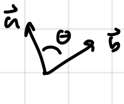
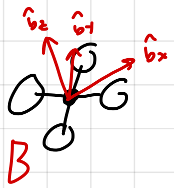
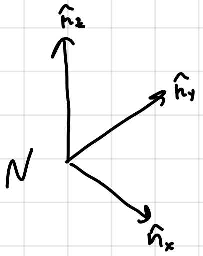
3/1 Recitation

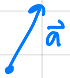
- quiz 6 solutions
- what is attitude
- attitude as a rotation
- wrapping our rotation in a quaternion
- quaternion math

A dot product between two vectors: $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos(\theta)$





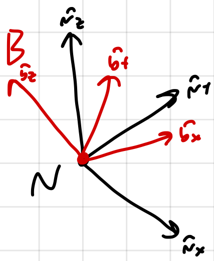




$$a_N = \begin{bmatrix} \vec{a} \cdot \hat{n}_x \\ \vec{a} \cdot \hat{n}_y \\ \vec{a} \cdot \hat{n}_z \end{bmatrix}$$

Attitude is all about expressing the relative orientations of two sets of basis. The most obvious way to do this is by storing all 9 of the dot products between the basis vectors in a Direction Cosine Matrix (DCM), also known as a rotation matrix.

Direction Cosine Matrix, or Rotation Matrix



$${}^{N}Q^{B} = \begin{array}{c|ccc|} & b_x & b_y & b_z \\ n_x & (\hat{n}_x \cdot b_x) & (\hat{n}_x \cdot b_y) & (\hat{n}_y \cdot b_z) \\ n_y & (\hat{n}_y \cdot b_x) & (\hat{n}_y \cdot b_y) & (\hat{n}_y \cdot b_z) \\ n_z & (\hat{n}_z \cdot b_x) & (\hat{n}_z \cdot b_y) & (\hat{n}_z \cdot b_z) \end{array} \in SO(3)$$

— row: $\hat{n}_x$ resolved in B frame

col: b_x expressed in N frame

"Special" - $\det(Q) = 1$

"orthogonal" - $Q^{-1} = Q^T$, $Q^T Q = I$

3×3 matrix

$$n_i \cdot b_j = \overset{1}{\cancel{|n_i|}}\,\overset{1}{\cancel{|b_j|}} \cos\theta$$

A dot product between two unit vectors is just the cos of the angle between them.

$${}^{B}Q^{N} = \left({}^{N}Q^{B}\right)^T$$

DCM's are orthogonal, so you can just transpose it to get the inverse. Q' = inv(Q).

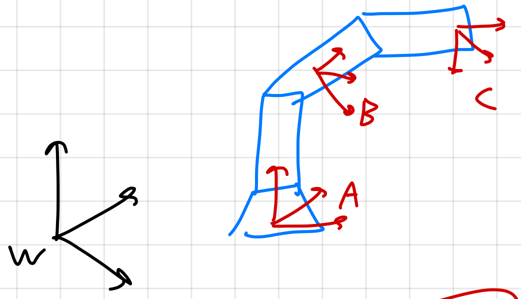
Resolve vectors in N or B

$$\underbrace{a_N}_{\vec{a}\text{ expressed in N frame}} = {}^{N}Q^{B} \underbrace{a_B}_{\vec{a}\text{ expressed in B frame}}$$

We can use the DCM to take vectors expressed in one basis, and resolve them in another.

Composing Attitude

Remember: $(ABC)^T = C^T B^T A^T$



$${}^WQ^C = {}^WQ^A\ {}^AQ^B\ {}^BQ^C$$

$${}^CQ^W = \left[{}^WQ^C\right]^T$$

$${}^CQ^W = ({}^BQ^C)^T({}^AQ^B)^T({}^WQ^A)^T$$

$${}^CQ^W = {}^CQ^B\ {}^BQ^A\ {}^AQ^W \quad \checkmark$$

We can compose DCM's with simple matrix multiplication. When we transpose it, we see that everything still makes sense. Always line up the frames the way they are here.

$$\xcancel{R_{A/B},\ R^{A}_{B},\ R_{A\to B}}$$

← you will see all sorts of garbage notation

Skew, cross product, matrix, hat map

$$\hat a = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \quad a \times b = \hat a b$$

$$\overset{\vee}{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

also have an "unhat" $\overset{\vee}{A} = a$ to turn the skew back to a vector

$$a_N = {}^NQ^B a_B$$

$$\hat a_N = {}^NQ^B\, \hat a_B ({}^NQ^B)^T$$ – Useless, but shows a nice connection to quaternion operation later

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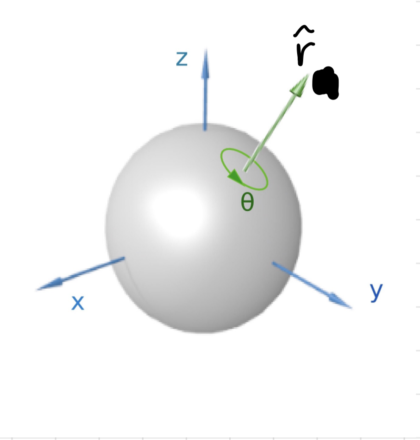
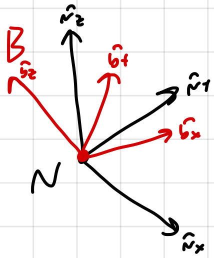
Store attitude as a rotation



euler:

Any 2 frames N and B, I can get from one to the other with a simple rotation about one axis.



"axis-angle"

$$\phi = \hat r \cdot \theta$$

$\phi \in \mathbb{R}^3$
$\hat r \in \mathbb{R}^3$
$\theta \in \mathbb{R}$

+ only 3 params
- what is $\hat r$ at $\theta = 0$?
- at 180°
- bad kinematics

Convert between

Q and ϕ using

matrix exp, matrix log

$$^{\sim}\phi^{B} = \overbrace{\left[\log\left(^{\sim}Q^{B}\right)\right]}$$

$$\Updownarrow$$

$$^{\sim}Q^{B} = \text{expm}\left(^{\sim}\hat{\phi}^{B}\right) \rightarrow \text{"Rodrigues formula"}$$

Quaternions

$$q = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ \hat{r} \cdot \sin\left(\frac{\theta}{2}\right) \end{bmatrix} \begin{matrix} \text{Scalar part} \\ \\ \text{Vector part} \end{matrix} \in \mathbb{R}^4$$

(technically $\mathbb{H}$ for quats, $\mathbb{S}^3$ for unit quaternions)

Compose

$$^{\sim}Q^{B} = {}^{\sim}Q^{A}\,{}^{A}Q^{B}$$ - Rotation matrices

$$^{\sim}q^{B} = {}^{\sim}q^{A} \odot {}^{A}q^{B}$$

$$\begin{bmatrix} s_1 \\ v_1 \end{bmatrix} \odot \begin{bmatrix} s_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} s_1 s_2 - v_1^T v_2 \\ s_1 v_2 + s_2 v_1 + v_1 \times v_2 \end{bmatrix}$$

Invert ("Conjugate")

$$^{\sim}Q^{B} = \left({}^{B}Q^{\sim}\right)^T$$

$$^{\sim}q^{B} = \left({}^{B}q^{\sim}\right)^{+} = \begin{bmatrix} s \\ -v \end{bmatrix}$$

Q multiply as matrix multiply

$$q_1 \odot q_2 = L(q_1) q_2$$

$$= \underbrace{\left[\begin{array}{c|c} s_1 & -v_1^T \\ \hline v_1 & s_1 I_3 + \hat{v}_1 \end{array}\right]}_{L(q_1)} \begin{bmatrix} s_2 \\ v_2 \end{bmatrix}$$

$$q_1 \odot q_2 = R(q_2) q_1$$

$$q_1 \odot q_2 = \underbrace{\begin{bmatrix} s_2 & -v_2^T \\ v_2 & s_2 I_3 - \hat{v}_2 \end{bmatrix}}_{R(q_2)} \begin{bmatrix} s_1 \\ v_1 \end{bmatrix}$$

$$q_1^{+} \odot q_2 = L(q_1)^T q_2$$

Resolve vectors with quats

$$a_N = {}^{\sim}Q^{B} a_B$$ - DCM's

$$\begin{bmatrix} a_N \\ 0 \end{bmatrix} = {}^{\sim}q^{B} \odot \begin{bmatrix} a_b \\ 0 \end{bmatrix} \odot \left({}^{\sim}q^{B}\right)^{+}$$

$$a_N = \underbrace{H^T L\left({}^{\sim}q^{B}\right) R\left({}^{\sim}q^{B}\right)^T H}_{^{\sim}Q^{B}} x_B$$

$$\hat{a}_N = {}^{\sim}Q^{B} \hat{a}_B \left({}^{\sim}Q^{B}\right)^T \qquad \hat{a} = \begin{bmatrix} 0 \\ a \end{bmatrix} \in \mathbb{R}^4$$

$$\hat{a}_N = {}^{\sim}q^{B} \odot \hat{a}_b \odot \left({}^{\sim}q^{B}\right)^{+}$$