Statistics of Solutions to A Stochastic Differential Equation Set

Chuan Lu

Information and Computing Science

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Outline

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Introduction

The SDEs

$$
\begin{cases}
\frac{du(t)}{dt} = (-\gamma(t) + i\omega)u(t) + b(t) + f(t) + \sigma W(t), \\
\frac{db(t)}{dt} = (-\gamma_b + i\omega_b)(b(t) - \hat{b}) + \sigma_b W_b(t), \\
\frac{d\gamma(t)}{dt} = -d_\gamma(\gamma(t) - \hat{\gamma}) + \sigma_\gamma W_\gamma(t)
\end{cases}
$$

The initial values are complex random variables, with their first-order and second-order statistics known.

Solution

With knowledge of ODEs, the solution of the SDE set is

$$\begin{cases} b(t) = \hat{b} + (b_0 - \hat{b})e^{\lambda_b(t-t_0)} + \sigma_b \int_{t_0}^{t} e^{\lambda_b(t-s)} dW_b(s) \\ \gamma(t) = \hat{\gamma} + (\gamma_0 - \hat{\gamma})e^{-d_\gamma(t-t_0)} + \sigma_\gamma \int_{t_0}^{t} e^{-d_\gamma(t-s)} dW_\gamma(s) \\ u(t) = e^{-J(t_0,t)+\hat{\lambda}(t-t_0)} u_0 + \int_{t_0}^{t} (b(s)+f(s)) e^{-J(s,t)+\hat{\lambda}(s-t_0)} ds \\ \qquad + \sigma \int_{t_0}^{t} e^{-J(s,t)+\hat{\lambda}(s-t_0)} dW(s) \end{cases}$$

with $\lambda_b = -\gamma_b + i\omega_b$, $\hat{\lambda} = -\hat{\gamma} + i\omega$, $J(s,t) = \int_s^t (\gamma(s') - \hat{\gamma}) ds'$.

Itô Isometry

$\forall f \in \mathscr{V}(S,T)$, B_t is a standard Brownian motion,

$$\mathsf{E}\left[\left(\int_S^T f(t,\omega)dB_t\right)^2\right] = \mathsf{E}\left[\int_S^T f^2(t,\omega)dt\right].$$

Itô Formula

Assume that X_t is a Itô process satisfying $dX_t = udt + vdB_t$, $g(t,x) \in C^2([0,\infty)\times\mathbb{R})$, then $Y_t = g(t,X_t)$ is also a Itô process satisfying

$$dY_t = \frac{\partial g}{\partial t}(t,X_t)dt + \frac{\partial g}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)\cdot(dX_t)^2,$$

with $dt\cdot dt = dt\cdot dB_t = dB_t\cdot dt = 0, dB_t\cdot dB_t = dt$.

Linear property of Itô integration

$$(1)\quad \int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t, \quad \text{a.e.}$$

$$(2)\quad \int_S^T (cf + g) dB_t = c\int_S^T f dB_t + \int_S^T g dB_t, \quad \text{a.e.}$$

$$(3)\quad \mathsf{E}\left[\int_S^T f dB_t\right] = 0.$$

Outline

Mean

With property of Itô integration (3), it is easy to know

$$
\begin{aligned}
\mathsf{E}(b(t)) &= \hat{b} + (\mathsf{E}[b_0] - \hat{b})e^{\lambda_b(t-t_0)} \\
\mathsf{E}(\gamma(t)) &= \hat{\gamma} + (\mathsf{E}[\gamma_0] - \hat{\gamma})e^{-d_\gamma(t-t_0)}
\end{aligned}
$$

Variance

According to definition,

$$\begin{aligned}\mathsf{Var}(b(t)) &= \mathsf{E}\left[(b(t)-\mathsf{E}[b(t)])(b(t)-\mathsf{E}[b(t)])^*\right] \\ &= e^{-2\gamma_b(t-t_0)}\mathsf{Var}(b_0)+\mathsf{E}\left[\sigma_b^2\int_{t_0}^{t} e^{\lambda_b(t-s)}dW_b(s)\left(\int_{t_0}^{t} e^{\lambda_b(t-s)}dW_b(s)\right)^*\right] \\ &= e^{-2\gamma_b(t-t_0)}\mathsf{Var}(b_0)+\sigma_b^2\mathsf{E}\left[\int_{t_0}^{t} e^{-2\gamma_b(t-s)}ds\right]\end{aligned}$$

The last step takes advantage of Itô isometry.

$$\mathsf{Var}(b(t)) = e^{-2\gamma_b(t-t_0)}\mathsf{Var}(b_0)+\frac{\sigma_b^2}{2\gamma_b}(1-e^{-2\gamma_b(t-t_0)})$$

$$\mathsf{Var}(\gamma(t)) = e^{-2d_\gamma(t-t_0)}\mathsf{Var}(\gamma_0)+\frac{\sigma_\gamma^2}{2d_\gamma}(1-e^{-2d_\gamma(t-t_0)})$$

Covariance

$$
\begin{aligned}
\mathsf{Cov}(b(t), b(t)^*) &= \mathsf{E}\left[(b(t) - \mathsf{E}[b(t)])(b(t)^* - \mathsf{E}[b(t)^*])\right] \\
&= \mathsf{E}\left[(b_0 - \mathsf{E}[b_0])(b_0^* - \mathsf{E}[b_0^*])e^{2\lambda_b(t-t_0)}\right] + \sigma_b \mathsf{E}\left[(b_0 - \mathsf{E}[b_0]) \int_{t_0}^{t} e^{\lambda_b(t-s)} dW_b(s)\right] \\
&+ \sigma_b \mathsf{E}\left[(b_0^* - \mathsf{E}[b_0^*])e^{\lambda_b(t-t_0)} \int_{t_0}^{t} e^{\lambda_b(t-s)} dW_b(s)\right] + \sigma_b^2 \mathsf{E}\left[\left(\int_{t_0}^{t} e^{\lambda_b(t-s)} dW_b(s)\right)^2\right]
\end{aligned}
$$

With property of Itô integration (3), the second and third term are both 0; with Itô isometry we know the last term is also 0.

$$
\mathsf{Cov}(b(t), b(t)^*) = \mathsf{E}[(b_0 - \mathsf{E}[b_0])(b_0^* - \mathsf{E}[b_0^*])]e^{2\lambda_b(t-t_0)} = \mathsf{Cov}(b_0, b_0^*)e^{2\lambda_b(t-t_0)}
$$

$$
\mathsf{Cov}(b(t), \gamma(t)) = \mathsf{E}[(b(t) - \mathsf{E}[b(t)])(\gamma(t) - \mathsf{E}[\gamma(t)])] = \mathsf{Cov}(b_0, \gamma_0)e^{(\lambda_b - d_\gamma)(t-t_0)}
$$

Outline

Mean

Using the same properties, it's obvious to have

$$\mathsf{E}[u(t)] = e^{\hat{\lambda}(t-t_0)}\mathsf{E}\left[e^{-J_0(t_0,t)}u_0\right] + \int_{t_0}^{t} e^{\hat{\lambda}(t-s)}\mathsf{E}\left[b(s)e^{-J(s,t)}\right] ds$$
$$+\sigma \int_{t_0}^{t} e^{\hat{\lambda}(t-s)} f(s)\mathsf{E}\left[e^{-J(s,t)}\right] ds.$$

It is necessary to compute expectations of terms like $\mathsf{E}[ze^{bx}]$, z is a complex-valued Gaussian random variable and x is a real-valued Gaussian variable. We propose two lemmas here.

Lemma 1

Lemma

$$E\left[ze^{ibx}\right] = (E[z] + ib\,Cov(z,x))e^{ibE[x] - \frac{1}{2}b^2 Var(x)}$$

with z being a complex-valued Gaussian, and x a real-valued Gaussian.

