Homework 2

In problems 3. - 5., references such as III.2.7 refer to Problem 7 in Section 2 of Chapter III in Conway's book.

If you use results from books including Conway's, please be explicit about what results you are using.

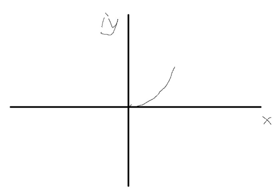
Homework 2 is due on ICON by Midnight, February 11.

1. Problem IV.1.5

   Sol. By Proposition 1.3,

$$\begin{aligned} V(\gamma) &= \int_0^1 |\gamma'(t)|dt = \int_0^1 |\frac{1-i}{t^2}e^{\frac{-1+i}{t}}|dt = \int_0^1 |\frac{1-i}{t^2}e^{-\frac{1}{t}}(\cos\frac{1}{t} + i\sin\frac{1}{t})|dt \\ &= \int_0^1 \frac{e^{-\frac{1}{t}}}{t^2}\sqrt{(\cos\frac{1}{t}+\sin\frac{1}{t})^2 + (\sin\frac{1}{t}-\cos\frac{1}{t})^2}dt = \int_0^1 \sqrt{2}\frac{e^{-\frac{1}{t}}}{t^2}dt \\ &= \sqrt{2}e^{-\frac{1}{t}}\Big|_0^1 = \sqrt{2}e^{-1}. \end{aligned}$$

   Hence γ is rectifiable. The trace looks like the graph below:



2. Problem IV.1.9

   Sol.

$$\int_\gamma \frac{1}{z}dz = \int_0^{2\pi} e^{-int}ine^{int}dt = 2\pi in.$$

3. Problem IV.1.12

   Sol.

$$I(r) = \int_\gamma \frac{e^{iz}}{z}dz = \int_0^{2\pi} \frac{e^{ire^{it}}}{re^{it}}ire^{it}dt = \int_0^{2\pi} ie^{ire^{it}}dt = \int_0^{2\pi} ie^{-r\sin t}(\cos(r\cos t) + i\sin(r\cos t))dt.$$

   Then

$$|I(r)| \le \int_0^{2\pi} |ie^{-r\sin t}(\cos(r\cos t) + i\sin(r\cos t))|dt = \int_0^{2\pi} e^{-r\sin t}dt = \int_0^{\pi} + \int_\pi^{2\pi} e^{-r\sin t}dt.$$

   Pick an arbitrary $\epsilon > 0$, the first term

$$\int_0^\pi e^{-r\sin t}dt = \int_0^\epsilon + \int_\epsilon^{\pi-\epsilon} + \int_{\pi-\epsilon}^{\pi} e^{-r\sin t}dt \le 2\epsilon + (\pi - 2\epsilon)e^{-r\sin\epsilon}.$$

   Then when $r \to \infty$,

$$\int_0^\pi e^{-r\sin t}dt \le 2\epsilon.$$

   By the arbitrariness of ϵ, $\int_0^\pi e^{-r\sin t}dt \to 0$ when $r \to \infty$. It is the same for the second term $\int_\pi^{2\pi}$. Hence $\lim\limits_{r\to\infty} I(r) = 0$.

4. Problem IV.1.13

   Sol (a).

$$\int_\gamma z^{-\frac{1}{2}} dz = \int_0^\pi e^{-\frac{1}{2}it} ie^{it} dt = 2e^{\frac{1}{2}it}\Big|_0^\pi = 2i - 2.$$

   (b).

$$\int_\gamma z^{-\frac{1}{2}} dz = \int_{2\pi}^\pi e^{-\frac{1}{2}it} ie^{it} dt = 2e^{\frac{1}{2}it}\Big|_{2\pi}^\pi = 2i + 2.$$

5. Problem IV.1.14

   Proof. First, assume φ is one-one. Then if φ is not strictly increasing, suppose there exists $x < y \in [a, b]$, s.t. $\varphi(x) \geq \varphi(y)$. If $\varphi(x) = \varphi(y)$, it contradicts with that φ is one-one. So $\varphi(x) > \varphi(y)$. Since $\varphi(x) > c$ (otherwise $\varphi(y) < c$ contradicts with $\varphi([a, b]) \geq c$), by continuity of φ, $\exists z \in [a, x]$, s.t. $\varphi(z) = \varphi(y)$, which makes a contradiction. Thus φ is strictly increasing.

