

Homework 4

Instructions: In problems the problems below, references such as III.2.7 refer to Problem 7 in Section 2 of Chapter III in Conway's book.

If you use results from books including Conway's, please be explicit about what results you are using.

Homework 4 is due in class at Midnight March 9.

Do the following problems:

1. IV.7.1

Sol. (Discussed with a college classmate) In fact, I don't think that this proposition is correct. For example, pick G the unit disk $B(0, 1)$, and $\gamma = \gamma(t) : [0, 1] \rightarrow B$, s.t. $\gamma(t) = t$ for $0 \leq t < 1$, and $\gamma(1) = 0$. Then γ is closed, and by simple calculation we know $V(\gamma) = 2$, which shows γ is rectifiable. Let $f = \frac{1}{z-1}$, then f is analytic in $B(0, 1)$. But when $t \rightarrow 1$, $f \circ \gamma(t) \rightarrow \infty$, hence it is not rectifiable.

2. IV.7.2

(a) Let $f(z) = z$, pick any $z_0 \in \{z \mid d(z, \partial G) < \frac{1}{2}r\}$, then since there is only one point $z = z_0$ satisfies $f(z) = z_0$, by Thm 7.2,

$$n(\gamma; z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - z_0} dz$$

Since $\frac{1}{z-z_0}$ is analytic on $\{z \mid d(z, \partial G) < \frac{1}{2}r\}$, by Prop 2.15, we know the integral is 0. Hence $\{z \mid d(z, \partial G) < \frac{1}{2}r\} \subset H$.

3. V.1.1

(a) Around $z = 0$,

$$\lim_{z \rightarrow 0} |zf(z)| = \lim_{z \rightarrow 0} |\sin(z)| = \frac{1}{2} \lim_{z \rightarrow 0} |e^{iz} - e^{-iz}| \leq \lim_{z \rightarrow 0} |z| = 0.$$

Hence by Thm 1.2, $z = 0$ is removable, and $f(0) = 1$ by power series expansion.

(b) At $z = 0$, $g(z) = \cos(z)$ is analytic, and $\cos(0) = 1$. Thus by Prop 1.4, $z = 0$ is a pole, and the singular part is $\frac{1}{z}$.

(c) At $z = 0$, $\lim_{z \rightarrow 0} zf(z) = \lim_{z \rightarrow 0} \cos z - 1 = 0$, then by Thm 1.2, 0 is removable, and $f(0) = 0$ by power series expansion.

(d) At $z = 0$,

$$f(z) = \sum_{n=0}^{-\infty} \frac{1}{(-n)!} z^n,$$

hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \{z \mid |z| > \exp(\frac{1}{\delta})\}$.

(e) At $z = 0$,

$$f(z) = \frac{1}{z^2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+2} z^n.$$

Hence 0 is a pole, and the singularity part is $\frac{1}{z}$.

(f) At $z = 0$,

$$f(z) = z \sum_{n=0}^{\infty} (-1)^n \frac{z^{-2n}}{n!} = z + \sum_{n=-1}^{-\infty} (-1)^{-n} \frac{z^{2n+1}}{(-n)!}$$

Hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \mathbb{C}$.

(g) Around $z = 0$, notice $\frac{z^2+1}{z-1}$ is analytic, hence 0 is a pole. Since $|z| < 1$,

$$f(z) = 1 - \frac{1}{z} + \frac{2}{z-1} = 1 - \frac{1}{z} - 2 \sum_{n=0}^{\infty} z^n,$$

we know the singular part is $-\frac{1}{z}$.

(h) For any $n > 0$,

$$\lim_{z \rightarrow 0} z^n f(z) = \lim_{z \rightarrow 0} z^n \frac{1}{\sum_{n=1}^{\infty} \frac{1}{n!} z^n} = \infty,$$

hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \{z \mid |z| > \frac{1}{1-e^\delta}\}$.

