

Numerical Analysis

Assignment 10

Chuan Lu

November 1, 2017

Problem 1. Problem 4.16, Page 242

Solution. For each $m < n$, by integration by parts,

$$\begin{aligned}\int_0^\infty e^{-x} x^m \varphi_n(x) dx &= \frac{(-1)^n}{n!} \int_0^\infty x^m \frac{d^n}{dx^n} (x^n e^{-x}) dx \\ &= \frac{(-1)^n}{n!} \left(x^m \frac{d^{n-1}}{dx^{n-1}} (x^n e^{-x}) \Big|_0^\infty - \int_0^\infty m x^{m-1} \frac{d^{n-1}}{dx^{n-1}} (x^n e^{-x}) dx \right)\end{aligned}$$

Since

$$x^m \frac{d^{n-1}}{dx^{n-1}} (x^n e^{-x}) = e^{-x} N(x),$$

where $N(x)$ is a polynomial of degree $n - 1 + m$, by L'Hospital's Rule we know the first term in the integration is 0. Then by induction we know

$$\begin{aligned}\int_0^\infty e^{-x} x^m \varphi_n(x) dx &= \frac{(-1)^{n+1} m}{n!} \int_0^\infty x^{m-1} \frac{d^{n-1}}{dx^{n-1}} (x^n e^{-x}) dx \\ &= \frac{(-1)^{n+m} m!}{n!} \int_0^\infty \frac{d^{n-m}}{dx^{n-m}} (x^n e^{-x}) dx \\ &= \frac{(-1)^{n+m} m!}{n!} \frac{d^{n-m}}{dx^{n-m}} \int_0^\infty x^n e^{-x} dx \\ &= \frac{(-1)^{n+m} m!}{n!} \frac{d^{n-m}}{dx^{n-m}} (n!) = 0.\end{aligned}$$

In the deduction we used the property that $f(x) = x^n e^{-x}$ is absolutely continuous. Since $\varphi_m(x)$ is a polynomial of degree $m < n$, we know

$$(\varphi_n(x), \varphi_m(x)) = 0, \text{ and } (\varphi_n(x), \varphi_n(x)) \neq 0.$$

Hence $\{\varphi_n(x)\}$ is a family of orthogonal polynomials.

Problem 2. Problem 4.18, Page 242

Solution. First, we derive c_n . Multiply both sides of (4.4.21) by $w(x)\varphi_{n-1}(x)$, and then integrate, we get

$$\int w \varphi_{n+1} \varphi_{n-1} dx = \int a_n w x \varphi_n \varphi_{n-1} + \int b_n w \varphi_n \varphi_{n-1} - c_n \int w \varphi_{n-1}^2.$$

Using the orthogonality of φ_n , the left side is 0, and the second term of right side is 0. Then

$$a_n \int w x \varphi_n \varphi_{n-1} = c_n \int w \varphi_{n-1}^2.$$

Since

$$a_n \int w x \varphi_n \varphi_{n-1} = a_n \int w \varphi_n (A_{n-1} x^n + B_{n-1} x^{n-1} + \cdots) = a_n \int w \varphi_n A_{n-1} x^n = a_n \frac{A_{n-1}}{A_n} \int w \varphi_n^2,$$

we have

$$c_n = \frac{a_n A_{n-1} \gamma_n}{A_n \gamma_{n-1}} = \frac{A_{n+1} A_{n-1} \gamma_n}{A_n^2 \gamma_{n-1}}.$$

Now we consider b_n . Multiply both sides of (4.4.21) by $w(x)\varphi_n(x)$, then integrate both sides, we get

$$\int w\varphi_{n+1}\varphi_n = \int a_n w x \varphi_n^2 + \int b_n w \varphi_n^2 - \int c_n w \varphi_{n-1}\varphi_n.$$

Using orthogonality, we get

$$\int a_n w x \varphi_n^2 + \int b_n w \varphi_n^2 = 0.$$

The first term can be wrote as

$$\begin{aligned} \int a_n w x \varphi_n^2 &= a_n \int w (A_n x^{n+1} + B_n x + \dots) \varphi_n = a_n \int w \left(\frac{A_n}{A_{n+1}} \varphi_{n+1} - \frac{A_n B_{n+1} - A_{n+1} B_n}{A_{n+1}} x^n + \dots \right) \varphi_n \\ &= a_n \int w \left(B_n - \frac{A_n}{A_{n+1}} B_{n+1} \right) x^n \varphi_n = a_n \int w \frac{1}{A_n} \left(B_n - \frac{A_n}{A_{n+1}} B_{n+1} \right) \varphi_n^2. \end{aligned}$$

