Introduction to Analysis I
Homework 7
Wednesday, November 8, 2017

Instructions: This and all subsequent homeworks must be submitted written in LaTeX.
Since by now, you should all be sufficiently familiar with LaTeXto compose on your own, I will forgo the formalities of the previous assignments and simply list the problems. I will expect, however, that you will cite relevant material in Royden or other sources properly and that you will acknowledge your collaborators.

Homework 7 is due by midnight, Saturday, November 18.

1. Problem 9, Chapter 6

Solution: Sufficiency. Suppose $m^*(E) = m > 0$. Since each point in E belongs to infinitely many of the I_k, then there exists $N > 0$, s.t.

$$\bigcup_{k=1}^{N} I_k \supset E,$$

otherwise there exists $x \in E$, for each N, $x \notin \bigcup\limits_{k=1}^{N} I_k$, and that means $x \notin \bigcup\limits_{k=1}^{\infty} I_k$, which leads to a contradiction. Suppose the smallest N such that $E \subset \bigcup\limits_{k=1}^{N} I_k$ to be N_1, then using the same method, there exists $N_2 > N_1$, s.t. $E \subset \bigcup\limits_{k=N_1+1}^{N_2} I_k$. By inductively constructing such N_j, we get a sequence $\{N_j\}$, s.t. $E \subset \bigcup\limits_{k=N_j+1}^{N_{j+1}} I_k$. Then by the definition of outer measure and finite addivity for open sets,

$$m = m^*(E) \leq m^*(\bigcup_{k=N_j+1}^{N_{j+1}} I_k) = \sum_{k=N_j+1}^{N_{j+1}} m(I_k).$$

Then for each $M > 0$,

$$\sum_{k=1}^{\infty} m(I_k) > Mm,$$

which makes a contradiction with $\sum_{k=1}^{\infty} m(I_k) < \infty$. Then $m^*(E) = 0$.
Necessity. With the definition of outer measure, for each $\epsilon_n = \epsilon 12^n$, there exist an open set $O_n = \bigcup\limits_{j=1}^{\infty} I_{n_j}$ being the construction of open intervals, s.t. $E \subset O_n$, and $m(O_n) = \sum\limits_{j=1}^{\infty} m(I_{n_j}) < \epsilon_n$. Let $\{I_n\} = \{I_{n_j}\}_{n,j}$, then for each $x \in E$, x belongs to infinitely many of these open intervals, and $\sum\limits_{n=1}^{\infty} m(I_n) = \epsilon < \infty$.

2. Problem 10, Chapter 6

Solution: For $x_2 > x_1$,

$$\begin{aligned} f(x_2) &= \sum_{k=1}^{\infty} l((c_k, d_k) \cap (-\infty, x_2)) = \sum_{k=1}^{\infty} l((c_k, d_k) \cap (-\infty, x_1)) + l((c_k, d_k) \cap (x_1, x_2)) \\ &= f(x_1) + \sum_{k=1}^{\infty} l((c_k, d_k) \cap (x_1, x_2)). \end{aligned}$$

Since (c_k, d_k) is an open set, and $\exists k$, s.t. $x_1 \in (c_k, d_k)$, thus $l((x_1, x_2) \cup (c_k, d_k)) > 0$. So $f(x_2) > f(x_1)$, which means f is increasing.

$\forall x \in E$, let $\{k_i\}$ be the infininte set where $x \in (c_{k_i}, d_{k_i})$. Let $f_k(x) = l((c_k, d_k) \cap (-\infty, x))$, then $f = \sum f_k$. Fix an $N > 0$, then since $x \in \bigcap_{i=1}^{N} (c_{k_i}, d_{k_i})$, and the intersection of finite open sets is also an open set, then there exists $t_N > 0$, s.t. $x + t_N \in \bigcap_{i=1}^{N} (c_{k_i}, d_{k_i})$. Thus

$$f_k(x + t_N) - f_k(x) = l(x, x + t_N) = t_N,$$

for each $k = k_i, 1 \le i \le N$. Thus since each f_k is nondecreasing, $f(x + t_N) - f(x) \ge N t_N$, we have $\overline{D} f(x) \ge N$. With the arbitrariness of $N > 0$, we know $\overline{D} f(x) = \infty$, which means f is not differentiable at x.

