Problem Set #1

Intro to Measure Theory
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Please note that in the following document I use $A - B$ to denote the collection of elements that are in A but not in B, i.e. $A - B = A \cap B^C$.

Exercise 1.3

$\mathcal{G}_1$ is not an algebra. $\mathcal{G}_1$ contains every open set in $\mathbb{R}$. Since the complement of an open set is a closed set, which is not in $\mathcal{G}_1$, hence $\mathcal{G}_1$ is not an algebra.
$\mathcal{G}_2$ is an algebra. $\emptyset \in \mathcal{G}_2$ is trivially true, since the union of zero interval of the form should be in $\mathcal{G}_2$. $\mathcal{G}_2$ is closed under complement, since $(a,b]^C = (-\infty, a] \cup (b, \infty) \in \mathcal{G}_2, (-\infty, b]^C = (b, \infty) \in \mathcal{G}_2$, and $(b, \infty)^C = (-\infty, b] \in \mathcal{G}_2$. By construction the finite union of such intervals is also in $\mathcal{G}_2$.
$\mathcal{G}_3$ is a σ-algebra. It can be verified that the above propertis also hold for $\mathcal{G}_3$ and can be extended to coutably infinite union.

Exercise 1.7

Since $\mathcal{A}$ is a σ-algebra, by definition $\emptyset \in \mathcal{A}$. Then $\emptyset^C = X \in \mathcal{A}$. It follows that $\{\emptyset, X\} \subset \mathcal{A}$. Since $\mathcal{P}(X)$ contains every subset of X, where $\mathcal{A}$ contains some subsets of X, it is obvious that $\mathcal{A} \subset \mathcal{P}(X)$.

Exercise 1.10

Proof. Since $\forall\alpha, \mathcal{S}_\alpha$ is a σ-algebra, $\emptyset \in \mathcal{S}_\alpha, \forall\alpha$. It follows that $\emptyset \in \bigcap_\alpha \mathcal{S}_\alpha$.
If $A \in \bigcap_\alpha \mathcal{S}_\alpha$, then $A \in \mathcal{S}_\alpha, \forall\alpha$. This implies $A^C \in \mathcal{S}_\alpha, \forall\alpha$. It follows that $A^C \in \bigcap_\alpha \mathcal{S}_\alpha$. So it is closed on complement.
Suppose $A_1, A_2, ... \in \bigcap_\alpha \mathcal{S}_\alpha$, then $A_1, A_2, ... \in \mathcal{S}_\alpha, \forall\alpha$. This implies $\bigcup_{i=1}^\infty A_i \in \mathcal{S}_\alpha, \forall\alpha$. Hence, $\bigcup_{i=1}^\infty A_i \in \bigcap_\alpha \mathcal{S}_\alpha, \forall\alpha$. So it is closed on countable union. □

Exercise 1.17

Proof. Let $A_1 = A, A_2 = B - A, A_3 = \emptyset, A_4 = \emptyset, ...$ Then $\bigcup_{i=1}^\infty A_i = B$ and $A_i \cap A_j = \emptyset$ when $i \neq j$. Hence,

$$\mu(B) = \mu(\bigcup_{i=1}^\infty A_i) = \sum_{i=0}^\infty \mu(A_i) = \mu(A_1) + \mu(A_2) + 0 + 0 + 0 + ... \geq \mu(A_1) = \mu(A).$$

Let $B_1 = A_1, B_2 = A_2 - A_1, ..., B_n = A_n - A_{n-1}, ...$ Then it follows that $\bigcup_{i=1}^\infty A_i = \bigcup_{i=1}^\infty B_i$ and when $i \neq j$, $B_i \cap B_j = \emptyset$. Also we have $\mu(B_i) \leq \mu(A_i)$ by the above conclusion. Hence,

$$\mu(\bigcup_{i=0}^\infty A_i) = \mu(\bigcup_{i=0}^\infty B_i) = \sum_{i=1}^\infty \mu(B_i) \leq \sum_{i=1}^\infty \mu(A_i).$$

□

Exercise 1.18

Proof. By construction, $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$.
Suppose $A_1, A_2, ... \in \mathcal{A}$ and they are disjoint. Observe that when $i \neq j, A_i \cap B$ and $A_j \cap B$ are disjoint. Hence, by construction,

$$\lambda(\bigcup_{i=0}^{\infty} A_i) = \mu(\bigcup_{i=0}^{\infty} A_i \cap B) = \mu((A_1 \cap B) \cup (A_2 \cap B) \cup ...) = \sum_{i=0}^{\infty} \mu(A_i \cup B) = \sum_{i=0}^{\infty} \lambda(A_i).$$

