

Homework 4

Instructions: In problems the problems below, references such as III.2.7 refer to Problem 7 in Section 2 of Chapter III in Conway's book.

If you use results from books including Conway's, please be explicit about what results you are using.

Homework 4 is due in class at Midnight March 9.

Do the following problems:

1. IV.7.1

Sol. (Discussed with a college classmate) In fact, I don't think that this proposition is correct. For example, pick G the unit disk $B(0, 1)$, and $\gamma = \gamma(t) : [0, 1] \rightarrow B$, s.t. $\gamma(t) = t$ for $0 \leq t < 1$, and $\gamma(1) = 0$. Then γ is closed, and by simple calculation we know $V(\gamma) = 2$, which shows γ is rectifiable. Let $f = \frac{1}{z-1}$, then f is analytic in $B(0, 1)$. But when $t \rightarrow 1$, $f \circ \gamma(t) \rightarrow \infty$, hence it is not rectifiable.

2. IV.7.2

(a) Let $f(z) = z$, pick any $z_0 \in \{z \mid d(z, \partial G) < \frac{1}{2}r\}$, then since there is only one point $z = z_0$ satisfies $f(z) = z_0$, by Thm 7.2,

$$n(\gamma; z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - z_0} dz$$

Since $\frac{1}{z-z_0}$ is analytic on $\{z \mid d(z, \partial G) < \frac{1}{2}r\}$, by Prop 2.15, we know the integral is 0. Hence $\{z \mid d(z, \partial G) < \frac{1}{2}r\} \subset H$.

3. V.1.1

(a) Around $z = 0$,

$$\lim_{z \rightarrow 0} |zf(z)| = \lim_{z \rightarrow 0} |\sin(z)| = \frac{1}{2} \lim_{z \rightarrow 0} |e^{iz} - e^{-iz}| \leq \lim_{z \rightarrow 0} |z| = 0.$$

Hence by Thm 1.2, $z = 0$ is removable, and $f(0) = 1$ by power series expansion.

(b) At $z = 0$, $g(z) = \cos(z)$ is analytic, and $\cos(0) = 1$. Thus by Prop 1.4, $z = 0$ is a pole, and the singular part is $\frac{1}{z}$.

(c) At $z = 0$, $\lim_{z \rightarrow 0} zf(z) = \lim_{z \rightarrow 0} \cos z - 1 = 0$, then by Thm 1.2, 0 is removable, and $f(0) = 0$ by power series expansion.

(d) At $z = 0$,

$$f(z) = \sum_{n=0}^{-\infty} \frac{1}{(-n)!} z^n,$$

hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \{z \mid |z| > \exp(\frac{1}{\delta})\}$.

(e) At $z = 0$,

$$f(z) = \frac{1}{z^2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+2} z^n.$$

Hence 0 is a pole, and the singularity part is $\frac{1}{z}$.

(f) At $z = 0$,

$$f(z) = z \sum_{n=0}^{\infty} (-1)^n \frac{z^{-2n}}{n!} = z + \sum_{n=-1}^{-\infty} (-1)^{-n} \frac{z^{2n+1}}{(-n)!}$$

Hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \mathbb{C}$.

(g) Around $z = 0$, notice $\frac{z^2+1}{z-1}$ is analytic, hence 0 is a pole. Since $|z| < 1$,

$$f(z) = 1 - \frac{1}{z} + \frac{2}{z-1} = 1 - \frac{1}{z} - 2 \sum_{n=0}^{\infty} z^n,$$

we know the singular part is $-\frac{1}{z}$.

(h) For any $n > 0$,

$$\lim_{z \rightarrow 0} z^n f(z) = \lim_{z \rightarrow 0} z^n \frac{1}{\sum_{n=1}^{\infty} \frac{1}{n!} z^n} = \infty,$$

hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \{z \mid |z| > \frac{1}{1-e^\delta}\}$.

(i)

$$f(z) = z \sum_{n=0}^{\infty} (-1)^n \frac{z^{-(2n+1)}}{(2n+1)!} = 1 + \sum_{n=-1}^{-\infty} (-1)^{-n} \frac{z^{2n-1}}{(-2n+1)!},$$

hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \{z \mid |z| < \delta\}$.

(j) Same with (i), 0 is an essential singularity, and $f(0 < |z| < \delta) = \{z \mid |z| < \delta^n\}$.

