Statistics of Solutions to A Stochastic Differential Equation Set

Chuan Lu

Information and Computing Science

復旦大學

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Outline

Introduction

The SDEs

$$
\begin{cases}
\dfrac{du(t)}{dt} = (-\gamma(t) + i\omega)u(t) + b(t) + f(t) + \sigma\, \dot{W}(t), \\
\dfrac{db(t)}{dt} = (-\gamma_b + i\omega_b)(b(t) - \hat{b}) + \sigma_b\, \dot{W}_b(t), \\
\dfrac{d\gamma(t)}{dt} = -d_\gamma(\gamma(t) - \hat{\gamma}) + \sigma_\gamma\, \dot{W}_\gamma(t)
\end{cases}
$$

The initial values are complex random variables, with their first-order and second-order statistics known.

Solution

With knowledge of ODEs, the solution of the SDE set is

$$\begin{cases} b(t) = \hat{b} + (b_0 - \hat{b})\, e^{\lambda_b (t-t_0)} + \sigma_b \int_{t_0}^{t} e^{\lambda_b (t-s)}\, dW_b(s) \\ \gamma(t) = \hat{\gamma} + (\gamma_0 - \hat{\gamma})\, e^{-d_\gamma (t-t_0)} + \sigma_\gamma \int_{t_0}^{t} e^{-d_\gamma (t-s)}\, dW_\gamma(s) \\ u(t) = e^{-J(t_0,t) + \hat{\lambda}(t-t_0)}\, u_0 + \int_{t_0}^{t} (b(s) + f(s))\, e^{-J(s,t) + \hat{\lambda}(s-t_0)}\, ds \\ \qquad + \sigma \int_{t_0}^{t} e^{-J(s,t) + \hat{\lambda}(s-t_0)}\, dW(s) \end{cases}$$

with $\lambda_b = -\gamma_b + i\omega_b$, $\hat{\lambda} = -\hat{\gamma} + i\omega$, $J(s,t) = \int_s^t (\gamma(s') - \hat{\gamma})\, ds'$.

Itô Isometry and Itô Formula

Itô Isometry

$\forall f \in \mathscr{V}(S,T)$, B_t is a standard Brownian motion,

$$\mathsf{E}\left[\left(\int_S^T f(t,\omega)\,dB_t\right)^2\right] = \mathsf{E}\left[\int_S^T f^2(t,\omega)\,dt\right].$$

Itô Formula

Assume that X_t is a Itô process satisfying $dX_t = udt + vdB_t$, $g(t,x) \in C^2([0,\infty)\times\mathbb{R})$, then $Y_t = g(t,X_t)$ is also a Itô process satisfying

$$dY_t = \frac{\partial g}{\partial t}(t,X_t)\,dt + \frac{\partial g}{\partial x}(t,X_t)\,dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)\cdot(dX_t)^2,$$

with $dt\cdot dt = dt\cdot dB_t = dB_t\cdot dt = 0$, $dB_t\cdot dB_t = dt$.

Linear property of Itô integration

$$(1)\quad \int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t, \quad \text{a.e.}$$

$$(2)\quad \int_S^T (cf + g)\, dB_t = c\int_S^T f dB_t + \int_S^T g dB_t, \quad \text{a.e.}$$

$$(3)\quad \mathsf{E}\left[\int_S^T f dB_t\right] = 0.$$

Outline

1. Introduction

2. Statistics of $b(t)$ and $\gamma(t)$

3. Statistics of $u(t)$

Mean

With property of Itô integration (3), it is easy to know

$$\mathsf{E}(b(t)) = \hat{b} + (\mathsf{E}[b_0] - \hat{b})\, e^{\lambda_b (t - t_0)}$$
$$\mathsf{E}(\gamma(t)) = \hat{\gamma} + (\mathsf{E}[\gamma_0] - \hat{\gamma})\, e^{-d_\gamma (t - t_0)}$$