Corollary

Under the condition of Lemma 1,

$$E\left[ze^{bx}\right] = (E[z] + b\,Cov(z,x))e^{bE[x] + \frac{1}{2}b^2 Var(x)}.$$

Proof of lemma 1 takes advantage of the characteristic function of multivariable Gaussian distribution.

Proof

Let $z = y + iw, \quad y, w \in \mathbb{R}$. Denote $\mathbf{v} = (x, y, w)$, then $\mathbf{v}$ satisfies the multivariable Gaussian distribution, with its characteristic function

$$\phi_{\mathbf{v}}(\mathbf{s}) = \exp(i\mathbf{s}^\top \mathsf{E}[\mathbf{v}] - \frac{1}{2}\mathbf{s}^\top \Sigma \mathbf{s}).$$

Let $g(\mathbf{v})$ being the PDF of $\mathbf{v}$, then one knows from that char. func. being Fourier transform of PDF,

$$\phi_{\mathbf{v}}(\mathbf{s}) = \frac{1}{(2\pi)^3} \int e^{i\mathbf{s}^\top \mathbf{v}} g(\mathbf{v}) d\mathbf{v}$$

According to the differential property of Fourier transform,

$$\frac{\partial \phi_{\mathbf{v}}(\mathbf{s})}{\partial s_2} = \frac{1}{(2\pi)^3} \int i y_0 e^{i\mathbf{s}^\top \mathbf{v}} g(\mathbf{v}) d\mathbf{v} = i\mathsf{E}\left[y_0 e^{i\mathbf{s}^\top \mathbf{v}}\right].$$

Let $\mathbf{v} = (b, 0, 0)^\top$,

$$\mathsf{E}[y_0 e^{ibx_0}] = -i \frac{\partial \phi_{\mathbf{v}}(s)}{\partial s_2}\bigg|_{\mathbf{s}=(b,0,0)^\top}$$

$$\mathsf{E}[y_0 e^{ibx_0}] = -i \frac{\partial \phi_{\mathbf{v}}(s)}{\partial s_2}\bigg|_{\mathbf{s}=(b,0,0)^\top}$$

Proof

From PDF of multivariable Gaussian distribution, one knows

$$\frac{\partial \phi_{\mathbf{v}}(\mathbf{s})}{\partial s_2} = (i\mathsf{E}[y_0] - \mathsf{Var}(y_0)s_2 - \mathsf{Cov}(x_0, y_0)s_1 - \mathsf{Cov}(y_0, w_0)s_3)\phi_{\mathbf{v}}(s)$$
$$\frac{\partial \phi_{\mathbf{v}}(\mathbf{s})}{\partial s_3} = (i\mathsf{E}[w_0] - \mathsf{Var}(w_0)s_3 - \mathsf{Cov}(x_0, w_0)s_1 - \mathsf{Cov}(y_0, w_0)s_2)\phi_{\mathbf{v}}(s)$$

Compute the partial derivatives at $\mathbf{s} = (b, 0, 0)^\top$,

$$\mathsf{E}\left[y_0 e^{ibx_0}\right] = (\mathsf{E}[y_0] + i\mathsf{Cov}(x_0, y_0)b)\exp(ib\mathsf{E}[x_0] - \frac{1}{2}\mathsf{Var}(x_0)b^2)$$
$$\mathsf{E}\left[w_0 e^{ibx_0}\right] = (\mathsf{E}[w_0] + i\mathsf{Cov}(x_0, w_0)b)\exp(ib\mathsf{E}[x_0] - \frac{1}{2}\mathsf{Var}(x_0)b^2)$$

Then

$$\mathsf{E}\left[z e^{ibx}\right] = (\mathsf{E}[z] + ib\mathsf{Cov}(z, x))e^{ib\mathsf{E}[x] - \frac{1}{2}b^2\mathsf{Var}(x)}.$$

□

Lemma 2

Lemma

$$E\left[zwe^{bx}\right] = \left[E[z]E[w] + Cov(z,w^*) + b(E[z]Cov(w,x)) + E[w]Cov(z,x) + b^2 Cov(z,x)Cov(w,x)\right] e^{bE[x]+\frac{b^2}{2}Var(x)}.$$

with z,w being complex-valued Gaussian, and x real-valued Gaussian.

