Homework 6

Instructions: In problems the problems below, references such as III.2.7 refer to Problem 7 in Section 2 of Chapter III in Conway's book.

If you use results from books including Conway's, please be explicit about what results you are using.

Homework 6 is due in class at Midnight Monday, April 23.

Do the following problems:

1. VI.1.4 Notice that this is related to a problem on the second midterm.

   Proof. If not, suppose for each $\epsilon > 0$, there is a polynomial $p(z)$, s.t. $\sup\{|p(z) - z^{-1}| : z \in A\} < \epsilon$. We claim that $|p(z) - z^{-1}| \le \epsilon$ for all $z \in A$. Otherwise, by the continuity of $|p(z)|$, $\exists z_0 \in O(z, r) \cap A$, s.t. $|p(z_0) - z^{-1}| > \epsilon$, which makes a contradiction.

   Now pick $\epsilon = \frac{1}{2R}$, then in A, $|p(z) - \frac{1}{z}| \le \frac{1}{2R}$, which means

   $$|zp(z) - 1| \le \frac{|z|}{|2R|} \le \frac{1}{2}$$

   on A, hence on $\{|z| = R\} \subset \partial A$. Hence, by Maximum Modulus Thm, since $g(z) = zp(z) - 1$ is analytic in $O(0, R)$, we know for each $z \in O(0, R)$, $|g(z)| < \max |g(\partial A)| \le \frac{1}{2}$. However, $|g(0)| = 1 > \frac{1}{2}$, which makes a contradiction.

2. VI.1.5

   Proof. Let $\{z_k\}_{k=1}^n$ be zeros of f in $B(0, \frac{1}{3}R)$, and

   $$g(z) = \frac{\prod_{k=1}^n z_k}{\prod_{k=1}^n (z - z_k)} f(z),$$

   then $g(z)$ is analytic in $B(0, R)$, and $g(0) = f(0) = a$. Hence by Max Modulus Thm,

   $$a = |g(0)| \le \left| \frac{\prod_{k=1}^n z_k}{\prod_{k=1}^n (z - z_k)} f(z) \right|_{z \in \{|z| = R\}} \le \frac{(\frac{1}{3}R)^n}{(\frac{2}{3}R)^n} M = \frac{1}{2^n} M.$$

   Then $n \le \frac{\log(M/a)}{\log 2}$.

3. VI.1.6

   Let $h(z) = \frac{f(z)}{g(z)}$, then since f, g never vanish in $B(0, R)$, h and $H = \frac{1}{h}$ are both analytic in $B(0, R)$. Then by M.M.T,

   $$|h(z)| \le \left| \frac{f(z)}{g(z)} \right|_{|z|=1} = 1,$$

   and

   $$|H(z)| = \left| \frac{1}{h(z)} \right| \le \left| \frac{g(z)}{f(z)} \right|_{|z|=1} = 1,$$

   we know $|h(z)| \equiv 1$ in $B(0, R)$, and by M.M.T, $h(z)$ is a constant λ, and $|\lambda| = 1$.

4. VI.2.5

   (a) Consider

   $$\varphi(z) = \varphi_{z_1}(z) \cdots \varphi_{z_n}(z),$$

   where $\varphi_{z_k}(z)$ is the Mobius transformation

   $$\varphi_{z_k}(z) = \frac{z - z_k}{1 - \bar{z}_k z}.$$

Then

$$g(z) = \frac{f(z)}{\varphi(z)}$$

is analytic in D. Now we only need to show that $|g| \le M$ in D. If not, assume there is a point $z_0 \in D$, s.t. $|g(z_0)| = M_0 > M$. Then for any $|z_0| < r < 1$, by max modulus thm,

$$\max_{|z|=r} |h(z)| \ge M_0.$$

Hence there is some $z_r \in \{|z| = r\}$, s.t.

$$M \ge |f(z_r)| \ge M_0 \varphi(z_r).$$

Let $r \to 1^-$, take limit on both side, we get

$$M \ge M_0 \lim_{r \to 1^-} \varphi(z_r) = M_0$$

as $|\varphi| = 1$ on ∂D, which leads to a contradiction. Hence, $|f| \le M|\varphi|$.

(b) With the same notation as (a), we notice

$$h(0) = \frac{f(0)}{\varphi(0)} = (-1)^n M,$$

which means $|h(0)| = M$. But we have shown $|h(z)| \le M$ on D, hence by M.M.T.,

$$h(z) \equiv (-1)^n M, \forall z \in D.$$

Hence

$$f(z) = h(z)\varphi(z) = (-1)^n M \varphi(z).$$

5. VI.2.8

Sol. Let $g(z) = \varphi_{\frac{1}{2}} \circ f(z)$, then

$$g(0) = 0,\ g'(0) = \frac{-f(0)f'(0) + \frac{5}{4}f'(0)}{(1 - \frac{1}{2}f(0))^2} = 1,$$

and g is analytic in D. Hence by Schwarz's Thm, there is some $|c| = 1$,

$$g(z) = cz = \varphi_{\frac{1}{2}} \circ f(z),$$

which means

$$f(z) = \varphi_{-\frac{1}{2}}(cz),$$

where $|c| = 1$. Thus the solution is not unique.

6. VII.2.1

Proof. If $f_n \to f$, then since γ is compact, we know $f_n \to f$ uniformly on γ.

Conversely, for each $z \in G$, let γ be a circle $C(z, r) \subset G$, then by Cauchy's theorem,

$$f_n(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(w)}{w - z} dw,$$

hence

$$|f(z) - f_n(z)| = \frac{1}{2\pi} \left| \int_\gamma \frac{f_n(w) - f(w)}{w - z} dw \right| \le 2 \sup_{w \in \gamma} \{|f(w) - f_n(w)|\}$$

For each $\epsilon > 0$, since $f_n \to f$ uniformly on γ, there is a $N > 0$, for each $n \ge N$ and each $w \in \gamma$, $|f_n(w) - f(w)| < \frac{1}{2}\epsilon$. Hence,

$$|f_n(z) - f(z)| < \epsilon.$$

Thus $f_n \to f$ on every $z \in G$.

7. VII.2.4

   Proof. Since $\{f_n\}$ is locally bounded, we know by Montel's theorem that $\{f_n\}$ is normal. Then each subsequence of $\{f_n\}$ has a subsequence that converges to an analytic function, and these functions are equal in A which has a limit point. Hence these limit functions are equal in G, which means $f_n \to f$.

8. VII.2.5

   (b) to (a) In fact it is trivial since uniform boundness leads to local boundness.

   (a) to (b) I don't think it is right. For example, let $f_n(z) = nz^n$ and G be the unit disk D, then of course $\{f_n\}$ is locally bounded by Lemma 2.8. Hence by Montel's theorem, $\{f_n\}$ is normal. However, since

   $$\lim_{z \to \partial D} |f_n(z)| = n,$$

   for each $\epsilon > 0$ and each $c > 0$, we can pick some n large enough and z sufficiently close to ∂D, and $|cf_n(z)| > \epsilon$.