Numerical Analysis
Assignment 13

Chuan Lu

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Problem 1. Problem 5.23

(a). Integrate (5.4.2) with respect to x on $[0,1]$, we get from the convergence of the function series,

$$\int_0^1 (\frac{t}{e^t-1}e^{tx} - \frac{t}{e^t-1})dx = \sum_{j=1}^{\infty} \frac{t^j}{j!}\int_0^1 B_j(x)dx,$$

thus

$$1 - \frac{t}{e^t-1} = \sum_{j=1}^{\infty} \frac{t^j}{j!}\int_0^1 B_j(x)dx,$$

with Taylor expansion of the left term of (5.4.5), we have $B_0 = 1$, thus

$$\sum_{j=1}^{\infty}(B_j + \int_0^1 B_j(x)dx)\frac{t^j}{j!} = 0.$$

The left term can be regarded as a Taylor series of a function f at $t = 0$ which satisfies $f(0) = 0$. By the uniqueness of Taylor expansion of a analytic function, we know $f \equiv 0$. Then for all $j > 0$,

$$B_j + \int_0^1 B_j(x)dx = 0.$$

Now we know $B_0 = 1,\ B_1 = -\frac{1}{2}$, thus from definition (5.4.5),

$$\frac{t}{e^t-1} + \frac{1}{2}t = 1 + \sum_{j=2}^{\infty} B_j\frac{t^j}{j!} = \frac{t}{2}\frac{e^t+1}{e^t-1} \equiv g(t).$$

We have

$$g(-t) = -\frac{t}{2}\frac{e^{-t}+1}{e^{-t}-1} = -\frac{t}{2}\frac{1+e^t}{1-e^t} = g(t),$$

hence $g(t)$ is an even function. Substitute $t = -t$ into last term,

$$1 + \sum_{j=2}^{\infty}(-1)^j B_j\frac{t^j}{j!} = g(-t) = g(t) = 1 + \sum_{j=2}^{\infty} B_j\frac{t^j}{j!}.$$

By simplification,

$$\sum_{j=1}^{\infty} B_{2j+1}\frac{t^{2j+1}}{(2j+1)!} = 0.$$

With the same arguments and the uniqueness of Taylor expansion, we have

$$B_{2j+1} = 0,\ \text{for all } j \geq 1.$$

(b). Take derivatives respective to x on both sides of (5.4.2),

$$\frac{t^2}{e^t-1}e^{tx} = \sum_{j=1}^{\infty} B_j'(x)\frac{t^j}{j!}.$$

Hence,

$$\sum_{j=1}^{\infty} B_j'(x)\frac{t^j}{j!} = t\frac{t(e^{tx}-1)}{e^t-1} + \frac{t^2}{e^t-1} = \sum_{j=1}^{\infty} B_j(x)\frac{t^{j+1}}{j!} + \sum_{j=0}^{\infty} B_j\frac{t^{j+1}}{j!} = \sum_{j=2}^{\infty} jB_{j-1}(x)\frac{t^j}{j!} + \sum_{j=1}^{\infty} jB_{j-1}\frac{t^j}{j!}$$

we have

$$tB_1'(x) + \sum_{j=2}^{\infty} B_j'(x)\frac{t^j}{j!} = B_0 t + \sum_{j=2}^{\infty} j(B_{j-1}(x) + B_{j-1})\frac{t^j}{j!}.$$

Since $tB_1'(x) = B_0 t$, with the uniqueness of Taylor expansion,

$$B_j'(x) = j(B_{j-1}'(x) + B_{j-1}),\ \text{for all } j \geq 2.$$

As proved in (a), $B_{2j+1} = 0$ for all $j \geq 1$, the conclusion holds.