(i)

$$f(z) = z \sum_{n=0}^{\infty} (-1)^n \frac{z^{-(2n+1)}}{(2n+1)!} = 1 + \sum_{n=-1}^{-\infty} (-1)^{-n} \frac{z^{2n-1}}{(-2n+1)!},$$

hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \{z \mid |z| < \delta\}$.

(j) Same with (i), 0 is an essential singularity, and $f(0 < |z| < \delta) = \{z \mid |z| < \delta^n\}$.

4. V.1.4

(a)

$$f(z) = \frac{1}{z} \left(\frac{1}{1-z} - \frac{1}{2(1-z/2)} \right) = \frac{1}{z} \left(\sum_{n=0}^{\infty} z^n - \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \right) = \frac{1}{2z} + \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+2}}\right) z^n$$

(b)

$$f(z) = \frac{1}{z} \left(\frac{1}{z-2} - \frac{1}{z-1} \right) = \frac{1}{z} \left(-\frac{1}{2} \frac{1}{1-\frac{z}{2}} - \frac{1}{1-\frac{1}{z}} \right) = \frac{1}{z} \left(-\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty} z^{-n-1} \right) = -\sum_{n=-1}^{\infty} \frac{z^n}{2^{n+2}} - \sum_{n=-\infty}^{-2} z^n$$

(c)

$$f(z) = \frac{1}{z} \left(\frac{\frac{1}{z}}{1-\frac{2}{z}} - \frac{\frac{1}{z}}{1-\frac{1}{z}} \right) = \frac{1}{z} \left(\sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n - \sum_{n=0}^{\infty} \frac{1}{z^n} \right) = \sum_{n=-\infty}^{-1} (2^{-(n+1)} - 1) z^n.$$

5. V.1.12

Proof. By (1.11), since f is analytic on $0 < |z| < \infty$,

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{\exp(\frac{1}{2}\lambda(z + \frac{1}{z}))}{z^{n+1}} dz$$

pick $\gamma = \exp(it)$, the unit circle, then

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} e^{\lambda \cos t} e^{-int} dt = \frac{1}{2\pi} \int_0^{2\pi} e^{\lambda \cos t} (\cos nt - i \sin nt) dt,$$

the real part is

$$\frac{1}{2\pi} \int_0^{2\pi} e^{\lambda \cos t} \cos nt dt = \frac{1}{2\pi} \left(\int_0^{\pi} e^{\lambda \cos t} \cos nt dt - \int_{\pi}^0 e^{\lambda \cos(2\pi-s)} \cos(2\pi-s) ds \right) = \frac{1}{\pi} \int_0^{\pi} e^{\lambda \cos t} \cos nt dt.$$

and the imaginary part is

$$-\frac{i}{2\pi} \int_0^{2\pi} e^{\lambda \cos t} \sin nt dt = -\frac{i}{2\pi} \left(\int_0^{\pi} e^{\lambda \cos t} \sin nt dt + \int_{\pi}^0 e^{\lambda \cos(2\pi-s)} \sin(2\pi-s) ds \right) = 0.$$

Hence $a_n = \frac{1}{\pi} \int_0^{\pi} e^{\lambda \cos t} \cos nt dt$.

$$b_n = \frac{1}{2\pi i} \int_{\gamma} \frac{\exp(\frac{1}{2}\lambda(z - \frac{1}{z}))}{z^{n+1}} dz$$

pick $\gamma = \exp(it)$,

$$b_n = \frac{1}{2\pi} \int_0^{2\pi} e^{\lambda i \sin t} e^{-int} dt = \frac{1}{2\pi} \int_0^{2\pi} \cos(nt - \lambda \sin t) - i \sin(nt - \lambda \sin t) dt.$$