3. Problem 13, Chapter 6

Solution: First, we show that under the conditions of the Vitali Covering Lemma, $\mathcal{F} \setminus \bigcup_{k=1}^{n} I_k$ is a Vitali covering of $E \setminus \bigcup_{k=1}^{n} I_k$. In fact, if it is not true, then there $\exists x \in E \setminus \bigcup_{k=1}^{n} I_k$ and $\epsilon > 0$, s.t. there does not exist an interval $I \in \mathcal{F} \setminus \bigcup_{k=1}^{n} I_k$, s.t. $x \in I$, and $m(I) < \epsilon$. However, since $\mathcal{F}$ is a Vitali covering of E, then for $\epsilon_i = \frac{\epsilon}{2^i}$, $i = 1, 2, \cdots$, there exists $I'_i \in \mathcal{F}$, s.t. $x \in I'_i$ and $m(I'_i) < \epsilon_i$. If there exists i_0, s.t. $I_{i_0} \notin \{I_k\}_{k=1}^{n}$, then $x \in I_{i_0}$ and $m(I_{i_0}) < \epsilon$, which makes a contradiction. Then $\{I'_i\} \in \{I_k\}_{k=1}^{n}$, but it contradicts with $n < \infty$. Hence, our claim is proved.

Then we show that if $\{I_k\}_{k=1}^{n}$ is a finite sequence of closed intervals, then we can find a pairwise disjoint subsequence $\{I_{k_j}\}$ s.t.

$$m(\bigcup_{j=1}^{m} I_{k_j}) \ge m(\bigcup_{k=1}^{n} I_k).$$

The proof of this proposition can be seen at http://www.personal.psu.edu/t20/papers/vitali-l2h/node5.html

Now we use these claims to construct a collection. Using Vitali Covering Lemma, for $\epsilon = \frac{3}{4} m(E)$, there exists a finite collection of disjoint intervals $\{I_k\}$, s.t. $m(E \setminus \bigcup I_k) < \epsilon$. Denote $A = \bigcup I_k$.

Since $\mathcal{F} \setminus A$ is a Vitali covering of $E \setminus I_k$, there is a finite set of intervals J_k, s.t. $J_k \subset E \setminus A$ (Otherwise just follow the proof of Vitali Covering Lemma on the textbook), and

$$m(E \setminus (A \cup \bigcup_{k=1}^{m} J_k)) < \frac{1}{12} m(E \setminus A).$$

Then using claim 2, there is a pairwise disjoint subset $\{J_{k_i}\}$ s.t.

$$m(\bigcup J_{k_i}) \ge \frac{1}{3} m(\bigcup J_k).$$

Then denote $B = \bigcup J_{k_i}$, and

$$m(E \setminus (A \cup B)) < \frac{2}{3} m(\bigcup J_k) + \frac{1}{12} m(E \setminus A) \le (\frac{2}{3} + \frac{1}{12}) m(E \setminus A) = \frac{3}{4} m(E \setminus A).$$

By constructing like above recursively, we can get a sequence of subsets $\{A_i\}$, s.t.

$$m(E \setminus \bigcup_{i=1}^{n} A_i) \le (\frac{3}{4})^n m(E).$$

By countable addivity,

$$m(E \setminus \bigcup_{i=1}^{\infty} A_i) = 0.$$

4. Problem 16, Chapter 6

Solution: First, we show that if $f_1,\ f_2$ are differentiable functions, then $f = f_1 + f_2$ is differentiable. In fact,

$$\begin{aligned}\overline{D}f(x) &= \lim_{h\to 0}\sup_{0<|t|<h}\frac{f(x+t)-f(x)}{t} = \lim_{h\to 0}\sup_{0<|t|<h}\left(\frac{f_1(x+t)-f_1(x)}{t}+\frac{f_2(x+t)-f_2(x)}{t}\right)\\ &\le \lim_{h\to 0}\sup_{0<|t|<h}\frac{f_1(x+t)-f_1(x)}{t}+\lim_{h\to 0}\sup_{0<|t|<h}\frac{f_2(x+t)-f_2(x)}{t}\end{aligned}$$

$$\begin{aligned}\underline{D}f(x) &= \lim_{h\to 0}\inf_{0<|t|<h}\frac{f(x+t)-f(x)}{t} = \lim_{h\to 0}\inf_{0<|t|<h}\left(\frac{f_1(x+t)-f_1(x)}{t}+\frac{f_2(x+t)-f_2(x)}{t}\right)\\ &\ge \lim_{h\to 0}\inf_{0<|t|<h}\frac{f_1(x+t)-f_1(x)}{t}+\lim_{h\to 0}\inf_{0<|t|<h}\frac{f_2(x+t)-f_2(x)}{t}\end{aligned}$$

But since f_1 and f_2 are both differentiable,

$$\lim_{h\to 0}\inf_{0<|t|<h}\frac{f_i(x+t)-f_i(x)}{t} = \lim_{h\to 0}\sup_{0<|t|<h}\frac{f_i(x+t)-f_i(x)}{t}$$

holds for $i = 1, 2$. Thus $\overline{D}f(x) = \underline{D}f(x)$, which means f is differentiable.

