Homework 2

Instructions:References such as III.2.7 refer to Problem 7 in Section 2 of Chapter III in Conway's book. If you use results from books including Conway's, please be explicit about what results you are using.
Reminder: Exam I will be from 6:30 to 10:30 on Thursday, February 15 in Room 40 of Schaeffer Hall.

Homework 3 is due in your Dropbox folder by 11:59, Sunday , February 19.

Working on this homework will help you with Exam I, so please don't put it off until after the exam.

1. Problem IV.2.4

(a) By Abel's transform, let $\{a_n\}, \{b_n\}$ be two sequences, and $B_k = \sum_{i=1}^{k} b_i$. Then

$$\sum_{k=1}^{n} a_k b_k = a_n B_n - \sum_{k=1}^{n-1} (a_{k+1} - a_k) B_k.$$

Hence for each fixed n, denote $\sum_{k=1}^{n} a_k$ by A_n

$$C_n = \lim_{r\to 1^-} \sum_{k=1}^{n} a_k r^k = r^n A_n - \sum_{k=1}^{n-1} r^k (r-1) A_k.$$

Since $\sum a_n (z-a)^n$ have radius of convergence 1,

$$\lim_{r\to 1^-} \sum_{n=1}^{\infty} a_n r^n < \infty,$$

then we can change the order of limits:

$$\lim_{r\to 1^-} \lim_{n\to\infty} \sum_{k=1}^{n} a_k r^k = \lim_{n\to\infty} \lim_{r\to 1^-} \sum_{k=1}^{n} a_k r^k = \lim_{n\to\infty} A_n$$

since each A_k is a finite number, which comes from $\sum a_n$ converges to A. Hence,

$$\lim_{r\to 1^-} \sum_{n=1}^{\infty} a_n r^n = \lim_{n\to\infty} A_n = A.$$

(b) Consider $a_n = \frac{(-1)^{n+1}}{n}$, then by Proposition III.1.4,

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim \frac{n+1}{n} = 1.$$

Hence the series $\sum a_n (z-a)^n$ have radius of convergence 1. Since the series $\sum a_n$ is a Leibniz series, then it converges to $A < \infty$.
Now consider the function $f(z) = \log z$, it is analytic on $|z-1| < 1$, and it has power series expansion

$$f(z) = \sum_{n=0}^{\infty} b_n (z-1)^n,$$

where $b_n = \frac{1}{n!} f^{(n)}(1) = \frac{(-1)^{n-1}}{n} (n \geq 1)$, and $b_0 = 0$. By this we can find $a_i = b_i$ for each $i \geq 0$, thus

$$\sum a_n = f(2) = \log 2$$

2. Problem IV.2.6

Sol. In the region where $f(z) = \sqrt{z}$ is analytic,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-1)^n,$$

where

$$a_n = \frac{1}{n!} f^{(n)}(1) = \frac{(-1)^{(n-1)}}{n!} \frac{(2n-3)!!}{2^n} (n \geq 1),\ a_0 = 1.$$

and since

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{2n+2}{2n-1} = 1,$$

we know the radius of convergence is 1.

3. Problem IV.2.9

(a) Let $f(z) = e^z - e^{-z}$, then by Corollary 2.13,

$$f^{(n-1)}(0) = \frac{(n-1)!}{2\pi i} \int_\gamma \frac{e^z - e^{-z}}{z^n} dz = 1 + (-1)^n.$$

Thus

$$\int_\gamma \frac{e^z - e^{-z}}{z^n} dz = \frac{2\pi i}{(n-1)!} (1 + (-1)^n).$$

(b) Let $f(z) = 1$, then

$$f^{(n-1)}(\frac{1}{2}) = \frac{(n-1)!}{2\pi i} \int_\gamma \frac{1}{(z-\frac{1}{2})^n} dz = \begin{cases} 1, n = 1 \\ 0, n \geq 2 \end{cases}$$

Thus

$$\int_\gamma \frac{1}{(z-\frac{1}{2})^n} dz = \begin{cases} 2\pi i, n = 1 \\ 0, n \geq 2 \end{cases}$$

(c) First, we have

$$\frac{1}{z^2+1} = \frac{1}{2i} (\frac{1}{z-i} - \frac{1}{z+i}).$$

Then let $f(z) = 1 = g(z)$, then

$$1 = f(i) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-i} dz = g(-i) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z+i} dz.$$

Hence

$$\int_\gamma \frac{1}{z^2+1} dz = \frac{1}{2i} (\int_\gamma \frac{dz}{z-i} - \int_\gamma \frac{dz}{z+i}) = 0.$$

(d) Let $f(z) = \sin z$, then f is analytic on $\mathbb{C}$.

$$f(0) = \frac{1}{2\pi i} \int_\gamma \frac{\sin z}{z} dz = 0.$$

Hence

$$\int_\gamma \frac{\sin z}{z} dz = 0.$$

(e) Let $f(z) = z^{1/m}$, then

$$f^{(m-1)}(1) = \frac{(m-1)!}{2\pi i} \int_\gamma \frac{z^{1/m}}{(z-1)^m} dz = \prod_{i=0}^{m-1} (\frac{1}{m} - i).$$

Hence

$$\int_\gamma \frac{z^{1/m}}{(z-1)^m} dz = \frac{2\pi i}{(m-1)!} \prod_{i=0}^{m-1} (\frac{1}{m} - i).$$

4. Problem IV.2.11

   First,
   $$f(z) = \frac{1}{2i}\log(\frac{1+iz}{1-iz}) = \frac{1}{2i}(\log(1+iz) - \log(1-iz)).$$
   We know $\log(z)$ is analytic on $\mathbb{C} \setminus l$, where l is a line starting from the origin. Thus f is analytic on $\mathbb{C} \setminus l\{z_1, z_2\}$, where $z_1 = -i,\ z_2 = i$. Hence $l : Re(z) = 0$.

   On a branch of f,
   $$\tan f(z) = \frac{1}{i}\frac{e^{if(z)} - e^{-if(z)}}{e^{if(z)} + e^{-if(z)}}.$$
   Since
   $$e^{if(z)} = e^{\frac{1}{2}\log(\frac{1+iz}{1-iz})} = (\frac{1+iz}{1-iz})^{1/2},$$
   we have
   $$\tan f(z) = \frac{1}{i}\frac{(\frac{1+iz}{1-iz})^{1/2} - (\frac{1+iz}{1-iz})^{-1/2}}{(\frac{1+iz}{1-iz})^{1/2} + (\frac{1+iz}{1-iz})^{-1/2}} = \frac{1}{i}\frac{2iz}{2} = z.$$
   By exercise III.3.19(d),
   $$f(z) = \frac{1}{2i}(\log\frac{1+iz}{1-iz}) = \frac{1}{2i}(\log\frac{z-i}{z+i} + \log i - \log(-i)) = -\frac{1}{2}\int_{-1}^{1}\frac{dt}{z-it} - \frac{\pi}{2}$$

5. Problem IV.3.3

6. Problem IV.3.6

7. Problem IV.3.14

8. Problem IV.4. 2

9. Problem IV. 4.3

10. Problem IV.5.7

11. Problem IV.5.9