**Homework 4**

**Instructions**: In problems the problems below, references such as III.2.7 refer to Problem 7 in Section 2 of Chapter III in Conway's book.

If you use results from books including Conway's, please be explicit about what results you are using.

*Homework 4 is due in class at Midnight March 9.*

Do the following problems:

1. IV.7.1

   **Sol. (Discussed with a college classmate)** In fact, I don't think that this proposition is correct. For example, pick $G$ the unit disk $B(0,1)$, and $\gamma = \gamma(t) : [0,1] \to B$, s.t. $\gamma(t) = t$ for $0 \le t < 1$, and $\gamma(1) = 0$. Then $\gamma$ is closed, and by simple calculation we know $V(\gamma) = 2$, which shows $\gamma$ is rectifiable. Let $f = \frac{1}{z-1}$, then $f$ is analytic in $B(0,1)$. But when $t \to 1$, $f \circ \gamma(t) \to \infty$, hence it is not rectifiable.

2. IV.7.2

   **(a)** Let $f(z) = z$, pick any $z_0 \in \{z \mid d(z, \partial G) < \frac{1}{2}r\}$, then since there is only one point $z = z_0$ satisfies $f(z) = z_0$, by Thm 7.2,
$$n(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}dz$$
   Since $\frac{1}{z-z_0}$ is analytic on $\{z \mid d(z, \partial G) < \frac{1}{2}r\}$, by Prop 2.15, we know the integral is 0. Hence $\{z \mid d(z, \partial G) < \frac{1}{2}r\} \subset H$.

3. V.1.1

   **(a)** Around $z = 0$,
$$\lim_{z\to 0}|zf(z)| = \lim_{z\to 0}|\sin(z)| = \frac{1}{2}\lim_{z\to 0}|e^{iz} - e^{-iz}| \le \lim_{z\to 0}|z| = 0.$$
   Hence by Thm 1.2, $z = 0$ is removable, and $f(0) = 1$ by power series expansion.

   **(b)** At $z = 0$, $g(z) = \cos(z)$ is analytic, and $\cos(0) = 1$. Thus by Prop 1.4, $z = 0$ is a pole, and the singular part is $\frac{1}{z}$.

   **(c)** At $z = 0$, $\lim_{z\to 0} zf(z) = \lim_{z\to 0}\cos z - 1 = 0$, then by Thm 1.2, 0 is removable, and $f(0) = 0$ by power series expansion.

   **(d)** At $z = 0$,
$$f(z) = \sum_{n=0}^{-\infty}\frac{1}{(-n)!}z^n,$$
   hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \{z \mid |z| > exp(\frac{1}{\delta})\}$.

   **(e)** At $z = 0$,
$$f(z) = \frac{1}{z^2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n} = \frac{1}{z} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+2}z^n.$$
   Hence 0 is a pole, and the singularity part is $\frac{1}{z}$.

   **(f)** At $z = 0$,
$$f(z) = z\sum_{n=0}^{\infty}(-1)^n\frac{z^{-2n}}{n!} = z + \sum_{n=-1}^{-\infty}(-1)^{-n}\frac{z^{2n+1}}{(-n)!}$$
   Hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \mathbb{C}$.

**(g)** Around $z = 0$, notice $\frac{z^2+1}{z-1}$ is analytic, hence 0 is a pole. Since $|z| < 1$,

$$f(z) = 1 - \frac{1}{z} + \frac{2}{z-1} = 1 - \frac{1}{z} - 2\sum_{n=0}^{\infty} z^n,$$

we know the singular part is $-\frac{1}{z}$.

**(h)** For any $n > 0$,

$$\lim_{z\to 0} z^n f(z) = \lim_{z\to 0} z^n \frac{1}{\sum_{n=1}^{\infty} \frac{1}{n!} z^n} = \infty,$$

hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \{z \mid |z| > \frac{1}{1-e^{\delta}}\}$.

**(i)**

$$f(z) = z \sum_{n=0}^{\infty} (-1)^n \frac{z^{-(2n+1)}}{(2n+1)!} = 1 + \sum_{n=-1}^{-\infty} (-1)^{-n} \frac{z^{2n-1}}{(-2n+1)!},$$

hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \{z \mid |z| < \delta\}$.

**(j)** Same with (i), 0 is an essential singularity, and $f(0 < |z| < \delta) = \{z \mid |z| < \delta^n\}$.

4. V.1.4

**(a)**

$$f(z) = \frac{1}{z}\left(\frac{1}{1-z} - \frac{1}{2(1-z/2)}\right) = \frac{1}{z}\left(\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right) = \frac{1}{2z} + \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+2}}\right) z^n$$

**(b)**

$$f(z) = \frac{1}{z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right) = \frac{1}{z}\left(-\frac{1}{2}\frac{1}{1-\frac{z}{2}} - \frac{1}{z}\frac{1}{1-\frac{1}{z}}\right) = \frac{1}{z}\left(-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty} z^{-n-1}\right) = -\sum_{n=-1}^{\infty}\frac{z^n}{2^{n+2}} - \sum_{n=-\infty}^{-2} z^n$$

**(c)**

$$f(z) = \frac{1}{z}\left(\frac{\frac{1}{z}}{1-\frac{2}{z}} - \frac{\frac{1}{z}}{1-\frac{1}{z}}\right) = \frac{1}{z}\left(\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \sum_{n=0}^{\infty}\frac{1}{z^n}\right) = \sum_{n=-\infty}^{-1}\left(2^{-(n+1)} - 1\right) z^n.$$

5. V.1.12

**Proof.** By (1.11), since $f$ is analytic on $0 < |z| < \infty$,

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{\exp(\frac{1}{2}\lambda(z + \frac{1}{z}))}{z^{n+1}} dz$$

pick $\gamma = \exp(it)$, the unit circle, then

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{\lambda\cos t} e^{-int} dt = \frac{1}{2\pi}\int_0^{2\pi} e^{\lambda\cos t}(\cos nt - i\sin nt) dt,$$

the real part is

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\lambda\cos t}\cos nt dt = \frac{1}{2\pi}\left(\int_0^{\pi} e^{\lambda\cos t}\cos nt dt - \int_\pi^0 e^{\lambda\cos(2\pi - s)}\cos(2\pi - s) ds\right) = \frac{1}{\pi}\int_0^{\pi} e^{\lambda\cos t}\cos nt dt.$$

and the imaginary part is

$$-\frac{i}{2\pi}\int_0^{2\pi} e^{\lambda\cos t}\sin nt dt = -\frac{i}{2\pi}\left(\int_0^{\pi} e^{\lambda\cos t}\sin nt dt + \int_\pi^0 e^{\lambda\cos(2\pi - s)}\sin(2\pi - s) ds\right) = 0.$$

Hence $a_n = \frac{1}{\pi}\int_0^{\pi} e^{\lambda\cos t}\cos nt dt$.

$$b_n = \frac{1}{2\pi i}\int_\gamma \frac{\exp(\frac{1}{2}\lambda(z-\frac{1}{z}))}{z^{n+1}}dz$$

pick $\gamma = \exp(it)$,

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} e^{\lambda i \sin t}e^{-int}dt = \frac{1}{2\pi}\int_0^{2\pi} \cos(nt-\lambda\sin t) - i\sin(nt-\lambda\sin t)dt.$$

The real part is

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\cos(nt-\lambda\sin t)dt &= \frac{1}{2\pi}\left(\int_0^{\pi}\cos(nt-\lambda\sin t)dt - \int_\pi^0 \cos(n(2\pi-s)-\lambda\sin(2\pi-s))ds\right)\\ &= \frac{1}{2\pi}\left(\int_0^{\pi}\cos(nt-\lambda\sin t)dt + \int_0^\pi \cos(-ns+\lambda\sin s)ds\right) = \frac{1}{\pi}\int_0^\pi \cos(nt-\lambda\sin t)dt.\end{aligned}$$

and the imaginary part is

$$\begin{aligned}-\frac{i}{2\pi}\int_0^{2\pi}\sin(nt-\lambda\sin t)dt &= -\frac{i}{2\pi}\left(\int_0^{\pi}\sin(nt-\lambda\sin t)dt - \int_\pi^0 \sin(n(2\pi-s)-\lambda\sin(2\pi-s))ds\right)\\ &= -\frac{i}{2\pi}\left(\int_0^{\pi}\sin(nt-\lambda\sin t)dt - \int_0^\pi \sin(-ns+\lambda\sin s)ds\right) = 0.\end{aligned}$$

Hence $b_n = \frac{1}{\pi}\int_0^\pi \cos(nt-\lambda\sin t)dt$.

6. V.1.13

**(a)** Suppose $f$ is entire and has a removable singularity at $\infty$, then $g(z) = f(\frac{1}{z})$ has a removable singularity at $z=0$, thus we can define $g(0) = a < \infty$, which means $f$ is bounded in the neighbourhood of $\infty$. Hence by Liouville Thm, $f$ is constant.

**(b)** By assumption, $g(z) = f(\frac{1}{z}) = \frac{1}{z^m}h(z)$, where $h(z)$ is analytic at $z=0$. Since $f(z) = z^m h(\frac{1}{z})$ is entire, it means at $z=0$, $h$ has a definition or has a removable singularity, and $h$ is entire. Hence, by (1) we know $h$ is a constant, which means $f$ is a polynomial of degree m.

**(c)** Let $f(z) = \frac{\prod(z-u_i)}{\prod(z-v_i)}$, then

$$g(z) = f(\frac{1}{z}) = \frac{\prod_{i=1}^n(\frac{1}{z}-u_i)}{\prod_{i=1}^m(\frac{1}{z}-v_i)}$$

has a removable singularity at $z=0$, which means $\lim_{z\to 0} g(z)$ exists and is not $\infty$. Notice

$$\lim_{z\to 0} g(z) = \lim_{z\to 0}\frac{z^m\prod_{i=1}^n(\frac{1}{z}-u_i)}{\prod_{i=1}^m(1-v_iz)},$$

if $m<n$, then $\lim_{z\to 0} g(z) = \infty$, which makes a contradiction. If $m \geq n$, then the limit is well-defined. Hence $f = \frac{p(z)}{q(z)}$, where $p, q$ are polynomials, and $\deg(p) \leq \deg(q)$.

**(d)** Let $f(z) = \frac{\prod(z-u_i)}{\prod(z-v_i)}$, then

$$g(z) = f(\frac{1}{z}) = \frac{\prod_{i=1}^n(\frac{1}{z}-u_i)}{\prod_{i=1}^k(\frac{1}{z}-v_i)}$$

has a pole of order m at $z=0$, which means $g(z) = \frac{h(z)}{z^m}$, where $h(z)$ is analytic at $z=0$. Then

$$z^m g(z) = \frac{z^m\prod_{i=1}^n(\frac{1}{z}-u_i)}{\prod_{i=1}^k(\frac{1}{z}-v_i)} = \frac{z^{m+k}\prod_{i=1}^n(\frac{1}{z}-u_i)}{\prod_{i=1}^m(1-v_iz)},$$

and we have $m+k=n$, otherwise the order of pole is not $m$. Hence $f = \frac{p}{q}$, and $\deg(p)-\deg(q) = m$.

7. V.1.17

   **Proof.** If not, first suppose $f$ has a pole of order $m$ at $a$. Then $f(z) = \frac{g(z)}{(z-a)^m}$, and $g$ is analytic on $G$.

   i) If $g(a) = 0$, then $f(z) = 0$, and we can define $f(a) = 0$, thus $a$ is a removable singularity.

   ii) If $g(a) \neq 0$, then according to the isolation of zeros, $\exists r > 0$, s.t. $g(z) \neq 0$ in $B(a, 2r)$. Consider $H = B(a, r)$, by max modulus theorem, $\min_H |g| = \min_{\partial H} |g|$, denote is as $c \neq 0$. Hence

$$\int\int_H |f(x+iy)|^2 dxdy = \int\int_H \frac{|g(x+iy)|^2}{|x+iy-a|^{2m}} dxdy \geq \int\int_H \frac{c^2}{|x+iy-a|^2} dxdy.$$

   Let $x = Re(a) + s\cos t, y = Im(a) + s\sin t$, then

$$\int\int_H |f(x+iy)|^2 dxdy \geq \int_0^r \int_0^{2\pi} \frac{c^2}{s^{2m}} sdsdt = 2\pi c^2 \int_0^r s^{1-2m} ds.$$

   If $m = 1$, then the integral becomes

$$\int_0^r s^{-1} ds = \ln(s)\Big|_0^r = \infty.$$

   If $m > 1$, then the integral is

$$\int_0^r s^{1-2m} ds = \frac{1}{2-2m} s^{2-2m}\Big|_0^r = \infty.$$

   Since the integrand is nonnegative,

$$\int_G |f|^2 \geq \int_H |f|^2 = \infty,$$

   which makes a contradiction. With the same method, we know $a$ is not an essential singularity. Hence $a$ is a removable one.

   By the deduction we know, if $0 < p < 2$, $a$ could be a removable singularity or a pole of order $m$, which satisfies $pm < 2$. If $p \geq 2$, then $a$ is a removable singularity.

8. V.2.1

   **(a)** Notice $(x^4 + x^2 + 1)(x^2 - 1) = x^6 - 1$, so $f(z) = \frac{z^2}{z^4+z^2+1}$ has four poles of order 1, and they are sixth roots of 1, which are $a_i = e^{i\theta_i}$ where

$$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}.$$

   Consider the two poles in the upper-half plane $a_1, a_2$, then

$$Res(f, a_1) = \lim_{z \to a_1} (z - a_1) f(z) = a_1^2 (a_1 - a_2)^{-1} (a_1 - a_3)^{-1} (a_1 - a_4)^{-1} = \frac{1}{2\sqrt{3}i} e^{\frac{\pi}{3}i},$$

$$Res(f, a_2) = \lim_{z \to a_2} (z - a_2) f(z) = a_2^2 (a_2 - a_1)^{-1} (a_2 - a_3)^{-1} (a_2 - a_4)^{-1} = -\frac{1}{2\sqrt{3}i} e^{\frac{2\pi}{3}i}.$$

   Let $\gamma$ be the closed path which is the boundary of upper half of disk of radius $R > 1$ with center 0, in counter-clockwise direction. Then by Residue Theorem,

$$\frac{1}{2\pi i} \int_\gamma f = Res(f, a_1) + Res(f, a_2) = \frac{1}{2\sqrt{3}i}$$

   And

$$\int_\gamma f = \int_{-R}^R \frac{x^2}{x^4 + x^2 + 1} dx + i \int_0^\pi \frac{R^3 e^{i3t}}{R^4 e^{i4t} + R^2 e^{i2t} + 1} dt.$$

Notice $|R^4e^{i4t}+R^2e^{i2t}+1| \geq R^4 - R^2 = 1$ when $R$ is sufficiently large, so

$$\left| i\int_0^{\pi} \frac{R^3e^{i3t}}{R^4e^{i4t}+R^2e^{i2t}+1}dt \right| \leq \frac{\pi R^3}{R^4-R^2-1} \to 0 \text{ as } R\to\infty.$$

Since $\frac{x^2}{x^4+x^2+1} \geq 0, \forall x\in\mathbb{R}$, we know

$$\int_0^{\infty} \frac{x^2}{x^4+x^2+1}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{x^4+x^2+1} = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{x^2}{x^4+x^2+1}dx = \frac{\pi}{\sqrt{3}}.$$

For the following problems, I will not write all the steps since I don't have enough time to do so. I will only write the key points.

**(b)** We know

$$\int_r^R \frac{\cos x - 1}{x^2}dx = \frac{1}{2}\left(\int_r^R + \int_{-R}^{-r} \frac{e^{ix}-1}{x^2}dx\right)$$

Consider $f = \frac{e^{iz}-1}{z^2}$, then $f$ has a simple pole at $z=0$. Let $\gamma$ be the curve in Example 2.7, then

$$0 = \int_r^R + \int_{-R}^{-r} \frac{e^{ix}-1}{x^2}dx + \int_{\gamma_r} + \int_{\gamma_R} \frac{e^{iz}-1}{z^2}dz$$

Since

$$\left|\int_{\gamma_R} \frac{e^{iz}-1}{z^2}dz\right| = \left|\int_0^{\pi} \frac{\exp(iRe^{it})-1}{Re^{it}}dt\right| \leq \frac{1}{R}\int_0^{\pi}\exp(-R\sin t)dt,$$

and by Example 2.7 we know the integral $\to 0$ as $R\to\infty$. And since $\frac{e^{iz}-1-iz}{z^2}$ has a removable singularity at $z=0$, by Example 2.7 we know

$$\lim_{\gamma\to 0}\left|\int_{\gamma_r} \frac{e^{iz}-1-iz}{z^2}dz\right| = 0,$$

but $\int_{\gamma} \frac{i}{z}dz = \pi$, hence

$$\lim_{\gamma\to 0}\left|\int_{\gamma_r} \frac{e^{iz}-1}{z^2}dz\right| = \pi.$$

Hence

$$\int_0^{\infty} \frac{\cos x - 1}{x^2}dx = \lim_{r\to 0, R\to\infty}\int_r^R \frac{\cos x - 1}{x^2}dx = -\frac{\pi}{2}.$$

**(c)** The same with Example 2.9, let $z=e^{i\theta}$, then

$$\int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}d\theta = -\frac{i}{4}\int_{\gamma} \frac{z^4+2z^2+1}{-az^2+(a^2+1)z-a}dz,$$

where $\gamma = \{|z|=1\}$. By residue theorem,

$$\int_{\gamma} \frac{z^4+2z^2+1}{-az^2+(a^2+1)z-a}dz = 2\pi i Res(f,a) = 2\pi i\frac{(a^2+1)^2}{1-a^2}.$$

Hence

$$\int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}d\theta = \frac{\pi}{2}\frac{(a^2+1)^2}{1-a^2}.$$

**(d)** Same with (c), let $z=e^{i\theta}$, then

$$\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = -2i\int_{\gamma} \frac{z}{(z^2+2az+1)^2}dz,$$

where $\gamma = \{|z| = 1\}$. By residue theorem,

$$\int_\gamma \frac{z}{(z^2+2az+1)^2}dz = \frac{1}{2}\pi i a(a^2-1)^{-\frac{3}{2}}.$$

Hence

$$\int_0^\pi \frac{d\theta}{(a+\cos\theta)^2} = \pi a(a^2-1)^{-\frac{3}{2}}.$$

9. V.2.2

**(a)**

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(x^2+a^2)^2}.$$

Consider $R > a$, let $\gamma = \{|z| = R, Im(z) > 0\}$, then by residue thm, since $f$ has a pole of order 2, $z = ai$,

$$\int_\gamma + \int_{-R}^R \frac{dz}{(z^2+a^2)^2} = 2\pi i Res(f, ai) = \frac{\pi}{2a^3}.$$

But

$$\left|\int_\gamma \frac{dz}{(z^2+a^2)^2}\right| \le \int_\gamma \left|\frac{dz}{(z^2+a^2)^2}\right| \le \int_0^\pi \frac{1}{|R^2-a^2|^2}dt \le \frac{\pi}{|R^2-a^2|^2} \to 0, \text{ as } R\to\infty.$$

Hence

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}.$$

**(b)** Define $G, l(z), \gamma$ as in Example 2.10, then

$$\int_\gamma \frac{(l(z))^3}{1+z^2}dz = \int_r^R \frac{(\log x)^3}{1+x^2}dx + \int_0^\pi \frac{(\log R + i\theta)^3}{1+R^2e^{2i\theta}} iRe^{i\theta}d\theta + \int_{-R}^{-r} \frac{(\log|x|+\pi i)^3}{1+x^2}dx + \int_\pi^0 \frac{(\log r + i\theta)^3}{1+r^2e^{2i\theta}} ire^{i\theta}d\theta$$

by residue thm,

$$\int_\gamma \frac{(l(z))^3}{1+z^2}dz = 2\pi i Res(f,i) = -\frac{1}{8}\pi^4 i.$$

And

$$\begin{aligned}\int_r^R \frac{(\log x)^3}{1+x^2}dx + \int_{-R}^{-r}\frac{(\log|x|+\pi i)^3}{1+x^2}dx &= \int_r^R \frac{(\log x)^3 + (\log x + \pi i)^3}{1+x^2}dx \\ &= 2\int_r^R \frac{(\log x)^3}{1+x^2} + 3\pi i \int_r^R \frac{(\log x)^2}{1+x^2}dx - 3\pi^2 \int_r^R \frac{\log x}{1+x^2}dx - \pi^3 i \int_r^R \frac{1}{1+x^2}dx\end{aligned}$$

we know from example 2.10, as $r\to 0, R\to\infty$, the third term is 0, and the last term is

$$-\pi^3 i \int_0^\infty \frac{1}{1+x^2}dx = -\frac{1}{2}\pi^4 i.$$

Now we consider the second term. Using the same notation $G, l(z), \gamma$,

$$\int_\gamma \frac{(l(z))^2}{1+z^2}dz = \int_r^R \frac{(\log x)^2}{1+x^2}dx + \int_0^\pi \frac{(\log R + i\theta)^2}{1+R^2e^{2i\theta}} iRe^{i\theta}d\theta + \int_{-R}^{-r} \frac{(\log|x|+\pi i)^2}{1+x^2}dx + \int_\pi^0 \frac{(\log r + i\theta)^2}{1+r^2e^{2i\theta}} ire^{i\theta}d\theta$$

by residue theorem,

$$\int_\gamma \frac{(l(z))^2}{1+z^2}dz = 2\pi i Res(f,i) = -\frac{1}{4}\pi^3.$$

And

$$\begin{aligned}\int_r^R \frac{(\log x)^2}{1+x^2}dx + \int_{-R}^{-r}\frac{(\log|x|+\pi i)^2}{1+x^2}dx &= \int_r^R \frac{(\log x)^2 + (\log x + \pi i)^2}{1+x^2}dx \\ &= 2\int_r^R \frac{(\log x)^2}{1+x^2} + 2\pi i \int_r^R \frac{\log x}{1+x^2}dx - \pi^2 \int_r^R \frac{1}{1+x^2}dx\end{aligned}$$

when $r \to 0, R \to \infty$, the second term is 0, and the last term is

$$-\pi^2 \int_0^\infty \frac{1}{1+x^2} dx = -\frac{1}{2}\pi^3.$$

Also, if $\rho > 0$ then

$$\begin{aligned}
\left| \rho \int_0^\pi \frac{(\log \rho + i\theta)^2}{1+\rho^2 e^{2i\theta}} e^{i\theta} d\theta \right| &\leq \rho \frac{|\log \rho|^2}{|1-\rho^2|} \int_0^\pi d\theta + 2\rho \frac{|\log \rho|}{|1-\rho^2|} \int_0^\pi \theta d\theta + \frac{\rho}{|1-\rho^2|} \int_0^\pi \theta^2 d\theta \\
&= \rho \frac{|\log \rho|^2}{|1-\rho^2|} \pi + \rho \frac{|\log \rho|}{|1-\rho^2|} \pi^2 + \frac{\rho}{|1-\rho^2|} \frac{\pi^3}{3}
\end{aligned}$$

let $\rho \to 0^+$ or $\rho \to \infty$, these terms will be 0. It is just the same for the three-order case, thus

$$\int_0^\infty \frac{(\log x)^2}{1+x^2} dx = \frac{1}{8}\pi^3,$$

hence

$$\int_0^\infty \frac{(\log x)^3}{1+x^2} dx = 0.$$

**(c)** First,

$$\int_0^\infty \frac{\cos ax}{(1+x^2)^2} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{e^{iax}}{(1+x^2)^2} dx.$$

Let $\gamma = \{|z| = R, Im(z) > 0\} \cup [-R, R]$, then by residue theorem,

$$\int_\gamma \frac{e^{iaz}}{(1+z^2)^2} dz = 2\pi i Res(f, i) = \frac{(a+1)e^{-a}\pi}{2}.$$

And

$$\begin{aligned}
\left| \int_{\gamma_R} \frac{e^{iaz}}{(1+z^2)^2} dz \right| &= \left| \int_0^\pi \frac{e^{-aR\sin\theta}(\cos(aR\cos\theta) + i\sin(aR\cos\theta))}{(1+R^2 e^{2i\theta})^2} iRe^{i\theta} d\theta \right| \\
&\leq \left| \int_0^\pi \frac{R}{|1-R^2|^2} d\theta \right| \to 0, \text{ when } R \to \infty.
\end{aligned}$$

Hence

$$\int_0^\infty \frac{\cos ax}{(1+x^2)^2} dx = \frac{(a+1)e^{-a}\pi}{4}.$$

**(d)** If $z = e^{2i\theta}$ then $\bar{z} = \frac{1}{z}$ and so

$$a + \sin^2\theta = a - \frac{1}{4}(e^{2i\theta} + e^{-2i\theta} - 2) = a - \frac{1}{4}(z + \bar{z} - 2) = -\frac{z^2 - (4a+2)z + 1}{4z}.$$

Hence

$$\int_0^{\pi/2} \frac{d\theta}{a + \sin^2\theta} = \frac{1}{2}\int_0^\pi \frac{d\theta}{a+\sin^2\theta} = i\int_\gamma \frac{1}{z^2 - (4a+2)z + 1} dz,$$

where $\gamma = \{|z| = 1\}$. But $z^2 - (4a+2)z + 1 = (z - z_1)(z - z_2)$ where $z_1 = (1+2a) + 2\sqrt{a^2+a}$, $z_2 = (1+2a) - 2\sqrt{a^2+a}$. Since $a > 0$ we know $|z_1| > 1$ and $|z_2| < 1$, by residue theorem,

$$\int_\gamma \frac{1}{z^2 - (4a+2)z + 1} dz = 2\pi i Res(f, z_1) = \frac{\pi i}{-2\sqrt{a^2+a}}.$$

Hence

$$\int_0^{\pi/2} \frac{d\theta}{a + \sin^2\theta} = \frac{\pi}{2\sqrt{a^2+a}}.$$

**(e)** Define $G, l(z), \gamma$ as in Example 2.10, then

$$\int_\gamma \frac{l(z)}{(1+z^2)^2}dz = \int_r^R \frac{\log z}{(1+x^2)^2}dx + \int_0^\pi \frac{\log R + i\theta}{(1+R^2e^{2i\theta})^2}iRe^{i\theta}d\theta + \int_{-R}^{-r}\frac{\log|x|+\pi i}{(1+x^2)^2}dx + \int_\pi^0 \frac{\log r + i\theta}{(1+r^2e^{2i\theta})^2}ire^{i\theta}d\theta$$

by residue theorem,

$$\int_\gamma \frac{l(z)}{(1+z^2)^2}dz = \frac{(\pi+2i)\pi i}{4}.$$

First,

$$\int_r^R \frac{\log z}{(1+x^2)^2}dx + \int_{-R}^{-r}\frac{\log|x|+\pi i}{(1+x^2)^2}dx = 2\int_r^R \frac{\log z}{(1+x^2)^2}dx + \frac{\pi i}{2}\left(\frac{R}{R^2+1} - \frac{r}{r^2+1} + \arctan R - \arctan r\right)$$

and

$$\left|\int_0^\pi \frac{\log R + i\theta}{(1+R^2e^{2i\theta})^2}iRe^{i\theta}d\theta\right| \le \frac{R|\log R|}{|1-R^2|^2}\int_0^\pi d\theta + \frac{R}{|1-R^2|^2}\int_0^\pi \theta d\theta \to 0,$$

as $R\to\infty$. Also, we know $r\log r \to 0$ as $r\to 0^+$, hence

$$\left|\int_\pi^0 \frac{\log r + i\theta}{(1+r^2e^{2i\theta})^2}ire^{i\theta}d\theta\right| \to 0.$$

Hence,

$$\int_0^\infty \frac{\log z}{(1+x^2)^2}dx = \frac{1}{2}\left(\frac{(\pi+2i)\pi i}{4} - \frac{\pi i}{2}\frac{\pi}{2}\right) = -\frac{\pi}{4}.$$

**(f)**

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{1+x^2}.$$

Let $\gamma = \{|z| = R, Im(z) > 0\} \cup [-R, R]$, then by residue theorem,

$$\int_\gamma \frac{dz}{1+z^2} = 2\pi i Res(f, i) = \pi.$$

But as $R\to\infty$,

$$\left|\int_{\gamma_R}\frac{dz}{1+z^2}\right| = \left|\int_0^\pi \frac{iRe^{i\theta}}{1+R^2e^{2i\theta}}d\theta\right| \le \frac{R}{|1-R^2|}\int_0^\pi d\theta \to 0,$$

hence

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

**(g)** Let $y = e^x$, then

$$\int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}dx = \int_0^\infty \frac{y^{-(1-a)}}{1+y}dy.$$

Since $0 < 1-a < 1$, by Example 2.12, we know

$$\int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}dx = \frac{\pi}{\sin(1-a)\pi} = \frac{\pi}{\sin a\pi}.$$

**(h)** First we have

$$\int_0^{2\pi}\log\sin^2 2\theta d\theta = 2\int_0^{2\pi}\log\sin 2\theta d\theta = 4\int_0^\pi \log\sin\theta d\theta.$$

10. V.2.3

**Sol.** We have proved that $f$ has an essential singularity at $z = 0$. Denote $G$ be the region surrounded by $\gamma$.

i) If $0 \notin G$, then by Cauchy's Thm, $\int_\gamma f = 0$.

ii) If $0 \in G$, then by Laurent expansion, $Res(f, 0) = 1$. But by a problem in last homework, we can construct a $\gamma$, for each $n \in \mathbb{N}$, $n(\gamma, 0) = n$. Hence $\int_\gamma f = 2\pi n i, n \in \mathbb{N}$.

11. V.2.4

**Sol.** In fact, in a neighbourhood of $a$, $f = \frac{c_{-1}}{z-a} + \sum_{n=0}^{\infty} c_n(z-a)^n$, and $g = \sum_{n=0}^{\infty} d_n(z-a)^n$. Hence by definition,

$$Res(fg, a) = c_{-1}d_0 = Res(f, a)g(a).$$

12. V.2.5

**Sol.** Notice in the last problem, each simple pole of $f$ is a simple pole of $fg$. By residue thm,

$$\frac{1}{2\pi i}\int_{\gamma} fg = \sum_{k=1}^{n} n(\gamma, a_k)Res(fg, a_k) = \sum_{k=1}^{n} n(\gamma, a_k)Res(f, a_k)g(a_k).$$