The proof of this lemma is the same as Lemma 1.

Mean

We now make use of Lemma 1 to obtain the mean of $u(t)$.

$$
\begin{aligned}
\mathsf{E}[u(t)] = & e^{\hat{\lambda}(t-t_0)}(\mathsf{E}[u_0] - \mathsf{Cov}(u_0, J(t_0,t)))e^{-\mathsf{E}[J(t_0,t)]+\frac{1}{2}\mathsf{Var}(J(t_0,t))} \\
& + \int_{t_0}^{t} e^{\hat{\lambda}(t-s)}(\hat{b} + e^{\lambda_b(s-t_0)}(\mathsf{E}[b_0] - \hat{b} - \mathsf{Cov}(b_0, J(s,t))))e^{-\mathsf{E}[J(s,t)]+\frac{1}{2}\mathsf{Var}(J(s,t))}ds \\
& + \int_{t_0}^{t} e^{\hat{\lambda}(t-s)}f(s)e^{-\mathsf{E}[J(s,t)]+\frac{1}{2}\mathsf{Var}(J(s,t))}ds
\end{aligned}
$$

The terms $\mathsf{Cov}(u_0, J(s,t))$, $\mathsf{Cov}(b_0, J(s,t))$, $\mathsf{E}[J(s,t)]$ and $\mathsf{Var}(J(s,t))$ can be found using Itô isometry.

Variance

Denote $u(t) = A + B + C$,

$$\begin{cases} A = e^{-J(t_0,t)+\hat{\lambda}(t-t_0)}u_0, \\ B = \int_{t_0}^{t}(b(s)+f(s))e^{-J(s,t)+\hat{\lambda}(t-s)}ds, \\ C = \sigma\int_{t_0}^{t}e^{-J(s,t)+\hat{\lambda}(t-s)}dW(s). \end{cases}$$

By definition we find $\mathrm{Var}(u(t)) = \mathsf{E}\left[|u(t)|^2\right] - |\mathsf{E}[u(t)]|^2$, with

$$\mathsf{E}\left[|u(t)|^2\right] = \mathsf{E}\left[|A|^2\right] + \mathsf{E}\left[|B|^2\right] + \mathsf{E}\left[|C|^2\right] + 2\mathrm{Re}\{\mathsf{E}[A^*B]\}.$$

We can obtain $\mathsf{E}\left[|A|^2\right]$ by Lemma 2, and $\mathsf{E}\left[|B|^2\right]$ by Itô isometry and noticing that

$$\mathrm{Cov}(J(s,t),J(r,t)) = \mathrm{Var}(J(s,t)) + \mathrm{Cov}(J(s,t),J(r,s)).$$

$\mathsf{E}\left[|C|^2\right]$ and $\mathrm{Re}\{\mathsf{E}[A^*B]\}$ can also be computed by Itô isometry and property of Itô integration.

Covariance

By definition,

$$\begin{aligned}
\mathsf{Cov}(u(t), u^*(t)) &= \mathsf{E}\left[u(t)^2\right] - \mathsf{E}[u(t)]^2 \\
\mathsf{Cov}(u(t), \gamma(t)) &= \mathsf{E}[u(t)(\gamma(t) - \hat{\gamma})] + \mathsf{E}[u(t)](\hat{\gamma} - \mathsf{E}[\gamma(t)]) \\
\mathsf{Cov}(u(t), b(t)) &= \mathsf{E}[u(t)b^*(t)] - \mathsf{E}[u(t)]\mathsf{E}[b(t)]^* \\
\mathsf{Cov}(u(t), b^*(t)) &= \mathsf{E}[u(t)b(t)] - \mathsf{E}[u(t)]\mathsf{E}[b(t)].
\end{aligned}$$

Each term can be obtained by Lemma 3 and Itô isometry.