□

Exercise 1.20

Proof. Suppose $B_1 = \emptyset, B_2 = A_1 - A_2, B_3 = A_1 - A_3, ...$ Then $B_1 \subset B_2 \subset B_3 ...$ By the conclusion of Theorem 1.19(1) we have $\lim_{n\to\infty} \mu(B_n) = \mu(\bigcup_{i=1}^{\infty} B_n)$. Observe that $\bigcup_{i=1}^{n} B_i = A_1 - \bigcap_{i=1}^{n} A_i = A_1 - A_n$ and $\bigcup_{i=1}^{\infty} B_i = A_1 - \bigcap_{i=1}^{\infty} A_i$. Also notice that if $M \subset N$, then $N - M$ and M are disjoint so $\mu(N) = \mu(N - M) + \mu(M)$, and hence $\mu(N - M) = \mu(N) - \mu(M)$. Thus,

$$\begin{aligned}
\mu(A_1) - \lim_{n\to\infty} \mu(A_n) &= \lim_{n\to\infty} (\mu(A_1) - \mu(A_n)) \\
&= \lim_{n\to\infty} \mu(A_1 - A_n) \\
&= \lim_{n\to\infty} \mu(B_n) \\
&= \mu(\bigcup_{i=1}^{\infty} B_i) \\
&= \mu(A_1 - \bigcap_{i=1}^{\infty} A_i) \\
&= \mu(A_1) - \mu(\bigcap_{i=1}^{\infty} A_i).
\end{aligned}$$

So we have $\lim_{n\to\infty} \mu(A_n) = \mu(\bigcap_{i=1}^{\infty} A_i)$. □

Exercise 2.10
By the definition of outer measure (countable subadditivity), we have $\mu^*(B) \leq \mu^*(B \cap E) + \mu^*(B \cap E^C)$. Combining the two inequalities yields the equality.

Exercise 2.14

Proof. By Caratheodory Extension Theory, we already know that $\sigma(\mathcal{A}) \subset \mathcal{M}$. Hence if we can show that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{O}) = \sigma(\mathcal{A})$, then we have $\mathcal{B}(\mathbb{R}) \subset \mathcal{M}$.
We need the following claim:

Claim 1. If $A \subset B$, then $\sigma(A) \subset \sigma(B)$. Moreover, $\sigma(\sigma(A)) = \sigma(A)$.

The proof of the claim is trivial.

Observe that

$$(a,b] = \bigcap_{n=1}^{\infty}(a, b+\frac{1}{n}),$$

$$(-\infty, a] = (\bigcup_{n=n_0}^{\infty}(-n, a-\epsilon)) \cup (\bigcap_{m=0}^{\infty}(a-2\epsilon, a+\frac{1}{m})), \text{for sufficiently large } n_0 \text{ and small } \epsilon,$$

$$(b, +\infty) = \bigcup_{n=n_0}^{\infty}(b, n), \text{for sufficiently large } n_0.$$

Therefore, for every set in $\mathcal{A}$, it is a finite union of disjoint intervals of the form $(a,b], (-\infty, a]$ and $(b, +\infty)$, each of which can be in turn written as a countable union and/or intersection of open intervals. Hence, every set in $\mathcal{A}$ can be written as a countable union and/or intersection of open intervals and therefore is in $\sigma(\mathcal{O})$. This shows that $\mathcal{A} \subset \sigma(\mathcal{O})$, which implies that $\sigma(\mathcal{A}) \subset \sigma(\sigma(\mathcal{O})) = \sigma(\mathcal{O})$.

For the other direction, pick a set $(a,b) \in \mathcal{O}$. Observe that

$$(a,b) = \bigcup_{n=n_0}^{\infty}(a, b-\frac{1}{n}],$$

when n_0 is large enough. So it can be written as a countable union of sets in $\mathcal{A}$, and therefore $(a,b) \in \sigma(\mathcal{A})$. This shows that $\mathcal{O} \subset \sigma(\mathcal{A})$, which implies that $\sigma(\mathcal{O}) \subset \sigma(\sigma(\mathcal{A})) = \sigma(\mathcal{A})$.
Combining the two directions above yields the equality. □

