Introduction to Analysis I
Homework 4
Wednesday, September 27, 2017

Instructions: This and all subsequent homeworks must be submitted written in LaTeX.
If you use results from books, Royden or others, please be explicit about what results you are using.

Homework 4 is due by midnight, Saturday, October 7.

1. (Problem 18, Page 63) Let I be a closed bounded interval and let f be a bounded measurable function defined on I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which
$$|h - f| < \epsilon \text{ on } F \text{ and } m(I \sim F) < \epsilon.$$

Collaborators: None

Solution: For $\epsilon > 0$, according to Lusin's Theorem, there exists a closed subset F of I, s.t. $m(I \backslash F) < \epsilon$, and f is continuous on F. Then for $x_0 \in F$, for this $\epsilon > 0$,
$$\exists \sigma > 0,\ \forall x \in (x_0 - \delta,\ x_0 + \delta) \cap F,\ |f(x) - f(x_0)| < \epsilon.$$
Since
$$\bigcup_{x_0 \in F} (x_0 - \delta_i,\ x_0 + \delta_i)$$
is an open cover of the closed set F, it has a finite subcover, denoted by
$$\bigcup_{i=1}^{n} (x_i - \delta_i,\ x_i + \delta_i).$$
If we define
$$h = \begin{cases} f(x_i), & \text{for } x \in (x_i - \delta_i,\ x_i + \delta_i), \\ 0, & \text{others} \end{cases}$$
then h takes constant values in finite intervals, and $h = 0$ in other parts, we know h is a step function. And from the construction above, on each interval $(x_i - \delta_i, x_i + \delta_i) \cap F$,
$$|h - f| < \epsilon \text{ on } (x_i - \delta_i, x_i + \delta_i) \cap F.$$
Thus h is the step function needed.

2. (Problem 22, Page 64) (Dini's Theorem) Let $\{f_n\}$ be an increasing sequence of continuous functions on $[a, b]$ which converges pointwise on $[a, b]$ to the continuous function f on $[a, b]$. Show that the convergence is uniform on $[a, b]$.

Collaborators: None

Solution: If the convergence is not uniform, then there $\exists \epsilon > 0, \forall N > 0,\ \exists n > N,\ \exists x \in [a, b],\ |f(x) - f_n(x)| > \epsilon$.
We can construct a sequence $\{x_n\}$ like this:
$$N = 1,\ \exists n_1 > 1,\ \exists x_1 \in [a, b],\ |f_{n_1}(x_1) - f(x_1)| \geqslant \epsilon.$$
$$N = n_1,\ \exists n_2 > n_1,\ \exists x_2 \in [a, b],\ |f_{n_2}(x_2) - f(x_2)| \geqslant \epsilon.$$
$$\cdots$$

$$N = n_{k-1},\ \exists n_k > n_{k-1},\ \exists x_k \in [a,b],\ |f_{n_k}(x_k) - f(x_k)| \geqslant \epsilon.$$
$$\cdots$$

Then since $x_i \in [a,b]$, according to Bolzano-Weierstrass Theorem, there is a convergent subsequence in $\{x_i\}$, and we may denote it by $\{y_i\}$ for convinence. Assume $y_i \to y \in [a,b]$, then since

$$\lim_{n\to\infty} f_n(y) = f(y),$$

for this $\epsilon > 0,\ \exists N$, s.t.

$$|f_N(y) - f(y)| < \epsilon.$$

since f_N is continuous, with $y_k \to y$, there $\exists K > 0$,

$$|f_N(y_k) - f(y_k)| < \epsilon$$

holds for all $k > K$. Notice that $\{f_n\}$ is an increasing sequence, when $n > N$ and $k > K$,

$$|f_n(y_k) - f(y_k)| \leqslant |f_N(y_k) - f(y_k)| < \epsilon.$$

Since $n_k \to \infty$ when $k \to \infty$, when k is sufficiently large we have $k > K,\ n_k > N$. Thus

$$|f_{n_k}(x_k) - f(x_k)| < \epsilon,$$

which makes a contradiction with the assumption. Thus the convergence is uniform.

3. (Problem 5, Page 364) Show that an extended real-valued function f on X is measurable if and only if for each rational number c, $\{x \in X \mid f(x) < c\}$ is a measurable set.

Collaborators: None

Solution: Notice that for each $r \in \mathbb{R} \supset \mathbb{Q}$, we have

$$\{x \in X \mid f(x) < r\} = \bigcup_{\substack{c<r \\ c\in\mathbb{Q}}} \{x \in X \mid f(x) < c\}.$$

Then since Lebesgue measurable sets make a $\sigma-$Algebra, we know $\{x \in X \mid f(x) < r\}$ is measurable. On the other hand, since each rational number is a real number, the other direction stands.

