Homework 3

Instructions:References such as III.2.7 refer to Problem 7 in Section 2 of Chapter III in Conway's book. If you use results from books including Conway's, please be explicit about what results you are using.
Reminder: Exam I will be from 6:30 to 10:30 on Thursday, February 15 in Room 40 of Schaeffer Hall.

Homework 3 is due in your Dropbox folder by 11:59, Sunday , February 19.

Working on this homework will help you with Exam I, so please don't put it off until after the exam.

1. Problem IV.2.4

(a) By Abel's transform, let $\{a_n\}, \{b_n\}$ be two sequences, and $B_k = \sum\limits_{i=1}^{k} b_i$. Then

$$\sum_{k=1}^{n} a_k b_k = a_n B_n - \sum_{k=1}^{n-1} (a_{k+1} - a_k) B_k.$$

Hence for each fixed n, denote $\sum\limits_{k=1}^{n} a_k$ by A_n

$$C_n = \lim_{r \to 1^-} \sum_{k=1}^{n} a_k r^k = r^n A_n - \sum_{k=1}^{n-1} r^k (r-1) A_k.$$

Since $\sum a_n (z-a)^n$ have radius of convergence 1,

$$\lim_{r \to 1^-} \sum_{n=1}^{\infty} a_n r^n < \infty,$$

then we can change the order of limits:

$$\lim_{r \to 1^-} \lim_{n \to \infty} \sum_{k=1}^{n} a_k r^k = \lim_{n \to \infty} \lim_{r \to 1^-} \sum_{k=1}^{n} a_k r^k = \lim_{n \to \infty} A_n$$

since each A_k is a finite number, which comes from $\sum a_n$ converges to A. Hence,

$$\lim_{r \to 1^-} \sum_{n=1}^{\infty} a_n r^n = \lim_{n \to \infty} A_n = A.$$

(b) Consider $a_n = \frac{(-1)^{n+1}}{n}$, then by Proposition III.1.4,

$$\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim \frac{n+1}{n} = 1.$$

Hence the series $\sum a_n (z-a)^n$ have radius of convergence 1. Since the series $\sum a_n$ is a Leibniz series, then it converges to $A < \infty$.
Now consider the function $f(z) = \log z$, it is analytic on $|z-1| < 1$, and it has power series expansion

$$f(z) = \sum_{n=0}^{\infty} b_n (z-1)^n,$$

where $b_n = \frac{1}{n!} f^{(n)}(1) = \frac{(-1)^{n-1}}{n} (n \geq 1)$, and $b_0 = 0$. By this we can find $a_i = b_i$ for each $i \geq 0$, thus

$$\sum a_n = f(2) = \log 2$$

2. Problem IV.2.6

Sol. In the region where $f(z) = \sqrt{z}$ is analytic,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-1)^n,$$

where

$$a_n = \frac{1}{n!} f^{(n)}(1) = \frac{(-1)^{(n-1)}}{n!} \frac{(2n-3)!!}{2^n} (n \geq 1),\ a_0 = 1.$$

and since

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{2n+2}{2n-1} = 1,$$

we know the radius of convergence is 1.

3. Problem IV.2.9

(a) Let $f(z) = e^z - e^{-z}$, then by Corollary 2.13,

$$f^{(n-1)}(0) = \frac{(n-1)!}{2\pi i} \int_\gamma \frac{e^z - e^{-z}}{z^n} dz = 1 + (-1)^n.$$

Thus

$$\int_\gamma \frac{e^z - e^{-z}}{z^n} dz = \frac{2\pi i}{(n-1)!} (1 + (-1)^n).$$

(b) Let $f(z) = 1$, then

$$f^{(n-1)}(\frac{1}{2}) = \frac{(n-1)!}{2\pi i} \int_\gamma \frac{1}{(z-\frac{1}{2})^n} dz = \begin{cases} 1, n = 1 \\ 0, n \geq 2 \end{cases}$$

Thus

$$\int_\gamma \frac{1}{(z-\frac{1}{2})^n} dz = \begin{cases} 2\pi i, n = 1 \\ 0, n \geq 2 \end{cases}$$

(c) First, we have

$$\frac{1}{z^2+1} = \frac{1}{2i}(\frac{1}{z-i} - \frac{1}{z+i}).$$

Then let $f(z) = 1 = g(z)$, then

$$1 = f(i) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-i} dz = g(-i) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z+i} dz.$$

Hence

$$\int_\gamma \frac{1}{z^2+1} dz = \frac{1}{2i}(\int_\gamma \frac{dz}{z-i} - \int_\gamma \frac{dz}{z+i}) = 0.$$

(d) Let $f(z) = \sin z$, then f is analytic on $\mathbb{C}$.

