Introduction to Analysis
Assignment 6

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Problem 1. Problem 29, Section 4.4, Page 89

Sol. Both are not true. We can construct a f like this:

$$f(x) = \begin{cases} 1 + \dfrac{1}{n^2},\ n \le x < n + \dfrac{1}{2},\ \forall n \in \mathbb{N} \\ -1,\ n + \dfrac{1}{2} \le x < n + 1,\ \forall n \in \mathbb{N} \end{cases}$$

Then f is measurable, and f is bounded on any bounded set, and

$$a_n = \int_n^{n+1} f = \frac{1}{2n^2}.$$

Clearly the series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{2n^2}$ converges absolutely, but

$$\int_1^{\infty} |f| = \sum_{n=1}^{\infty} (1 + \frac{1}{2n^2}) = \infty,$$

which means f is not integrable on $[1, \infty)$.

Problem 2. Problem 33, Section 4.4, Page 90

Proof. First,

$$|f_n - f| \le |f| + |f_n|,\ \forall n.$$

Then since f is integrable on E, if $\lim\limits_{n\to\infty} \int_E |f_n| = \int_E |f|$, we know $|f_n| + |f|$ converges pointwise a.e. to $2|f|$, and

$$\lim_{n\to\infty} \int_E (|f_n| + |f|) = 2 \int_E |f| < \infty,$$

with General Lebesgue Dominated Convergence Theorem, notice $|f_n - f|$ converges pointwise a.e. to 0,

$$\lim_{n\to\infty} \int_E |f_n - f| = \int_E 0 = 0.$$

On the other hand, notice

$$|f_n| - |f| \le |f_n - f|,\ \forall n.$$

with the same method, since $\int_E |f - f_n| \to 0$, and $|f - f_n|$ converges pointwise to 0,

$$\lim_{n\to\infty} \int_E |f_n - f| = \int_E 0 = 0,$$

we know from $|f_n| - |f|$ converges pointwise a.e. to 0,

$$\lim_{n\to\infty} \int_E |f_n| - |f| = \int_E 0 = 0.$$

Hence

$$\lim_{n\to\infty} \int_E |f_n| = \int_E |f|.$$

Problem 3. Problem 35, Section 4.4, Page 90

Proof. Denote $f_n(x) = f(x, a_n)$, in which $\{a_n\}$ is any series which converges to 0. Then from the condition we know $f_n(x)$ converges pointwise to $f(x)$, and $|f_n(x)| \le g(x)$. Then using Lebesgue Dominated Convergence Theorem, since g is integrable on $[0, 1]$, we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx.$$

It shows that

$$\limsup_{y\to 0} \int_0^1 f(x,y)dx = \liminf_{y\to 0} \int_0^1 f(x,y)dx = \int_0^1 f(x)dx,$$

whic means

$$\lim_{y\to 0} \int_0^1 f(x,y)dx = \int_0^1 f(x)dx.$$

For the continuity of h, we need to show that $\forall y_0 \in [0, 1]$, $\forall \epsilon > 0$, $\exists \delta > 0$, when $|y - y_0| < \delta$, we have $h(y) - h(y_0) = |\int_0^1 f(x,y)dx - \int_0^1 f(x,y_0)|dx < \epsilon$. Since $f(x,y)$ is continuous in y for each x, then for each fixed x, $\exists \delta_1$, when $|y - y_0| < \delta_1$, $|f(x,y) - f(x,y_0)| < \epsilon$. Then

$$|h(y) - h(y_0)| = \left|\int_0^1 f(x,y)dx - \int_0^1 f(x,y_0)dx\right| \le \int_0^1 |f(x,y) - f(x,y_0)|dx < \epsilon.$$

We know the continuity of h since we can pick $\delta = \delta_1$.

Problem 4. Problem 36, Section 4.4, Page 90

Proof. For any fixed $y \in [0, 1]$, suppose $\{h_n\}$ is a sequence with $h_n \to 0$. Let

$$f_n(x) = \frac{f(x, y + h_n) - f(x,y)}{h_n}$$

Since $\partial f/\partial y$ exists,

$$\lim_{n\to\infty} f_n(x) = \lim_{h\to 0} \frac{f(x, y+h) - f(x,y)}{h} = \frac{\partial f}{\partial y}(x,y).$$

It means $f_n(x)$ converges pointwise to $\frac{\partial f}{\partial y}(x,y)$. Thus

$$\exists N > 0,\ \forall n > N, \left|f_n(x) - \frac{\partial f}{\partial y}(x,y)\right| < 1.$$

Since

$$\left|\frac{\partial f}{\partial y}(x,y)\right| \le g(x)$$

we have

$$|f_n(x)| \le g(x) + 1,$$

and $g(x) + 1$ is integrable on $[0, 1]$. By Lebesgue Dominated Convergence Theorem,

$$\int_0^1 f_n(x)dx \to \int_0^1 \frac{\partial f}{\partial y}(x,y)dx.$$

Since $\{h_n\}$ is arbitrary, and f_n is integrable, we know

$$\limsup_{n\to\infty} \int_0^1 f_n(x)dx = \liminf_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{h\to 0} \frac{1}{h}\left(\int_0^1 f(x, y+h)dx - \int_0^1 f(x,y)dx\right) = \frac{d}{dy}\int_0^1 f(x,y)dx.$$

Problem 5. Problem 38, Section 4.5, Page 91

(i).

$$\lim_{n\to\infty} \int_0^n f dx = \lim_{n\to\infty} \sum_{m=1}^{n} \frac{(-1)^m}{m} = -\ln 2,$$

But

$$\int_1^\infty f^+ = \sum_{m=1}^\infty \frac{1}{2m} = \infty,$$

So f is not integrable.
(ii).

$$\lim_{n\to\infty} \int_1^n f = \int_1^\infty \frac{\sin x}{x} dx,$$

with Dirichlet's Criterion, we know this integral converges. But

$$\int_1^\infty |f| \geq \int_1^\infty \frac{1}{2x} dx - \int_1^\infty \frac{\cos 2x}{x} dx,$$

and the second term converges with Dirichlet's Criterion, but the first term $\to \infty$, we know this integral diverges to ∞. Thus f is not integrable.
This two counterexamples do not contradict to the continuity: f is not integrable over the whole set $E = [1, \infty)$.

Problem 9. Problem 39, Section 4.5, Page 91

Proof (i). Denote

$$F_1 = E_1,\ F_n = E_n \setminus \bigcup_{m=1}^{n-1} E_m,\ n \geq 2.$$

Then $\{F_i\}$ is a sequence of disjoint measurable subsets of E. Then using Theorem 20,

$$\int_{\cup_{n=1}^\infty E_n} f = \sum_{n=1}^\infty \int_{F_n} f = \lim_{n\to\infty} \sum_{m=1}^n \int_{F_m} f = \lim_{n\to\infty} \int_{E_n} f.$$

Proof of (ii). Using the same method with (i), only changing F to

$$F_1 = E_1,\ F_n = E_1 \setminus \bigcup_{m=1}^{n-1} E_m,\ n \geq 2.$$

The other parts of proof is just the same.