

Statistics of Solutions to Test Models for SPEKF

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Outline

Introduction

Stochastic Parameterization Extended Kalman Filter (SPEKF)

Itô Integration

Statistics of $b(t)$ and $\gamma(t)$, Gaussian

Mean

Variance

Covariance

Statistics of $u(t)$, Non-Gaussian

Mean

Variance

Covariance

Numerical Simulation

Parameters and Algorithms

Results

Signals from nature can be modeled by Langevin equation:

Langevin Equation

$$\frac{du(t)}{dt} = -\gamma(t)u(t) + i\omega u(t) + \sigma\dot{W}(t) + f(t),$$

where $\dot{W}(t)$ is a Brownian motion, and $f(t)$ is the external force.

B. Gershgorin, *et al* proposed the Stochastic Parameterization Extended Kalman Filter (SPEKF) to cope with model errors.

Test Model

$$\begin{cases} \frac{du(t)}{dt} = (-\gamma(t) + i\omega)u(t) + b(t) + f(t) + \sigma W(t), \\ \frac{db(t)}{dt} = (-\gamma_b + i\omega_b)(b(t) - \hat{b}) + \sigma_b W_b(t), \\ \frac{d\gamma(t)}{dt} = -d_\gamma(\gamma(t) - \hat{\gamma}) + \sigma_\gamma W_\gamma(t) \end{cases}$$

The initial values are complex random variables, with their first-order and second-order statistics known.

Introduction

Solution

With knowledge of ODEs, solution of the SDE set is

$$\left\{ \begin{array}{l} b(t) = \hat{b} + (b_0 - \hat{b})e^{\lambda_b(t-t_0)} + \sigma_b \int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s) \\ \gamma(t) = \hat{\gamma} + (\gamma_0 - \hat{\gamma})e^{-d_\gamma(t-t_0)} + \sigma_\gamma \int_{t_0}^t e^{-d_\gamma(t-s)} dW_\gamma(s) \\ u(t) = e^{-J(t_0,t) + \hat{\lambda}(t-t_0)} u_0 + \int_{t_0}^t (b(s) + f(s)) e^{-J(s,t) + \hat{\lambda}(s-t_0)} ds \\ \quad + \sigma \int_{t_0}^t e^{-J(s,t) + \hat{\lambda}(s-t_0)} dW(s) \end{array} \right.$$

with $\lambda_b = -\gamma_b + i\omega_b$, $\hat{\lambda} = -\hat{\gamma} + i\omega$, $J(s,t) = \int_s^t (\gamma(s') - \hat{\gamma}) ds'$.

Introduction

Itô Isometry

$\forall f \in \mathcal{V}(S, T)$, B_t is a standard Brownian motion,

$$\mathbb{E} \left[\left(\int_S^T f(t, \omega) dB_t \right)^2 \right] = \mathbb{E} \left[\int_S^T f^2(t, \omega) dt \right].$$

Linear property of Itô integration

$$(1) \quad \int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t, \text{ a.e.}$$

$$(2) \quad \int_S^T (cf + g) dB_t = c \int_S^T f dB_t + \int_S^T g dB_t, \text{ a.e.}$$

$$(3) \quad \mathbb{E} \left[\int_S^T f dB_t \right] = 0.$$

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Mean of $b(t)$, $\gamma(t)$

Solution

$$\begin{cases} b(t) = \hat{b} + (b_0 - \hat{b})e^{\lambda_b(t-t_0)} + \sigma_b \int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s) \\ \gamma(t) = \hat{\gamma} + (\gamma_0 - \hat{\gamma})e^{-d_\gamma(t-t_0)} + \sigma_\gamma \int_{t_0}^t e^{-d_\gamma(t-s)} dW_\gamma(s) \end{cases}$$

With property of Itô integration (3), the last term in $E[b(t)]$ is 0.

