Homework 13

Chuan Lu, 13300180056, chuanlu13@fudan.edu.cn

Problem 1. Prove 4.2.16

Proof. The truncation error at point (t_n, x_i) is

$$\begin{aligned} R_i^n &= \frac{u(t_{n+1}, x_i) - u(t_{n-1}, x_i)}{2\tau} - a\Delta_h u(t_n, x_i) - f(t_n, x_i) \\ &= R_t + R_x, \end{aligned}$$

where

$$\begin{aligned} R_t &= \frac{u(t_{n+1}, x_i) - u(t_{n-1}, x_i)}{2\tau} - \frac{\partial u}{\partial t}(t_n, x_i) \\ &= \frac{\tau^2}{6}\frac{\partial^3 u}{\partial t^3}(t_n, x_i) + O(\tau^4), \\ R_x &= -a\Delta_h u_i^n + a\frac{\partial^2 u}{\partial x^2}(t_n, x_i) \\ &= -\frac{ah^2}{12}\frac{\partial^4 u}{\partial x^4}(t_n, x_i) + O(h^4) \end{aligned}$$

Adding the two terms we can get (4.2.16). □

Problem 2. Problem 1, Page 195

The schemes. The iteration of θ scheme is

$$(\boldsymbol{I} - a\tau\theta\Delta_h)\boldsymbol{u}^{n+1} = (\boldsymbol{I} + a\tau(1-\theta)\Delta_h)\boldsymbol{u}^n + \tau(\theta\boldsymbol{f}^{n+1} + \boldsymbol{f}^n),$$

where $\boldsymbol{u}$ and $\boldsymbol{f}$ are vectors of grid points.
The exact solution is

$$u(t, x) = \sin \pi x e^{-\pi^2 t}.$$

Result. The results are as follows:

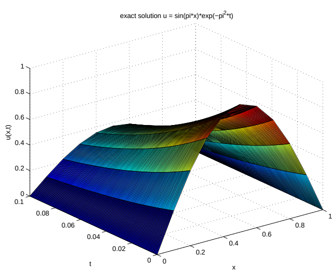

Figure 1: Exact sol.

When $t \to \infty$, the computational result should be the same as the exact solution.

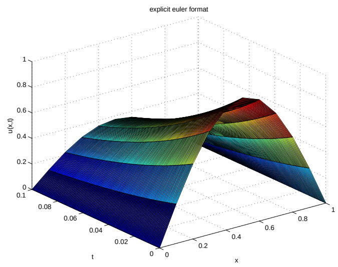

Figure 2: $\theta = 0$

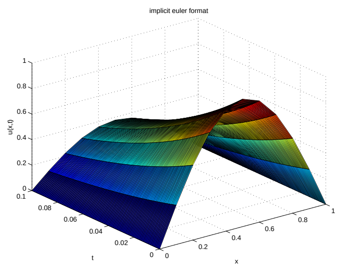

Figure 3: $\theta = 1$

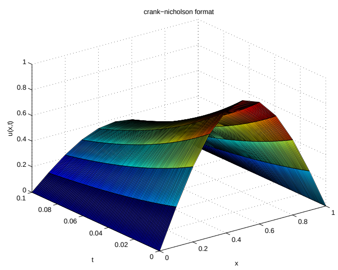

Figure 4: $\theta = \frac{1}{2}$

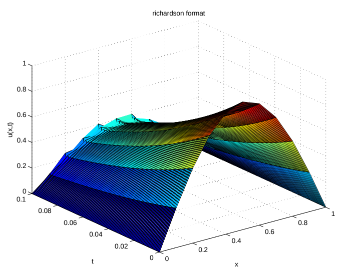

Figure 5: Richardson