Introduction to Analysis I
Homework 5
Sunday, October 8, 2017

Instructions: This and all subsequent homeworks must be submitted written in LaTeX.
If you use results from books, Royden or others, please be explicit about what results you are using.

Homework 5 is due by midnight, Saturday, October 21.

1. (Problem 24, Page 64) Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be increasing. Show that f is measurable by first showing that, for each natural number n, the strictly increasing function $x \to f(x) + x/n$ is measurable, and then taking pointwise limits.

   Collaborators: None

   Solution: Denote $f_n(x) = f(x) + \frac{x}{n}$. Then since $f : I \to \mathbb{R}$ is increasing, $f_n(x)$ is strictly increasing on $I \in \{[a,b], [a,b), (a,b], (a,b)\}$.

   For each fixed number c, if $\exists x_0 \in I$, s.t. $f(x_0) = c$, then the set

   $$\{x \mid f(x) < c\} = (a, x_0),$$

   and the left side is the same as the set I, and it is an interval in $\mathbb{R}$, hence is measurable. Thus f_n is measurable for every n.

   For each $x \in I$,

   $$\lim_{n\to\infty} f_n(x) = f(x) + \lim_{n\to\infty} \frac{x}{n} = f(x),$$

   hence f_n converges to f pointwise. Using Proposition 9, we know f is measurable.

2. (Problem 8, Page 343) Let $(X, \mathcal{M}, \mu)$ be a measure space. The measure μ is said to be **semifinite** provided each measurable set of infinite measure contains measurable sets of arbitrarily large finite measure.

   (a) Show that each σ-finite measure is semifinite.

   (b) For $E \in \mathcal{M}$, define $\mu_1(E) = \mu(E)$, if $\mu(E) < \infty$, and if $\mu(E) = \infty$, define $\mu_1(E) = \infty$ if E contains measurable sets of arbitrarily large finite measure and $\mu_1(E) = 0$ otherwise. Show μ_1 is a semifinite measure: it is called the semifinite part of μ.

   (c) Find a measure μ_2 on $\mathcal{M}$ that only takes the valures 0 and ∞ and $\mu = \mu_1 + \mu_2$.

   Collaborators: None

   Solution: (a) If μ is a $\sigma-$finite measure, then $X = \bigcup\limits_{n=1}^{\infty} X_i$, where X_i has finite measure. Assume E is a measurable set with infinite measure, then since $E \subset X$, $E = E \cap X = \bigcup\limits_{n=1}^{\infty} E \cap X_n$. For $\forall M > 0$, there must exist $N > 0$, s.t. $\mu\left(\bigcup\limits_{n=1}^{N} E \cap X_i\right) > M$, otherwise $\mu(E) \leq \sum\limits_{n=1}^{\infty} \mu(E \cap X_n) < M_0$ for some $M_0 > 0$, which makes a contradiction with $\mu(E) = \infty$. Besides, we know from $\mu(X_i) < \infty$, that $\mu\left(\sum\limits_{n=1}^{N} (E \cap X_n)\right) < \infty$. Thus μ is semifinite.

   (b) Suppose E is a measurable set of infinite measure in the measure space $(X, \mathcal{M}, \mu_1)$. Then according to the definite of μ_1, E contains measurable sets of arbitrarily large finite measure. Thus μ_1 is semifinite.

   (c) Define μ_2 like this: If E can be represented as countable union of measurable sets with finite measure, then $\mu_2(E) = 0$. Otherwise $\mu_2(E) = \infty$.

First, we show that $\mu = \mu_1 + \mu_2$. If $\mu(E) < \infty$, then $E = \bigcup\limits_{n=1}^{\infty} E_n$, in which $E_n = \varnothing$ for $n \geq 2$. Thus $\mu_2(E) = 0 = \mu(E) - \mu_1(E)$. Now we assume $\mu(E) = \infty$. If E can be represented as countable union of measurable sets with finite measure, that is, $E = \bigcup\limits_{n=1}^{\infty} E_n$ with $\mu(E_n) < \infty$, then with the continuity of measure, $\forall M > 0$, $\exists N > 0$, s.t. $\mu \left(\bigcup\limits_{n=1}^{N} E_n \right) > M$. Since $\mathcal{M}$ is a $\sigma-$algebra, it shows that E contains measurable sets of arbitrarily large finite measure. Thus $\mu(E) = \infty = \mu_1(E) + 0 = \mu_1(E) + \mu_2(E)$. If E can't be represented as countable union of measurable sets with finite measure, then $\mu(E) = \infty = \mu_1(E) + \mu_2(E)$.