Thus

$$a_n \left(\frac{B_n}{A_n} - \frac{B_{n+1}}{A_{n+1}} \right) \gamma_n + b_n \gamma_n = 0,$$

we know

$$b_n = a_n \left(\frac{B_{n+1}}{A_{n+1}} - \frac{B_n}{A_n} \right).$$

Problem 3. Problem 4.21, Page 243

Proof. Denote $\varphi_n(x) = A_n x^n + B_n x^{n-1} + \dots$, and $A_n > 0$. Then by (4.4.21),

$$\varphi_{n+1}(x) = (a_n x + b_n) \varphi_n(x) - c_n \varphi_{n-1}(x).$$

We add a $\varphi_0(x)$ to this series, and $\varphi_0(x) = A_0 > 0$. First, when $n = 1$, since

$$\int_a^b w(x) \varphi_1(x) \varphi_0(x) dx = \int_a^b A_0 w(x) \varphi_1(x) dx = 0 = A_0 \varphi_1(\xi) \int_a^b w(x) dx,$$

we know $\varphi_1(\xi) = 0$, $\xi \in (a, b)$. Then we show $\varphi_2(x)$ has two different roots in (a, b) . First,

$$\varphi_2(\xi) = (a_1 \xi + b_1) \varphi_1(\xi) - c_1 \varphi_0(\xi) = -c_1 \varphi_0(\xi) < 0.$$

Suppose $\varphi_2(x)$ does not change sign in (a, b) , then

$$\int w(x) \varphi_2(x) \varphi_0(x) = A_0 \int w(x) \varphi_2(x) < 0.$$

It makes a contradiction with orthogonality. Then there $\exists x_1 \in (a, b)$, s.t. $\varphi_2(x_1) = 0$. Since $A_2 > 0$, x_1 cannot be a double root. If $\varphi_2(x)$ has only one root in (a, b) , then

$$\varphi_2(x)(x - x_1) = q(x)(x - x_1)^2,$$

integrate by $w(x)$, then since $(x - x_1)$ is of degree 1, left side is 0. But we know $q(x)$ has no root in (a, b) , it does not change sign in (a, b) , thus the integration is not 0. It makes a contradiction. Thus $\varphi_2(x)$ has two different roots in (a, b) , and with $\varphi_2(\xi) < 0$ and $A_2 > 0$, we know the two roots are in (a, ξ) and (ξ, b) separately. Using the same method, we know that $\varphi_n(x)$ has n different roots in (a, b) .

Now we assume this proposition holds for $\varphi_m(x)$, $m \leq n$. Denote roots of $\varphi_n(x)$ to be x_i , then since

$$\varphi_{n+1}(x_i) = (a_n x_i + b_n) \varphi_n(x_i) - c_n \varphi_{n-1}(x_i) = -c_n \varphi_{n-1}(x_i),$$

$$\varphi_{n+1}(x_{i+1}) = (a_n x_{i+1} + b_n) \varphi_n(x_{i+1}) - c_n \varphi_{n-1}(x_{i+1}) = -c_n \varphi_{n-1}(x_{i+1}),$$

from the assumption we know $\varphi_{n-1}(x_i)$ and $\varphi_{n-1}(x_{i+1})$ has different signs, which means $\varphi_n(x)$ has a root in each of this intervals. Then from φ_n has n different roots in (a, b) , from induction we get the result.

Problem 4. Problem 4.23, Page 243

Solution. We know the orthogonal functions on $[-1, 1]$ with weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ are Chebyshev polynomials (normalized):

$$T_0(x) = \frac{1}{\sqrt{\pi}}, \quad T_1(x) = \sqrt{\frac{2}{\pi}} x, \quad T_2(x) = \sqrt{\frac{2}{\pi}} (2x^2 - 1).$$

For any $p(x) = a_0 + a_1x + a_2x^2$ of degree 2 that minimizes the distance, we have

$$a_j = (f, T_j), \quad j = 0, 1, 2.$$

Thus

$$\begin{aligned} a_0 &= (f, T_0) = \frac{1}{\sqrt{\pi}} \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \cos^{-1} x dx = \frac{1}{\sqrt{\pi}} \int_{\pi}^0 -y dy = \frac{(\pi)^{\frac{3}{2}}}{2}, \\ a_1 &= (f, T_1) = \sqrt{\frac{2}{\pi}} \int_{-1}^1 \frac{x}{\sqrt{1-x^2}} \cos^{-1} x dx = \sqrt{\frac{2}{\pi}} \int_0^{\pi} y \cos y dy = \sqrt{\frac{2}{\pi}} (y \sin y \Big|_0^{\pi} - \int_0^{\pi} \sin y dy) = -\sqrt{\frac{8}{\pi}}, \\ a_2 &= (f, T_2) = \sqrt{\frac{2}{\pi}} \int_{-1}^1 \frac{2x^2-1}{\sqrt{1-x^2}} \cos^{-1} x dx = \sqrt{\frac{2}{\pi}} \int_0^{\pi} y \cos 2y dy = \sqrt{\frac{2}{\pi}} \left(\frac{y \sin 2y}{2} \Big|_0^{\pi} - \frac{1}{2} \int_0^{\pi} \sin 2y dy \right) = 0, \end{aligned}$$

Thus

$$p_2(x) = -\frac{4}{\pi}x + \frac{\pi}{2}.$$