4. (Problem 13, Page 365) Let $\{f_n\}$ be a sequence of real-valued functions on X such that for each natural number n, $\mu\{x \in X \mid |f_n(x) - f_{n+1}(x)| > 1/2^n\} < 1/2^n$. Show that $\{f_n\}$ is pointwise convergent a.e. on X.

Collaborators: None

Solution: Denote

$$E_n = \{x \in X \mid |f_n(x) - f_{n+1}(x)| > \frac{1}{2^n}\},$$

Then

$$\bigcup_{n=1}^{\infty} \mu(E_n) < \sum_{n=1}^{\infty} \mu(E_n) < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

According to Borel-Cantelli Lemma, $\exists E_0,\ \mu(E_0) = 0$, s.t. $\forall x \in X \setminus E_0$, x belongs to at most finite number of E_n. That means, $\forall x \in X \setminus E_0$, $\exists N > 0$, $\forall n > N$, $x \in E_n$. So

$$|f_n(x) - f_{n+1}(x)| < \frac{1}{2^n}.$$

Thus

$$\forall m > n, |f_m(x) - f_n(x)| < \sum_{i=n}^{m-1} |f_{i+1}(x) - f_i(x)| < \frac{1}{2^{n-1}}.$$

This means that $f_n(x)$ is a Cauchy sequence, then it converges. Thus $\{f_n\}$ is pointwise convergent on $X \setminus E_0$, and thus is pointwise convergent a.e. on X.

5. (Problem 15, Page 365) A sequence $\{f_n\}$ of measurable real-valued functions on X is said to *converge in measure* to a measurable function f provided that for each $\eta > 0$,

$$\lim_{n\to\infty} \mu\{x \in X \mid |f_n(x) - f(x)| > \eta\} = 0.$$

A sequence $\{f_n\}$ of measurable functions is said to be *Cauchy in measure* provided that for each $\epsilon > 0$ and $\eta > 0$, there is an index N such that for each $m, n \geq N$,

$$\mu\{x \in X \mid |f_n(x) - f_m(x)| > \eta\} < \epsilon.$$

(a) Show that if $\mu(X) < \infty$ and if $\{f_n\}$ converges pointwise a.e. on X to a measurable function f, then $\{f_n\}$ converges to f in measure.

(b) Show that if $\{f_n\}$ converges to f in measure, then there is a subsequence of $\{f_n\}$ that converges pointwise a.e. to f.

(c) Show that if $\{f_n\}$ is Cauchy in measure, then there is a measurable function f to which $\{f_n\}$ converges in measure.

Collaborators: None, but (b) and (c) checked a proof from my Real Analysis textbook in college (Written by Prof. Zhaobo Huang, Fudan University).

Solution: (a) Since $\{f_n\}$ converges pointwise a.e. on X, there $\exists E_0,\ \mu(E_0) = 0$, s.t. $\lim\limits_{n\to\infty} f_n = f$ on $E_1 = X \setminus E_0$. Then $\forall \epsilon > 0$,

$$E_1 = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{x \in E_1 \mid |f_n(x) - f(x)| \leq \epsilon\} = \varliminf_{n\to\infty} \{x \in E_1 \mid |f_n(x) - f(x)| \leq \epsilon\}.$$

Thus, according to the continuity of measure,

$$\mu(E_1) \leq \varliminf_{n\to\infty} \mu(\{x \in E_1 \mid |f_n(x) - f(x)| \leq \epsilon\}).$$

Since $\mu(X) < \infty$, we have

$$\begin{aligned}\lim_{n\to\infty} \mu(\{x \in E_1 \mid |f_n(x) - f(x)| > \epsilon\}) &\leq \varlimsup_{n\to\infty} \mu(\{x \in E_1 \mid |f_n(x) - f(x)| > \epsilon\}) \\ &= \mu(X) - \varliminf_{n\to\infty} \mu(\{x \in E_1 \mid |f_n(x) - f(x)| \leq \epsilon\}) \leq 0.\end{aligned}$$

Thus $\{f_n\}$ converges to f in measure.

(b) Since $\{f_n\}$ converges to f in measure, $\forall k \in \mathbb{N}$, $\exists n_k > n_{k-1} \in \mathbb{N}$, s.t. when $n \geq n_k$,

$$\mu(\{x \in X \mid |f_n(x) - f(x)| \geq \frac{1}{2^k}\}) < \frac{1}{2^k}.$$

Denote $E_k = \{x \in X \mid |f_{n_k}(x) - f(x)| \geq \frac{1}{2^k}\}$, then $\mu(E_k) < \frac{1}{2^k}$. Let

$$F_k = \bigcap_{i=k}^{\infty} (X \setminus E_i) = \{x \in X \mid |f_{n_i}(x) - f(x)| \leq \frac{1}{2^i}, i = k, k+1, \cdots\}$$

then $\{f_{n_k}\}$ converges uniformly to f on F_k, thus is convergent on $F = \bigcup\limits_{k=1}^{\infty} F_k$. On the other hand,

$$X \setminus F = \bigcap_{k=1}^{\infty} (X \setminus F_k) = \bigcap_{k=1}^{\infty} \bigcup_{k=1}^{\infty} E_i = \varlimsup_{i\to\infty} E_i,$$

and

$$\sum_{i=1}^{\infty} \mu(E_i) \leq \sum_{i=1}^{\infty} \frac{1}{2^i} = 1,$$

thus $\mu(X \setminus F) = 0$, $\{f_{n_k}\}$ converges pointwise a.e. on X.