Now we show that μ_2 is a measure. In fact, we only need to show that $\mu_2(\varnothing) = 0$, and μ_2 is countably additive. First, from the definition we know $\mu(\varnothing) = 0 < \infty$, so $\mu_2(\varnothing) = 0$. Suppose $\{E_n\}_{n=1}^{\infty}$ is a sequence of disjoint measurable sets. If $\exists i$, s.t. $\mu_2(E_i) = \infty$. Then E_i can't be represented as countable union of measurable sets with finite measure, thus $\bigcup\limits_{n=1}^{\infty} E_n$ can't be represented as countable union of measurable sets with finite measure. Then

$$\mu_2 \left(\bigcup_{n=1}^{\infty} E_n \right) = \infty = \sum_{n=1}^{\infty} \mu_2(E_n).$$

Now we suppose all $\mu_2(E_n) < \infty$. In this case, since μ_2 takes value only in $\{0, \infty\}$, we have $\mu_2(E_n) = 0$. In this case, all E_n is a countable union of measurable sets with finite measure, then $\bigcup\limits_{n=1}^{\infty} E_n$ is a countable union of measurable sets with finite measure. Thus

$$\mu_2 \left(\bigcup_{n=1}^{\infty} E_n \right) = 0 = \sum_{n=1}^{\infty} \mu_2(E_n).$$

3. (Problem 9, Page 343) Prove Proposition 3, that is, show that $\mathcal{M}_0$ is a σ-algebra, μ_0 is properly defined and $(X, \mathcal{M}_0, \mu_0)$ is complete. In what sense is $\mathcal{M}_0$ minimal?

Collaborators: None

Solution: **(1)** We show $\mathcal{M}_0$ is a $\sigma-$algebra. **(1.1)** First, since $X \in \mathcal{M}, \emptyset \in \mathcal{M}$, and $X = X \cup \emptyset$, and $\mu(\emptyset) = 0$, we know $X \in \mathcal{M}_0$. **(1.2)** Suppose $E \in \mathcal{M}_0$, then $E = A \cup B$ where $B \in \mathcal{M}$ and $A \subset C$ for some $C \in \mathcal{M}$, and $\mu(C) = 0$. Then $E^c = A^c \cap B^c = (B^c \cap C^c) \cup (C \setminus A)$. Since $\mathcal{M}$ is a $\sigma-$algebra, we know $B^c, C^c, B^c \cup C^c \in \mathcal{M}$, and $C \setminus A \in \mathcal{M}$, and $(C \setminus A) \subset C$, with $\mu(C) = 0$. Hence $E^c \in \mathcal{M}_0$. **(1.3)** Suppose $\{E_n\}$ is a sequence of sets in $\mathcal{M}_0$. We may assume $E_n = A_n \cup B_n$, where $B_n \in \mathcal{M}$, and $A_n \subset C_n$ for some $C_n \in \mathcal{M}$ with $\mu(C) = 0$. Then

$$\bigcup_{n=1}^{\infty} E_n = \left(\bigcup_{n=1}^{\infty} A_n \right) \bigcup \left(\bigcup_{n=1}^{\infty} B_n \right),$$

and $\bigcup\limits_{n=1}^{\infty} B_n \in \mathcal{M}$. With the countable addivity of measure, $\bigcup\limits_{n=1}^{\infty} A_n \subset \bigcup\limits_{n=1}^{\infty} C_n \in \mathcal{M}$, and $\mu \left(\bigcup\limits_{n=1}^{\infty} C_n \right) = \sum\limits_{n=1}^{\infty} \mu(C_n) = 0$. Thus $\bigcup\limits_{n=1}^{\infty} E_n \in \mathcal{M}$.