Thus we find

$$E[b(t)] = \hat{b} + (E[b_0] - \hat{b})e^{\lambda_b(t-t_0)}$$

$$E[\gamma(t)] = \hat{\gamma} + (E[\gamma_0] - \hat{\gamma})e^{-d_\gamma(t-t_0)}$$

Variance of $b(t)$, $\gamma(t)$

According to definition,

$$\begin{aligned}\text{Var}(b(t)) &= \mathbb{E}[(b(t) - \mathbb{E}[b(t)])(b(t) - \mathbb{E}[b(t)])^*] \\ &= e^{-2\gamma_b(t-t_0)} \text{Var}(b_0) + \mathbb{E} \left[\sigma_b^2 \int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s) \left(\int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s) \right)^* \right] \\ &= e^{-2\gamma_b(t-t_0)} \text{Var}(b_0) + \sigma_b^2 \mathbb{E} \left[\int_{t_0}^t e^{-2\gamma_b(t-s)} ds \right]\end{aligned}$$

The last step takes use of Itô isometry. Compute the integration in last term, we have

$$\begin{aligned}\text{Var}(b(t)) &= e^{-2\gamma_b(t-t_0)} \text{Var}(b_0) + \frac{\sigma_b^2}{2\gamma_b} (1 - e^{-2\gamma_b(t-t_0)}) \\ \text{Var}(\gamma(t)) &= e^{-2d_\gamma(t-t_0)} \text{Var}(\gamma_0) + \frac{\sigma_\gamma^2}{2d_\gamma} (1 - e^{-2d_\gamma(t-t_0)})\end{aligned}$$

Covariance of $b(t)$, $\gamma(t)$

According to definition,

$$\begin{aligned}\text{Cov}(b(t), b(t)^*) &= \mathbb{E}[(b(t) - \mathbb{E}[b(t)])(b(t)^* - \mathbb{E}[b(t)^*])] \\ &= \mathbb{E} \left[(b_0 - \mathbb{E}[b_0])(b_0^* - \mathbb{E}[b_0^*])e^{2\lambda_b(t-t_0)} \right] + \sigma_b \mathbb{E} \left[(b_0 - \mathbb{E}[b_0]) \int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s) \right] \\ &\quad + \sigma_b \mathbb{E} \left[(b_0^* - \mathbb{E}[b_0^*])e^{\lambda_b(t-t_0)} \int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s) \right] + \sigma_b^2 \mathbb{E} \left[\left(\int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s) \right)^2 \right]\end{aligned}$$

With property of Itô integration (3), the second and third term are both 0; with Itô isometry we know the last term is also 0. Accordingly,

$$\text{Cov}(b(t), b(t)^*) = \mathbb{E}[(b_0 - \mathbb{E}[b_0])(b_0^* - \mathbb{E}[b_0^*])]e^{2\lambda_b(t-t_0)} = \text{Cov}(b_0, b_0^*)e^{2\lambda_b(t-t_0)}$$

$$\text{Cov}(b(t), \gamma(t)) = \mathbb{E}[(b(t) - \mathbb{E}[b(t)])(\gamma(t) - \mathbb{E}[\gamma(t)])] = \text{Cov}(b_0, \gamma_0)e^{(\lambda_b - d_\gamma)(t-t_0)}$$

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Mean of $u(t)$

Using the same properties, it is easy to find

$$\begin{aligned} \mathbb{E}[u(t)] &= e^{\hat{\lambda}(t-t_0)} \mathbb{E} \left[e^{-J_0(t_0,t)} u_0 \right] + \int_{t_0}^t e^{\hat{\lambda}(t-s)} \mathbb{E} \left[b(s) e^{-J(s,t)} \right] ds \\ &\quad + \sigma \int_{t_0}^t e^{\hat{\lambda}(t-s)} f(s) \mathbb{E} \left[e^{-J(s,t)} \right] ds. \end{aligned}$$

We find it necessary to compute expectations of terms like

$$\mathbb{E}[ze^{bx}],$$

where z is a complex-valued Gaussian random variable and x is a real-valued Gaussian variable. We propose two lemmas here.

