Homework 2

Instructions:References such as III.2.7 refer to Problem 7 in Section 2 of Chapter III in Conway's book. If you use results from books including Conway's, please be explicit about what results you are using.
Reminder: Exam I will be from 6:30 to 10:30 on Thursday, February 15 in Room 40 of Schaeffer Hall.

Homework 3 is due in your Dropbox folder by 11:59, Sunday , February 19.

Working on this homework will help you with Exam I, so please don't put it off until after the exam.

1. Problem IV.2.4

(a) By Abel's transform, let $\{a_n\}, \{b_n\}$ be two sequences, and $B_k = \sum_{i=1}^{k} b_i$. Then

$$\sum_{k=1}^{n} a_k b_k = a_n B_n - \sum_{k=1}^{n-1} (a_{k+1} - a_k) B_k.$$

Hence for each fixed n, denote $\sum_{k=1}^{n} a_k$ by A_n

$$C_n = \lim_{r \to 1^-} \sum_{k=1}^{n} a_k r^k = r^n A_n - \sum_{k=1}^{n-1} r^k (r-1) A_k.$$

Since $\sum a_n (z-a)^n$ have radius of convergence 1,

$$\lim_{r \to 1^-} \sum_{n=1}^{\infty} a_n r^n < \infty,$$

then we can change the order of limits:

$$\lim_{r \to 1^-} \lim_{n \to \infty} \sum_{k=1}^{n} a_k r^k = \lim_{n \to \infty} \lim_{r \to 1^-} \sum_{k=1}^{n} a_k r^k = \lim_{n \to \infty} A_n$$

since each A_k is a finite number, which comes from $\sum a_n$ converges to A. Hence,

$$\lim_{r \to 1^-} \sum_{n=1}^{\infty} a_n r^n = \lim_{n \to \infty} A_n = A.$$

(b) Consider $a_n = \frac{(-1)^{n+1}}{n}$, then by Proposition III.1.4,

$$\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim \frac{n+1}{n} = 1.$$

Hence the series $\sum a_n (z-a)^n$ have radius of convergence 1. Since the series $\sum a_n$ is a Leibniz series, then it converges to $A < \infty$.
Now consider the function $f(z) = \log z$, it is analytic on $|z-1| < 1$, and it has power series expansion

$$f(z) = \sum_{n=0}^{\infty} b_n (z-1)^n,$$

where $b_n = \frac{1}{n!} f^{(n)}(1) = \frac{(-1)^{n-1}}{n} (n \geq 1)$, and $b_0 = 0$. By this we can find $a_i = b_i$ for each $i \geq 0$, thus

$$\sum a_n = f(2) = \log 2$$

2. Problem IV.2.6

   Sol. In the region where $f(z) = \sqrt{z}$ is analytic,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-1)^n,$$

   where

$$a_n = \frac{1}{n!} f^{(n)}(1) = \frac{(-1)^{(n-1)}}{n!} \frac{(2n-3)!!}{2^n} (n \geq 1),\ a_0 = 1.$$

   and since

$$\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \frac{2n+2}{2n-1} = 1,$$

   we know the radius of convergence is 1.

3. Problem IV.2.9
4. Problem IV.2.11
5. Problem IV.3.3
6. Problem IV.3.6
7. Problem IV.3.14
8. Problem IV.4. 2
9. Problem IV. 4.3
10. Problem IV.5.7
11. Problem IV.5.9