Numerical Analysis
Assignment 11

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November 4, 2017

Problem 1. Problem 4.30

Solution. For Legendre approximation, first we have

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1),\ P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

Then

$$(f, P_0) = \sqrt{\frac{1}{2}} \int_{-1}^{1} f(x)P_0(x)dx = 0,$$

$$(f, P_1) = \sqrt{\frac{3}{2}} \int_{-1}^{1} f(x)P_1(x)dx = \frac{4\sqrt{6}}{\pi^2},$$

$$(f, P_2) = \sqrt{\frac{5}{2}} \int_{-1}^{1} f(x)P_2(x)dx = 0,$$

$$(f, P_3) = \sqrt{\frac{7}{2}} \int_{-1}^{1} f(x)P_3(x)dx = \sqrt{\frac{7}{2}} \frac{48}{\pi^2}(1 - \frac{10}{\pi^2}),$$

thus

$$\begin{aligned} p_3(x) &= c_0P_0 + c_1P_1 + c_2P_2 + c_3P_3 \\ &= \sqrt{\frac{7}{2}} \frac{120}{\pi^2}(1 - \frac{10}{\pi^2})x^3 + \frac{8}{\pi^2}(\sqrt{\frac{3}{2}} - 9\sqrt{\frac{7}{2}}(1 - \frac{10}{\pi^2}))x \end{aligned}$$

For Chebyshev approximation,

$$T_0(x) = 1,\ T_1(x) = x,\ T_2(x) = 2x^2 - 1,\ T_3(x) = 4x^3 - 3x$$

Then if we use Simpson's rule to compute the numerical integration (we cannot use midpoint rule or trapezoidal in this case, which will lead to large error),

$$(f, T_0) = \frac{1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} f(x)dx = \frac{1}{\pi} \int_{0}^{\pi} f(\cos\theta)d\theta = \frac{1}{6\pi}(f(\cos 0) + f(\cos\frac{\pi}{2}) + f(\cos\pi))\pi = 0,$$

$$(f, T_1) = \frac{2}{\pi} \int_{-1}^{1} \frac{x}{\sqrt{1 - x^2}} f(x)dx = \frac{2}{\pi} \int_{0}^{\pi} \cos\theta f(\cos\theta)d\theta = \frac{1}{3\pi}(\cos 0 f(\cos 0) + \cos\frac{\pi}{2} f(\cos\frac{\pi}{2}) \cos\pi f(\cos\pi))\pi = \frac{1}{3},$$

$$(f, T_2) = \frac{2}{\pi} \int_{-1}^{1} \frac{2x^2 - 1}{\sqrt{1 - x^2}} f(x)dx = \frac{2}{\pi} \int_{0}^{\pi} \cos 2\theta f(\cos\theta)d\theta = \frac{1}{3\pi}(\cos 0 f(\cos 0) + \cos\pi f(\cos\frac{\pi}{2}) \cos 2\pi f(\cos\pi))\pi = 0,$$

$$(f, T_3) = \frac{2}{\pi} \int_{-1}^{1} \frac{4x^3 - 3x}{\sqrt{1 - x^2}} f(x)dx = \frac{2}{\pi} \int_{0}^{\pi} \cos 3\theta f(\cos\theta)d\theta = \frac{1}{3\pi}(\cos 0 f(\cos 0) + \cos\frac{3\pi}{2} f(\cos\frac{\pi}{2}) + \cos 3\pi f(\cos\pi))\pi = \frac{1}{3},$$

thus

$$t_3(x) = \frac{4}{3}x^3 - \frac{2}{3}x.$$

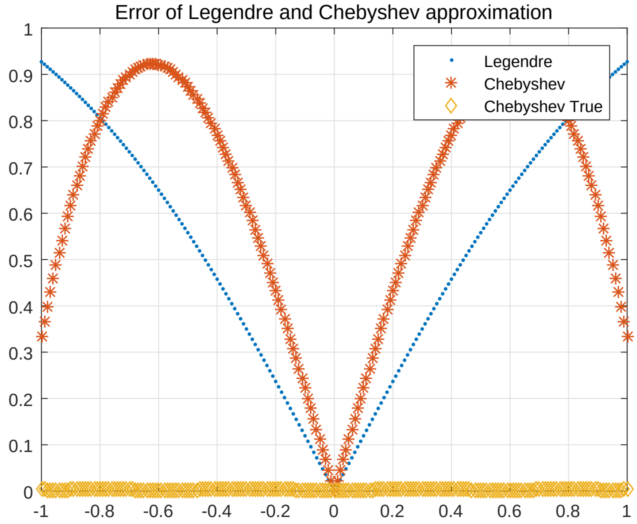
In fact, if we integrate the coefficients using higher order method, we know

$$c_0 = 0,\ c_1 = 1.134,\ c_2 = 0,\ c_3 = -0.138,$$

thus

$$\hat{t_3}(x) = -0.552x^3 + 1.548x.$$

The error of approximation is shown as follows:



Use the result of True Chebyshev approximation $\hat{t_3}(x)$, we can find maximum of errors satisfying Theorem 9:

x	error
-1.0	-0.004
-0.86	0.004
-0.38	-0.004
0.39	0.004
0.88	-0.004
1.0	0.004

And the maxima of error using True Chebyshev approximation is 0.0047, it shows

$$0.004 \leq \rho_3(f) \leq 0.0047.$$

Problem 2. Problem 4.31

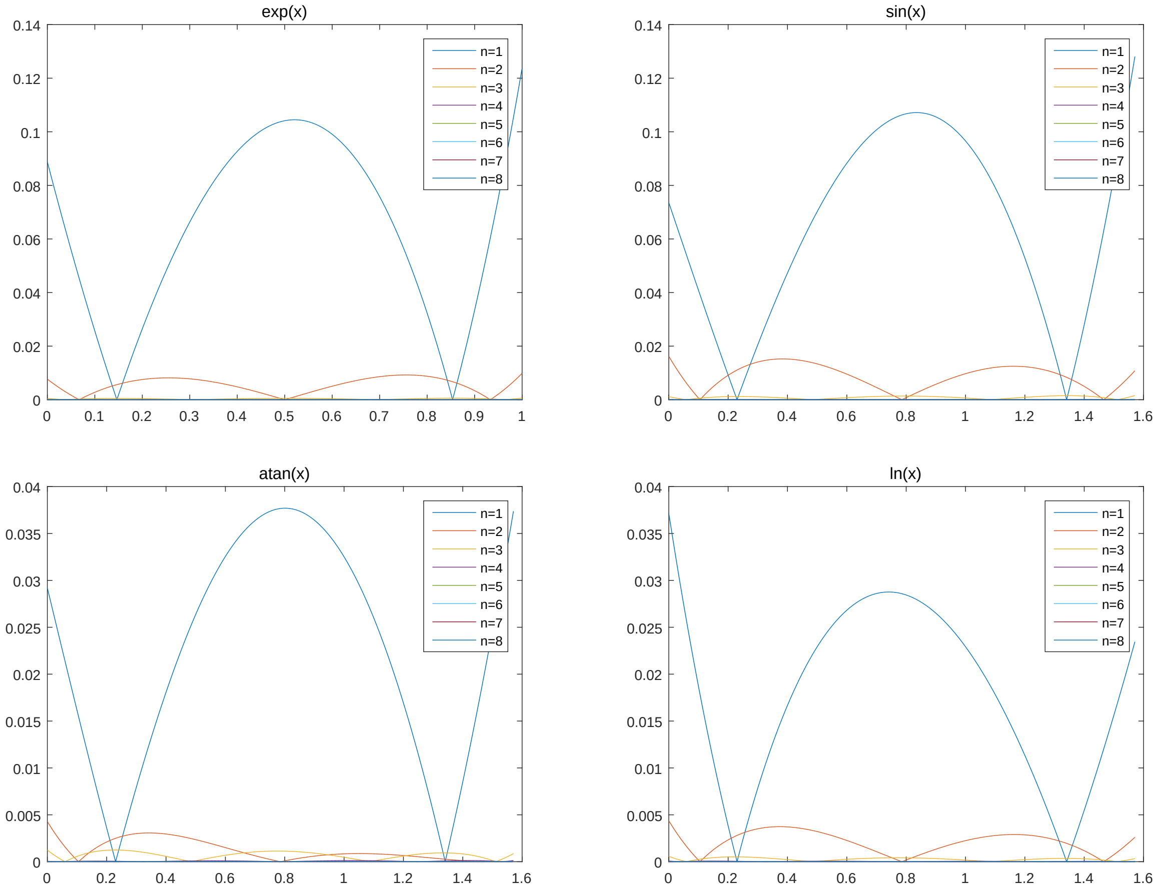
code. By Lagrange form, we can compute the near minimax approximations using the code:

```
function res = near_minimax(f, g, n, xx)
% near minimax approximation using lagrange form
% f: the function to approximate
% g: the map of [0, 1] to the desired interval
% n: order
% xx: eval points

x = feval(g, cos((2*(0:n)+1)./(2*n+2)*pi));
xx = feval(g, xx);
res = zeros(1, length(xx));

for i = 1:(n+1)
        tmp = zeros(1, length(xx))+1;
        for j = 1:(n+1)
            if j ~= i
                tmp = tmp.*(xx - x(j))./(x(i)-x(j));
            end
        end
        res = res + tmp*feval(f, x(i));
end
```

```
f1 = @(x) exp(x);
g1 = @(x) (x+1)/2;
f2 = @(x) sin(x);
g2 = @(x) (x+1)*pi/4;
```

```
f3 = @(x) atan(x);
g3 = @(x) (x+1)/2;
f4 = @(x) log(x);
g4 = @(x) (x+3)/2;

xx = -1:1e-4:1;

%%%%%%%%%% (a) %%%%%%%%
figure(1);
error1 = zeros(8, 1);
for i = 1:8
    y1 = f1(g1(xx));
    y2 = near_minimax(f1, g1, i, xx);
    plot(g1(xx), abs(y1-y2));
    error1(i) = max(abs(y1-y2));
    hold on;
end
legend('n=1', 'n=2', 'n=3', 'n=4', 'n=5', 'n=6', 'n=7', 'n=8');
title('exp(x)');

%%%%%%%%% (b) %%%%%%%
figure(2);
error2 = zeros(8, 1);
for i = 1:8
    y1 = f2(g2(xx));
    y2 = near_minimax(f2, g2, i, xx);
    plot(g2(xx), abs(y1-y2));
    error2(i) = max(abs(y1-y2));
    hold on;
end
legend('n=1', 'n=2', 'n=3', 'n=4', 'n=5', 'n=6', 'n=7', 'n=8');
title('sin(x)');

%%%%%%%%% (c) %%%%%%
figure(3);
error3 = zeros(8, 1);
for i = 1:8
    y1 = f3(g3(xx));
    y2 = near_minimax(f3, g3, i, xx);
    plot(g2(xx), abs(y1-y2));
    error3(i) = max(abs(y1-y2));
    hold on;
end
legend('n=1', 'n=2', 'n=3', 'n=4', 'n=5', 'n=6', 'n=7', 'n=8');
title('atan(x)');

%%%%%%%% (d) %%%%%%%%%
figure(4);
error4 = zeros(8, 1);
for i = 1:8
    y1 = f4(g4(xx));
    y2 = near_minimax(f4, g4, i, xx);
    plot(g2(xx), abs(y1-y2));
    error4(i) = max(abs(y1-y2));
    hold on;
end
legend('n=1', 'n=2', 'n=3', 'n=4', 'n=5', 'n=6', 'n=7', 'n=8');
title('ln(x)');
```

