Assignment 8 **17.5.6**

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Problem 1. Explain that when $c(x) < 0$, the extremum principle may not stand.

Proof. Consider the 1-dimension Helmholtz equation:

$$\frac{d^2u}{dx^2} + k^2u = 1, \quad u(0) = u(1) = 0.$$

We can find the solution of this equation has a form like

$$u(x) = \frac{\cos k + 1}{k^2 \sin k} \cos kx - \frac{1}{k^2} \sin kx + \frac{1}{k^2}.$$

If we let $k = \frac{\pi}{2}$, then $u'(\frac{1}{2}) = 0$, but $u(\frac{1}{2}) = \frac{4(1-\sqrt{2})}{\pi^2} < 0$. So this time the extremum principle does not stand.

Problem 2. If $u(0) = u(1) = 0$, and $\frac{d^2u}{dx^2} = f(x)$. Prove

$$u(x) = \int_0^1 G(x; x_0) f(x_0) dx_0$$

with Green function.

Proof. In fact, we just check if $u(x)$ satisfies the conditions in the problem.
First,

$$\begin{aligned} u(0) &= \int_0^1 x_0 \times 0 \times f(x_0) dx_0 = 0, \\ u(1) &= \int_0^1 x_0 \times 0 \times f(x_0) dx_0 = 0. \end{aligned} \tag{1}$$

Second,

$$\begin{aligned} u'(x) &= \int_0^x (1 - x_0) f(x_0) dx_0 + (1 - x) x f(x) + \int_x^1 -x_0 f(x_0) dx_0 - x(1 - x) f(x) \\ &= \int_0^x f(x_0) dx_0 - \int_0^1 x_0 f(x_0) dx_0, \end{aligned} \tag{2}$$

So $u''(x) = f(x)$, with uniqueness of Cauchy problem, we can prove the $u(x)$ is the function we need.

Problem 3. Consider the equation

$$-a\frac{d^2u}{dx^2}+b\frac{du}{dx}+cu=1,$$

with boundary conditions $u(0)=0$, $\frac{du}{dx}(1)+u(1)=0$, and parameters $a>0, c\geqslant 0$ in $(0,1)$. Solve the equation in accurate form and by difference scheme.

Accurate. The characteristic equation of this differential equation is

$$-a\lambda^2+b\lambda+c=0.$$

Case 1: $b=c=0$. In this case, $u(x)=\frac{-x^2+2}{2a}$.
Case 2: $c=0, b\neq 0$. The roots of the characteristic equation are $x_1=0, x_2=-\frac{a}{b}$. So the general solution is $u(x)=k_1e^{bx/a}+k_2$ for the homogeneous problem. And a special solution of the nonhomogeneous problem is $u=\frac{x}{b}$, and the accurate solution is

$$u(x)=-\frac{a}{b^2}e^{bx/a}-\frac{1}{b}e^{b/a}+\frac{x}{b}-\frac{a}{b^2}.$$

considering the boundary conditions.
Case 3: $b,c\neq 0$. Using the same techniques, the accurate solution is

$$u(x)=\frac{e^{\lambda_2}-(1+\lambda_1)}{c((1+\lambda_2)e^{\lambda_1}-(1+\lambda_1)e^{\lambda_2})}e^{\lambda_1 x}+\frac{e^{\lambda_1}-(1+\lambda_2)}{c((1+\lambda_1)e^{\lambda_2}-(1+\lambda_2)e^{\lambda_1})}e^{\lambda_2 x}+\frac{1}{c},$$

in which λ_1,λ_2 are roots of $-a\lambda^2+b\lambda+c=0$.

Numerical. The computing scheme of this problem is

$$-a\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}+b\frac{u_{i+1}-u_{i-1}}{2h}+cu_i=f_i,\quad i=1,2,\cdots,N-1. \tag{3}$$

Then the linear equation can be written as

$$\mathbf{Au}+\mathbf{Bu}+\mathbf{Cu}=\mathbf{f},$$

with

$$\mathbf{A}=\text{tridiag}(1,-2,1),$$
$$\mathbf{B}=\text{tridiag}(-1,0,1),$$
$$\mathbf{C}=\text{diag}(1),$$
$$\mathbf{f}=\begin{pmatrix}1\\1\\\cdots\\\cdots\\1\\1\end{pmatrix}+\begin{pmatrix}\frac{a}{h^2}u_0-\frac{b}{2h}u_0\\0\\\cdots\\\cdots\\0\\\frac{a}{h^2}u_N-\frac{b}{2h}u_N\end{pmatrix}.$$

We can simply give the solution by

$$\mathbf{u}=(\mathbf{A}+\mathbf{B}+\mathbf{C})^{-1}\mathbf{f}.$$

The numerical simulation is as follows, in which we choose $a=1, b=10, c=10$. Relative error is as follows:

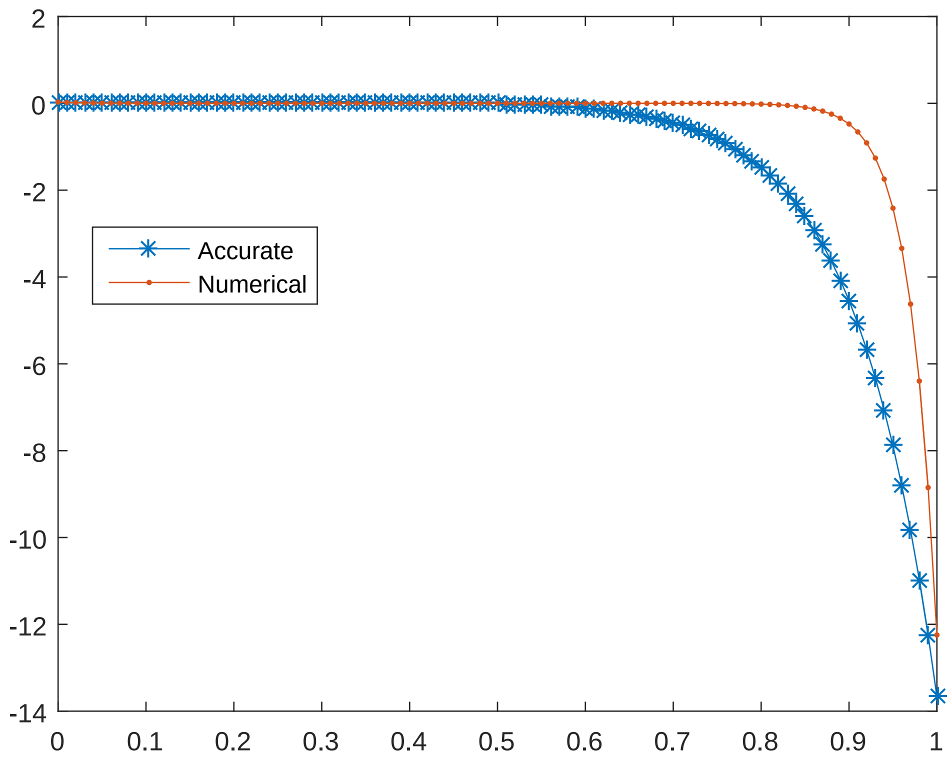

Figure 1: Simulation and Accurate results

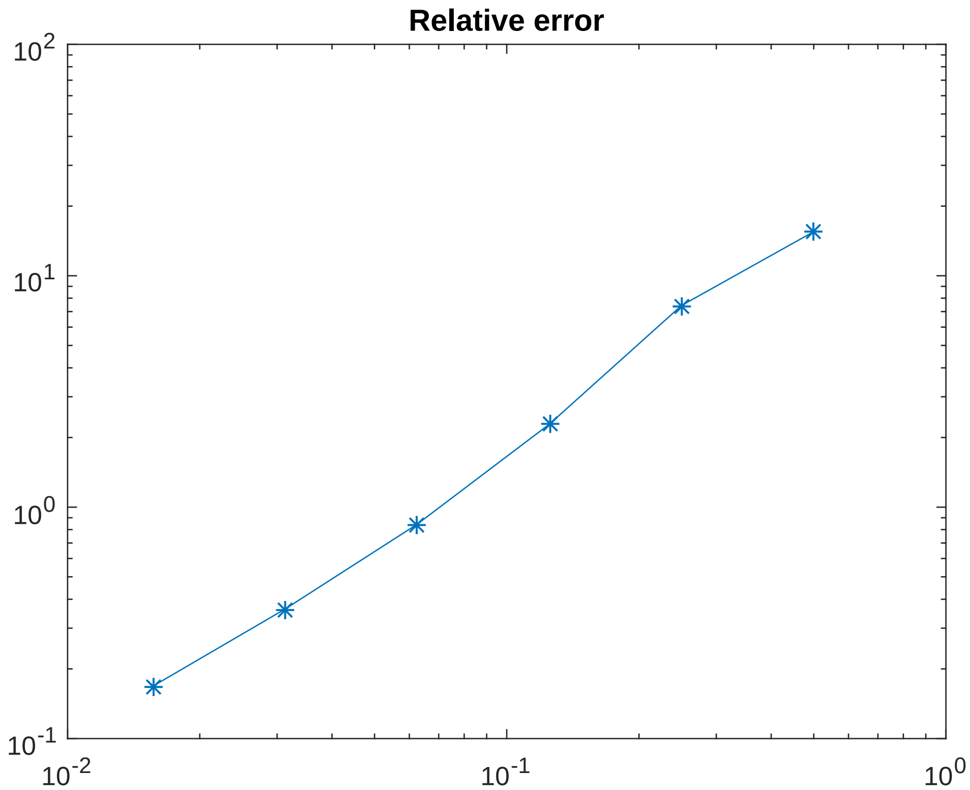

Figure 2: Relative error