Problem Set #1

Dynamic Programming Exercise
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Exercise 1

1. State variable: B_t, the amount of oil left in the oil field after time t, where $B_0 - B$.

2. Control variable: q_t, the amount of oil she chooses to sell at time t.

3. Transition equation: $B_t = q_{t+1} + B_{t+1}$.

4. Problem: let $\beta = \frac{1}{1+r}$, $\max_{\{q_1,q_2,...\}} \sum_{t=1}^{\infty} \beta^{t-1} p_t q_t$
Let $V_t(B_t)$ be the value function that represents the maximum present value of sales at time t, with B_t barrels of oil left in the field. Since we have an infinite time horizon, we can drop the subscript t. Then the Bellman equation is

$$V(B) = \max_q \{pq + V(B - q)\}.$$

5. Euler equation: Since r is fixed and p_t are exogenous and are known to us, we can define the discounted price $p'_t = \beta^{t-1} p_t$. Hence our objective function becomes

$$\max_{\{q_1,q_2,...\}} p'_1 q_1 + p'_2 q_2 + p'_3 q_3 + ...$$

Obviously we should choose to sell all B barrels at the time when p'_t is the largest. Hence for the Euler equation, $\forall i \in \mathbb{N}$, if $\exists j \in \mathbb{N}$ such that $p'_j = \beta^{j-1} p_j \geq p'_i = \beta^{i-1} p_i$, then $q_i = 0$.

6. If $\forall t, p_{t+1} = p_t$, then it is obvious that $p'_1 \geq p'_2 \geq p'_3 \geq ...$ Hence by the Euler Equation above we should choose to sell all oil at the first day, i.e. $V(B) = p_1 B$.

7. If $\forall t, p_{t+1} > (1 + r)p_t$, then it follows that $p'_1 < p'_2 < p'_3 < ...$ In this way, the owner should wait until forever to sell the oil.

8. Since marginal benefits in each period are constant, the only case when there is an interior solution is $\forall t, p_{t+1} = (1 + r)p_t$. In this case, the owner is indifferent to how much to sell in each period.

Exercise 2

1. State variables: k_t or y_t, both indicate the total production that allows us to allocate in the current period.

2. Control variables: c_t or i_t.

3. Bellman Equation:
$$V(y) = \max_c \{u(c) + \beta \mathbb{E} V(y')\},$$
where $y' = z'[(1-\delta)k + (y-c)]^\alpha$.

4. Solution: see python code in a separate file.

Exercise 3

1. Bellman Equation:
$$V(y) = \max_c \{u(c) + \beta \mathbb{E} V(y')\}.$$

2. Solution: see code in a separate file.

Exercise 4

1. Bellman Equation:
$$V(w) = \max_{\{0,1\}} \{V^0(w), V^1(w)\},$$
where $V^0(w) = b + \beta V(w')$, and $V^1(w) = \sum_{t=0}^{\infty} \beta^t w = \frac{w}{1-\beta}$.

2. Solution: see code in a separate file

3. As we can see from the figure, the reservation wage increases as the unemployment benefit increases. This is intuitively clear. Since if the unemployment benefit is high, then people have less motivation to accept a job.