(c) If $\{f_n\}$ is Cauchy in measure, then for each $n_i \in \mathbb{N}$, there $\exists n_{i+1} > n_i$, s.t.

$$\mu(\{x \in X \mid |f_{n_i}(x) - f_{n_{i+1}}(x)| \geq \frac{1}{2^i}\}) \leq \frac{1}{2^i}.$$

We can denote the subsequence $\{f_{n_i}\}$ as $\{g_i\}$, and $E_i = \{x \in X \mid |f_{n_i}(x) - f_{n_{i+1}}(x)| \geq \frac{1}{2^i}\}$. Let $F_k = \bigcup\limits_{i=k}^{\infty} E_i$, then $\mu(F_k) \leq \sum\limits_{i=k}^{\infty} \frac{1}{2^i} \leq \frac{1}{2^{k-1}}$. Denote $F = \bigcap\limits_{i=1}^{\infty} F_i$, then $\mu(F) = 0$.

If we choose $x \notin F_k$, and $i \geq j \geq k$, then

$$|g_i(x) - g_j(x)| \leq \sum_{l=j}^{i-1} |g_{l+1}(x) - g_l(x)| \leq \frac{1}{2^j},$$

which means $g_i(x)$ is Cauchy respective to i on $X \setminus F$, thus converges on $X \setminus F$. Thus we can construct a function f like this:

$$f(x) = \begin{cases} 0, \ x \in F \\ \lim\limits_{i \to \infty} f_i(x), \ x \in X \setminus F. \end{cases}$$

Then $\{g_i(x)\}$ converges to f on $X \setminus F$, which means $\{g_i\}$ converges pointwise a.e. on X. Now we show $\{f_n\}$ converges to f in measure. Notice

$$\mu(\{x \in X \mid |f_n - f| \geq \xi\}) \leq \mu(\{x \mid |f_n - g_m| \geq \frac{\xi}{2}\}) + \mu(\{x \mid |g_m - f| \geq \frac{\xi}{2}\}), \ m > n$$

and when $n \to \infty$, the two terms in the right can be arbitrarily small, so

$$\lim_{n \to \infty} \mu(\{x \in X \mid |f_n - f| \geq \xi\}) = 0, \ \forall \xi > 0.$$

The $\{f_n\}$ converges in measure.

6. (Problem 16, Page 365) Assume $\mu(X) < \infty$. Show that $\{f_n\}$ converges to f in measure if and only if each subsequence of $\{f_n\}$ has a further subsequence that converges pointwise a.e. on X to f. Use this to show that for two sequences that converge in measure, the product sequence also converges in measure to the product of the limits.

Collaborators: None

Solution: **(1)** Necessity: From $\{f_n\}$ converges to f in measure and the property of convergence, we know that each subsequence of $\{f_n\}$ converges to f in measure. From Problem 5(b) we know $\{f_{n_k}\}$ has a subsequence $\{f_{n_{k_j}}\}$ converges pointwise a.e. on X to f.

Sufficiency: If each subsequence $\{f_{n_k}\}$ has a further subsequence $\{f_{n_{k_j}}\}$ converges pointwise a.e. on X to f, then since $\mu(X) < \infty$, from Problem 5(a) we know $\{f_{n_k}\}$ converges in measure to f. If $\{f_n\}$ does not converge to f in measure, then there $\exists \epsilon > 0$,

$$\begin{aligned}
&\text{for } n_1 = 1, \ \exists n_2 > n_1, \ \mu(\{x \mid |f_{n_2} - f| \geq \epsilon\}) \geq \epsilon, \\
&\text{for } n_2, \ \exists n_3 > n_2, \ \mu(\{x \mid |f_{n_3} - f| \geq \epsilon\}) \geq \epsilon, \\
&\cdots, \\
&\text{for } n_k, \ \exists n_{k+1} > n_k, \ \mu(\{x \mid |f_{n_{k+1}} - f| \geq \epsilon\}) \geq \epsilon, \\
&\cdots.
\end{aligned}$$

Then we get a subsequence $\{f_{n_k}\}$ which does not converge to f in measure. It makes a contratiction.