Result. The graph of error, and the max error of infinite norm are as below, and for each n and function f, using Theorem 4.9, $\rho_n(f)$ is less or equal than the infinite norm of error. The left bound is just too difficult to find.




```
error1 =

   0.123795135763495
   0.009865548570435
   0.000600007042634
   0.000029454776569
   0.000001211208768
   0.000000042830298
   0.000000001328131
   0.000000000036665

error2 =

   0.128085593199313
   0.016221586055544
   0.001558351287206
   0.000120525677443
   0.000007798442861
   0.000000433620299
   0.000000021134835
   0.000000000916854

error3 =

   0.037696680264831
   0.004285211299804
   0.001249669440946
   0.000128958381678
   0.000031926143639
   0.000006935414046
   0.000000610998106
```

```
    0.000000258964760

error4 =

    0.037165432256798
    0.004372493413419
    0.000572167228374
    0.000079420776488
    0.000011447097561
    0.000001693662660
    0.000000255467302
    0.000000039109056
```

Problem 3. Problem 4.34

Sol.

$$f(x) - I_n(x) = f(x) - C_n(x) + C_n(x) - I_n(x) = f(x) - C_n(x) - \sum_{j=0}^{n}(f(x_j) - C_n(x_j))l_j(x)$$
$$= g(x) - \sum_{j=0}^{n} g(x_j)l_j(x),$$

where

$$g(x) = f(x) - C_n(x) = c_{n+1}T_{n+1}(x) + \sum_{j=n+2}^{\infty} c_j T_j(x).$$

Since x_j are roots of $T_{n+1}(x)$, and

$$\sum_{j=n+2}^{\infty} c_j T_j(x) \ll c_{n+1}T_{n+1}(x),$$

then there exists $\alpha_n,\ \beta_n$, s.t.

$$\alpha_n|T_{n+1}| \le |f - I_n| \le \beta_n|T_{n+1}|.$$

Using (4.7.28),

$$\rho_n(f) \le \|f - I_n\|_\infty \le \frac{1}{(n+1)!2^n}\|f^{(n+1)}\|_\infty \le \frac{1}{(n+1)!2^n}.$$

On the other hand, using (4.7.30),

$$\rho_n(f) \ge \frac{1}{2 + \frac{2}{\pi}\log(n+1)}\|f - I_n\|_\infty \ge \frac{\alpha_n}{2 + \frac{2}{\pi}\log(n+1)} \max|T_{n+1}| = \frac{\alpha_n}{2 + \frac{2}{\pi}\log(n+1)}.$$