(2) We show μ_0 is properly defined, which means μ_0 satisfies the properties of measure. **(2.1)** First, for all $E \in \mathcal{M}_0$, suppose $E = A \cup B$ where $A \in \mathcal{M}$ and $B \subset C \in \mathcal{M}$ with $\mu(C) = 0$. Then $\mu_0(E) = \mu(A) \geq 0$. **(2.2)** Since $\emptyset \in \mathcal{M}$, and $\emptyset = \emptyset \cap \emptyset$ with $\mu(\emptyset) = 0$, we know $\mu_0(\emptyset) = \mu(\emptyset) = 0$. **(2.3)** Suppose $\{E_n\}$ is a countable collection of disjoint sets in $\mathcal{M}_0$, denote $E_n = A_n \cup B_n$, and $A_n \in \mathcal{M}$, $B_n \subset C_n \in \mathcal{M}$, with $\mu(C_n) = 0$. Then the same with **(1.3)**, we know

$$\bigcup_{n=1}^{\infty} E_n = \left(\bigcup_{n=1}^{\infty} A_n \right) \bigcup \left(\bigcup_{n=1}^{\infty} B_n \right),$$

with $\bigcup\limits_{n=1}^{\infty} B_n \subset \bigcup\limits_{n=1}^{\infty} C_n \in \mathcal{M}$, and $\mu\left(\bigcup\limits_{n=1}^{\infty} C_n\right) = \sum\limits_{n=1}^{\infty} \mu(C_n) = 0$. Then

$$\mu_0\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} \mu_0(E_n).$$

(3) We show $(X, \mathcal{M}_0, \mu_0)$ is complete. If $E \in \mathcal{M}_0$ and $\mu_0(E) = 0$, then $E = A \cup B$ where $A \in \mathcal{M}$ and $B \subset C \in \mathcal{M}$ and $\mu(C) = 0$, and $\mu(A) = \mu_0(E) = 0$. Then each subset $F \subset E$ has a form $E = \emptyset \cup F$, and $F \subset (A \cup C)$ where $\mu_0(A \cup C) = \mu(A) + \mu(C) = 0$, and $A \cup C, \emptyset \in \mathcal{M}$. Thus $F \in \mathcal{M}_0$.

4. (Problem 10, Page 343) If $(X, \mathcal{M}, \mu)$ is a measure space, we say that a subset E of X is **locally measurable** provided for each $B \in \mathcal{M}$ with $\mu(B) < \infty$, the intersection $E \cap B$ belongs to $\mathcal{M}$. The measure μ is called **saturated** provided every locally measurable set is measurable.
   (a) Show that each σ-finite measure is saturated.
   (b) Show that the collection $\mathcal{C}$ of locally measurable sets is a σ-algebra.
   (c) Let $(X, \mathcal{M}, \mu)$ be a measure space and $\mathcal{C}$ the σ of locally measurable sets. For $E \in \mathcal{C}$, define $\overline{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\overline{\mu}(E) = \infty$ if $E \notin \mathcal{M}$. Show that $(X, \mathcal{C}, \overline{\mu})$ is a saturated measure space.
   (d) If μ is semifinite and $E \in \mathcal{C}$, set $\underline{\mu}(E) = \sup\{\mu(B) \mid B \in \mathcal{M},\, B \subseteq E\}$. Show that $(X, \mathcal{C}, \underline{\mu})$ is a saturated measure space and that $\underline{\mu}$ is an extension of μ. Give an example to show that $\overline{\mu}$ and $\underline{\mu}$ may be different.

Collaborators: None

Solution: **(a)** Suppose μ is a $\sigma-$finite measure. Then $X = \bigcup\limits_{n=1}^{\infty} X_n$ with X_n measurable and $\mu(X_n) < \infty$. Suppose E is a locally measurable set, then for all X_n, $E \cap X_n \in \mathcal{M}$. Thus $E = \bigcup\limits_{n=1}^{\infty} (E \cap X_n) \in \mathcal{M}$.