4. V.1.4

(a)

$$f(z) = \frac{1}{z} \left(\frac{1}{1-z} - \frac{1}{2(1-z/2)} \right) = \frac{1}{z} \left(\sum_{n=0}^{\infty} z^n - \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \right) = \frac{1}{2z} + \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+2}}\right) z^n$$

(b)

$$f(z) = \frac{1}{z} \left(\frac{1}{z-2} - \frac{1}{z-1} \right) = \frac{1}{z} \left(-\frac{1}{2} \frac{1}{1-\frac{z}{2}} - \frac{1}{z} \frac{1}{1-\frac{1}{z}} \right) = \frac{1}{z} \left(-\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty} z^{-n-1} \right) = -\sum_{n=-1}^{\infty} \frac{z^n}{2^{n+2}} - \sum_{n=-\infty}^{-2} z^n$$

(c)

$$f(z) = \frac{1}{z} \left(\frac{\frac{1}{z}}{1-\frac{2}{z}} - \frac{\frac{1}{z}}{1-\frac{1}{z}} \right) = \frac{1}{z} \left(\sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n - \sum_{n=0}^{\infty} \frac{1}{z^n} \right) = \sum_{n=-\infty}^{-1} (2^{-(n+1)} - 1) z^n.$$

5. V.1.12

Proof. By (1.11), since f is analytic on $0 < |z| < \infty$,

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{\exp(\frac{1}{2}\lambda(z + \frac{1}{z}))}{z^{n+1}} dz$$

pick $\gamma = \exp(it)$, the unit circle, then

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} e^{\lambda \cos t} e^{-int} dt = \frac{1}{2\pi} \int_0^{2\pi} e^{\lambda \cos t} (\cos nt - i \sin nt) dt,$$

the real part is

$$\frac{1}{2\pi} \int_0^{2\pi} e^{\lambda \cos t} \cos nt dt = \frac{1}{2\pi} \left(\int_0^{\pi} e^{\lambda \cos t} \cos nt dt - \int_{\pi}^0 e^{\lambda \cos(2\pi-s)} \cos(2\pi-s) ds \right) = \frac{1}{\pi} \int_0^{\pi} e^{\lambda \cos t} \cos nt dt.$$

and the imaginary part is

$$-\frac{i}{2\pi} \int_0^{2\pi} e^{\lambda \cos t} \sin nt dt = -\frac{i}{2\pi} \left(\int_0^{\pi} e^{\lambda \cos t} \sin nt dt + \int_{\pi}^0 e^{\lambda \cos(2\pi-s)} \sin(2\pi-s) ds \right) = 0.$$

Hence $a_n = \frac{1}{\pi} \int_0^{\pi} e^{\lambda \cos t} \cos nt dt$.

$$b_n = \frac{1}{2\pi i} \int_{\gamma} \frac{\exp(\frac{1}{2}\lambda(z - \frac{1}{z}))}{z^{n+1}} dz$$

pick $\gamma = \exp(it)$,

$$b_n = \frac{1}{2\pi} \int_0^{2\pi} e^{\lambda i \sin t} e^{-int} dt = \frac{1}{2\pi} \int_0^{2\pi} \cos(nt - \lambda \sin t) - i \sin(nt - \lambda \sin t) dt.$$

The real part is

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \cos(nt - \lambda \sin t) dt &= \frac{1}{2\pi} \left(\int_0^{\pi} \cos(nt - \lambda \sin t) dt - \int_{\pi}^0 \cos(n(2\pi - s) - \lambda \sin(2\pi - s)) ds \right) \\ &= \frac{1}{2\pi} \left(\int_0^{\pi} \cos(nt - \lambda \sin t) dt + \int_0^{\pi} \cos(-ns + \lambda \sin s) ds \right) = \frac{1}{\pi} \int_0^{\pi} \cos(nt - \lambda \sin t) dt. \end{aligned}$$

and the imaginary part is

$$\begin{aligned} -\frac{i}{2\pi} \int_0^{2\pi} \sin(nt - \lambda \sin t) dt &= -\frac{i}{2\pi} \left(\int_0^{\pi} \sin(nt - \lambda \sin t) dt - \int_{\pi}^0 \sin(n(2\pi - s) - \lambda \sin(2\pi - s)) ds \right) \\ &= -\frac{i}{2\pi} \left(\int_0^{\pi} \sin(nt - \lambda \sin t) dt - \int_0^{\pi} \sin(-ns + \lambda \sin s) ds \right) = 0. \end{aligned}$$

Hence $b_n = \frac{1}{\pi} \int_0^{\pi} \cos(nt - \lambda \sin t) dt$.

6. V.1.13

7. V.1.17

8. V.2.1

9. V.2.2

10. V.2.3

11. V.2.4

12. V.2.5