Variance

According to definition,

$$
\begin{aligned}
\mathsf{Var}(b(t)) &= \mathsf{E}\left[(b(t)-\mathsf{E}[b(t)])(b(t)-\mathsf{E}[b(t)])^*\right] \\
&= e^{-2\gamma_b(t-t_0)}\mathsf{Var}(b_0) + \mathsf{E}\left[\sigma_b^2 \int_{t_0}^{t} e^{\lambda_b(t-s)}\, dW_b(s) \left(\int_{t_0}^{t} e^{\lambda_b(t-s)}\, dW_b(s)\right)^*\right] \\
&= e^{-2\gamma_b(t-t_0)}\mathsf{Var}(b_0) + \sigma_b^2 \mathsf{E}\left[\int_{t_0}^{t} e^{-2\gamma_b(t-s)}\, ds\right]
\end{aligned}
$$

The last step takes advantage of Itô isometry.

$$
\mathsf{Var}(b(t)) = e^{-2\gamma_b(t-t_0)}\mathsf{Var}(b_0) + \frac{\sigma_b^2}{2\gamma_b}(1-e^{-2\gamma_b(t-t_0)})
$$

$$
\mathsf{Var}(\gamma(t)) = e^{-2d_\gamma(t-t_0)}\mathsf{Var}(\gamma_0) + \frac{\sigma_\gamma^2}{2d_\gamma}(1-e^{-2d_\gamma(t-t_0)})
$$

Covariance

$$
\begin{aligned}
\mathsf{Cov}(b(t), b(t)^*) &= \mathsf{E}\left[(b(t) - \mathsf{E}[b(t)])(b(t)^* - \mathsf{E}[b(t)^*])\right] \\
&= \mathsf{E}\left[(b_0 - \mathsf{E}[b_0])(b_0^* - \mathsf{E}[b_0^*])e^{2\lambda_b(t-t_0)}\right] + \sigma_b \mathsf{E}\left[(b_0 - \mathsf{E}[b_0])\int_{t_0}^{t} e^{\lambda_b(t-s)}\, dW_b(s)\right] \\
&+ \sigma_b \mathsf{E}\left[(b_0^* - \mathsf{E}[b_0^*])e^{\lambda_b(t-t_0)}\int_{t_0}^{t} e^{\lambda_b(t-s)}\, dW_b(s)\right] + \sigma_b^2 \mathsf{E}\left[(\int_{t_0}^{t} e^{\lambda_b(t-s)}\, dW_b(s))^2\right]
\end{aligned}
$$

With property of Itô integration (3), the second and third term are both 0; with Itô isometry we know the last term is also 0.

$$\mathsf{Cov}(b(t), b(t)^*) = \mathsf{E}[(b_0 - \mathsf{E}[b_0])(b_0^* - \mathsf{E}[b_0^*])]e^{2\lambda_b(t-t_0)} = \mathsf{Cov}(b_0, b_0^*)e^{2\lambda_b(t-t_0)}$$

$$\mathsf{Cov}(b(t), \gamma(t)) = \mathsf{E}[(b(t) - \mathsf{E}[b(t)])(\gamma(t) - \mathsf{E}[\gamma(t)])] = \mathsf{Cov}(b_0, \gamma_0)e^{(\lambda_b - d_\gamma)(t-t_0)}$$

Outline

Mean

Using the same properties, it's obvious to have

$$\mathsf{E}[u(t)] = e^{\hat{\lambda}(t-t_0)}\mathsf{E}\left[e^{-J_0(t_0,t)}u_0\right] + \int_{t_0}^{t} e^{\hat{\lambda}(t-s)}\mathsf{E}\left[b(s)e^{-J(s,t)}\right]ds$$
$$+\sigma\int_{t_0}^{t} e^{\hat{\lambda}(t-s)}f(s)\mathsf{E}\left[e^{-J(s,t)}\right]ds.$$

It is necessary to compute expectations of terms like $\mathsf{E}[ze^{bx}]$, z is a complex-valued Gaussian random variable and x is a real-valued Gaussian variable. We propose two lemmas here.

Lemma 1

Lemma

$$E\left[ze^{ibx}\right] = (E[z] + ibCov(z,x))\, e^{ibE[x] - \frac{1}{2}b^2 Var(x)}$$

with z being a complex-valued Gaussian, and x a real-valued Gaussian.

