Introduction to Analysis
Assignment 8

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Problem 1. Problem 37, Page 123

Sol. (i) Let

$$f(x) = \begin{cases} x\sin(\frac{1}{x}),\ 0 < x \leq 1, \\ 0,\ x = 0 \end{cases}$$

Then f is continuous on $(0,1]$. Since $\lim_{x\to 0^+} f(x) = 0 = f(0)$, f is continuous on $[0,1]$. For $x_1, x_2 \in [\epsilon, 1]$ where $\epsilon > 0$,

$$\begin{aligned} |f(x_1) - f(x_2)| &= |x_1 \sin\frac{1}{x_1} - x_2 \sin\frac{1}{x_2}| = |(x_1 - x_2)\sin\frac{1}{x_1} + x_2(\sin\frac{1}{x_1} - \sin\frac{1}{x_2})| \\ &\leq |x_1 - x_2| \sin\frac{1}{x_1} + 2x_2|\cos\frac{1}{2}(\frac{1}{x_1} + \frac{1}{x_2})\sin\frac{1}{2}(\frac{1}{x_1} - \frac{1}{x_2})| \\ &\leq |x_1 - x_2|\sin\frac{1}{x_1} + 2x_2|\sin\frac{x_2 - x_1}{2x_1x_2}| \leq |x_1 - x_2|\sin\frac{1}{x_1} + 2x_2|\frac{x_2 - x_1}{2x_1x_2}| \\ &= |x_1 - x_2|(\sin\frac{1}{x_1} + \frac{1}{x_1}) \leq |x_1 - x_2|(1 + \frac{1}{\epsilon}). \end{aligned}$$

Thus f is Lipschitz, with Proposition 7 we know f is absolutely continuous.
However, with Problem 35 we know f is not of bounded variation on $[0,1]$, and with Remark on Page 122 we know f is not absolutely continuous on $[0,1]$.
(ii) If not, then there is a $\epsilon > 0$, s.t. for each $\delta > 0$, there is a finite disjoint collection $\{(a_k, b_k)\}$ of open intervals satisfying $\sum_{k=1}^{n}(b_k - a_k) < \delta$, s.t. $\sum_{k=1}^{n}|f(b_k) - f(a_k)| \geq \epsilon$. As suggested in conditions, for each $c > 0$, f is absolutely continuous on $[c,1]$. Then these open intervals must lie in $[0,c]$ for every c. With the continuity of f on $[0,1]$, there exists $c > 0$, s.t. $0 < f(c) - f(0) < \epsilon$. If we take $\delta = c$, because f is increasing, $\sum_{k=1}^{n}(b_k - a_k) < f(c) - f(0) < \epsilon$. It contradicts with our assumption. Hence f is absolutely continuous on $[0,1]$.
(iii) First we show f is absolutely continuous by showing that it satisfies the condition in (ii). For each $c > 0$, since on $[c,1]$ we have

$$\sqrt{x_1} - \sqrt{x_2}| = \frac{|x_1 - x_2|}{\sqrt{x_1} + \sqrt{x_2}} \leq \frac{|x_1 - x_2|}{2\sqrt{c}},$$

we know for each $\epsilon > 0$, pick $\delta = 2\sqrt{c}\epsilon$, then for each collection $\{(a_k, b_k)\}$ satisfying $\sum_{k=0}^{n}|b_k - a_k| < \delta$,

$$\sum_{k=1}^{n}|f(b_k) - f(a_k)| < \sum_{k=1}^{n}\frac{|x_1 - x_2|}{2\sqrt{c}} < \epsilon.$$

Hence f is absolutely continuous on $[c,1]$. Since f is increasing, we know f is absolutely continuous on $[0,1]$.
On the other hand, if there exists $\lambda > 0$, s.t.

$$|f(x_1) - f(x_2)| < \lambda|x_1 - x_2|$$

for each x_1, x_2, then if we pick $\max(x_1, x_2) < \frac{1}{4\lambda^2}$, from the argument above we know

$$|f(x_1) - f(x_2)| = \frac{|x_1 - x_2|}{\sqrt{x_1} + \sqrt{x_2}} > \lambda|x_1 - x_2|.$$

Hence f is not Lipschitz.

