Length of projected $\bar{w}$)(Length of $\bar{v}$)

pointing in the opposite direction

= 0 (Two are mutually perpendicular)

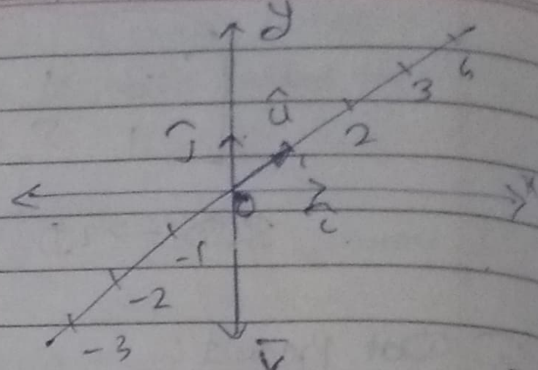
$\bar{v}\cdot\bar{w}$ = (Length of $\bar{v}$)(Length of projected $\bar{w}$)

*Order doesn't matter

If we take a line of evenly spaced dots and apply a transformation, a linear transformation will keep those dots evenly spaced in the output space (number line)

$\begin{bmatrix} u_x & u_y \end{bmatrix}$

↑ where $\hat{i}$ lands ↑ where $\hat{j}$ lands



$$\begin{bmatrix} u_x & u_y \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = u_x \cdot x + u_y\, y$$

matrix-vector Product

⇕

Dot Product

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Transform $\bar{v}$

· Dot product

· (Dual vector)

If there is not a unit vector, eg: $3\hat{u}$

$\begin{bmatrix} 3u_x & 3u_y \end{bmatrix}$

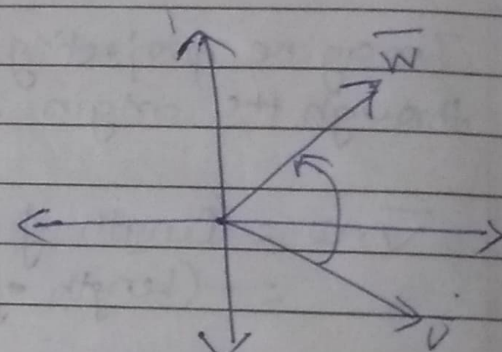
Project then scale

⑥ ~~Ve~~ <u>Cross Product</u>:

$\bar{v} \times \bar{w}$ = Area of ||gm

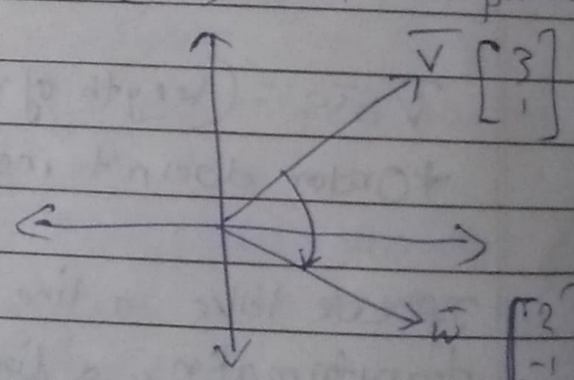
$$\boxed{\bar{v} \times \bar{w} = -\bar{w} \times \bar{v}}$$

($\bar{v}$ is to the right of $\bar{w}$) ($\bar{v}$ is to the left of $\bar{w}$)



(+ve) ~~area~~ cross product

Since $\hat{i}$ is on the right of $\hat{j}$



$$\bar{v} \times \bar{w} = \det\left(\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}\right)$$

(-ve) cross product

More Perpendicular ⇒ $\bar{v} \times \bar{w}$ is bigger

$$(3\bar{v}) \times \bar{w} = 3(\bar{v} \times \bar{w})$$

$\bar{v} \times \bar{w} = \underbrace{\bar{p}}_{\text{vector}}$ (with length 2.P)

Perpendicular to the 1/^m (given by right hand rule)

(1) Define a 3D to 1D linear transformation in terms of $\bar{v}$ and $\bar{w}$

(2) find its dual vector

(3) show that its dual is ~~equal~~ $\bar{v} \times \bar{w}$

The real cross product takes in two vector and gives a single vector as output. ~~not~~

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \det\left(\begin{array}{c} \begin{array}{ccc} & \bar{v} & \bar{w} \end{array} \\ \begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix} \\ \begin{array}{ccc} \text{variable} & & \end{array} \end{array}\right)$$

for any input vector x, y, z, you consider a parallelepiped defined by this vector v and w then you return its volume

This function is linear

$$\begin{bmatrix} ? & ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\begin{array}{c} \begin{array}{ccc} & \bar{v} & \bar{w} \end{array} \\ \begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix} \end{array}\right)$$

1x3 matrix encoding the 3D to 1D linear transformation.

$$\underbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}}_{\bar{p}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

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$$p_1 \cdot x + p_2 \cdot y + p_3 \cdot z = x(V_2 W_3 - V_3 W_2) + y(V_3 W_1 - V_1 W_3) + z(V_1 W_2 - V_2 W_1)$$