Problem 2. Problem 5.31

```matlab
function T = romberg(f, a, b, n)
% Romberg extrapolation
% f: function
% x: interval: [a, b]
% n: order
T = zeros(n, n);
T(1, 1) = (feval(f, a)+feval(f, b))/2*(b-a);

for i = 2:n
    k = 2^(i-1);
    h = (b-a)/k;
    eval_points = a + (2*linspace(1, k, k)-1)*h/2;
    sums = sum(feval(f, eval_points));
    T(i, 1) = 0.5*(h*sums + T(i-1, 1));
    m = 1;
    for j = 2:i
        m = m*4;
        T(i, j) = (m*T(i, j-1)-T(i-1, j-1))/(m-1);
    end
end

T = diag(T);
```

```matlab
format long g;

f1 = @(x) exp(-x.^2);
f2 = @(x) x.^2.5;
f3 = @(x) 1./(1+x.^2);
f4 = @(x) 1./(2+cos(x));
f5 = @(x) exp(x).*cos(4*x);

a1 = 0; b1 = 1;
a2 = 0; b2 = 1;
a3 = -4; b3 = 4;
a4 = 0; b4 = 2*pi;
a5 = 0; b5 = pi;

N = 20;
```

```
T1 = romberg(f1, a1, b1, N);
T2 = romberg(f2, a2, b2, N);
T3 = romberg(f3, a3, b3, N);
T4 = romberg(f4, a4, b4, N);
T5 = romberg(f5, a5, b5, N);

res = [T1 T2 T3 T4 T5];

dlmwrite('res2.m', res, 'delimiter', ' ', 'precision', '%2.9f');
```

```
0.683939721 0.500000000 0.470588235 2.094395102 37.920111314
0.731045203 0.339463097 1.223529412 2.792526803 -0.705452807
0.739446087 0.310849407 1.993202614 3.297841177 3.229586000
0.743193648 0.298084229 2.365757648 3.468111523 2.177291245
0.745016009 0.291875140 2.508348805 3.547746390 1.737097813
0.745920955 0.288791713 2.579942644 3.587716961 1.519545872
0.746372654 0.287252625 2.615800086 3.607662729 1.410943245
0.746598407 0.286483408 2.633718795 3.617631337 1.356665152
0.746711272 0.286098841 2.642677198 3.622615109 1.329529004
0.746767702 0.285906563 2.647156280 3.625106928 1.315961292
0.746795918 0.285810424 2.649395806 3.626352829 1.309177482
0.746810025 0.285762355 2.650515567 3.626975779 1.305785582
0.746817079 0.285738320 2.651075447 3.627287254 1.304089633
0.746820606 0.285726303 2.651355387 3.627442991 1.303241659
0.746822369 0.285720294 2.651495357 3.627520860 1.302817671
0.746823251 0.285717290 2.651565342 3.627559794 1.302605678
0.746823692 0.285715788 2.651600335 3.627579261 1.302499681
0.746823912 0.285715037 2.651617831 3.627588995 1.302446683
0.746824023 0.285714661 2.651626579 3.627593862 1.302420183
0.746824078 0.285714473 2.651630953 3.627596295 1.302406934
```

Cmp. When comparing with Simpson's rule and trapezoidal rule, we can find that romberg extrapolation converges quicker than the corresponding order of the two rules, but slower than the 2^N order of the two rules.

Problem 3. Problem 5.37

```
function res = gauss_laguerre(f, n)
% Gauss-Laguerre Quadrature
% f: func
% n: num of points

%%%% Reference: MATLAB documentation %%%%
syms t;
w(t) = exp(-t);
F = laguerreL(n, t);
x = vpasolve(F);

y = sym('y', [n, 1]);
sys = sym(zeros(n));
for k = 0:n-1
    sys(k+1) = sum(y.*(x.^k)) == int(t^k*w(t), t, 0, inf);
```

```
end
a = vpasolve(sys, y);
a = structfun(@double, a);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
res = sum(a.*f(x));
```

Sol. (a) By (5.6.11),

$$\int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-x}e^{x-x^2}dx.$$

(b) By (5.6.11),

$$\int_0^\infty \frac{x}{(1+x^2)^2}dx = \int_0^\infty e^{-x}\frac{xe^x}{(1+x^2)^2}dx.$$

(c) By (5.6.11),

$$\int_0^\infty \frac{\sin x}{x}dx = \int_0^\infty e^{-x}\frac{e^x \sin x}{x}dx.$$

The result is as follows, from left to right are (a), (b), (c); the first part is the result, and the second part is the error.

```
1.0879867064 0.5927453354 1.0961083700
0.8476788354 0.4816145789 1.2060830918
0.8791464499 0.4870482698 1.1347650995
0.8925955133 0.4974949629 1.0269639487
```

```
0.1653274309 0.0927453354 0.4746879568
0.4056353019 0.0183854211 0.3647132350
0.3741676874 0.0129517302 0.4360312273
0.3607186240 0.0025050371 0.5438323781
```