Problem 4. Problem 4.37

Sol. We can get the result from (4.7.48) and (4.7.39). Denote $f(x) = x^6 - x^3$.
When $n = 0$,

$$x_0 = 1,\ x_1 = -1,\ E_0 = \frac{1}{2}(f(x_0) - f(x_1)) = -1.$$

When $n = 1$,

$$x_0 = 1,\ x_1 = 0,\ x_2 = -1,\ E_1 = \frac{1}{2}(\frac{1}{2}(f(x_0) + f(x_2)) - f(x_1)) = \frac{1}{2}.$$

When $n = 2$,

$$x_0 = 1,\ x_1 = \frac{1}{2},\ x_2 = -\frac{1}{2},\ x_3 = -1,\ E_2 = \frac{1}{3}(\frac{1}{2}(f(x_0) - f(x_3)) - f(x_1) + f(x_2)) = -\frac{1}{4}.$$

When $n = 3$,

$$x_0 = 1,\ x_1 = \frac{\sqrt{2}}{2},\ x_2 = 0,\ x_3 = -\frac{\sqrt{2}}{2},\ x_4 = -1,\ E_3 = \frac{1}{4}(\frac{1}{2}(f(x_0) + f(x_4)) - f(x_1) + f(x_2) - f(x_3)) = \frac{3}{16}.$$

When $n = 4$,

$$x_0 = 1,\ x_1 = \cos(\pi/5),\ x_2 = \cos(2\pi/5),\ x_3 = \cos(3\pi/5),\ x_4 = \cos(4\pi/5),\ x_5 = -1,$$

$$E_4 = \frac{1}{5}(\frac{1}{2}(f_0 - f_5) - f_1 + f_2 - f_3 + f_4) = 0.$$

When $n = 5$,

$$x_0 = 1,\ x_1 = \frac{\sqrt{3}}{2},\ x_2 = \frac{1}{2},\ x_3 = 0,\ x_4 = -\frac{1}{2},\ x_5 = \frac{\sqrt{3}}{2},\ x_6 = -1,$$

$$E_5 = \frac{1}{6}(\frac{1}{2}(f_0 + f_6) - f_1 + f_2 - f_3 + f_4 - f_5) = \frac{1}{32}.$$

Thus $\alpha = 0$, and

$$c_{4,0} = \frac{2}{5}(\frac{1}{2}(f_0 \cos 0 + f_5 \cos 0) + f_1 \cos 0 + f_2 \cos 0 + f_3 \cos 0 + f_4 \cos 0) = \frac{5}{8},$$

$$c_{4,1} = \frac{2}{5}(\frac{1}{2}(f_0 \cos 0 + f_5 \cos \pi) + f_1 \cos(\pi/5) + f_2 \cos(2\pi/5) + f_3 \cos(3\pi/5) + f_4 \cos(4\pi/5)) = -\frac{3}{4},$$

$$c_{4,2} = \frac{2}{5}(\frac{1}{2}(f_0 \cos 0 + f_5 \cos 2\pi) + f_1 \cos(2\pi/5) + f_2 \cos(4\pi/5) + f_3 \cos(6\pi/5) + f_4 \cos(8\pi/5)) = \frac{15}{32},$$

$$c_{4,3} = \frac{2}{5}(\frac{1}{2}(f_0 \cos 0 + f_5 \cos 3\pi) + f_1 \cos(3\pi/5) + f_2 \cos(6\pi/5) + f_3 \cos(9\pi/5) + f_4 \cos(12\pi/5)) = -\frac{1}{4},$$

$$c_{4,4} = \frac{2}{5}(\frac{1}{2}(f_0 \cos 0 + f_5 \cos 4\pi) + f_1 \cos(4\pi/5) + f_2 \cos(8\pi/5) + f_3 \cos(12\pi/5) + f_4 \cos(16\pi/5)) = \frac{7}{32}$$

$$c_{4,5} = \frac{2}{5}(\frac{1}{2}(f_0 \cos 0 + f_5 \cos 5\pi) + f_1 \cos(5\pi/5) + f_2 \cos(10\pi/5) + f_3 \cos(15\pi/5) + f_4 \cos(20\pi/5)) = 0.$$

Thus

$$p_5(x) = \frac{5}{8} - \frac{3}{4}x + \frac{15}{32}(2x^2 - 1) - \frac{1}{4}(4x^3 - 3x) + \frac{7}{32}(8x^4 - 8x^2 + 1) = \frac{7}{4}x^4 - x^3 - \frac{13}{16}x^2 + \frac{3}{8}.$$