Let

$$g^+ = \max\{g(x),\ 0\},\ g^- = \min\{g(x),\ 0\},$$

Then $g^+ \ge 0,\ g^- \le 0$, and $g(x) = g^+(x) + g^-(x)$. Define

$$f_1(x) = \int_a^x g^+,\ f_2(x) = \int_a^x g^-,$$

then $f_1,\ f_2$ are both monotone functions, using Lebesgue's Theorem, $f_1,\ f_2$ are both differentiable a.e. on (a, b), respectively differentiable on $(a, b)\setminus E_1$ and $(a, b)\setminus E_2$ where $m(E_1) = m(E_2) = 0$. Then using the conclusion above, $f = f_1 + f_2$ is differentiable on $(a, b)\setminus(E_1 \cup E_2)$, and hence differentiable a.e. on (a, b).

5. Problem 26, Chapter 6

Solution: No. In fact, let $\{x_i\}$ be the sequence of rational numbers in $[0, 1]$, then for each i, there is an irrational number y_i, s.t. $x_i < y_i < y_{i+1}$. Then

$$TV(f) \ge \sum_{i=1}^{\infty}(|f(x_{i+1}) - f(y_i)| + |f(y_i) - f(x_i)|) = \sum_{i=1}^{\infty} 2 = \infty.$$

Thus f is not of bounded variation.

6. Problem 29, Chapter 6

Solution:

(a). No. In fact, consider the subinterval $[0, 1]$, let $x_i = \frac{1}{\sqrt{n\pi}}$, and $P_n = \{0, x_{2n+1}, x_{2n}, \cdots, x_1\}$, then

$$V(f, P_n) = \frac{1}{\pi}(1 + \frac{2}{2n+1} + \frac{2}{2n} + \cdots + \frac{2}{2} + \frac{2}{1} - \cos(1)).$$

Since the sum is a harmonic series, which diverges, we know f is not of bounded variation.

(b). Yes. Consider only the interval $[0, 1]$ since g is an even function. Since $g \in \mathcal{C}^1$ and is bounded on (0, 1), then

$$g'(x) = 2x\cos\frac{1}{x} + \sin\frac{1}{x},$$

and $|g'(x)| \le 3$, thus by the definition of Riemann integral, for any partition P,

$$V(g, P) \le \int_0^1 |g'| dx.$$

Thus

$$TV(g) \le \int_0^1 |g'| dx \le 3.$$

7. Problem 35, Chapter 6

Solution: First, we have

$$f'(x) = \alpha x^{\alpha-1} \sin(\frac{1}{x^\beta}) - \beta x^{\alpha-\beta-1} \cos(\frac{1}{x^\beta}).$$

Since we know $f \in \mathcal{C}^1$ and $|f| \le 1$ on $(0, 1)$, then according to the definition of Riemann integrals, for any partition P of $[0, 1]$,

$$V(f, P) \le \int_0^1 |f'| dx \le \int_0^1 \alpha x^{\alpha-1} + \beta x^{\alpha-\beta-1} dx.$$

Since $\alpha > \beta > 0$, we have $\alpha - 1 > -1$, $\alpha - \beta - 1 > -1$, thus

$$TV(f) \le \int_0^1 \alpha x^{\alpha-1} + \beta x^{\alpha-\beta-1} dx < \infty.$$

When $\alpha \le \beta$, let $P_n = \{0, (\frac{n\pi}{2})^{-\beta}, \cdots, (\frac{\pi}{2})^{-\beta}\}$ be a partition of $[0, 1]$. Then

$$V(f, P_n) = \sum_{i=1}^{n} (\frac{i\pi}{2})^{-\frac{\alpha}{\beta}} \ge \sum_{i=1}^{n} \frac{2}{i}\pi.$$

Since this series is a harmonic series, thus it diverges, which means

$$TV(f) = \infty.$$