Introduction to Analysis I
Homework 2
Thursday, August 31, 2017

Instructions: You may submit this homework "the old fashion" way, i.e., using paper and pencil (or pen), but if you do so, please use at least one sheet of $(8\frac{1}{2} \times 11)$ paper per problem. Write your name at the top of each sheet you use. Please write neatly. Staple the sheets together or use a paper clip.

However, I encourage you to do at least some of the problems using LaTeX. As of the third assignment, you will have to submit your homework in LaTeX.
If you use results from books, Royden or others, please be explicit about what results you are using.

Homework 2 is due by the start of class on Wednesday, September 13.

1. Recall the unconventional terminology I introduced in class: If J is an infinite set and $f : J \to \mathbb{R}$ is a real-valued function, then the symbol $\sum_{j\in J} f(j)$ is the series determined by J, with terms $f(j)$. We say $\sum_{j\in J} f(j)$ *converges to* S *unconditionally* if for every $\epsilon > 0$ there is a finite subset $F_0 \subset J$ such that if F is any finite subset of J that contains F_0, then $|\sum_{j\in F} f(j) - S| < \epsilon$. Show that if $\sum_{j\in J} f(j)$ converges to some S unconditionally, then $\{j \in J \mid f(j) \neq 0\}$ is countable.

   Collaborators:

   Solution: Proof. According to the assumption, $\forall\epsilon > 0$, there is a finite subset F_0, $\forall$ finite $F \supset F_0$, $-\epsilon - (\sum_{j\in F_0} f(j) - S) < \sum_{j\in F\setminus F_0} f(j) < \epsilon - (\sum_{j\in F_0} f(j) - S)$. Denote $\alpha = \epsilon - (\sum_{j\in F_0} f(j) - S)$.
   Suppose that $E = \{j \in J | f(j) \neq 0\}$ is uncountable, denote $E_1 = \{j \in J | f(j) > 0\}$, $E_2 = \{j \in J | f(j) < 0\}$, then $E = E_1 + E_2$. Then either E_1 or E_2 is uncountable, otherwise E is countable. We may assume that E_1 is uncountable. We can construct a sequence of subsets $\{F_n\}$ like this:
   First, $F_1 = \{j \in J | f(j) > \alpha\}$ is countable, otherwise $F_1 \setminus F_0$ is uncountable, which means we can find a finite set from $F_1 \setminus F_0$, s.t. $\sum_{j\in F_1\setminus F_0} f(j) > \alpha$. Similarly, $F_{n+1} = \{j \in J | f(j) > \frac{\alpha}{2^n}\}$ are all countable, which means $G_n = E_1 - F_{n+1}$ is uncountable. However, since the countable sum of countable sequences is also countable, so $\lim_{n\to\infty} G_n = \{j \in J | f(j) \leq 0\}$ is uncountable, and it contradicts to the assumption. So there must exist a $k > 0$, F_k is uncountable, and thus $F_k \setminus F_0$ is uncountable. Thus we can select 2^k elements from F_k, and their sum is larger then α, and it also contradicts with our assumption. ∎

2. Assume now that J is countably infinite and show that the following assertions about a series $\sum_{j\in J} f(j)$ are equivalent:

   (a) The series $\sum_{j\in J} f(j)$ converges unconditionally.

   (b) For each bijection $\phi : \mathbb{N} \to J$, the series $\sum_{k=1}^{\infty} f \circ (\phi(k))$ converges in the sense discussed in the text, and that the sum is the sum of $\sum_{j\in J} f(j)$.

   (c) The series of absolute values $\sum_{j\in J} |f(j)|$ converges.

   Collaborators:

   Solution: Proof. Since J is countably infinite, we may assume $J = \{j_i\}_{i=0}^{\infty}$.
   (a) $\to$ (b): Since $\sum_{j\in J} f(j)$ converges unconditionally, $\forall\epsilon > 0$, $\exists$ a finite set F_0, $\forall F \supset F_0$,

$$\left| \sum_{j\in F} f(j) - \sum_{i=1}^{\infty} f(j_i) \right| < \epsilon.$$

Since ϕ is a bijection, there $\exists N > 0$, $\phi(\{1, 2, \cdots, N\}) \supset F_0$. Denote $\phi(\{1, 2, \cdots, N\}) = F$, then

$$\left|\sum_{i=1}^{N} f(\phi(i)) - \sum_{i=1}^{\infty} f(j_i)\right| = \left|\sum_{i=1}^{N} f(\phi(i)) - \sum_{i=1}^{\infty} f(\phi(j_i))\right| < \epsilon.$$

Thus the series converges, and the sum is $\sum_{j\in J} f(j)$.

(b) $\rightarrow$ (c): According to (b) we have $\sum_{k=1}^{\infty} f(j_k)$ converges. Suppose $\sum_{j\in J} |f(j)|$ does not converge, we first prove that $\sum_{k=1}^{\infty} f(j_k)^+$ and $\sum_{k=1}^{\infty} f(j_k)^-$ do not converge, where $f(j_k)^+, f(j_k)^-$ means the absolute value of positive and negative terms in $f(j)$, seperately.