Exercise 3.1

Proof. Pick A, a countable subset of $\mathbb{R}$. Since A is countable, its items can be enumerated: $A = \{a_1, a_2, a_3, ...\}$, where each a_i is a real number. With some small $\epsilon > 0$, we can construct a sequance of open intervals $\{(m_i, n_i)\}_{i=0}^{\infty}$, where $m_i = a_i - \frac{\epsilon}{2^i}$, and $n_i = a_i + \frac{\epsilon}{2^i}$ such that every point is covered by one corresponding interval and these intervals are disjoint. Observe that the length of the first interval is ϵ, the length of the second interval is $\frac{\epsilon}{2}$, and so on. Hence,

$$\begin{aligned}\mu(A) \leq \mu(\{a_1, a_2, a_3, ...\}) &= \sum_{i=0}^{\infty}\mu((a_i, b_i)) \\ &= \sum_{i=0}^{\infty}\frac{\epsilon}{2^i} = 2\epsilon.\end{aligned}$$

Since we can make ϵ arbitrarily small, we conclude that $\mu(A) = 0$. □

Exercise 3.4

Proof. Since $\{x \in X : f(x) < a\} = \{x \in X : f(x) \geq a\}^C$, by the definition of σ-algebra, (*) being in $\mathcal{M}$ implies (3) being in $\mathcal{M}$, and vice versa. Hence we have shown that (*) and (3) are equivalent. In the same way, we can see that (1) and (2) are equivalent. So it suffices to show that (1) and (*) are equivalent.

Now assume (*), i.e. $\forall a, \{x \in X : f(x) < a\} \in \mathcal{M}$. Fix some $a \in \mathbb{R}$. Let $a_n = a + \frac{1}{n}$, then $\forall n \in \mathbb{N}, P_n = \{x \in X : f(x) < a + \frac{1}{n}\} \in \mathcal{M}$. Moreover, observe that $\bigcap_{n=1}^{\infty} P_n = \{x \in X : f(x) \leq a\}$. Since $\forall n, P_n \in \mathcal{M}$, by the definition of σ-algebra, $\{x \in X : f(x) \leq a\} = \bigcap_{n=1}^{\infty} P_n \in \mathcal{M}$. Hence (*) implies (1).

For the other direction, assume (1), i.e. $\forall a, \{x \in X : f(x) \leq a\} \in \mathcal{M}$. Fix some a, we see that $\forall n \in \mathbb{N}, Q_n = \{x \in X : f(x) \leq a - \frac{1}{n}\} \in \mathcal{M}$. Hence $\{x \in X : f(x) < a\} = \bigcup_{n=1}^{\infty} \{x \in X : f(x) \leq a - \frac{1}{n}\} \in \mathcal{M}$. So (1) implies (*).

Hence (1) and (*) are equivalent, and therefore so are (*), (1), (2), and (3). □

Exercise 3.7

Proof.

1. $f + g$
   Take $F(f, g) = f + g$. Result then follows from (4).

2. $f \cdot g$
   Take $F(f, g) = f * g$. Result then follows from (4).

3. $\max(f, g)$
   Observe that

   $$\{x \in X : \max(f(x), g(x)) > a\} = \{x \in X : f(x) > a\} \cap \{x \in X : g(x) > a\}.$$

   By measurability of f and g, the latter two sets are measurable. Hence so does the first set.

4. $\min(f, g)$
   Observe that

   $$\{x \in X : \min(f(x), g(x)) < a\} = \{x \in X : f(x) < a\} \cap \{x \in X : g(x) < a\}.$$

   The rest is the same.

5. $|f|$
   Consider the set $\{x \in X : |f(x)| < a\}$. If $a \leq 0$ we get $\emptyset$, which is in $\mathcal{M}$. Now fix some $a > 0$.

   $$\begin{aligned} \{x \in X : |f(x)| < a\} &= \{x \in X : 0 \leq f(x) < a\} \cup \{x \in X : -a < f(x) \leq 0\} \\ &= (\{x \in X : f(x) < a\} - \{x \in X : f(x) < 0\}) \\ &\quad \cup (\{x \in X : f(x) \leq 0\} - \{x \in X : f(x) \leq -a\}), \end{aligned}$$

   where each of the set above is in $\mathcal{M}$. So $\{x \in X : |f(x)| < a\} \in \mathcal{M}$.

□

Exercise 3.14

Proof. Here we only consider the case when $f \geq 0$ and f is bounded above by some $M > 0$. For general f we can apply the same decomposition as is described in the proof of the theorem.