The real part is

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \cos(nt - \lambda \sin t) dt &= \frac{1}{2\pi} \left(\int_0^{\pi} \cos(nt - \lambda \sin t) dt - \int_{\pi}^0 \cos(n(2\pi - s) - \lambda \sin(2\pi - s)) ds \right) \\ &= \frac{1}{2\pi} \left(\int_0^{\pi} \cos(nt - \lambda \sin t) dt + \int_0^{\pi} \cos(-ns + \lambda \sin s) ds \right) = \frac{1}{\pi} \int_0^{\pi} \cos(nt - \lambda \sin t) dt. \end{aligned}$$

and the imaginary part is

$$\begin{aligned} -\frac{i}{2\pi} \int_0^{2\pi} \sin(nt - \lambda \sin t) dt &= -\frac{i}{2\pi} \left(\int_0^{\pi} \sin(nt - \lambda \sin t) dt - \int_{\pi}^0 \sin(n(2\pi - s) - \lambda \sin(2\pi - s)) ds \right) \\ &= -\frac{i}{2\pi} \left(\int_0^{\pi} \sin(nt - \lambda \sin t) dt - \int_0^{\pi} \sin(-ns + \lambda \sin s) ds \right) = 0. \end{aligned}$$

Hence $b_n = \frac{1}{\pi} \int_0^{\pi} \cos(nt - \lambda \sin t) dt$.

6. V.1.13

(a) Suppose f is entire and has a removable singularity at ∞ , then $g(z) = f(\frac{1}{z})$ has a removable singularity at $z = 0$, thus we can define $g(0) = a < \infty$, which means f is bounded in the neighbourhood of ∞ . Hence by Liouville Thm, f is constant.

(b) By assumption, $g(z) = f(\frac{1}{z}) = \frac{1}{z^m} h(z)$, where $h(z)$ is analytic at $z = 0$. Since $f(z) = z^m h(\frac{1}{z})$ is entire, it means at $z = 0$, h has a definition or has a removable singularity, and h is entire. Hence, by (1) we know h is a constant, which means f is a polynomial of degree m .

(c) Let $f(z) = \frac{\prod(z-u_i)}{\prod(z-v_i)}$, then

$$g(z) = f\left(\frac{1}{z}\right) = \frac{\prod_{i=1}^n (\frac{1}{z} - u_i)}{\prod_{i=1}^m (\frac{1}{z} - v_i)}$$

has a removable singularity at $z = 0$, which means $\lim_{z \rightarrow 0} g(z)$ exists and is not ∞ . Notice

$$\lim_{z \rightarrow 0} g(z) = \lim_{z \rightarrow 0} \frac{z^m \prod_{i=1}^n (\frac{1}{z} - u_i)}{\prod_{i=1}^m (1 - v_i z)},$$

if $m < n$, then $\lim_{z \rightarrow 0} g(z) = \infty$, which makes a contradiction. If $m \geq n$, then the limit is well-defined. Hence $f = \frac{p(z)}{q(z)}$, where p, q are polynomials, and $\deg(p) \leq \deg(q)$.

(d) Let $f(z) = \frac{\prod(z-u_i)}{\prod(z-v_i)}$, then

$$g(z) = f\left(\frac{1}{z}\right) = \frac{\prod_{i=1}^n (\frac{1}{z} - u_i)}{\prod_{i=1}^k (\frac{1}{z} - v_i)}$$

has a pole of order m at $z = 0$, which means $g(z) = \frac{h(z)}{z^m}$, where $h(z)$ is analytic at $z = 0$. Then

$$z^m g(z) = \frac{z^m \prod_{i=1}^n (\frac{1}{z} - u_i)}{\prod_{i=1}^k (\frac{1}{z} - v_i)} = \frac{z^{m+k} \prod_{i=1}^n (\frac{1}{z} - u_i)}{\prod_{i=1}^m (1 - v_i z)},$$

and we have $m + k = n$, otherwise the order of pole is not m . Hence $f = \frac{p}{q}$, and $\deg(p) - \deg(q) = m$.

7. V.1.17

Proof. If not, first suppose f has a pole of order m at a . Then $f(z) = \frac{g(z)}{(z-a)^m}$, and g is analytic on G .

- i) If $g(a) = 0$, then $f(z) = 0$, and we can define $f(a) = 0$, thus a is a removable singularity.
- ii) If $g(a) \neq 0$, then according to the isolation of zeros, $\exists r > 0$, s.t. $g(z) \neq 0$ in $B(a, 2r)$. Consider $H = B(a, r)$, by max modulus theorem, $\min_H |g| = \min_{\partial H} |g|$, denote is as $c \neq 0$. Hence

$$\int \int_H |f(x+iy)|^2 dx dy = \int \int_H \frac{|g(x+iy)|^2}{|x+iy-a|^{2m}} dx dy \geq \int \int_H \frac{c^2}{|x+iy-a|^2} dx dy.$$

Let $x = \operatorname{Re}(a) + s \cos t, y = \operatorname{Im}(a) + s \sin t$, then

$$\int \int_H |f(x+iy)|^2 dx dy \geq \int_0^r \int_0^{2\pi} \frac{c^2}{s^{2m}} s ds dt = 2\pi c^2 \int_0^r s^{1-2m} ds.$$

If $m = 1$, then the integral becomes

$$\int_0^r s^{-1} ds = \ln(s) \Big|_0^r = \infty.$$

If $m > 1$, then the integral is

$$\int_0^r s^{1-2m} ds = \frac{1}{2-2m} s^{2-2m} \Big|_0^r = \infty.$$

Since the integrand is nonnegative,

$$\int_G |f|^2 \geq \int_H |f|^2 = \infty,$$

which makes a contradiction. With the same method, we know a is not an essential singularity. Hence a is a removable one.

By the deduction we know, if $0 < p < 2$, a could be a removable singularity or a pole of order m , which satisfies $pm < 2$. If $p \geq 2$, then a is a removable singularity.

8. V.2.1

(a) Notice $(x^4 + x^2 + 1)(x^2 - 1) = x^6 - 1$, so $f(z) = \frac{z^2}{z^4 + z^2 + 1}$ has four poles of order 1, and they are sixth roots of 1, which are $a_i = e^{i\theta_i}$ where

$$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}.$$

Consider the two poles in the upper-half plane a_1, a_2 , then

$$\operatorname{Res}(f, a_1) = \lim_{z \rightarrow a_1} (z - a_1) f(z) = a_1^2 (a_1 - a_2)^{-1} (a_1 - a_3)^{-1} (a_1 - a_4)^{-1} = \frac{1}{2\sqrt{3}i} e^{\frac{\pi}{3}i},$$

$$\operatorname{Res}(f, a_2) = \lim_{z \rightarrow a_2} (z - a_2) f(z) = a_2^2 (a_2 - a_1)^{-1} (a_2 - a_3)^{-1} (a_2 - a_4)^{-1} = -\frac{1}{2\sqrt{3}i} e^{\frac{2\pi}{3}i}.$$

Let γ be the closed path which is the boundary of upper half of disk of radius $R > 1$ with center 0, in counter-clockwise direction. Then by Residue Theorem,

$$\frac{1}{2\pi i} \int_{\gamma} f = \operatorname{Res}(f, a_1) + \operatorname{Res}(f, a_2) = \frac{1}{2\sqrt{3}i}.$$

And

$$\int_{\gamma} f = \int_{-R}^R \frac{x^2}{x^4 + x^2 + 1} dx + i \int_0^{\pi} \frac{R^3 e^{i3t}}{R^4 e^{i4t} + R^2 e^{i2t} + 1} dt.$$

Notice $|R^4 e^{i4t} + R^2 e^{i2t} + 1| \geq R^4 - R^2 = 1$ when R is sufficiently large, so

$$\left| i \int_0^\pi \frac{R^3 e^{i3t}}{R^4 e^{i4t} + R^2 e^{i2t} + 1} dt \right| \leq \frac{\pi R^3}{R^4 - R^2 - 1} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