$\therefore$ $p_1 = V_2 W_3 - V_3 W_2$

$p_2 = V_3 W_1 - V_1 W_3$

$p_3 = V_1 W_2 - V_2 W_1$

(Area of ||gm) $\times$ (Component of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\perp$ to $\bar{V}$ and $\bar{W}$)

We define vector p to be such a vector that for any given vector [x y z] dot product of (p) and (x, y, z) is equal to determinant of matrix

$$\begin{bmatrix} x & V1 & W1 \\ y & V2 & W2 \\ z & V3 & W3 \end{bmatrix} = M$$

Determinant is equal to the volume of the parallelepiped

$\therefore$ Volume of parallelepiped = det (m) = vector(p) . vector([x y z]) — (1)

Volume of parallelepiped

= Area of base $\times$ height

Area of base = $(\bar{V} \times \bar{W})$ .. (Area of ||gm)

Height = portion of (x y z) which is $\perp$ to ||gm

(obtained by taking dot product of (x y z) with unit vector which is $\perp$ to ||gm) = u

Now our new formula for Volume of ||gm

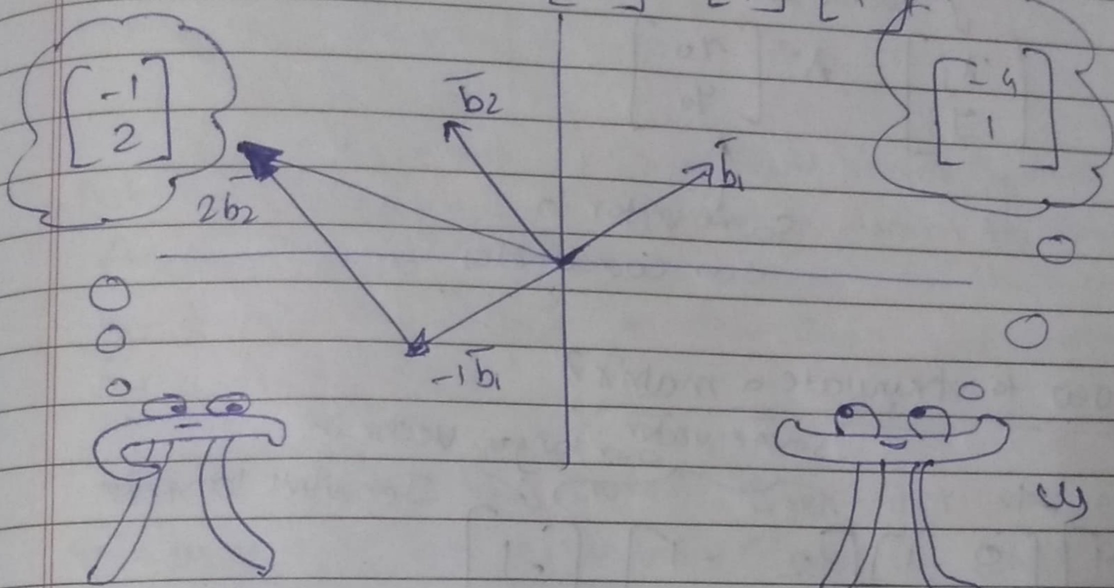
= (Area of ||gm) $\times$ (u . [x y z]) — (2)

from (1) and (2),
Area of ||^{gm} $\times$ vector(u) = vector(p)

$\therefore$ vector p is $\perp$ to ~~the~~ both $\bar{v}$ and $\bar{w}$ and has a magnitude = area of ||^{gm} formed by $\bar{v}$ and $\bar{w}$.

(7) <u>Change of Basis</u>:

$$-1\begin{bmatrix}2\\1\end{bmatrix}+2\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}-4\\1\end{bmatrix}$$



Jennifer

$$\bar{b}_1=\begin{bmatrix}2\\1\end{bmatrix};\ \bar{b}_2=\begin{bmatrix}-1\\1\end{bmatrix}$$

Our grid $\longrightarrow$ Jennifer's grid

$$\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}$$

Our language $\longleftarrow$ Jennifer's language

Jennifer's grid $\longrightarrow$ Our grid

Inverse $\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}^{-1}=\begin{bmatrix}1/3 & 1/3\\-1/3 & 2/3\end{bmatrix}$

Jennifer's language $\longleftarrow$ Our language

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \qquad A \begin{bmatrix} x_j \\ y_j \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Jennifer's basis vectors written in our coordinates

Vector in her coordinates

Same vector in our coordinates

vector in her coordinates

$$\begin{bmatrix} x_j \\ y_j \end{bmatrix} = A^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Same vector in our coordinates

* How to translate a matrix?

Same vector in our language

Vector in Jennifer's language

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Inverse change of basis matrix

Transformation matrix

Change of basis

Transformed vector in our language

Transformed vector in her language

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix}$$

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3) Eigenvectors and eigenvalues:

Most of the vectors get knocked off their span during the transformation.
Some vectors do remain on their own span meaning the effect that the matrix has on such a vector is just to stretch it or squish it (like a scalar)

for eg: $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

$\hat{i}$ basis vector is a special vector which does not get knocked off their span during the transformation.
Another vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ also remains on their span.
and all the vectors on this span get stretched out by a factor of 2.

Such vectors are called Eigen vectors and each eigen vector has eigen value (factor by which it is stretched or squashed during the transformation).
eigen value can be (-ve) too (vector ~~gets squished~~ gets flipped)

Eigen vector $\Rightarrow$ Axis of rotation
Eigen value = 1 (for rotation around an axis) since it does not stretch or squash anything

$$\underbrace{A\vec{v}}_{\text{Eigenvector}} = \lambda\vec{v}$$

A: Transformation matrix
λ: eigenvalue

Scaling by λ
$\Updownarrow$
matrix multiplication by
$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$\therefore A\vec{v} = (\lambda I)\vec{v}$
$\therefore (A - \lambda I)\vec{v} = 0$

We want a non zero solution for $\vec{v}$

$\therefore \det(A - \lambda I) = 0$ (Squishification)

For eg: $\det\Bigg(\underbrace{\begin{bmatrix} 2-1.00 & 2 \\ 1 & 3-1.00 \end{bmatrix}}_{A-\lambda I}\Bigg) = 0.00$

When $\lambda = 1$, the matrix $(A - \lambda I)$ squishes space onto a line which means there is ~~an no~~ a non-zero vector $\bar{v}$.

$\therefore (A - \lambda I)\vec{v} = 0$

$\boxed{A\vec{v} = \lambda\vec{v}}$

$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \bar{v} = 1\bar{v}$

eigen value

($\bar{v}$ would stay fixed in place)

* A matrix might not have any eigenvalue.
  For example: rotation matrix ~~fot~~

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Subtract λ $\therefore \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = (-\lambda)(-\lambda) - (-1)(1)$
$= \lambda^2 + 1 = 0$

$\therefore \lambda = \pm i$ (No real number solutions)

* A single eigenvalue can have more than a line full of eigenvectors

Eg: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ Scales everything by 2

* What if both basis vectors are eigenvalues?

Eg: Consider $\hat{i}$ scaled by (-1) and $\hat{j}$ scaled by 2

$\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ is the matrix associated with the transformation.

↳ Diagonal matrix

All the basis vectors are eigen vectors with diagonal ~~vectors~~ entries of this matrix ~~begin~~ being eigenvalues.

Expressing a vectors into eigen vectors:

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}_{\text{change of basis matrix}} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Eigen basis: Set of basis vectors which are also eigen vectors

Compute $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}^{100}$

~~Change the given basis vectors~~

change the given matrix to eigen basis, calculate the 100th ~~pop~~ power and then transform the resultant matrix back to our old basis vectors.

* Not all matrices can become diagonal

Eg: A shear matrix

Determinant and Eigen vectors don't depend on the coordinate system.

* formal definition of linearity:

Additivity: $L(\bar{v}+\bar{w}) = L(\bar{v}) + L(\bar{w})$
Scaling: $L(c\bar{v}) = cL(\bar{v})$

Linear tranformations preserve addition and Scalar multiplication.

Derivative is linear
$L(\bar{v}+\bar{w}) = L(\bar{v}) + L(\bar{w})$

$$\frac{d}{dx}(x^3+x^2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2)$$

$L(c\bar{v}) = cL(\bar{v})$

$$\frac{d}{dx}(4x^3) = 4\frac{d}{dx}(x^3)$$

$$\frac{d}{dx}(x^3+5x^2+4x+5) = 3x^2+10x+4$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \\ 0 & 0 & 0 & 0 & \\ \vdots & \vdots & & & \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 5 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1\cdot 4 \\ 2\cdot 5 \\ 3\cdot 1 \\ 0 \\ \vdots \end{bmatrix}$$

Linear algebra concepts	Alternate names when applied to functions
(1) Linear transformations	Linear operations
(2) Dot products	Inner products
(3) Eigenvectors	Eigenfunctions

* Rules for Vectors addition and Scaling (Axioms):

1. $\overline{u} + (\overline{v} + \overline{w}) = (\overline{u} + \overline{v}) + \overline{w}$

2. $\overline{v} + \overline{w} = \overline{w} + \overline{v}$

3. There is a vector 0 such that $0 + \overline{v} = \overline{v}$ for all $\overline{v}$

4. For every vector $\overline{v}$ there is a vector $-\overline{v}$ so that $\overline{v} + (-\overline{v}) = 0$

5. $a(b\overline{v}) = (ab)\overline{v}$

6. $1\overline{v} = \overline{v}$

7. $a(\overline{v} + \overline{w}) = a\overline{v} + a\overline{w}$

8. $(a+b)\overline{v} = a\overline{v} + b\overline{v}$