Suppose $0 \leq f(x) < M, \forall x \in \mathbb{R}$. Let $N_1 \in \mathbb{N}$ and $N_1 > M$, then $f(x) < N_1, \forall x \in \mathbb{R}$. Hence $\forall x \in \mathbb{R}, x \notin E_{\infty}^{N_1}$. Moreever, $\forall \epsilon > 0, \exists N_2 \in \mathbb{N}$ and $N_2 > N_1$ such that $\frac{1}{2^{N_2}} < \epsilon$. Now, if $n > N_2$ and $n \in \mathbb{N}$, $\forall x_0 \in \mathbb{R}, \exists i_0 \in \{1, 2, 3, ..., n2^n\}$ such that $x \in E_{i_0}^n$. This means that $f(x) \in [\frac{i_0-1}{2^n}, \frac{i_0}{2^n})$, while $s_n(x) = \frac{i_0-1}{2^n}$. So $|f(x) - s_n(x)| < \frac{1}{2^n} < \frac{1}{2^{N_2}} < \epsilon$. Note that since we can arbitrarily choose $x_0 \in \mathbb{R}$, and the value of n does not depend on x_0, the convergence is hence uniform. □

Exercise 4.14

Proof. To show that $f \in \mathcal{L}^1(\mu, E)$, it suffices to show that $\int_E f^+ d\mu < \infty$ and $\int_E f^- d\mu < \infty$.

Since $\forall x \in E, |f(x)| < M$, we know $0 \leq f^+(x) < M, \forall x \in E$. By definition,

$$\int_E f^+ d\mu = \sup_s \{\int_E s d\mu : 0 \leq s(x) \leq f(x), \text{s is a simple, measurable function}\}.$$

For each simple function $s(x)$, observe that

$$\int_E s d\mu = \sum_{i=1}^{N} c_i \mu(E \cap E_i) \leq \sum_{i=1}^{N} M\mu(E \cap E_i) = M \sum_{i=1}^{N} \mu(E \cap E_i) = M\mu(E) < \infty.$$

So $M\mu(E)$ is an upper bound for the set $\{\int_E s d\mu : 0 \leq s(x) \leq f(x), \text{s is a simple, measurable function}\}$. Since $\int_E f^+ d\mu$ is the smallest upper bound, it follows that $\int_E f^+ d\mu \leq M\mu(E) < \infty$.

Similarly one can show that $\int_E f^- d\mu < \infty$. Combining the two yields the result that $f \in \mathcal{L}^1(\mu, E)$.

□

Exercise 4.14

Proof. Here we only consider function $f \geq 0$. General functions can be first decomposed as above and then treated in the same way. Since $f \in \mathcal{L}^1(\mu, E)$, we know that $\int_E f d\mu = I < \infty$. Suppose by contradiction that $F = \{x \in E : \forall M > 0, f(x) < M\} \subset E$, and $\mu(F) = m > 0$. We can construct a simple function $s_0(x) \leq f(x)$ as usual on $E - F$ and take 0 on F. Then $s_0 \in \{s(x) : 0 \leq s(x) \leq f(x)$ s is a simple, measurable function$\}$. Thus, $\int_E s_0 d\mu \leq \sup_s \{\int_E s d\mu : 0 \leq s(x) \leq f(x), \text{s is a simple, measurable function}\}$. Now, let $s_1 = s_0 + \chi_F \mu(F)$, $s_2 = s_0 + 2\chi_F \mu(F)$, ... and $s_n = s_0 + n\chi_F \mu(F)$. Observe that $\forall x \in E, s_0(x) < s_1(x) < s_2(x) < ...$ and $\forall n \in \mathbb{N}, s_n(x) \leq f(x)$ when $x \in E$. The latter is because when $x \in E, f(x) > n, \forall n$. Hence, we have a sequence of simple functions $\{s_n\}$. Now, observe that $\forall n \in \mathbb{N}, \int_E s_{n+1} d\mu - \int_E s_n d\mu = (n+1)\mu(F) - n\mu(F) = \mu(F) = m > 0$. So $\{\int_E s_n d\mu\}$ is an increasing sequence and does not converge. Since every term in the sequence is in $\{\int_E s d\mu : 0 \leq s(x) \leq f(x), \text{s is a simple, measurable function}\}$, we conclude that $\sup_s \{\int_E s d\mu : 0 \leq s(x) \leq f(x), \text{s is a simple, measurable function}\} = \infty$, which contradicts the fact that f is integrable. Thus, the set of points where f is not finite must have measure of zero, i.e. f is almost finite everywhere. □

Exercise 4.14

Proof. Write $f = f^+ - f^-$ and $g = g^+ - g^-$, which are defined as in Definition 4.3. Now, by definition,

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu = \sup_s\{\int_E s d\mu, 0 \leq s \leq f^+\} - \sup_s\{\int_E s d\mu, 0 \leq s \leq f^-\}$$

$$\int_E g d\mu = \int_E g^+ d\mu - \int_E g^- d\mu = \sup_s\{\int_E s d\mu, 0 \leq s \leq g^+\} - \sup_s\{\int_E s d\mu, 0 \leq s \leq g^-\}$$

where s is a simple, measurable function.