Mean of $u(t)$

Lemma (1)

$$E\left[ze^{ibx}\right] = (E[z] + ib\text{Cov}(z,x))e^{ibE[x] - \frac{1}{2}b^2\text{Var}(x)},$$

with z being a complex-valued Gaussian, and x a real-valued Gaussian.

Corollary (1)

Under the condition of Lemma 1,

$$E\left[ze^{bx}\right] = (E[z] + b\text{Cov}(z,x))e^{bE[x] + \frac{1}{2}b^2\text{Var}(x)}.$$

Proof of lemma 1 takes advantage of characteristic function of multivariable Gaussian distribution.

Proof

Let $z = y + iw$, $y, w \in \mathbb{R}$. Denote $\mathbf{v} = (x, y, w)^\top$, then \mathbf{v} satisfies the multivariable Gaussian distribution, with its characteristic function

$$\phi_{\mathbf{v}}(\mathbf{s}) = \exp(i\mathbf{s}^\top \mathbf{E}[\mathbf{v}] - \frac{1}{2}\mathbf{s}^\top \Sigma \mathbf{s}).$$

Let $g(\mathbf{v})$ being the PDF of \mathbf{v} , then one knows from that char. func. being Fourier transform of PDF,

$$\phi_{\mathbf{v}}(\mathbf{s}) = \frac{1}{(2\pi)^3} \int e^{i\mathbf{s}^\top \mathbf{v}} g(\mathbf{v}) d\mathbf{v}.$$

According to the differential property of Fourier transform (Proposition 2.10),

$$\frac{\partial \phi_{\mathbf{v}}(\mathbf{s})}{\partial s_2} = \frac{1}{(2\pi)^3} \int i y_0 e^{i\mathbf{s}^\top \mathbf{v}} g(\mathbf{v}) d\mathbf{v} = i \mathbf{E} \left[y_0 e^{i\mathbf{s}^\top \mathbf{v}} \right].$$

Let $\mathbf{v} = (b, 0, 0)^\top$,

$$\mathbf{E}[y_0 e^{ibx_0}] = -i \frac{\partial \phi_{\mathbf{v}}(s)}{\partial s_2} \Big|_{\mathbf{s}=(b,0,0)^\top}, \quad \mathbf{E}[w_0 e^{ibx_0}] = -i \frac{\partial \phi_{\mathbf{v}}(s)}{\partial s_3} \Big|_{\mathbf{s}=(b,0,0)^\top}$$

Proof

From PDF of multivariable Gaussian distribution, one knows

$$\frac{\partial \phi_{\mathbf{v}}(\mathbf{s})}{\partial s_2} = (i\mathbb{E}[y_0] - \text{Var}(y_0)s_2 - \text{Cov}(x_0, y_0)s_1 - \text{Cov}(y_0, w_0)s_3)\phi_{\mathbf{v}}(\mathbf{s})$$

$$\frac{\partial \phi_{\mathbf{v}}(\mathbf{s})}{\partial s_3} = (i\mathbb{E}[w_0] - \text{Var}(w_0)s_3 - \text{Cov}(x_0, w_0)s_1 - \text{Cov}(y_0, w_0)s_2)\phi_{\mathbf{v}}(\mathbf{s})$$

Compute the partial derivatives at $\mathbf{s} = (b, 0, 0)^\top$,

$$\mathbb{E}\left[y_0 e^{ibx_0}\right] = (\mathbb{E}[y_0] + i\text{Cov}(x_0, y_0)b) \exp(ib\mathbb{E}[x_0] - \frac{1}{2}\text{Var}(x_0)b^2)$$

$$\mathbb{E}\left[w_0 e^{ibx_0}\right] = (\mathbb{E}[w_0] + i\text{Cov}(x_0, w_0)b) \exp(ib\mathbb{E}[x_0] - \frac{1}{2}\text{Var}(x_0)b^2)$$

Then

$$\mathbb{E}\left[ze^{ibx}\right] = (\mathbb{E}[z] + ib\text{Cov}(z, x))e^{ib\mathbb{E}[x] - \frac{1}{2}b^2\text{Var}(x)}.$$

□

Lemma (2)

$$E\left[ze^{bx}\right] = [E[z]E[w] + \text{Cov}(z, w^*) + b(E[z]\text{Cov}(w, x)) + E[w]\text{Cov}(z, x) + b^2\text{Cov}(z, x)\text{Cov}(w, x)] e^{bE[x] + \frac{b^2}{2}\text{Var}(x)}.$$

with z, w being complex-valued Gaussian, and x real-valued Gaussian.

