Homework 4

Instructions: In problems the problems below, references such as III.2.7 refer to Problem 7 in Section 2 of Chapter III in Conway's book.

If you use results from books including Conway's, please be explicit about what results you are using.

Homework 4 is due in class at Midnight March 9.

Do the following problems:

1. IV.7.1

   Sol. (Discussed with a college classmate) In fact, I don't think that this proposition is correct. For example, pick G the unit disk $B(0,1)$, and $\gamma = \gamma(t) : [0,1] \to B$, s.t. $\gamma(t) = t$ for $0 \le t < 1$, and $\gamma(1) = 0$. Then γ is closed, and by simple calculation we know $V(\gamma) = 2$, which shows γ is rectifiable. Let $f = \frac{1}{z-1}$, then f is analytic in $B(0,1)$. But when $t \to 1$, $f \circ \gamma(t) \to \infty$, hence it is not rectifiable.

2. IV.7.2

   (a) Let $f(z) = z$, pick any $z_0 \in \{z \mid d(z, \partial G) < \frac{1}{2}r\}$, then since there is only one point $z = z_0$ satisfies $f(z) = z_0$, by Thm 7.2,

$$n(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} dz$$

   Since $\frac{1}{z-z_0}$ is analytic on $\{z \mid d(z, \partial G) < \frac{1}{2}r\}$, by Prop 2.15, we know the integral is 0. Hence $\{z \mid d(z, \partial G) < \frac{1}{2}r\} \subset H$.

3. V.1.1

   (a) Around $z = 0$,

$$\lim_{z\to 0} |zf(z)| = \lim_{z\to 0} |\sin(z)| = \frac{1}{2} \lim_{z\to 0} |e^{iz} - e^{-iz}| \le \lim_{z\to 0} |z| = 0.$$

   Hence by Thm 1.2, $z = 0$ is removable, and $f(0) = 1$ by power series expansion.

   (b) At $z = 0$, $g(z) = \cos(z)$ is analytic, and $\cos(0) = 1$. Thus by Prop 1.4, $z = 0$ is a pole, and the singular part is $\frac{1}{z}$.

   (c) At $z = 0$, $\lim_{z\to 0} zf(z) = \lim_{z\to 0} \cos z - 1 = 0$, then by Thm 1.2, 0 is removable, and $f(0) = 0$ by power series expansion.

   (d) At $z = 0$,

$$f(z) = \sum_{n=0}^{-\infty} \frac{1}{(-n)!} z^n,$$

   hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \{z \mid |z| > exp(\frac{1}{\delta})\}$.

   (e) At $z = 0$,

$$f(z) = \frac{1}{z^2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+2} z^n.$$

   Hence 0 is a pole, and the singularity part is $\frac{1}{z}$.

   (f) At $z = 0$,

$$f(z) = z \sum_{n=0}^{\infty} (-1)^n \frac{z^{-2n}}{n!} = z + \sum_{n=-1}^{-\infty} (-1)^{-n} \frac{z^{2n+1}}{(-n)!}$$

   Hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \mathbb{C}$.

(g) Around $z=0$, notice $\frac{z^2+1}{z-1}$ is analytic, hence 0 is a pole. Since $|z|<1$,

$$f(z) = 1 - \frac{1}{z} + \frac{2}{z-1} = 1 - \frac{1}{z} - 2\sum_{n=0}^{\infty} z^n,$$

we know the singular part is $-\frac{1}{z}$.

(h) For any $n>0$,

$$\lim_{z\to 0} z^n f(z) = \lim_{z\to 0} z^n \frac{1}{\sum_{n=1}^{\infty}\frac{1}{n!}z^n} = \infty,$$

hence 0 is an essential singularity, and $f(0<|z|<\delta) = \{z \mid |z| > \frac{1}{1-e^{\delta}}\}$.

(i)

$$f(z) = z\sum_{n=0}^{\infty}(-1)^n \frac{z^{-(2n+1)}}{(2n+1)!} = 1 + \sum_{n=-1}^{-\infty}(-1)^{-n}\frac{z^{2n-1}}{(-2n+1)!},$$

hence 0 is an essential singularity, and $f(0<|z|<\delta) = \{z \mid |z| < \delta\}$.

(j) Same with (i), 0 is an essential singularity, and $f(0<|z|<\delta) = \{z \mid |z| < \delta^n\}$.