   Now assume φ is strictly increasing, then φ is an injection. Besides, for each $y \in [c, d]$, by continuity of φ, there is a $x \in [a, b]$, s.t. $\varphi(x) = y$. Hence φ is a bijection.

6. Problem IV.1.20

   Sol.

$$\int_\gamma \frac{1}{z^2 - 1} dz = \int_0^{2\pi} \frac{1}{(e^{it} + 1)^2 - 1} ie^{it} dt = \int_0^{2\pi} \frac{i}{2 + e^{it}} dt = \frac{1}{2} i(t + i\ln(2 + e^{it}))\Big|_0^{2\pi} = \pi i$$

7. Problem IV.2.1

   Proof. We need to show that g is continuous at each $t_0 \in [c, d]$. In fact, $\forall \epsilon > 0$, since φ is continuous, $\exists \delta > 0$, when $|t_1 - t_0| < \delta$, we have $|\varphi(s, t_1) - \varphi(s, t_0)| < \frac{\epsilon}{b-a}$. Thus when $|t_1 - t_0| < \delta$,

$$|g(t_1) - g(t_0)| \leq \int_a^b |\varphi(s, t_1) - \varphi(s, t_2)| ds < \int_a^b \frac{\epsilon}{b - a} ds = \epsilon.$$

   Hence g is continuous at t_0.

8. Problem IV.2.2 (Please note. This problem will be used a number of places in the theory we will develop.)

   Proof. First, by the same deduction of last problem, we can show g is continuous. Thus, if we prove that g is differentiable with g' given by

$$g'(z) = \int_\gamma \frac{\partial\varphi}{\partial z}(w, z) dz,$$

   then g' is continuous since $\frac{\partial\varphi}{\partial z}$ is continuous. Hence we only need to prove the formula for g'.

   For a fixed point z_0 in G, let $\epsilon > 0$. Denote $\frac{\partial\varphi}{\partial z}$ by ϕ. Pick a closed set $F \subset G$ s.t. $z_0 \in F$, then since γ is rectifiable, ϕ is uniformly continuous on $\gamma \times F$. Thus, there is a $\delta > 0$ s.t. $|\phi(w', z') - \phi(w, z)| < \epsilon$ when $(w - w')^2 + (z - z')^2 < \delta^2$. In particular,

$$|\phi(w, z) - \phi(w, z_0)| < \epsilon$$

   when $|z - z_0| < \delta$. Hence,

$$|\int_{z_0}^z \phi(w, z) - \phi(w, z_0) dw| \leq \epsilon|z - z_0|.$$

   For a fixed $w \in \gamma$, $\Phi(z) = \varphi(w, z) - z\phi(w, z_0)$ is a primitive of $\phi(w, z) - \phi(w, z_0)$. By Fundamental Thm of Calculus,

$$|\varphi(w, z) - \varphi(w, z_0) - (z - z_0)\phi(w, z_0)| \leq \epsilon|z - z_0|$$

   for each w when $|z - z_0| < \delta$. Hence

$$|\frac{g(z) - g(z_0)}{z - z_0} - \int_\gamma \phi(w, z_0) dw| \leq \epsilon \int_\gamma 1 ds.$$

   when $0 < |z - z_0| < \delta$. Hence by arbitrariness of ϵ we proved the proposition.

9. Problem IV.2.3

Proof. Let $\phi = \frac{\varphi(w)}{w-z}$, then $\frac{\partial \phi}{\partial z} = (w-z)^{-2}\varphi(w)$ is continuous on $\mathbb{C} - \{\gamma\}$. Thus by exercise 2 we know g is analytic, and

$$g'(z) = \int_\gamma \varphi(w)(w-z)^{-2} dw.$$

Now suppose for $k \leq n$,

$$g^{(k)}(z) = n! \int_\gamma \varphi(w)(w-z)^{-(n+1)} dw,$$

then

$$g^{(n+1)}(z) = (n+1)n! \int_\gamma \varphi(w)(w-z)^{-(n+2)} dw = (n+1)! \int_\gamma \varphi(w)(w-z)^{-(n+2)} dw.$$

By induction we know the result is correct.