$$f(0) = \frac{1}{2\pi i} \int_\gamma \frac{\sin z}{z} dz = 0.$$

Hence

$$\int_\gamma \frac{\sin z}{z} dz = 0.$$

(e) Let $f(z) = z^{1/m}$, then

$$f^{(m-1)}(1) = \frac{(m-1)!}{2\pi i} \int_\gamma \frac{z^{1/m}}{(z-1)^m} dz = \prod_{i=0}^{m-1} (\frac{1}{m} - i).$$

Hence

$$\int_\gamma \frac{z^{1/m}}{(z-1)^m} dz = \frac{2\pi i}{(m-1)!} \prod_{i=0}^{m-1} (\frac{1}{m} - i).$$

4. Problem IV.2.11

   First,
   $$f(z) = \frac{1}{2i}\log(\frac{1+iz}{1-iz}) = \frac{1}{2i}(\log(1+iz) - \log(1-iz)).$$
   We know $\log(z)$ is analytic on $\mathbb{C} \setminus l$, where l is a line starting from the origin. Thus f is analytic on $\mathbb{C} \setminus l\{z_1, z_2\}$, where $z_1 = -i,\ z_2 = i$. Hence $l : Re(z) = 0$.

   On a branch of f,
   $$\tan f(z) = \frac{1}{i}\frac{e^{if(z)} - e^{-if(z)}}{e^{if(z)} + e^{-if(z)}}.$$
   Since
   $$e^{if(z)} = e^{\frac{1}{2}\log(\frac{1+iz}{1-iz})} = (\frac{1+iz}{1-iz})^{1/2},$$
   we have
   $$\tan f(z) = \frac{1}{i}\frac{(\frac{1+iz}{1-iz})^{1/2} - (\frac{1+iz}{1-iz})^{-1/2}}{(\frac{1+iz}{1-iz})^{1/2} + (\frac{1+iz}{1-iz})^{-1/2}} = \frac{1}{i}\frac{2iz}{2} = z.$$
   When $|z| < 1$, $f(z)$ is analytic. Hence by Theorem 2.8,
   $$\begin{aligned}\log(1+iz) - \log(1-iz) &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{((1+iz)-1)^n}{n} - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{((1-iz)-1)^n}{n} \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(iz)^n}{n} - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-iz)^n}{n} = 2i\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{2n+1}.\end{aligned}$$
   Hence $f(z) = \sum\limits_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{2n+1}$.

5. Problem IV.3.3

   Sol. The only function is $f(z) = \exp(z)$.

   First, it is clear that $f(z) = \exp(z)$ is a solution. Now suppose there is another function g, s.t. $g(x) = \exp(x), \forall x \in \mathbb{R}$, then for each point $x_0 \in \mathbb{R}$, let $\{x_n\} = \{x_0 + \frac{1}{n}\}$, then $f(x_i) = g(x_i)$ for each i, and $\{x_n\} \to x_0$. Hence, x_0 is a limit point of the set $\{f = g\}$, thus by Corollary 3.8, $f = g$ on $\mathbb{C}$.

6. Problem IV.3.6

   Proof. If $f(a) = 0$, then the proposition holds. Now suppose $f(a) \neq 0$, hence $|f(a)| > 0$. Let $g(z) = \frac{1}{f(z)}$, then since f is analytic, g is analytic in G. Hence $|g(a)| = \left|\frac{1}{f(a)}\right| = \frac{1}{|f(a)|} \geq |g(z)|$ for each $z \in G$. Then by Maximum Modulus Thm, g is constant, which means f is constant.

7. Problem IV.4.2

   Sol. Checked https://math.stackexchange.com/questions/2422165/finding-a-closed-rectifiable-curve-with-prescribed-winding-numbers/2422214

   Let γ: $[0,1] \to \mathbb{C}$ be a union of circles:
   $$\gamma([\frac{1}{2} - \frac{1}{2^n}, \frac{1}{2} - \frac{1}{2^{n+1}}]) = \gamma_1 = B(\frac{1}{2^n}, \frac{1}{2^n}),\ n \geq 1,$$
   and
   $$\gamma([1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}]) = \gamma_2 = B(-\frac{1}{2^n}, \frac{1}{2^n}), n \geq 1.$$
   And the direction of γ_1 is counterclockwise and γ_2 is clockwise.

   Then γ is rectifiable since length of γ is $2\sum_{n=1}^{\infty} 2\pi\frac{1}{2^n} = 4\pi$. Now we show that γ is continuous. In fact, we only need to show that γ is continuous at the point $t = \frac{1}{2}$ and $t = 1$ since circles are continuous. First we know $\lim_{t\to 1}\gamma(t) = 0$, hence γ is closed. For $\forall\epsilon > 0$, pick n, s.t. $\frac{1}{2^n} < \frac{\epsilon}{2}$, then $d(\gamma(t), 0) < 2 \times \frac{1}{2^n} < \epsilon$ for all $t \in [1 - \frac{1}{2^n}, 1]$. Hence γ is continuous at $t = 1$, and similarly, γ is continuous at $t = \frac{1}{2}$.

For any integer k, if $k > 0$, pick a point a inside the k^{th} largest circle and outside the $(k+1)^{th}$ circle of γ_1, then by Proposition 4.1, $n(\gamma; a) = k$. If $k = 0$, pick a outside the largest circle of both γ_1 and γ_2, then $n(\gamma; a) = 0$. If $k < 0$, then pick a inside the k^{th} largest circle and outside the $(k+1)^{th}$ circle of γ_2, by Prop 4.1 and Prop 4.3 $n(\gamma; a) = k$.

8. Problem IV.4.3

   Proof. By Corollary 3.6, since p is a polynomial of degree n, we know

$$p(z) = c(z - z_1)(z - z_2) \cdots (z - z_n),$$

   where z_i are roots of p and z_i could equal to z_j. Then

$$\frac{p'(z)}{p(z)} = \frac{a \sum_{i=1}^n \prod_{j \neq i}^n (z - z_j)}{a \prod_{i=1}^n (z - z_j)} = \sum_{i=1}^n \frac{1}{z - z_i},$$

   with $z_i \in B(0, R)$. Hence,

$$\int_\gamma \frac{p'(z)}{p(z)} dz = \int_\gamma \sum_{i=1}^n \frac{1}{z - z_i} dz = \sum_{i=1}^n \int_\gamma \frac{1}{z - z_i} dz = \sum_{i=1}^n 2\pi i = 2\pi i n$$

   by Prop 4.3 and Thm 4.4.

9. Problem IV.5.7

   Sol. Let $f(z) = z^n$. Then by Corollary 5.9,

$$\int_\gamma \frac{z^n}{(z-1)^n} dz = \int_\gamma \frac{f(z)}{(z-1)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(1) n(\gamma; 1) = 2\pi i n \times n(\gamma; 1) = 2\pi i n.$$

10. Problem IV.5.9

   Proof. With Morera's Theorem, we need to show for each triangular T in $\mathbb{C}$, the integral of f on T is 0. We separate it into three cases, and use A, B, C to denote the vertices of T.

   1) T does not intersect $[-1, 1]$. Then by Cauchy's Theorem, since f is analytic on $\mathbb{C} \setminus [-1, 1]$, we know $\int_T f = 0$.

   2) T intersects $[-1, 1]$ on one point x_0. Then f must intersect the x-axis on another point x_1. Without loss of generality, assume A is on the upper half plane and B, C are on lower half plane. Then

$$\int_T f = \int_{A \to x_0} + \int_{x_0 \to B} + \int_{B \to C} + \int_{C \to x_1} + \int_{x_1 \to A} + \int_{x_0 \to x_1} + \int_{x_1 \to x_0} f = \int_{A x_0 x_1 A} f + \int_{x_0 B C x_1 x_0} f.$$

   Since f is continuous, we know f is uniformly continuous in the closed region with T being its border. Then $\forall \epsilon > 0$, $\exists \delta > 0$, $\forall |z_1 - z_2| < \delta$, $|f(z_1) - f(z_2)| < \epsilon$. And f is bounded inside the region, suppose $|f(z)| \leq M$.

   Pick z_0 on $T_{A \to x_0}$ and z_1 on $T_{x_1 \to A}$, s.t. $Im(z_0) = Im(z_1)$, and $|z_0 - x_0| < \delta, |z_1 - x_1| < \delta$. Let $y_0 = Re(z_0), y_1 = Re(z_1)$, then $|x_0 - y_0| < \delta, |x_1 - y_1| < \delta$, and $Im(z_0) = Im(z_1) < \delta$. Denote $g(x) = f(x + Im(z_0)), x \in \mathbb{R}$, then

$$\left| \int_{x_0}^{x_1} f - \int_{z_0}^{z_1} f \right| = \left| \int_{x_0}^{y_0} + \int_{y_0}^{y_1} + \int_{y_1}^{x_1} f - \int_{y_0}^{y_1} g \right| < 2\delta M + \left| \int_{y_0}^{y_1} (f - g) \right| < 2\delta M + \epsilon |y_1 - y_0|.$$

   Hence

$$\left| \int_{A x_0 x_1 A} f \right| = \left| \int_{A z_0 z_1 A} f + \int_{z_0}^{x_0} + \int_{x_0}^{x_1} + \int_{x_1}^{z_1} + \int_{z_1}^{z_0} f \right| < 2\delta M + 2\delta M + \epsilon |y_1 - y_0|.$$

   With the arbitrariness of ϵ and corrosponding δ(we can make δ arbitrarily small), we know

$$\int_{A x_0 x_1 A} f = 0.$$

Similarly,

$$\int_T f = \int_{Ax_0x_1A} + \int_{x_0BCx_1x_0} f = 0.$$

3) An edge of T overlaps with $[-1, 1]$. In fact, from the discussion in 2), we know

$$\int_T f = 0.$$

Hence, f is analytic on $\mathbb{C}$, which means f is entire.