Problem 2. Problem 39, Page 123

Sol. Suppose E is a measurable set, then for each $\epsilon' > 0$, there exists an open set $O \supset E$, and $m(O \setminus E) < \epsilon'$. Let $O = \bigcup_{k=1}^{\infty} (a_k, b_k)$ be the open decomposition of O, then (a_i, b_i) are pairwise disjoint.

If f is absolutely continuous, by Problem 38, for each $\epsilon > 0$, there is a $\delta > 0$, for each $\{(a_k, b_k)\}$ satisfying $\sum_{k=1}^{\infty} (b_k - a_k) < \delta$, $\sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \epsilon$. In fact, if we take $\delta_1 < \delta - \epsilon'$, then since f is increasing, $f(E) \in f(\cup\{(a_k, b_k)\})$ when $E \in \cup\{(a_k, b_k)\}$, thus

$$m^*(f(E)) < m(f(\cup\{(a_k, b_k)\})) < \delta.$$

On the other hand, since open sets are measurable, with Problem 39 we know the reverse holds.

Problem 3. Problem 41, Page 123

Sol. (i) With the continuity of f and the compactness of $[a, b]$, we know the maximum and minimum of f on $[a, b]$ exists, thus f maps $[a, b]$ to a closed set. Thus f maps a F_σ set to a F_σ set.
(ii) Problem 40 tells f maps a set of measure zero to a set of measure zero. Since each measurable set could be represented as a union of a measure-zero set and a F_σ set, we know f maps a measurable set to a measurable set.

Problem 4. Problem 49, Page 128

Sol. Since f is differentiable a.e. on (a, b), we first show that $\{\text{Diff}_{1/n} f\}$ converges pointwise a.e. to f' on (a, b). Suppose f is differentiable on $E \in (a, b)$, with $m((a, b) \setminus E) = 0$. For $\forall x$ in E, we know for any $n > 0$, $\underline{D}f(x) \leq \text{Diff}_{1/n} f(x) \leq \overline{D}f(x)$, so

$$\underline{D}f(x) \leq \lim_{n\to\infty} \text{Diff}_{1/n} f(x) \leq \overline{D}f(x).$$

Since $\underline{D}f(x) = \overline{D}f(x)$, we know

$$\lim_{n\to\infty} \text{Diff}_{1/n} f(x) = f'(x), \ \forall x \in E.$$

Thus

$$\int_a^b \lim_{n\to\infty} \text{Diff}_{1/n} f = \int_a^b f'.$$

By fundamental theorem of integral (or by (29)), we know

$$\lim_{n\to\infty} \int_a^b \text{Diff}_{1/n} f = f(b) - f(a).$$

Hence,

$$\int_a^b \lim_{n\to\infty} \text{Diff}_{1/n} f = \lim_{n\to\infty} \int_a^b \text{Diff}_{1/n} f$$

is equivlent to

$$\int_a^b f' = f(b) - f(a).$$

Problem 5. Problem 56, Page 129

(i). Since O is an open set, let

$$O = \bigcup_{n=1}^{\infty} (a_i, b_i)$$

be its open decomposition, where $\{(a_i, b_i)\}$ are pairwise disjoint. Then since g is absolutely continuous thus continuous, and g is strictly increasing, g maps an open interval to an open interval (in fact, this has been proved in a midterm exam). Hence, for any open interval (a, b), by Corollary 12 we have

$$\int_a^b g'(x) = g(b) - g(a) = m(g((a, b))).$$

By countable addivity of measure and integral,

$$\int_O g' = \sum_{i=1}^{\infty} \int_{a_i}^{b_i} g' = \sum_{i=1}^{\infty} m(g((a_i, b_i))) = m(g(O)).$$

(ii). We know that the intersection of two open sets is an open set, then by induction we know

$$m(g(\cap_{i=1}^n O_i)) = \int_{\cap_{i=1}^n O_i} g', \ \forall n > 0.$$

Let $E = \cap_{i=1}^\infty O_i$, then first we have

$$m(g(E)) = m(g(\cap_{i=1}^\infty O_i)) \le m(g(\cap_{i=1}^n O_i)) = \int_{\cap_{i=1}^n O_i} g', \ \forall n.$$

