Introduction to Analysis I
Homework 4
Wednesday, September 27, 2017

Instructions: This and all subsequent homeworks must be submitted written in LaTeX.
If you use results from books, Royden or others, please be explicit about what results you are using.

Homework 4 is due by midnight, Saturday, October 7.

1. (Problem 18, Page 63) Let I be a closed bounded interval and let f be a bounded measurable function defined on I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which

$$|h - f| < \epsilon \text{ on } F \text{ and } m(I \sim F) < \epsilon.$$

Collaborators: None

Solution: For $\epsilon > 0$, according to Lusin's Theorem, there exists a closed subset F of I, s.t. $m(I \backslash F) < \epsilon$, and f is continuous on F. Then for $x_0 \in F$, for this $\epsilon > 0$,

$$\exists \sigma > 0,\ \forall x \in (x_0 - \delta,\ x_0 + \delta) \cap F,\ |f(x) - f(x_0)| < \epsilon.$$

Since

$$\bigcup_{x_0 \in F} (x_0 - \delta_i,\ x_0 + \delta_i)$$

is an open cover of the closed set F, it has a finite subcover, denoted by

$$\bigcup_{i=1}^{n} (x_i - \delta_i,\ x_i + \delta_i).$$

If we define

$$h = \begin{cases} f(x_i), \text{ for } x \in (x_i - \delta_i,\ x_i + \delta_i), \\ 0, \text{ others} \end{cases}$$

then h takes constant values in finite intervals, and $h = 0$ in other parts, we know h is a step function. And from the construction above, on each interval $(x_i - \delta_i, x_i + \delta_i) \cap F$,

$$|h - f| < \epsilon \text{ on } (x_i - \delta_i, x_i + \delta_i) \cap F.$$

Thus h is the step function needed.

2. (Problem 22, Page 64) (Dini's Theorem) Let $\{f_n\}$ be an increasing sequence of continuous functions on $[a, b]$ which converges pointwise on $[a, b]$ to the continuous function f on $[a, b]$. Show that the convergence is uniform on $[a, b]$.

Collaborators: None

Solution: If the convergence is not uniform, then there $\exists \epsilon > 0, \forall N > 0,\ \exists n > N,\ \exists x \in [a, b],\ |f(x) - f_n(x)| > \epsilon$.

We can construct a sequence $\{x_n\}$ like this:

$$N = 1,\ \exists n_1 > 1,\ \exists x_1 \in [a, b],\ |f_{n_1}(x_1) - f(x_1)| \geqslant \epsilon.$$
$$N = n_1,\ \exists n_2 > n_1,\ \exists x_2 \in [a, b],\ |f_{n_2}(x_2) - f(x_2)| \geqslant \epsilon.$$
$$\cdots$$

$$N = n_{k-1},\ \exists n_k > n_{k-1},\ \exists x_k \in [a,b],\ |f_{n_k}(x_k) - f(x_k)| \geqslant \epsilon.$$

$$\cdots$$

Then since $x_i \in [a,b]$, according to Bolzano-Weierstrass Theorem, there is a convergent subsequence in $\{x_i\}$, and we may denote it by $\{y_i\}$ for convinence. Assume $y_i \to y \in [a,b]$, then since

$$\lim_{n\to\infty} f_n(y) = f(y),$$

for this $\epsilon > 0,\ \exists N$, s.t.

$$|f_N(y) - f(y)| < \epsilon.$$

since f_N is continuous, with $y_k \to y$, there $\exists K > 0$,

$$|f_N(y_k) - f(y_k)| < \epsilon$$

holds for all $k > K$. Notice that $\{f_n\}$ is an increasing sequence, when $n > N$ and $k > K$,

$$|f_n(y_k) - f(y_k)| \leqslant |f_N(y_k) - f(y_k)| < \epsilon.$$

Since $n_k \to \infty$ when $k \to \infty$, when k is sufficiently large we have $k > K,\ n_k > N$. Thus

$$|f_{n_k}(x_k) - f(x_k)| < \epsilon,$$

which makes a contradiction with the assumption. Thus the convergence is uniform.

3. (Problem 5, Page 364) Show that an extended real-valued function f on X is measurable if and only if for each rational number c, $\{x \in X \mid f(x) < c\}$ is a measurable set.

   Collaborators:

   Solution:

4. (Problem 13, Page 365) Let $\{f_n\}$ be a sequence of real-valued functions on X such that for each natural number n, $\mu\{x \in X \mid |f_n(x) - f_{n+1}(x)| > 1/2^n\} < 1/2^n$. Show that $\{f_n\}$ is pointwise convergent a.e. on X.

   Collaborators:

   Solution:

5. (Problem 15, Page 365) A sequence $\{f_n\}$ of measurable real-valued functions on X is said to *converge in measure* to a measurable function f provided that for each $\eta > 0$,

$$\lim_{n\to\infty} \mu\{x \in X \mid |f_n(x) - f(x)| > \eta\} = 0.$$

   A sequence $\{f_n\}$ of measurable functions is said to be *Cauchy in measure* provided that for each $\epsilon > 0$ and $\eta > 0$, there is an index N such that for each $m, n \geq N$,

$$\mu\{x \in X \mid |f_n(x) - f_m(x)| > \eta\} < \epsilon.$$

   (a) Show that if $\mu(X) < \infty$ and if $\{f_n\}$ converges pointwise a.e. on X to a measurable function f, then $\{f_n\}$ converges to f in measure.
   (b) Show that if $\{f_n\}$ converges to f in measure, then there is a subsequence of $\{f_n\}$ that converges pointwise a.e. to f.

(c) Show that if $\{f_n\}$ is Cauchy in measure, then there is a measurable function f to which $\{f_n\}$ converges in measure.

6. (Problem 16, Page 365) Assume $\mu(X) < \infty$. Show that $\{f_n\}$ converges to f in measure if and only if each subsequence of $\{f_n\}$ has a further subsequence that converges pointwise a.e. on X to f. Use this to show that for two sequences that converge in measure, the product sequence also converges in measure to the product of the limits.

7. Show that if f is an lower semicontinuous (resp. upper semicontinuous) function on an interval $[a, b]$, then there is a family $\{f_\alpha\}$ of continuous functions on the interval $[a, b]$ such that $f(x) = \sup\{f_\alpha(x) \mid \alpha \in A\}$ (resp. $f(x) = \inf\{f_\alpha(x) \mid \alpha \in A\}$) for all $x \in [a, b]$.

   Collaborators:

   Solution: