Introduction to Analysis I
Homework 4
Wednesday, September 27, 2017

Instructions: This and all subsequent homeworks must be submitted written in LaTeX.
If you use results from books, Royden or others, please be explicit about what results you are using.

Homework 4 is due by midnight, Saturday, October 7.

1. (Problem 18, Page 63) Let I be a closed bounded interval and let f be a bounded measurable function defined on I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which
$$|h - f| < \epsilon \text{ on } F \text{ and } m(I \sim F) < \epsilon.$$

   Collaborators:

   Solution:

2. (Problem 22, Page 64) (Dini's Theorem) Let $\{f_n\}$ be an increasing sequence of continuous functions on $[a, b]$ which converges pointwise on $[a, b]$ to the continuous function f on $[a, b]$. Show that the convergence is uniform on $[a, b]$.

   Collaborators:

   Solution:

3. (Problem 5, Page 364) Show that an extended real-valued function f on X is measurable if and only if for each rational number c, $\{x \in X \mid f(x) < c\}$ is a measurable set.

   Collaborators:

   Solution:

4. (Problem 13, Page 365) Let $\{f_n\}$ be a sequence of real-valued functions on X such that for each natural number n, $\mu\{x \in X \mid |f_n(x) - f_{n+1}(x)| > 1/2^n\} < 1/2^n$. Show that $\{f_n\}$ is pointwise convergent a.e. on X.

   Collaborators:

   Solution:

5. (Problem 15, Page 365) A sequence $\{f_n\}$ of measurable real-valued functions on X is said to *converge in measure* to a measurable function f provided that for each $\eta > 0$,
$$\lim_{n\to\infty} \mu\{x \in X \mid |f_n(x) - f(x)| > \eta\} = 0.$$
A sequence $\{f_n\}$ of measurable functions is said to be *Cauchy in measure* provided that for each $\epsilon > 0$ and $\eta > 0$, there is an index N such that for each $m, n \geq N$,
$$\mu\{x \in X \mid |f_n(x) - f_m(x)| > \eta\} < \epsilon.$$

(a) Show that if $\mu(X) < \infty$ and if $\{f_n\}$ converges pointwise a.e. on X to a measurable function f, then $\{f_n\}$ converges to f in measure.

(b) Show that if $\{f_n\}$ converges to f in measure, then there is a subsequence of $\{f_n\}$ that converges pointwise a.e. to f.

(c) Show that if $\{f_n\}$ is Cauchy in measure, then there is a measurable function f to which $\{f_n\}$ converges in measure.

6. (Problem 16, Page 365) Assume $\mu(X) < \infty$. Show that $\{f_n\}$ converges to f in measure if and only if each subsequence of $\{f_n\}$ has a further subsequence that converges pointwise a.e. on X to f. Use this to show that for two sequences that converge in measure, the product sequence also converges in measure to the product of the limits.

7. Show that if f is an lower semicontinuous (resp. upper semicontinuous) function on an interval $[a, b]$, then there is a family $\{f_\alpha\}$ of continuous functions on the interval $[a, b]$ such that $f(x) = \sup\{f_\alpha(x) \mid \alpha \in A\}$ (resp. $f(x) = \inf\{f_\alpha(x) \mid \alpha \in A\}$) for all $x \in [a, b]$.

Collaborators:

Solution: