Math Homework #1

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Questions from the Book

Problem 3.6: Proof:

Contradiction: Assume that $\exists A$ s.t. $P(A) \neq \sum_{i \in I} P(A \cap B)$. This implies that $\exists a \in A$ s.t. $a \notin \bigcup_{i \in I} B_i = \Omega$, as since all B_j are disjoint. Here we have a contradiction, as $A \in F$ and $\exists a \in A, a \notin \Omega$.

Problem 3.8: Proof

$$1 - \prod_{k=1}^{n}(1 - P(E_k)) = 1 - \prod_{k=1}^{n}(P(E_k^c)) = 1 - P(\bigcap_{k=1}^{n} E_k^c) = P(\bigcup_{k=1}^{n} E_k) \tag{1}$$

Problem 3.11:

We have that $P(s = crime|test = +)$ cannot be solved directly. Hence, we use Bayes Rule:

$$\begin{aligned} P(s = crime|test = +) &= \frac{P(test = +|s = crime)P(s = crime)}{P(test = +|s = crime)P(s = crime) + P(test = +|s = innocent)P(s = innocent)} \\ &= \frac{(1)(\frac{1}{250{,}000{,}000})}{(1)(\frac{1}{250{,}000{,}000}) + \left(\frac{1}{3{,}000{,}000}\right)\left(\frac{249{,}999{,}999}{250{,}000{,}000}\right)} \\ &= 0.0118 \end{aligned} \tag{2}$$

Problem 3.12:

Let w_1 = the contestant's chosen window, of the set $W = \{w_1, w_2, w_3\}$. Each has an equal 1/3 chance of being correct, and 2/3 chance of being a goat. Window w_3 is exposed as a goat (this can be done without loss of generality). While w_1 still has a 1/3 chance of containing the prize, w_2 now represents the entire original chance of either w_2 and w_3 having a combined 2/3 chance of having the prize behind one of them. Hence, it is wiser to switch.

For a larger problem, let the chosen window be w_1 of the set of windows $W = \{w_1, w_2, ..., w_9, w_{10}\}$. If 8 are opened to reveal goats, leaving w_1 and w_2 (also possible without loss of generality, as $\binom{9}{8} = 9$), we have that you have an 9/10 chance of winning by switching and a 1/10 chance of winning by staying.

Problem 3.16: Proof:

$$V(X) = E[(X - \mu)^2] = E[X^2 - 2\mu X + \mu^2] = E[X^2] - 2\mu E[X] + \mu^2 = E[X^2] - \mu^2 \tag{3}$$

Problem 3.33: Proof:

We observe that the binomial random variable B has a mean $\mu = np$ and variance $\sigma^2 = np(1-p)$. However, this is simply a sum of iid Bernoulli random variables X_k with $\mu_s = p$, $\sigma_s^2 = p(1-p)$, and $\sum_{k=1}^{n} X_k = B$. If we then divide B by n, as long as n is sufficiently large we can use the Weak Law of Large Numbers to show:

$$\begin{aligned} P\left(\left|\frac{\sum_{k=1}^{n} X_k}{n} - \mu_s\right| \geq \epsilon\right) &= \frac{\sigma_s^2}{n\epsilon^2} \\ P\left(\left|\frac{\sum_{k=1}^{n} X_k}{n} - p\right| \geq \epsilon\right) &= \frac{p(1-p)}{n\epsilon^2} \\ P\left(\left|\frac{B}{n} - p\right| \geq \epsilon\right) &= \frac{p(1-p)}{n\epsilon^2} \end{aligned} \tag{4}$$

Problem 3.36: Proof:

Let X_i = a Bernoulli trial (with $\mu = 0.801$ and $\sigma \approx 0.1594$) determining whether student i will attend the University, with $i \in I = \{1, 2, 3, ..., 6241, 6242\}$, $\forall i$. From this, we can infer that $S = \sum_{i \in I} X_i$ is a Binomial random variables with $n = 6242$. We may then use the Central Limit Theorem to determine the probability that over 5500 students will attend:

$$1 - P(S < 5500) = 1 - P\left(\frac{S - n\mu}{\sqrt{n}\sigma} < \frac{5500 - (6242)(0.801)}{(79)(0.1594)}\right) = 1 - P(Z < 39.7) \approx 0 \tag{5}$$

Hence, we predict with almost absolute certainty that the University will not have over 5500 students enroll for the coming academic year.

Question #2

Part A:

Let $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$, where $P(i) = \frac{1}{8} \forall i \in \Omega$. Let A = {1,3,5,7}, B = {2,4,6,8}, and C = {1,2,3,4}. We then have that:

$$\begin{aligned} P(A \cap B) = P(1,3,5,7)P(2,4,6,8) &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \\ P(B \cap C) = P(2,4,6,8)P(1,2,3,4) &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \\ (A \cap C) = P(1,3,5,7)P(1,2,3,4) &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \end{aligned} \tag{6}$$

However, we have that:

$$0 = P(A \cap B \cap C) \neq P(1,3,5,7)P(2,4,6,8)P(1,2,3,4) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8} \tag{7}$$

Part B:

Let $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$, where $P(i) = \frac{1}{8} \forall i \in \Omega$. Let A = {1,3,5,8}, B = {1,4,6,8}, and C = {2,4,7,8}. We then have that:

$$\begin{aligned} P(A \cap B) = P(1,3,5,8)P(2,4,7,8) &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \\ P(B \cap C) = P(1,4,6,8)P(2,4,7,8) &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \\ P(A \cap B \cap C) = P(1,3,5,8)P(1,4,6,8)P(2,4,7,8) &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8} \end{aligned} \tag{8}$$

However, we have that:

$$\frac{1}{8} = (A \cap C) \neq P(1,3,5,8)P(2,4,7,8) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \tag{9}$$

Question #3

Proof: Let B be a "Bedford random variable", whose domain is $\Omega = \{1,2,3,4,5,6,7,8,9\}$. In order for a distribution to qualify as a valid discrete distribution, B must follow the following properties:

$$\begin{gathered} \forall \omega \in \Omega,\ P(\omega) \geq 0 \\ \sum_{i \in I} P(\omega_i) = 1 \end{gathered} \tag{10}$$

where I is the index set $\forall \omega_i \in \Omega$. We see that both of these properties exist, as both the probabilities for each element of Ω is non-negative, and all add to 1. This is observable in the table below:

First significant digit	Predicted frequency
1	0.301
2	0.176
3	0.125
4	0.097
5	0.079
6	0.067
7	0.058
8	0.051
9	0.046
Total	1.00

Hence, we have that Bedford's Law follows a well-defined discrete distribution.

Question #4

Part A Proof:

$$E[X] = \sum_{n=1}^{\infty} 2^n \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \left(\frac{2}{2}\right)^n = \sum_{n=1}^{\infty} 1 = +\infty \tag{11}$$

Part B Proof:

$$E[X] = \sum_{n=1}^{\infty} \ln(2^n) \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} n \ln(2) \left(\frac{1}{2}\right)^n = \ln(2) \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = 2\ln(2) \tag{12}$$

Question #5

If one investor bought an asset in the country they are from, they would neither gain nor lose money, and have an estimated return of 0%. However, if they invest in the other country, they would have an estimated return of $(0.5)(1.25)\ +\ (0.5)(0.80) = 1.025$ units of their home currency. Hence, each investor should buy the risk-free investment in the other country they do not currently reside in.

Question #6

Part A

Let $X : \Omega \to \mathbb{R}$ be a continuous random variable. We assuming some valid mapping between our sample space and $\mathbb{R}$. Because $E[x^2] = V[X] + E[X]^2 = \infty$, and we know $E[X]$ must be finite, and subsequently $V[X]$ must be infinite. This allows us to identify our random variable as following the Pareto distribution with parameter $\alpha \in (1, 2]$:

$$X = \begin{cases} \frac{\alpha x_m^\alpha}{X^{\alpha+1}} & \text{if } x_m \leq x \\ 0 & \text{if } x_m > x \end{cases}$$

Part B

Let $X \sim Exponential(\lambda = \frac{3}{2})$ and $Y \sim Unif(0, 1)$. As $E[X] = \frac{1}{2} < \frac{2}{3} = E[Y]$, and the following is true:

$$P(X > Y) = 1 - P(X < Y) = 1 - P(X - Y < 0) = 0.8512 \tag{13}$$

Hence, we have that the conditions are satisfied.

Part C

Let $X, Y, Z \sim Unif(-1, 1)$. All have an identical expected value of 0, and have a strictly positive probability of being larger than one another. Hence, the conditions are satisfied.

Question #7

Let $X \sim N(0, 1)$, $P(Z = 1) = P(= -1) = \frac{1}{2}$, and $Y = XZ$

Park A

We seek to demonstrate $Y \sim N(0, 1)$. This can be done by equating its CDF to that of X, which is a standard normal (due to CDFs being uniquely defined). This is as follows:

$$\begin{aligned} P(Y < y) &= P(X * Z < y) \\ &= P(X < y|Z = 1)P(Z = 1) + P(-X < y|Z = 1)P(Z = -1) \\ &= P(X < y)\left(\frac{1}{2}\right) + P(X > -y)\left(\frac{1}{2}\right) \\ &= \left(\frac{1}{2}\right)\left[P(X < y) + P(X < y)\right] \text{ (as standard normal is symmetrical about 0)} \\ &= P(X < y) \end{aligned} \tag{14}$$

Part B

For any value of X, $P(Y = X) = P(-Y = X) = \frac{1}{2}$. However, if we have the absolute value of both sides, both are constrained to be positive. Hence, we have $P(|Y| = |X|) = P(|-Y| = |X|) = 1$.