(2) Suppose $\{f_n\}$, $\{g_n\}$ are two sequences that converge in measure to respectively f and g. For each subsequence $\{f_{n_k} g_{n_k}\}$ of $\{f_n g_n\}$, from (1) we know $\{f_{n_k}\}$ has a subsequence $\{f_{n_{k_l}}\}$ converges pointwise a.e. to f, and $\{g_{n_{k_l}}\}$ has a subsequence $\{g_{n_{k_{l_j}}}\}$ converges pointwise a.e. to g. Since the union of two measure-zero sets is a measure-zero set, we know $\{f_{n_{k_{l_j}}} g_{n_{k_{l_j}}}\}$ converges pointwise a.e. to fg according to the linear property of limits. Thus according to (1) we know $\{f_n g_n\}$ converges in measure to the product of limits.

7. Show that if f is an lower semicontinuous (resp. upper semicontinuous) function on an interval $[a, b]$, then there is a family $\{f_\alpha\}$ of continuous functions on the interval $[a, b]$ such that $f(x) = \sup\{f_\alpha(x) \mid \alpha \in A\}$ (resp. $f(x) = \inf\{f_\alpha(x) \mid \alpha \in A\}$) for all $x \in [a, b]$.

Collaborators: Checked https://people.math.gatech.edu/˜loss/16FALLTEA/NOTES/semicontinuity.pdf for help.

Solution: We first show a lemma.

Lemma 1 *If $\{f_\alpha(x)\}$ is a family of upper semi-continuous functions, then $\inf\limits_{\alpha\in I} f_\alpha(x)$ is also upper semi-continuous. Conversely, lower semi-continuous is preserved under monotone increasing limits.*

Proof. In fact, suppose $\{f_\alpha(x)\}_{\alpha\in I}$ is a family of upper semi-continuous functions, then

$$\{x \in [a, b] \mid f_\alpha(x) \geq t\} = \bigcap_{\alpha\in I} \{x \in [a, b] \mid f_\alpha(x) \geq t\}.$$

With the continuity of $f_\alpha(x)$ and the compactness of $[a, b]$, we know $\{x \in [a, b] \mid f_\alpha(x) \geq t\}$ is closed, so $\{x \in [a, b] \mid f_\alpha(x) \geq t\}$ is also closed. Thus $\inf\limits_{\alpha\in I} f_\alpha(x)$ is upper semi-continuous. ■

We just show the case when f is a lower semicontinuous function. Let

$$f_\epsilon(x) = \inf_{y\in[a,b]} \left(f(y) + \frac{|x - y|}{\epsilon} \right),$$

and we show that $f_\epsilon(x)$ is continuous and converges pointwise to f from below when $\epsilon \to 0^+$.

First, we have $f_\epsilon(x) \leq f(x)$ by setting $y = x$ instead of taking the infimum. For each fixed $y \in [a, b]$, the map

$$x \mapsto f(y) + \frac{|x - y|}{\epsilon}$$

is continuous, hence according to the lemma, $f_\epsilon(x)$ is an upper semi-continuous function.

Since $[a, b]$ is conpact, f is lower semicontinuous, and the map $y \mapsto |x - y|$ is continuous, the infimum is attained, and there exists $y(\epsilon, x) \in [a, b]$, s.t.

$$f_\epsilon(x) = f(y(\epsilon, x)) + \frac{d(x, y(\epsilon, x))}{\epsilon}.$$

Pick $\{x_n\} \to x$, then since $[a, b]$ is compact we may assume $y(\epsilon, x) \to y_0$. Then since $f_\epsilon(x)$ is upper semicontinuous,

$$f_\epsilon(x) \geq \overline{\lim_{n\to\infty}} f_\epsilon(x_n) = \overline{\lim_{n\to\infty}} f(y(\epsilon, x_n)) + \frac{|x - y_0|}{\epsilon} \geq f(y_0) + \frac{|x - y_0|}{\epsilon} \geq f_\epsilon(x)$$

Thus $f_\epsilon(x)$ is continuous. When $\epsilon \to 0$, we know $y(\epsilon, x) \to x$, otherwise $\frac{d(x,y(\epsilon,x))}{\epsilon} \to \infty$. And we have

$$f(x) \geq \overline{\lim_{\epsilon\to 0}} f_\epsilon(x) = \overline{\lim_{\epsilon\to 0}} \left(f(y(\epsilon, x)) + \frac{d(x, y(\epsilon, x))}{\epsilon} \right) \geq f(x) + \overline{\lim_{\epsilon\to 0}} \frac{d(x, y(\epsilon, x))}{\epsilon}.$$

Which means

$$\frac{d(x, y(\epsilon, x))}{\epsilon} \to 0,$$

and $f_\epsilon(x) \to f(x)$.