Suppose $\sum_{k=1}^{\infty} f(j_k)+$ converges, then since

$$\sum_{k=1}^{\infty} f(j_k)^- = \sum_{k=1}^{\infty} f(j_k)^+ - \sum_{k=1}^{\infty} f(j_k),$$

we know that $\sum_{k=1}^{\infty} f(j_k)^-$ converges, which means

$$\sum_{k=1}^{\infty} |f(k_j)| = \sum_{k=1}^{\infty} f(k_j)^+ + \sum_{k=1}^{\infty} f(k_j)^-$$

also converges, which lead to a contradiction.

Thus we can rearrange the order of $\sum_{j\in J} f(j)$ like this: for each $f(a_i) > 0$, put enough $f(b_i) < 0$ after it, making $f(a_i) + \sum_i f(b_i) < 1$. The possibility of this map comes from that $\sum_j f(j)^- \rightarrow \infty$. Then we constructed a bijection from $\mathbb{N} \rightarrow J$. However, $\sum_j \phi(f(j)) < \sum_{j_i} -1 \rightarrow -\infty$, which makes a contradiction. Thus $\sum_j |f(j)|$ converges.

(c) $\rightarrow$ (a): Since the series of absolute values $\sum_{j\in J} |f(j)|$ converges, we have the original series $\sum_{j\in J} |f(j)|$ converges, and mark that sum as S. Then according to the definition, for $\forall \epsilon > 0$, there $\exists N > 0$, $\sum_{i=N}^{\infty} |f(j_i)| < \epsilon$. If we take $F_0 = \{j_1, j_2, \cdots, j_{N-1}\}$, then $\forall$ finite $F \supset F_0$, $\sum_{j\in F\setminus F_0} f(j) \leq \sum_{i=N}^{\infty} |f(j_i)| < \epsilon$. That means (a) holds. $\blacksquare$

3. (a) Show that the series

$$\sum_{n\geq 1, n\neq m} \frac{1}{m^2 - n^2}$$

is convergent and has sum equal to $-\frac{3}{4m^2}$ (decompose the rational fraction $1/(m^2 - x^2)$).

(b) Let $u_{mn} = \frac{1}{m^2-n^2}$ if $m \neq n$, and let $u_{nn} = 0$. Show that

$$\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} u_{mn}\right) = -\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} u_{mn}\right) \neq 0.$$

(c) Explain what, if anything, the computations of parts a) and b) of this problem have to do with Problem 2.

Collaborators:

Solution: Proof. (a)

$$
\begin{aligned}
\sum_{n\neq m}\frac{1}{m^2-n^2} &= -\frac{1}{2m}\sum_{n\neq m}\left(\frac{1}{n-m}-\frac{1}{n+m}\right)\\
&= -\frac{1}{2m}(-\frac{1}{m-1}-\frac{1}{m+1}-\cdots-1-\frac{1}{2m-1}\\
&\quad +1-\frac{1}{2m+1}+\cdots+\frac{1}{m}-\frac{1}{3m}\\
&\quad +\frac{1}{m+1}-\frac{1}{3m+1}+\cdots+\frac{1}{2m}-\frac{1}{4m}+\cdots)\\
&= -\frac{1}{2m}\left(\lim_{n\to\infty}(-\sum_{i=-m}^{m}\frac{1}{n+i})+\frac{1}{m}+\frac{1}{2m}\right)\\
&= -\frac{3}{4m}.
\end{aligned}
$$

(b) According to (a),

$$
\sum_{n=0}^{\infty}u_{mn} = -\frac{3}{4m^2}+\frac{1}{m^2}=\frac{1}{4m^2},\quad \sum_{m=0}^{\infty}u_{mn}=-\frac{1}{4n^2}.
$$

Thus

$$
\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}u_{mn}\right) = -\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}u_{mn}\right)=\sum_{n=1}^{\infty}\frac{1}{4n^2}\neq 0.
$$

(c) The computation of (b) has the same principle as (c)→(b) in prob. 2. ■

4. (Problems 17, 18, and 19, page 43) These problems tie well together, so I thought I would make them one big problem.

   (a) Show that a set E is measurable if and only if for each $\epsilon > 0$, there is a closed set F and an open set $\mathcal{O}$ for which $F \subseteq E \subseteq \mathcal{O}$ and $m^*(\mathcal{O} \sim F) < \epsilon$.

   (b) Suppose E is a set with finite outer measure. Show that there is a G_δ set G such that $E \subseteq G$ and $m^*(E) = m(G)$. Show that E is measurable if and only if there is an F_σ set F contained in E such that $m(F) = m^*(E)$.

