

# Statistics of Solutions to A Stochastic Differential Equation Set

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Introduction

Statistics of  $b(t)$  and  $\gamma(t)$

## The SDEs

$$\begin{cases} \frac{du(t)}{dt} = (-\gamma(t) + i\omega)u(t) + b(t) + f(t) + \sigma W(t), \\ \frac{db(t)}{dt} = (-\gamma_b + i\omega_b)(b(t) - \hat{b}) + \sigma_b W_b(t), \\ \frac{d\gamma(t)}{dt} = -d_\gamma(\gamma(t) - \hat{\gamma}) + \sigma_\gamma W_\gamma(t) \end{cases}$$

The initial values are complex random variables, with their first-order and second-order statistics known.

## Solution

With knowledge of ODEs, the solution of the SDE set is

$$\left\{ \begin{array}{l} b(t) = \hat{b} + (b_0 - \hat{b})e^{\lambda_b(t-t_0)} + \sigma_b \int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s) \\ \gamma(t) = \hat{\gamma} + (\gamma_0 - \hat{\gamma})e^{-d_\gamma(t-t_0)} + \sigma_\gamma \int_{t_0}^t e^{-d_\gamma(t-s)} dW_\gamma(s) \\ u(t) = e^{-J(t_0,t) + \hat{\lambda}(t-t_0)} u_0 + \int_{t_0}^t (b(s) + f(s)) e^{-J(s,t) + \hat{\lambda}(s-t_0)} ds \\ \quad + \sigma \int_{t_0}^t e^{-J(s,t) + \hat{\lambda}(s-t_0)} dW(s) \end{array} \right.$$

with  $\lambda_b = -\gamma_b + i\omega_b$ ,  $\hat{\lambda} = -\hat{\gamma} + i\omega$ ,  $J(s,t) = \int_s^t (\gamma(s') - \hat{\gamma}) ds'$ .

# Itô Isometry and Itô Formula

## Itô Isometry

$\forall f \in \mathcal{V}(S, T)$ ,  $B_t$  is a standard Brownian motion,

$$\mathbb{E} \left[ \left( \int_S^T f(t, \omega) dB_t \right)^2 \right] = \mathbb{E} \left[ \int_S^T f^2(t, \omega) dt \right].$$

## Itô Formula

Assume that  $X_t$  is a Itô process satisfying  $dX_t = udt + vdB_t$ ,  
 $g(t, x) \in C^2([0, \infty) \times \mathbb{R})$ , then  $Y_t = g(t, X_t)$  is also a Itô process satisfying

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X_t) \cdot (dX_t)^2,$$

with  $dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, dB_t \cdot dB_t = dt$ .

# Outline

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