4. V.1.4

(a)

$$f(z) = \frac{1}{z}\left(\frac{1}{1-z} - \frac{1}{2(1-z/2)}\right) = \frac{1}{z}\left(\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right) = \frac{1}{2z} + \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+2}}\right)z^n$$

(b)

$$f(z) = \frac{1}{z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right) = \frac{1}{z}\left(-\frac{1}{2}\frac{1}{1-\frac{z}{2}} - \frac{1}{z}\frac{1}{1-\frac{1}{z}}\right) = \frac{1}{z}\left(-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty} z^{-n-1}\right) = -\sum_{n=-1}^{\infty}\frac{z^n}{2^{n+2}} - \sum_{n=-\infty}^{-2} z^n$$

(c)

$$f(z) = \frac{1}{z}\left(\frac{\frac{1}{z}}{1-\frac{2}{z}} - \frac{\frac{1}{z}}{1-\frac{1}{z}}\right) = \frac{1}{z}\left(\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \sum_{n=0}^{\infty}\frac{1}{z^n}\right) = \sum_{n=-\infty}^{-1}\left(2^{-(n+1)} - 1\right)z^n.$$

5. V.1.12

Proof. By (1.11), since f is analytic on $0<|z|<\infty$,

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{\exp(\frac{1}{2}\lambda(z+\frac{1}{z}))}{z^{n+1}}dz$$

pick $\gamma = \exp(it)$, the unit circle, then

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{\lambda\cos t}e^{-int}dt = \frac{1}{2\pi}\int_0^{2\pi} e^{\lambda\cos t}(\cos nt - i\sin nt)dt,$$

the real part is

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\lambda\cos t}\cos nt dt = \frac{1}{2\pi}\left(\int_0^{\pi} e^{\lambda\cos t}\cos nt dt - \int_{\pi}^{0} e^{\lambda\cos(2\pi-s)}\cos(2\pi-s)ds\right) = \frac{1}{\pi}\int_0^{\pi} e^{\lambda\cos t}\cos nt dt.$$

and the imaginary part is

$$-\frac{i}{2\pi}\int_0^{2\pi} e^{\lambda\cos t}\sin nt dt = -\frac{i}{2\pi}\left(\int_0^{\pi} e^{\lambda\cos t}\sin nt dt + \int_{\pi}^{0} e^{\lambda\cos(2\pi-s)}\sin(2\pi-s)ds\right) = 0.$$

Hence $a_n = \frac{1}{\pi}\int_0^{\pi} e^{\lambda\cos t}\cos nt dt$.

$$b_n = \frac{1}{2\pi i}\int_\gamma \frac{\exp(\frac{1}{2}\lambda(z-\frac{1}{z}))}{z^{n+1}}dz$$

pick $\gamma = \exp(it)$,

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} e^{\lambda i \sin t}e^{-int}dt = \frac{1}{2\pi}\int_0^{2\pi}\cos(nt-\lambda\sin t) - i\sin(nt-\lambda\sin t)dt.$$

The real part is

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\cos(nt-\lambda\sin t)dt &= \frac{1}{2\pi}\left(\int_0^{\pi}\cos(nt-\lambda\sin t)dt - \int_\pi^0 \cos(n(2\pi-s)-\lambda\sin(2\pi-s))ds\right)\\ &= \frac{1}{2\pi}\left(\int_0^{\pi}\cos(nt-\lambda\sin t)dt + \int_0^{\pi}\cos(-ns+\lambda\sin s)ds\right) = \frac{1}{\pi}\int_0^{\pi}\cos(nt-\lambda\sin t)dt.\end{aligned}$$

and the imaginary part is

$$\begin{aligned}-\frac{i}{2\pi}\int_0^{2\pi}\sin(nt-\lambda\sin t)dt &= -\frac{i}{2\pi}\left(\int_0^{\pi}\sin(nt-\lambda\sin t)dt - \int_\pi^0 \sin(n(2\pi-s)-\lambda\sin(2\pi-s))ds\right)\\ &= -\frac{i}{2\pi}\left(\int_0^{\pi}\sin(nt-\lambda\sin t)dt - \int_0^{\pi}\sin(-ns+\lambda\sin s)ds\right) = 0.\end{aligned}$$

Hence $b_n = \frac{1}{\pi}\int_0^\pi \cos(nt-\lambda\sin t)dt$.

6. V.1.13

(a) Suppose f is entire and has a removable singularity at ∞, then $g(z) = f(\frac{1}{z})$ has a removable singularity at $z=0$, thus we can define $g(0) = a < \infty$, which means f is bounded in the neighbourhood of ∞. Hence by Liouville Thm, f is constant.