The left side of this inequity is independent of n, thus

$$m(g(E)) \le \int_{\cap_{i=1}^\infty O_i} g' = \int_E g'.$$

On the other hand, we know

$$\int_E g' \le m(g(E))$$

by the same arguments. Hence

$$m(g(E)) = \int_E g'.$$

(iii). In a midterm exam we have proved that a strictly increasing and continuous function maps a null set to a null set, thus $m(g(E)) = 0$. By Theorem 10 we know g' is integrable on $[a, b]$, then

$$\int_E g' = 0.$$

(iv). For any measurable subset A, by Theorem 11 of Chapter 2, there is a G_δ set $G \supset A$, and $m(G \setminus A) = 0$. Let $E = G \setminus A$, by (ii) and (iii), we have

$$m(g(E)) = \int_E g', \ m(g(G)) = \int_G g'.$$

Since $E \cap G = \emptyset$, by addivity of measure and integral,

$$m(g(A)) = m(g(E)) + m(g(G)) = \int_{E \cup G} g' = \int_A g'.$$

(v). Since φ is a simple function, assume $\varphi = \sum_{k=1}^n c_k 1_{E_k}$. Since E_k are measurable sets, we know $g((a, b)) \cap E_k$ are all measurable. Denote $D_k = g((a, b)) \cap E_k$. Since g is strictly increasing and continuous, we know $g^{-1}(D_k)$ is measurable, then by addivity of domain and (iv),

$$\int_a^b \varphi(g(x))g'(x)dx = \sum_{k=1}^n \int_{g^{-1}(D_k)} c_k g'(x)dx = \sum_{k=1}^n c_k m(g^{-1}(D_k)) = \int_c^d c_k 1_{g^{-1}(D_k)} = \int_c^d \varphi(x)dx.$$

(vi). By Simple Approximation Lemma, for any $\epsilon > 0$, there are simple functions φ and ψ, s.t.

$$0 \le \varphi \le f \le \psi, \ 0 \le \psi - \varphi < \epsilon.$$

Since g is strictly increasing and absolutely continuous, g' exists a.e., integrable on $[a, b]$, and $g' > 0$. Then

$$\int_a^b \varphi(g(x))g'(x)dx \le \int_a^b f(g(x))g'(x)dx \le \int_a^b \psi(g(x))g'(x)dx.$$

By (v) we know

$$\int_c^d \varphi(y)dy \le \int_a^b f(g(x))g'(x)dx \le \int_c^d \psi(y)dy,$$

and by montonicity of integrals,

$$\int_c^d \varphi(y)dy \le \int_c^d f(y)dy \le \int_c^d \psi(y)dy,$$

and

$$\int_c^d \psi(y)dy - \int_c^d \varphi(y)dy = \int_c^d (\psi - \varphi)dy \le \epsilon(d-c).$$

By the arbitrariness of ϵ,

$$\int_c^d \psi(y)dy - \int_c^d \varphi(y)dy = 0.$$

Hence

$$\int_a^b f(g(x))g'(x)dx = \int_c^d f(y)dy.$$

Problem 6. Problem 59, Page 129

Sol. Define $F(x) = \int_a^x f(s)ds$, then F is absolutely continuous and thus its derivative exists a.e. on (a,b), and $F' = f(x)$ a.e. by Theorem 14. Then

$$\begin{aligned}\frac{d}{dx}\int_{g(a)}^{g(x)} f(s)ds = \lim_{n\to\infty} n\int_{g(x)}^{g(x+\frac{1}{n})} f(s)ds &= \lim_{n\to\infty} n(F(g(x+1/n)) - F(g(x)))\\ &= \lim_{n\to\infty} nf(g(x+\xi))(g(x+1/n)-g(x)) = \lim_{n\to\infty} f(g(x+\xi))g'(x+\eta),\ \ \xi,\eta\in[0,1/n]\\ &= f(g(x))g'(x),\ \ a.e.\end{aligned}$$

In the deduction we used the mean value theorem, and used the property that F, g are absolutely continuous and their derivatives exists a.e.. Hence,

$$\frac{d}{dx}(\int_{g(a)}^{g(x)} f(s)ds - \int_a^x f(g(t))g'(t)dt) = f(g(x))g'(x) - f(g(x))g'(x) = 0,\ \ a.e.$$

Thus by Lemma 13,

$$\int_{g(a)}^{g(x)} f(s)ds = \int_a^x f(g(t))g'(t)dt.$$