The proof of this lemma is the same as Lemma 1.

Mean of $u(t)$

We now make use of Lemma 1 to obtain mean of $u(t)$.

$$\begin{aligned} E[u(t)] &= e^{\hat{\lambda}(t-t_0)} (E[u_0] - \text{Cov}(u_0, J(t_0, t))) e^{-E[J(t_0, t)] + \frac{1}{2} \text{Var}(J(t_0, t))} \\ &\quad + \int_{t_0}^t e^{\hat{\lambda}(t-s)} (\hat{b} + e^{\lambda_b(s-t_0)} (E[b_0] - \hat{b} - \text{Cov}(b_0, J(s, t)))) e^{-E[J(s, t)] + \frac{1}{2} \text{Var}(J(s, t))} ds \\ &\quad + \int_{t_0}^t e^{\hat{\lambda}(t-s)} f(s) e^{-E[J(s, t)] + \frac{1}{2} \text{Var}(J(s, t))} ds \end{aligned}$$

The terms $\text{Cov}(u_0, J(s, t))$, $\text{Cov}(b_0, J(s, t))$, $E[J(s, t)]$ and $\text{Var}(J(s, t))$ can be found using Itô isometry.

Variance of $u(t)$

Denote $u(t) = A + B + C$,

$$\begin{cases} A = e^{-J(t_0,t) + \hat{\lambda}(t-t_0)} u_0, \\ B = \int_{t_0}^t (b(s) + f(s)) e^{-J(s,t) + \hat{\lambda}(t-s)} ds, \\ C = \sigma \int_{t_0}^t e^{-J(s,t) + \hat{\lambda}(t-s)} dW(s). \end{cases}$$

By definition we find $\text{Var}(u(t)) = \mathbb{E}[|u(t)|^2] - |\mathbb{E}[u(t)]|^2$, with

$$\mathbb{E}[|u(t)|^2] = \mathbb{E}[|A|^2] + \mathbb{E}[|B|^2] + \mathbb{E}[|C|^2] + 2\text{Re}\{\mathbb{E}[A^*B]\}.$$

We can obtain $\mathbb{E}[|A|^2]$ by Lemma 2, and $\mathbb{E}[|B|^2]$ by Itô isometry. Noticing that

$$\text{Cov}(J(s,t), J(r,t)) = \text{Var}(J(s,t)) + \text{Cov}(J(s,t), J(r,s)).$$

$\mathbb{E}[|C|^2]$ and $\text{Re}\{\mathbb{E}[A^*B]\}$ can also be computed by Itô isometry and property of Itô integration.

By definition,

$$\text{Cov}(u(t), u^*(t)) = \mathbb{E}[u(t)^2] - \mathbb{E}[u(t)]^2$$

$$\text{Cov}(u(t), \gamma(t)) = \mathbb{E}[u(t)(\gamma(t) - \hat{\gamma})] + \mathbb{E}[u(t)](\hat{\gamma} - \mathbb{E}[\gamma(t)])$$

$$\text{Cov}(u(t), b(t)) = \mathbb{E}[u(t)b^*(t)] - \mathbb{E}[u(t)]\mathbb{E}[b(t)]^*$$

$$\text{Cov}(u(t), b^*(t)) = \mathbb{E}[u(t)b(t)] - \mathbb{E}[u(t)]\mathbb{E}[b(t)].$$

Each term can be obtained by Lemma 3 and Itô isometry.