Observe that since $f \leq g$ on E, $f^+ \leq g^+$ and $g^- \leq f^-$ on E. For any simple function s such that $0 \leq s \leq f^+$, it follows that $0 \leq s \leq f^+ \leq g^+$, and similarly $0 \leq s \leq g^-$ implies $0 \leq s \leq f^-$. Thus,

$$\{\int_E s d\mu, 0 \leq s \leq f^+\} \subset \{\int_E s d\mu, 0 \leq s \leq g^+\}$$

and

$$\{\int_E s d\mu, 0 \leq s \leq g^-\} \subset \{\int_E s d\mu, 0 \leq s \leq f^-\}.$$

Take sup on both sides,

$$\sup_s\{\int_E s d\mu, 0 \leq s \leq f^+\} \leq \sup_s\{\int_E s d\mu, 0 \leq s \leq g^+\}$$

$$\sup_s\{\int_E s d\mu, 0 \leq s \leq g^-\} \leq \sup_s\{\int_E s d\mu, 0 \leq s \leq f^-\}.$$

Hence, $\sup_s\{\int_E s d\mu, 0 \leq s \leq f^+\} - \sup_s\{\int_E s d\mu, 0 \leq s \leq f^-\} \leq \sup_s\{\int_E s d\mu, 0 \leq s \leq g^+\} - \sup_s\{\int_E s d\mu, 0 \leq s \leq g^-\}$. It then follows immediately that $\int_E f d\mu \leq \int_E g d\mu$. $\square$

Exercise 4.15

Proof. Since $A \subset E$, for any measurable set X, $(X \cap A \subset (X \cap E)$, and therefore $\mu(X \cap A) \leq \mu(X \cap E)$. Pick any simple, measurable function s. Observe that $\int_A s d\mu = \sum_{i=0}^N c_i \mu(E_i \cap A) \leq \sum_{i=1}^N c_i \mu(E_i \cap E) = \int_E s d\mu$.

By definition,

$$\int_E f d\mu = \sup_s\{\int_E s d\mu, 0 \leq s \leq f^+\} - \sup_s\{\int_E s d\mu, 0 \leq s \leq f^-\},$$

$$\int_A f d\mu = \sup_s\{\int_A s d\mu, 0 \leq s \leq f^+\} - \sup_s\{\int_A s d\mu, 0 \leq s \leq f^-\},$$

where s is a simple, measurable function. Now, pick any $\int_A s d\mu \in \{\int_A s d\mu, 0 \leq s \leq f^+\}$. Fix this simple function s. Then $\int_E s d\mu$ must be in $\{\int_E s d\mu, 0 \leq s \leq f^+\}$, and

$\int_A sd\mu \leq \int_E sd\mu$. That is to say, $\forall x \in \{\int_A sd\mu, 0 \leq s \leq f^+\}, \exists y \in \{\int_E sd\mu, 0 \leq s \leq f^+\}$, such that $x \leq y$. This implies that

$$\sup_s \{\int_A sd\mu, 0 \leq s \leq f^+\} \leq \sup_s \{\int_E sd\mu, 0 \leq s \leq f^+\},$$

and similarly one can show that

$$\sup_s \{\int_A sd\mu, 0 \leq s \leq f^-\} \leq \sup_s \{\int_E sd\mu, 0 \leq s \leq f^-\}.$$

It immediately follows that $\int_A fd\mu \leq \int_E fd\mu < \infty$. So $f \in \mathcal{L}^1(\mu, A)$. $\square$

Exercise 4.21
It is obvious that since $\mu(A - B) = 0$,

$$\int_{A-B} fd\mu = 0.$$

By the above theorem we know that Lesbegue integral satisfies countable additivity. Hence

$$\int_B fd\mu = \int_B fd\mu + 0 = \int_B fd\mu + \int_{A-B} fd\mu = \int_A fd\mu \geq \int_A fd\mu.$$