(b) **(b.1)** First, for all $B \in \mathcal{M}$, $\mu(B) < \infty$, $X \cap B = B \in \mathcal{M}$. Hence $X \in \mathcal{C}$. **(b.2)** Suppose $E \in \mathcal{C}$, then for each $B \in \mathcal{M}$ with $\mu(B) < \infty$, $E \cap B \in \mathcal{M}$. Then $E^c \cap B = B \setminus (E \cap B) \in \mathcal{M}$. **(b.3)** Suppose $\{E_n\} \in \mathcal{C}$ is a sequence of local measurable sets. Then for each $B \in \mathcal{M}$ and $\mu(B) < \infty$, for all n, $E_n \cap B \in \mathcal{M}$. Then fix B, we get

$$\left(\bigcup_{n=1}^{\infty} E_n\right) \cap B = \bigcup_{n=1}^{\infty} (E_n \cap B) \in \mathcal{M}.$$

Hence $\mathcal{C}$ is a $\sigma-$algebra.

(c) **(c.1)** First we show $\overline{\mu}$ is a measure. First, $\overline{\mu}(E) \geq \mu(E) \geq 0$. As $\emptyset \in \mathcal{M}$, $\overline{\mu}(\emptyset) = \mu(\emptyset) = 0$. Suppose $\{E_n\} \in \mathcal{C}$ is a sequence of disjoint sets, then if $\exists i$, s.t. $E_i \notin \mathcal{M}$, then $\bigcup\limits_{n=1}^{\infty} E_n \notin \mathcal{M}$. Thus

$$\overline{\mu}\left(\bigcup_{n=1}^{\infty} E_n\right) = \infty = \sum_{n=1}^{\infty} \overline{\mu}(E_n).$$

Now suppose $E_n \in \mathcal{M}$. Thus the countably additive property of $\overline{\mu}$ is the same as μ. **(c.2)** Now suppose $B \in \mathcal{C}$ and $\overline{\mu}(B) < \infty$, then $B \in \mathcal{M}$, otherwise $\overline{\mu}(B) = \infty$. Suppose E is a local measurable set, then $E \cap B \in \mathcal{C}$, thus according to the definition of $\mathcal{C}$, we know $E \in \mathcal{C}$. Thus $(X, \mathcal{C}, \overline{\mu})$ is a saturated measure space.

(d) **(d.1)** First we show $\underline{\mu}$ is a measure. $\underline{\mu}(E) \geq \mu(E) \geq 0$, $\underline{\mu}(\emptyset) = \sup\{\mu(B) \mid B \in \mathcal{M}, B \subset \emptyset\} = 0$. Suppose $\{E_n\} \in \mathcal{C}$ is a sequence of disjoint sets, then if all $E_n \in \mathcal{M}$, we have $\underline{\mu}(E_n) = \mu(E_n)$. Using the countable addivity of μ we can get the countable addivity of $\underline{\mu}$. Now suppose $E_n \notin \mathcal{M}$. Then using the countable addivity of μ,

$$\underline{\mu}\left(\bigcup_{n=1}^{\infty} E_n\right) = \sup\left\{\mu(B) \mid B \in \mathcal{M}, B \subset \bigcup_{n=1}^{\infty} E_n\right\} = \sum_{n=1}^{\infty} \sup\{\mu(B) \mid B \in \mathcal{M}, B \subset E_n\} = \sum_{n=1}^{\infty} \underline{\mu}(E_n).$$

(d.2) In **(d.1)** we have shown $\underline{\mu}$ is an extension of μ. Now we show $(X, \mathcal{C}, \underline{\mu})$ is a saturated measure space. Suppose E is a locally measurable set, then for each $B \in \mathcal{C}$, $\underline{\mu}(B) < \infty$, $E \cap B \in \mathcal{C}$. For all $F \in \mathcal{F}$ with $\mu(F) < \infty$, we have $\underline{\mu}(F) = \mu(F) < \infty$. Thus $E \cap F \in \mathcal{C}$, and it means E is measurable in $(X, \mathcal{C}, \underline{\mu})$. Hence, $(X, \mathcal{C}, \underline{\mu})$ is a saturated measure space.

5. (Problem 18, Page 373) Let $\{u_n\}$ be a sequence of nonnegative measurable functions on X. For $x \in X$, define $f(x) = \sum_{n=1}^{\infty} u_n(x)$. Show that

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \left[\int_X u_n \, d\mu\right].$$