Since $\frac{x^2}{x^4+x^2+1} \geq 0, \forall x \in \mathbb{R}$, we know

$$\int_0^\infty \frac{x^2}{x^4+x^2+1} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x^2}{x^4+x^2+1} = \frac{1}{2} \lim_{R \rightarrow \infty} \int_{-R}^R \frac{x^2}{x^4+x^2+1} dx = \frac{\pi}{\sqrt{3}}.$$

For the following problems, I will not write all the steps since I don't have enough time to do so. I will only write the key points.

(b) We know

$$\int_r^R \frac{\cos x - 1}{x^2} dx = \frac{1}{2} \left(\int_r^R + \int_{-R}^{-r} \frac{e^{ix} - 1}{x^2} dx \right)$$

Consider $f = \frac{e^{iz}-1}{z^2}$, then f has a simple pole at $z = 0$. Let γ be the curve in Example 2.7, then

$$0 = \int_r^R + \int_{-R}^{-r} \frac{e^{ix} - 1}{x^2} dx + \int_{\gamma_r} + \int_{\gamma_R} \frac{e^{iz} - 1}{z^2} dz$$

Since

$$\left| \int_{\gamma_R} \frac{e^{iz} - 1}{z^2} dz \right| = \left| \int_0^\pi \frac{\exp(iRe^{it}) - 1}{Re^{it}} dt \right| \leq \frac{1}{R} \int_0^\pi \exp(-R \sin t) dt,$$

and by Example 2.7 we know the integral $\rightarrow 0$ as $R \rightarrow \infty$. And since $\frac{e^{iz}-1-iz}{z^2}$ has a removable singularity at $z = 0$, by Example 2.7 we know

$$\lim_{\gamma \rightarrow 0} \left| \int_{\gamma_r} \frac{e^{iz} - 1 - iz}{z^2} dz \right| = 0,$$

but $\int_\gamma \frac{i}{z} dz = \pi$, hence

$$\lim_{\gamma \rightarrow 0} \left| \int_{\gamma_r} \frac{e^{iz} - 1}{z^2} dz \right| = \pi.$$

Hence

$$\int_0^\infty \frac{\cos x - 1}{x^2} dx = \lim_{r \rightarrow 0, R \rightarrow \infty} \int_r^R \frac{\cos x - 1}{x^2} dx = -\frac{\pi}{2}.$$

(c) The same with Example 2.9, let $z = e^{i\theta}$, then

$$\int_0^\pi \frac{\cos 2\theta}{1 - 2a \cos \theta + a^2} d\theta = -\frac{i}{4} \int_\gamma \frac{z^4 + 2z^2 + 1}{-az^2 + (a^2 + 1)z - a} dz,$$

where $\gamma = \{|z| = 1\}$. By residue theorem,

$$\int_\gamma \frac{z^4 + 2z^2 + 1}{-az^2 + (a^2 + 1)z - a} dz = 2\pi i \operatorname{Res}(f, a) = 2\pi i \frac{(a^2 + 1)^2}{1 - a^2}.$$

Hence

$$\int_0^\pi \frac{\cos 2\theta}{1 - 2a \cos \theta + a^2} d\theta = \frac{\pi}{2} \frac{(a^2 + 1)^2}{1 - a^2}.$$

(d) Same with (c), let $z = e^{i\theta}$, then

$$\int_0^\pi \frac{d\theta}{(a + \cos \theta)^2} = -2i \int_\gamma \frac{z}{(z^2 + 2az + 1)^2} dz,$$

where $\gamma = \{|z| = 1\}$. By residue theorem,

$$\int_{\gamma} \frac{z}{(z^2 + 2az + 1)^2} dz = \frac{1}{2} \pi i a (a^2 - 1)^{-\frac{3}{2}}.$$