Problem 4. Problem 5.40

(a). Let $n \geq 1,\ h = b/n,\ x_j = jh$ for $j = 0, 1, \cdots, n$. For $x_{j-1} \leq x \leq x_j$, define

$$f_n(x) = \frac{1}{h}((x_j - x)f(x_{j-1}) + (x - x_{j-1})f(x_j)),$$

then

$$I_n(f) = \sum_{j=1}^{n}\int_{x_{j-1}}^{x_j}\frac{1}{x^\alpha}\frac{1}{h}((x_j - x)f(x_{j-1}) + (x - x_{j-1})f(x_j))dx = \sum_{k=0}^{n} w_k f(x_k),$$

with

$$w_0 = \frac{1}{h}\int_{x_0}^{x_1}\frac{1}{x^\alpha}(x_1 - x)dx,\ w_n = \frac{1}{h}\int_{x_{n-1}}^{x_n}\frac{1}{x^\alpha}(x - x_{n-1})dx,$$
$$w_j = \frac{1}{h}\int_{x_{j-1}}^{x_j}(x - x_{j-1})\frac{1}{x^\alpha}dx + \frac{1}{h}\int_{x_j}^{x_{j+1}}(x_{j+1} - x)\frac{1}{x^\alpha}dx.$$

To simplify, let $x - x_{j-1} = uh$,

$$\frac{1}{h}\int_{x_{j-1}}^{x_j}(x - x_{j-1})\frac{1}{x^\alpha}dx = \frac{1}{h^{\alpha-1}}\int_0^1\frac{u}{(j-1+u)^\alpha}du,$$

$$\frac{1}{h}\int_{x_{j-1}}^{x_j}(x_j - x)\frac{1}{x^\alpha}dx = \frac{1}{h^{\alpha-1}}\int_0^1\frac{1-u}{(j-1+u)^\alpha}du.$$

Define

$$\phi_1(k) = \int_0^1\frac{u}{(k-1+u)^\alpha}dx,\ \phi_2(k) = \int_0^1\frac{1-u}{(k-1+u)^\alpha}dx,$$

then

$$w_0 = \frac{1}{h^{\alpha-1}}\phi_2(1),\ w_n = \frac{1}{h^{\alpha-1}}\phi_1(n),\ w_j = \frac{1}{h^{\alpha-1}}(\phi_1(j) + \phi_2(j+1)).$$

From (5.6.22) we know,

$$|I(f) - I_n(f)| \leq \frac{h^2}{8}\|f''\|_\infty\int_0^b\frac{1}{x^\alpha}dx.$$

(b). The code and result are as follows. For (ii), after a change of variable, we get

$$\int_0^1 \frac{\pi/(2\sin u)}{u^{1-\pi/2}} du,$$

and $f = \frac{1}{\sin u}$ still has a singularity on $u = 0$.

```
function res = product_trapezoidal(f, b, n, alpha)
% Product trapezoidal rule;
% Compute integral of ‘f‘ on ‘[0, b]‘ with order ‘n‘;

phi1 = @(k) integral(@(u) u./(k-1+u).^alpha, 0, 1);
phi2 = @(k) integral(@(u) (1-u)./(k-1+u).^alpha, 0, 1);

h = b/n;
x = 0:h:b;
w = zeros(n+1, 1); w(1) = phi2(1); w(n+1) = phi1(n);
for i = 2:n
    w(i) = phi1(i-1)+phi2(i);
end
res = f(x)*w/(h^(alpha-1));
```

```
f1 = @(x) exp(x);
alpha1 = 1/pi;
b1 = 1;

f2 = @(x) pi./(2.*sin(x));
alpha2 = 1-pi/2;
b2 = 1;

n = 10;
res = zeros(n, 2);
for i = 2:n
%       disp(product_trapezoidal(f1, b1, i, alpha1))
    res(i, 1) = product_trapezoidal(f1, b1, i, alpha1);
    res(i, 2) = product_trapezoidal(f2, b2, i, alpha2);
end

dlmwrite('res4.m', res, 'delimiter', ' ', 'precision', '%2.10f');
```

```
0.0000000000 0.0000000000
2.3504642772 Inf
2.3247123774 Inf
2.3156515667 Inf
2.3114419298 Inf
2.3091488426 Inf
2.3077632089 Inf
2.3068623347 Inf
2.3062438291 Inf
2.3058008946 Inf
```