Outline

Parameters

External forcing $f(t) = \frac{3}{2}e^{0.1it}$, and parameters of the equation set are given

$$\begin{cases} d = 1.5, & d_\gamma = 0.01d \\ \sigma = 0.1549, & \omega = 1.78 \\ \sigma_\gamma = 5\sigma, & \gamma_b = 0.1d \\ \sigma_b = 5\sigma, & \omega_b = \omega \\ \hat{b} = 0, & \hat{\gamma} = 0 \end{cases}$$

We assume the initial values satisfying

$$\begin{cases} \mathrm{Re}(u_0), \mathrm{Im}(u_0) \sim \mathcal{N}(0, \frac{1}{\sqrt{2}}), & \text{i.i.d.} \\ \mathrm{Re}(b_0), \mathrm{Im}(b_0) \sim \mathcal{N}(0, \frac{1}{\sqrt{2}}), & \text{i.i.d.} \\ \gamma_0 \sim \mathcal{N}(0,1) \quad \text{i.i.d.} \end{cases}$$

Parameters

The statistics between initial values are

$$\begin{cases} \text{Cov}(u_0, u_0^*) = 0 \\ \text{Cov}(u_0, \gamma_0) = 0 \\ \text{Cov}(u_0, b_0) = 0 \\ \text{Cov}(u_0, b_0^*) = 0 \end{cases}$$

Euler-Maruyama Scheme

Itô integration can be simulated by E-M scheme:

$$X_j = X_{j-1} + f(X_{j-1})\Delta t + g(X_{t-1})(W(\tau_j) - W(\tau_{j-1})),$$

with

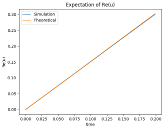
$$W(\tau_j) - W(\tau_{j-1}) = \sum_{k=jR-R+1}^{jR} dW_k,$$

and R being the step length of E-M scheme,

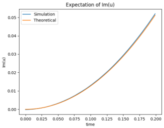
$$dW = \sqrt{\Delta t} * \texttt{randn}().$$

Simulation

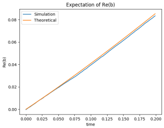
Simulate 100,000 times with $R = 1$, the results are as follows:



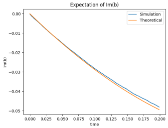
(a) E(Re(u))



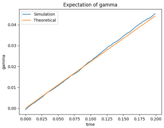
(b) E(Im(u))



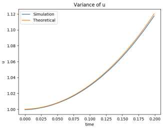
(c) E(Re(b))



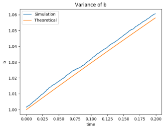
(d) E(Im(b))



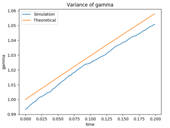
(e) E(γ)



(f) Var(u)



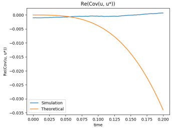
(g) Var(b)



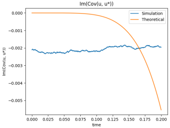
(h) Var(γ)

Figure: Simulation of Expectations and Variances

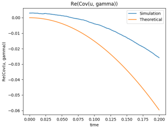
Simulation



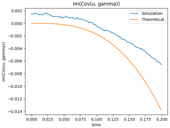
(a) Re(Cov(u, u^*))



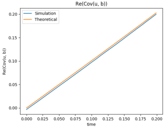
(b) Im(Cov(u, u^*))



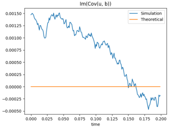
(c) Re(Cov(u, γ))



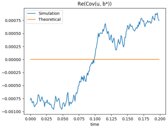
(d) Im(Cov(u, γ))



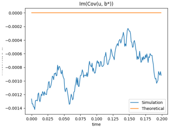
(e) Re(Cov(u, b))



(f) Im(Cov(u, b))



(g) Re(Cov(u, b^*))



(h) Im(Cov(u, b^*))

Figure: Simulation of Covariances

Discussion

From the simulation results we find that the simulation of expectations and variances fit the theoretical results properly; but the simulation of covariances have an error of $O(10^{-3})$. It could be caused by the condition number of multiplication of complex numbers.

Thank you!