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Parameters

As a real-world example, we choose a periodic external force

$$f(t) = \frac{3}{2}e^{\frac{1}{10}it},$$

and parameters of the equation set are given

$$\begin{cases} d = 1.5, & d_\gamma = 0.01d \\ \sigma = 0.1549, & \omega = 1.78 \\ \sigma_\gamma = 5\sigma, & \gamma_b = 0.1d \\ \sigma_b = 5\sigma, & \omega_b = \omega \\ \hat{b} = 0, & \hat{\gamma} = 0 \end{cases}$$

We assume that initial values satisfy

$$\begin{cases} \operatorname{Re}(u_0), \operatorname{Im}(u_0) \sim \mathcal{N}(0, \frac{1}{\sqrt{2}}), \text{ i.i.d.} \\ \operatorname{Re}(b_0), \operatorname{Im}(b_0) \sim \mathcal{N}(0, \frac{1}{\sqrt{2}}), \text{ i.i.d.} \\ \gamma_0 \sim \mathcal{N}(0, 1) \end{cases}$$

Euler-Maruyama Scheme

Itô integration can be simulated by E-M scheme:

$$X_j = X_{j-1} + f(X_{j-1})\Delta t + g(X_{j-1})(W(\tau_j) - W(\tau_{j-1})),$$

with

$$W(\tau_j) - W(\tau_{j-1}) = \sum_{k=jR-R+1}^{jR} dW_k,$$

and R being the step length of E-M scheme,

$$dW = \sqrt{\Delta t} \times \text{randn}().$$

Revised Euler Scheme

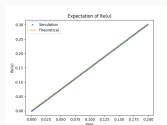
Since we need to simulate a SDE set with three relevant equations, we choose the Revised Euler Scheme from numerical solution to ODE for simulating $u(t)$:

$$x_n = x_{n-1} + \Delta t \phi\left(t_{n-1} + \frac{\Delta t}{2}, x_{n-1} + \frac{\Delta t}{2} \phi(t_{n-1}, x_{n-1})\right),$$

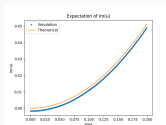
it is a second-order Runge-Kutta scheme with a locally quadratic convergence.

Result

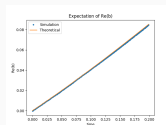
Simulate 10^6 times with $R = 1$, the results are as follows:



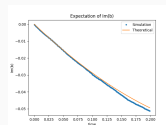
(a) $E(\text{Re}(u))$



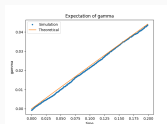
(b) $E(\text{Im}(u))$



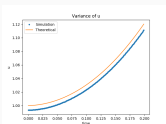
(c) $E(\text{Re}(b))$



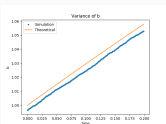
(d) $E(\text{Im}(b))$



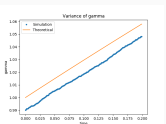
(e) $E(\gamma)$



(f) $\text{Var}(u)$



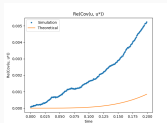
(g) $\text{Var}(b)$



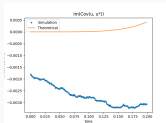
(h) $\text{Var}(\gamma)$

Figure 1: Simulation of Expectations and Variances, $n = 10^6$

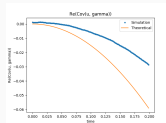
Result



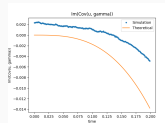
(a) $\text{Re}(\text{Cov}(u, u^*))$



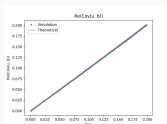
(b) $\text{Im}(\text{Cov}(u, u^*))$



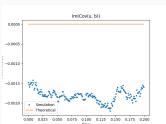
(c) $\text{Re}(\text{Cov}(u, \gamma))$



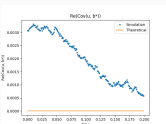
(d) $\text{Im}(\text{Cov}(u, \gamma))$



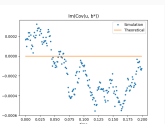
(e) $\text{Re}(\text{Cov}(u, b))$



(f) $\text{Im}(\text{Cov}(u, b))$



(g) $\text{Re}(\text{Cov}(u, b^*))$

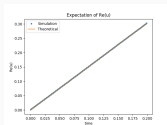


(h) $\text{Im}(\text{Cov}(u, b^*))$

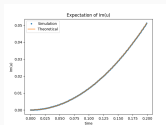
Figure 2: Simulation of Covariances, $n = 10^6$