(b) By assumption, $g(z) = f(\frac{1}{z}) = \frac{1}{z^m}h(z)$, where $h(z)$ is analytic at $z=0$. Since $f(z) = z^m h(\frac{1}{z})$ is entire, it means at $z=0$, h has a definition or has a removable singularity, and h is entire. Hence, by (1) we know h is a constant, which means f is a polynomial of degree m.

(c) Let $f(z) = \frac{\prod(z-u_i)}{\prod(z-v_i)}$, then

$$g(z) = f(\frac{1}{z}) = \frac{\prod_{i=1}^n(\frac{1}{z}-u_i)}{\prod_{i=1}^m(\frac{1}{z}-v_i)}$$

has a removable singularity at $z=0$, which means $\lim_{z\to 0} g(z)$ exists and is not ∞. Notice

$$\lim_{z\to 0} g(z) = \lim_{z\to 0}\frac{z^m\prod_{i=1}^n(\frac{1}{z}-u_i)}{\prod_{i=1}^m(1-v_i z)},$$

if $m<n$, then $\lim_{z\to 0} g(z) = \infty$, which makes a contradiction. If $m \geq n$, then the limit is well-defined. Hence $f = \frac{p(z)}{q(z)}$, where p, q are polynomials, and $\deg(p) \leq \deg(q)$.

(d) Let $f(z) = \frac{\prod(z-u_i)}{\prod(z-v_i)}$, then

$$g(z) = f(\frac{1}{z}) = \frac{\prod_{i=1}^n(\frac{1}{z}-u_i)}{\prod_{i=1}^k(\frac{1}{z}-v_i)}$$

has a pole of order m at $z=0$, which means $g(z) = \frac{h(z)}{z^m}$, where $h(z)$ is analytic at $z=0$. Then

$$z^m g(z) = \frac{z^m\prod_{i=1}^n(\frac{1}{z}-u_i)}{\prod_{i=1}^k(\frac{1}{z}-v_i)} = \frac{z^{m+k}\prod_{i=1}^n(\frac{1}{z}-u_i)}{\prod_{i=1}^m(1-v_i z)},$$

and we have $m+k=n$, otherwise the order of pole is not m. Hence $f = \frac{p}{q}$, and $\deg(p) - \deg(q) = m$.

7. V.1.17

   Proof. If not, first suppose f has a pole of order m at a. Then $f(z) = \frac{g(z)}{(z-a)^m}$, and g is analytic on G.

   i) If $g(a) = 0$, then $f(z) = 0$, and we can define $f(a) = 0$, thus a is a removable singularity.

   ii) If $g(a) \neq 0$, then according to the isolation of zeros, $\exists r > 0$, s.t. $g(z) \neq 0$ in $B(a, 2r)$. Consider $H = B(a, r)$, by max modulus theorem, $\min_H |g| = \min_{\partial H} |g|$, denote is as $c \neq 0$. Hence

   $$\int\int_H |f(x+iy)|^2 dxdy = \int\int_H \frac{|g(x+iy)|^2}{|x+iy-a|^{2m}} dxdy \geq \int\int_H \frac{c^2}{|x+iy-a|^2} dxdy.$$

   Let $x = Re(a) + s\cos t, y = Im(a) + s\sin t$, then

   $$\int\int_H |f(x+iy)|^2 dxdy \geq \int_0^r \int_0^{2\pi} \frac{c^2}{s^{2m}} sdsdt = 2\pi c^2 \int_0^r s^{1-2m} ds.$$

   If $m = 1$, then the integral becomes

   $$\int_0^r s^{-1} ds = \ln(s) \Big|_0^r = \infty.$$

   If $m > 1$, then the integral is

   $$\int_0^r s^{1-2m} ds = \frac{1}{2-2m} s^{2-2m} \Big|_0^r = \infty.$$

   Since the integrand is nonnegative,

   $$\int_G |f|^2 \geq \int_H |f|^2 = \infty,$$

   which makes a contradiction. With the same method, we know a is not an essential singularity. Hence a is a removable one.

   By the deduction we know, if $0 < p < 2$, a could be a removable singularity or a pole of order m, which satisfies $pm < 2$. If $p \geq 2$, then a is a removable singularity.

8. V.2.1

9. V.2.2

10. V.2.3

11. V.2.4

12. V.2.5