Corollary

Under the condition of Lemma 1,

$$E\left[ze^{bx}\right] = (E[z] + bCov(z,x))\, e^{bE[x] + \frac{1}{2}b^2 Var(x)}.$$

Proof of lemma 1 takes advantage of the characteristic function of multivariable Gaussian distribution.

Proof

Let $z = y + iw, \quad y, w \in \mathbb{R}$. Denote $\mathbf{v} = (x, y, w)$, then $\mathbf{v}$ satisfies the multivariable Gaussian distribution, with its characteristic function

$$\phi_{\mathbf{v}}(\mathbf{s}) = \exp(i\mathbf{s}^\top \mathsf{E}[\mathbf{v}] - \frac{1}{2}\mathbf{s}^\top \Sigma \mathbf{s}).$$

Let $g(\mathbf{v})$ being the PDF of $\mathbf{v}$, then one knows from that char. func. being Fourier transform of PDF,

$$\phi_{\mathbf{v}}(\mathbf{s}) = \frac{1}{(2\pi)^3} \int e^{i\mathbf{s}^\top \mathbf{v}} g(\mathbf{v})\, d\mathbf{v}$$

According to the differential property of Fourier transform,

$$\frac{\partial \phi_{\mathbf{v}}(\mathbf{s})}{\partial s_2} = \frac{1}{(2\pi)^3} \int i y_0\, e^{i\mathbf{s}^\top \mathbf{v}} g(\mathbf{v})\, d\mathbf{v} = i\mathsf{E}\left[y_0\, e^{i\mathbf{s}^\top \mathbf{v}}\right].$$

Let $\mathbf{v} = (b, 0, 0)^\top$,

$$\mathsf{E}[y_0\, e^{ibx_0}] = -i \frac{\partial \phi_{\mathbf{v}}(s)}{\partial s_2}\bigg|_{\mathbf{s}=(b,0,0)^\top}$$

$$\mathsf{E}[y_0\, e^{ibx_0}] = -i \frac{\partial \phi_{\mathbf{v}}(s)}{\partial s_2}\bigg|_{\mathbf{s}=(b,0,0)^\top}$$

Proof

From PDF of multivariable Gaussian distribution, one knows

$$\frac{\partial \phi_{\mathbf{v}}(\mathbf{s})}{\partial s_2} = (i\mathsf{E}[y_0] - \mathsf{Var}(y_0)s_2 - \mathsf{Cov}(x_0, y_0)s_1 - \mathsf{Cov}(y_0, w_0)s_3)\phi_{\mathbf{v}}(s)$$
$$\frac{\partial \phi_{\mathbf{v}}(\mathbf{s})}{\partial s_3} = (i\mathsf{E}[w_0] - \mathsf{Var}(w_0)s_3 - \mathsf{Cov}(x_0, w_0)s_1 - \mathsf{Cov}(y_0, w_0)s_2)\phi_{\mathbf{v}}(s)$$

Compute the partial derivatives at $\mathbf{s} = (b, 0, 0)^\top$,

$$\mathsf{E}\left[y_0 e^{ibx_0}\right] = (\mathsf{E}[y_0] + i\mathsf{Cov}(x_0, y_0)b)\exp(ib\mathsf{E}[x_0] - \frac{1}{2}\mathsf{Var}(x_0)b^2)$$
$$\mathsf{E}\left[w_0 e^{ibx_0}\right] = (\mathsf{E}[w_0] + i\mathsf{Cov}(x_0, w_0)b)\exp(ib\mathsf{E}[x_0] - \frac{1}{2}\mathsf{Var}(x_0)b^2)$$

Then

$$\mathsf{E}\left[ze^{ibx}\right] = (\mathsf{E}[z] + ib\mathsf{Cov}(z, x))\, e^{ib\mathsf{E}[x] - \frac{1}{2}b^2\mathsf{Var}(x)}.$$

□

Lemma 2

Lemma

$$E\left[zwe^{bx}\right] = [E[z]E[w] + Cov(z, w^*) + b(E[z]Cov(w,x)) + E[w]Cov(z,x) + b^2 Cov(z,x)Cov(w,x)]\, e^{bE[x] + \frac{b^2}{2} Var(x)}.$$

with z, w being complex-valued Gaussian, and x real-valued Gaussian.