Result

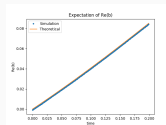
Simulate 10^7 times with $R = 1$, the results are as follows:



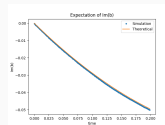
(a) $E(\text{Re}(u))$



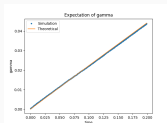
(b) $E(\text{Im}(u))$



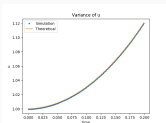
(c) $E(\text{Re}(b))$



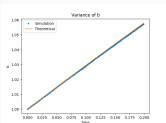
(d) $E(\text{Im}(b))$



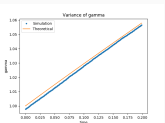
(e) $E(\gamma)$



(f) $\text{Var}(u)$



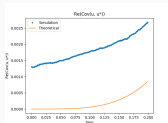
(g) $\text{Var}(b)$



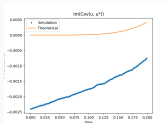
(h) $\text{Var}(\gamma)$

Figure 3: Simulation of Expectations and Variances, $n = 10^7$

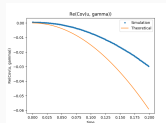
Result



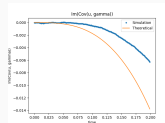
(a) $\text{Re}(\text{Cov}(u, u^*))$



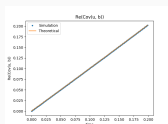
(b) $\text{Im}(\text{Cov}(u, u^*))$



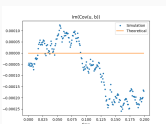
(c) $\text{Re}(\text{Cov}(u, \gamma))$



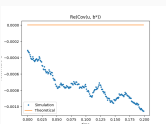
(d) $\text{Im}(\text{Cov}(u, \gamma))$



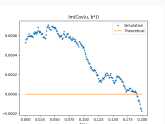
(e) $\text{Re}(\text{Cov}(u, b))$



(f) $\text{Im}(\text{Cov}(u, b))$



(g) $\text{Re}(\text{Cov}(u, b^*))$



(h) $\text{Im}(\text{Cov}(u, b^*))$

Figure 4: Simulation of Covariances, $n = 10^7$

From the simulation results we find that the simulation of expectations fit the theoretical results satisfyingly.

For the results of Variances and Covariances,

- When $n = 10^6$, the error of variances is about $O(10^{-2}) \sim O(10^{-3})$, error of covariances is about $O(10^{-2}) \sim O(10^{-3})$.
- When $n = 10^7$, the error of $\text{Var}(u)$, $\text{Var}(b)$ is about $O(10^{-5})$, and error of $\text{Var}(\gamma)$ is about $O(10^{-3}) \sim O(10^{-4})$, error of $\text{Cov}(u, u^*)$, $\text{Cov}(u, \gamma)$ is about $O(10^{-3})$, error of $\text{Cov}(u, b)$, $\text{Cov}(u, b^*)$ is $O(10^{-4})$.

One can find the simulation errors decrease with the growth of simulations. Thus one could believe that when $n \rightarrow \infty$, the simulation would converge to the theoretical result. In fact, error of simulation comes from that initial values are all random variables; according to Law of large numbers, the errors would converge to 0 when $n \rightarrow \infty$.

We can get third-order and fourth-order statistics of u, b, γ with the same methods, which could be written in a form of multiple integrals of statistics of initial values.

Thank you!