Part C

As we are simply attempting to show X and Y are *not* independent, it is sufficient to show that they lack a necessary property of independence. This can be done with the following:

$$\begin{aligned} P(X<x|Y<x) =& P(X<x|XZ<x) \\ =& P(X<x|X<x)P(Z=1) + P(X<x|-X<x)P(Z=-1) \\ =& P(X<x)\left(\frac{1}{2}\right) + P(X<x|-X<x)\left(\frac{1}{2}\right) \\ \neq& P(X<x) \end{aligned} \tag{15}$$

Part D

$$\begin{aligned} Cov[X,Y] =& E[(X-\mu_x)(Y-\mu_y)] = E[XY] + \mu_x\mu_y = (0) + (0)(0) = 0 \\ & \text{as} \\ E[XY] =& E[XXZ] = (1)\left(\frac{1}{2}\right)E[X^2] + (-1)\left(\frac{1}{2}\right)E[X^2] = 0 \end{aligned} \tag{16}$$

Part E

This contradicts our answers for Parts C and D; hence, this is not true, and extends from the fact that $Cov[X,Y]=0$ is not a sufficient condition for independence of X and Y.

Question #8

Let $F(X)=x$ be the CDF of the continuous uniform distribution between 0 and 1. For some $x\in(0,1)$, the probability that at least one variable X_i is larger than x simply the product of $nF(X)$s; this then represents distribution of the maximum M. Likewise, the probability that at least one random variables is below x is $1-[1-F(X)]^n$ (of which $[1-F(X)]^n$ represents the probability that at all variables are above x). Given $\forall i, X_i \sim U(0,1)$, we can identify the CDFs of M and m as follows:

$$\begin{aligned} M &\sim F(x)^n = \ x^n \\ m &\sim 1-[1-F(x)]^n = \ 1-(1-x)^n \end{aligned} \tag{17}$$

The PDFs of the minimum and maximum are then simply the derivative of these functions, and their expected values are easily computable:

$$\begin{aligned} M &\sim \frac{\partial}{\partial x}[x^n] = \ nx^{n-1}, \quad E[M] = \int_0^1 nx^n = \frac{n}{n+1} \\ m &\sim \frac{\partial}{\partial x}\big[1-[1-x]^n\big] = \ n[1-x]^{n-1}, \quad E[m] = \int_0^1 xn[1-x]^n = \frac{1}{n+1} \end{aligned} \tag{18}$$

Question #9

We define $X_i =$ a random variable denoting the state of the economy, with $P(X=0)=P(X=1)=\frac{1}{2}$. Hence each X_i is a Bernoulli trial with $\mu=\frac{1}{2}$ and $\sigma^2=\frac{1}{4}$. Further, let $\Upsilon = \{X_i\}_{i=1}^n$, and $S_n = \sum_{k=1}^n X_k$.

Part A

Let $n = 1000$. We then use the Central Limit Theorem:

$$\begin{aligned} 1 - P(S < 490) - P(S > 510) &= P(S < 510) - P(S < 490) \\ &= P\left(\frac{S - n\mu}{\sqrt{n}\sigma} < \frac{510 - (0.5)(1000)}{\sqrt{250}}\right) - P\left(\frac{S - n\mu}{\sqrt{n}\sigma} < \frac{490 - (0.5)(1000)}{\sqrt{250}}\right) \\ &= P(Z < 0.632) - P(Z < -0.632) \\ &= 0.7363 - 0.2637 \\ &= 0.4726 \end{aligned} \tag{19}$$

Part B

We can use the Weak Law of Large Numbers:

$$P\left(|\frac{S}{n} - \mu| \geq \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2} \quad \Rightarrow \quad P\left(|\frac{S}{n} - 0.5| \geq 0.005\right) \leq \frac{10,000}{n} \tag{20}$$

Hence, we see that $n = 1,000,000$.

Question #10

Contradiction: Assume $\theta < 0$. We observe that $e^{\theta X}$ for some $\theta \neq 0$ is a convex function; hence, we can use Jensen's inequality to show:

$$\begin{aligned} e^{\theta E[X]} &\leq E[e^{\theta X}] = 1 \\ ln[e^{\theta E[X]}] &\leq 0 \\ \theta E[X] &\leq 0 \end{aligned} \tag{21}$$

This leads to a contradiction because as $\theta < 0$, $\theta E[X] > 0$ as both θ and $E[X]$ are strictly negative. Thus, as $\theta \neq 0$ by hypothesis, $\theta > 0$.