   (c) Suppose E is a set with finite outer measure. Show that if E is *not* measurable then there is an open set $\mathcal{O}$ containing E that has finite outer measure for which

$$
m^*(\mathcal{O} \sim E) > m^*(\mathcal{O}) - m^*(E).
$$

Collaborators:

Solution: Proof. (a):

Sufficiency: For $\forall A \subset \mathbb{R}$, first we have $m^*(A) \leq m^*(A\cap E) + m^*(A \cap E^c)$. So we only need to show that $m^*(A) \geq m^*(A\cap E)+m^*(A\cap E^c)$. According to assumption, $m^*(A\cap E)+m^*(A\cap E^c) \leq m^*(A\cap O)+m^*(A\cap F^c) < m^*(A\cap O)+m^*(A\cap O^c)+\epsilon$. Since the open set O is measurable, we have $m^*(A\cap) + m^*(A\cap O^c) = m^*(O)$. Thus by the arbitrariness of ϵ, we know that $m^*(A) = m^*(A\cap E)+m^*(A\cap E^c)$.

Necessity: We first show that $\forall \epsilon > 0$, we can find an open set $O \supset E$, s.t. $m^*(O \setminus E) < \epsilon$.
First, if $m^*(E) < \infty$. Then according to definition of outer measure, $\forall \epsilon > 0$, there is an open cover $\{O_i\}$ of E, s.t.

$$
\sum_i m^*(O_i) < m^*(E) + \epsilon.
$$

Let $O = \bigcup_i O_i$, then $m^*(O) < m^*(E) + \epsilon$, and O is an open set. Thus

$$m^*(O \setminus E) = m^*(E) - m^*(E \cap O) = m^*(O) - m^*(E) < \epsilon.$$

If $m^*(E) = \infty$, define $E_n = \{x \in E : |x| \le n\}$. Then E_n is measurable with finite outer measure, the argument above shows that there is an open set $O_k \supset E_k$, s.t.

$$m^*(O_k \setminus E_k) < \frac{\epsilon}{2^k}.$$

Then $O = \bigcup_k O_k$ is an open set that contains E, and

$$m^*(O \setminus E) = m^*\left(\bigcup_k O_k \setminus E\right) \le \sum_k m^*(O_k \setminus E) \le \sum_k m^*(O_k \setminus E_k) < \epsilon.$$

So there is an open set $O \supset E$, s.t. $m^*(O \setminus E) < \frac{\epsilon}{2}$. Meanwhile, since E^c is measurable, there is an open set $G \supset E^c$, s.t. $m^*(G \setminus E^c) < \frac{\epsilon}{2}$. Thus we have a closed set $F = G^c$, and $O \supset E \supset F$,

$$m^*(O \setminus F) \le m^*(O \setminus E) + m^*(E \setminus F) = m^*(O \setminus E) + m^*(G \setminus E^c) < \epsilon.$$

(b): From (a) we know that $\forall \epsilon > 0$, there exists an open set G and a closed set F, s.t. $G \supset E \supset F$. So

$$m(G \setminus E) \le m(G \setminus F) < \epsilon,\ m(E \setminus F) \le m(G \setminus F) < \epsilon.$$

Thus we can construct two series of sets $\{G_k\}$ and $\{F_k\}$:

$$G_k \supset E \supset F_k,\ m(G_k \setminus E) < \frac{\epsilon}{2^k},\ m(E \setminus F_k) < \frac{\epsilon}{2^k}.$$

Thus $G = \bigcap G_k$ is a G_δ set, and $F = \bigcup F_k$ is a F_σ set, and

$$m(G) - m^*(E) \le m(G \setminus E) < \frac{\epsilon}{2^n},\ m^*(E) - m(F) \le m(E \setminus F) < \frac{\epsilon}{2^n}.$$

Let $n \to \infty$, we have $m^*(E) = m(G) = m(F)$.

On the other hand, suppose there exists a F_σ set F contained in E, s.t. $m(F) = m^*(E)$. Then similar to the process of (a), $\forall \epsilon > 0$, there is an open set $G \supset E$, s.t. $m^*(G \setminus E) < \epsilon$. Thus $m^*(G \setminus F) < m^*(G \setminus E) + m^*(E \setminus F) < \epsilon$. Then E is a measurable set.

(c): Fisrt there exist an open set $O \supset E$ which has finite outer measure. Suppose for each open set O containing E that has finite outer measure,

$$m^*(O \setminus E) \le m^*(O) - m^*(E).$$

Then we have

$$m^*(O \setminus E) = m^*(O) - m^*(E)$$

according to the properties of outer measures. So according to the definition of outer measures, $\forall \epsilon > 0$, there is an open set $O \supset E$, s.t. $m^*(O) - m^*(E) < \epsilon$. This leads to $m^*(O \setminus E) < epsilon$. Accoding to Theorem 11 on Page 40, E is measurable, which makes an contradiction. ■

5. (Problem 28, Page 47) Show that continuity of measure together with finite additivity of measure implies countable additivity of measure.

Collaborators:

Solution: Proof. Let $\{A_k\}_{k=1}^\infty$ be a set of disjoint measurable sets. Denote $B_k = \bigcup_{i=1}^k A_i$, then $\{B_k\}$ is an ascending sequence of measurable sets, and

$$m(B_k) = \sum_{i=1}^k m(A_i).$$

According to the continuity of measure,

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = m\left(\bigcup_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k) = \lim_{k\to\infty} \sum_{n=1}^{n} m(A_k) = \sum_{k=1}^{\infty} m(A_k).$$

It shows the countable additivity of measure. ■