Introduction to Analysis I
Homework 5
Sunday, October 8, 2017

Instructions: This and all subsequent homeworks must be submitted written in LaTeX.
If you use results from books, Royden or others, please be explicit about what results you are using.

Homework 5 is due by midnight, Saturday, October 21.

1. (Problem 24, Page 64) Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be increasing. Show that f is measurable by first showing that, for each natural number n, the strictly increasing function $x \to f(x) + x/n$ is measurable, and then taking pointwise limits.

   Collaborators: None

   Solution: Denote $f_n(x) = f(x) + \frac{x}{n}$. Then since $f : I \to \mathbb{R}$ is increasing, $f_n(x)$ is strictly increasing on $I \in \{[a, b], [a, b), (a, b], (a, b)\}$.

   For each fixed number c, if $\exists x_0 \in I$, s.t. $f(x_0) = c$, then the set

   $$\{x \mid f(x) < c\} = (a, x_0),$$

   and the left side is the same as the set I, and it is an interval in $\mathbb{R}$, hence is measurable. Thus f_n is measurable for every n.

   For each $x \in I$,

   $$\lim_{n\to\infty} f_n(x) = f(x) + \lim_{n\to\infty} \frac{x}{n} = f(x),$$

   hence f_n converges to f pointwise. Using Proposition 9, we know f is measurable.

2. (Problem 8, Page 343) Let $(X, \mathcal{M}, \mu)$ be a measure space. The measure μ is said to be **semifinite** provided each measurable set of infinite measure contains measurable sets of arbitrarily large finite measure.

   (a) Show that each σ-finite measure is semifinite.

   (b) For $E \in \mathcal{M}$, define $\mu_1(E) = \mu(E)$, if $\mu(E) < \infty$, and if $\mu(E) = \infty$, define $\mu_1(E) = \infty$ if E contains measurable sets of arbitrarily large finite measure and $\mu_1(E) = 0$ otherwise. Show μ_1 is a semifinite measure: it is called the semifinite part of μ.

   (c) Find a measure μ_2 on $\mathcal{M}$ that only takes the valures 0 and ∞ and $\mu = \mu_1 + \mu_2$.

   Collaborators:

   Solution:

3. (Problem 9, Page 343) Prove Proposition 3, that is, show that $\mathcal{M}_0$ is a σ-algebra, μ_0 is properly defined and $(X, \mathcal{M}_0, \mu_0)$ is complete. In what sense is $\mathcal{M}_0$ minimal?

   Collaborators:

   Solution:

4. (Problem 10, Page 343) If $(X, \mathcal{M}, \mu)$ is a measure space, we say that a subset E of X is **locally measurable** provided for each $B \in \mathcal{M}$ with $\mu(B) < \infty$, the intersection $E \cap B$ belongs to $\mathcal{M}$. The measure μ is called **saturated** provided every locally measurable set is measurable.

   (a) Show that each σ-finite measure is saturated.

   (b) Show that the collection $\mathcal{C}$ of locally measurable sets is a σ-algebra.

   (c) Let $(X, \mathcal{M}, \mu)$ be a measure space and $\mathcal{C}$ the σ of locally measurable sets. For $E \in \mathcal{C}$, define $\overline{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\overline{\mu}(E) = \infty$ if $E \notin \mathcal{M}$. Show that $(X, \mathcal{C}, \overline{\mu})$ is a saturated measure space.

   (d) If μ is semifinite and $E \in \mathcal{C}$, set $\underline{\mu}(E) = \sup\{\mu(B) \mid B \in \mathcal{M},\, B \subseteq E\}$. Show that $(X, \mathcal{C}, \underline{\mu})$ is a saturated measure space and that $\underline{\mu}$ is an extension of μ. Give an example to show that $\overline{\mu}$ and $\underline{\mu}$ may be different.

   Collaborators:

   Solution:

5. (Problem 18, Page 373) Let $\{u_n\}$ be a sequence of nonnegative measurable functions on X. For $x \in X$, define $f(x) = \sum_{n=1}^{\infty} u_n(x)$. Show that

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \left[\int_X u_n \, d\mu \right].$$