Hence

$$\int_0^{\pi} \frac{d\theta}{(a + \cos \theta)^2} = \pi a (a^2 - 1)^{-\frac{3}{2}}.$$

9. V.2.2

(a)

$$\int_0^{\infty} \frac{dx}{(x^2 + a^2)^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^2}.$$

Consider $R > a$, let $\gamma = \{|z| = R, \operatorname{Im}(z) > 0\}$, then by residue thm, since f has a pole of order 2, $z = ai$,

$$\int_{\gamma} + \int_{-R}^R \frac{dz}{(z^2 + a^2)^2} = 2\pi i \operatorname{Res}(f, ai) = \frac{\pi}{2a^3}.$$

But

$$\left| \int_{\gamma} \frac{dz}{(z^2 + a^2)^2} \right| \leq \int_{\gamma} \left| \frac{dz}{(z^2 + a^2)^2} \right| \leq \int_0^{\pi} \frac{1}{|R^2 - a^2|^2} dt \leq \frac{\pi}{|R^2 - a^2|^2} \rightarrow 0, \text{ as } R \rightarrow \infty.$$

Hence

$$\int_0^{\infty} \frac{dx}{(x^2 + a^2)^2} = \frac{\pi}{4a^3}.$$

(b) Define $G, l(z), \gamma$ as in Example 2.10, using the same method of Example 2.10, we can show

$$\int_0^{\infty} \frac{(\log x)^2}{1 + x^2} dx = 0.$$

$$\int_{\gamma} \frac{(l(z))^3}{1 + z^2} dz = \int_r^R \frac{(\log x)^3}{1 + x^2} dx + \int_0^{\pi} \frac{(\log R + i\theta)^3}{1 + R^2 e^{2i\theta}} i R e^{i\theta} d\theta + \int_{-R}^{-r} \frac{(\log |x| + \pi i)^3}{1 + x^2} dx + \int_{\pi}^0 \frac{(\log r + i\theta)^3}{1 + r^2 e^{2i\theta}} i r e^{i\theta} d\theta$$

by residue thm,

$$\int_{\gamma} \frac{(l(z))^3}{1 + z^2} dz = 2\pi i \operatorname{Res}(f, i) = -\frac{1}{8} \pi^4 i.$$

And

$$\int_r^R \frac{(\log x)^3}{1 + x^2} dx + \int_{-R}^{-r} \frac{(\log |x| + \pi i)^3}{1 + x^2} dx =$$

10. V.2.3

Sol. We have proved that f has an essential singularity at $z = 0$. Denote G be the region surrounded by γ .

i) If $0 \notin G$, then by Cauchy's Thm, $\int_{\gamma} f = 0$.

ii) If $0 \in G$, then by Laurent expansion, $\operatorname{Res}(f, 0) = 1$. But by a problem in last homework, we can construct a γ , for each $n \in \mathbb{N}$, $n(\gamma, 0) = n$. Hence $\int_{\gamma} f = 2\pi n i, n \in \mathbb{N}$.

11. V.2.4

Sol. In fact, in a neighbourhood of a , $f = \frac{c_{-1}}{z-a} + \sum_{n=0}^{\infty} c_n (z-a)^n$, and $g = \sum_{n=0}^{\infty} d_n (z-a)^n$. Hence by definition,

$$\operatorname{Res}(fg, a) = c_{-1} d_0 = \operatorname{Res}(f, a) g(a).$$

12. V.2.5

Sol. Notice in the last problem, each simple pole of f is a simple pole of fg . By residue thm,

$$\frac{1}{2\pi i} \int_{\gamma} fg = \sum_{k=1}^n n(\gamma, a_k) \operatorname{Res}(fg, a_k) = \sum_{k=1}^n n(\gamma, a_k) \operatorname{Res}(f, a_k) g(a_k).$$