Statistics of Solutions to A Stochastic Differential Equation Set

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Outline

The SDEs

$$\begin{cases} \dfrac{du(t)}{dt} = (-\gamma(t) + i\omega)u(t) + b(t) + f(t) + \sigma W(t), \\ \dfrac{db(t)}{dt} = (-\gamma_b + i\omega_b)(b(t) - \hat{b}) + \sigma_b W_b(t), \\ \dfrac{d\gamma(t)}{dt} = -d_\gamma(\gamma(t) - \hat{\gamma}) + \sigma_\gamma W_\gamma(t) \end{cases}$$

The initial values are complex random variables, with their first-order and second-order statistics known.

Solution

With knowledge of ODEs, the solution of the SDE set is

$$
\begin{cases}
b(t) = \hat{b} + (b_0 - \hat{b})\, e^{\lambda_b (t-t_0)} + \sigma_b \int_{t_0}^{t} e^{\lambda_b (t-s)}\, dW_b(s) \\
\gamma(t) = \hat{\gamma} + (\gamma_0 - \hat{\gamma})\, e^{-d_\gamma (t-t_0)} + \sigma_\gamma \int_{t_0}^{t} e^{-d_\gamma (t-s)}\, dW_\gamma(s) \\
u(t) = e^{-J(t_0,t)+\hat{\lambda}(t-t_0)}\, u_0 + \int_{t_0}^{t} (b(s) + f(s))\, e^{-J(s,t)+\hat{\lambda}(s-t_0)}\, ds \\
\qquad + \sigma \int_{t_0}^{t} e^{-J(s,t)+\hat{\lambda}(s-t_0)}\, dW(s)
\end{cases}
$$

with $\lambda_b = -\gamma_b + i\omega_b$, $\hat{\lambda} = -\hat{\gamma} + i\omega$, $J(s,t) = \int_s^t (\gamma(s') - \hat{\gamma})\, ds'$.

Itô Isometry and Itô Formula

Itô Isometry

$\forall f \in \mathscr{V}(S,T)$, B_t is a standard Brownian motion,

$$\mathsf{E}\left[\left(\int_S^T f(t,\omega)\,dB_t\right)^2\right] = \mathsf{E}\left[\int_S^T f^2(t,\omega)\,dt\right].$$

Itô Formula

Assume that X_t is a Itô process satisfying $dX_t = udt + vdB_t$, $g(t,x) \in C^2([0,\infty)\times\mathbb{R})$, then $Y_t = g(t,X_t)$ is also a Itô process satisfying

$$dY_t = \frac{\partial g}{\partial t}(t,X_t)\,dt + \frac{\partial g}{\partial x}(t,X_t)\,dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)\cdot(dX_t)^2,$$

with $dt\cdot dt = dt\cdot dB_t = dB_t\cdot dt = 0, dB_t\cdot dB_t = dt.$

Outline

1. Introduction

2. Statistics of $b(t)$ and $\gamma(t)$