Introduction to Analysis
Assignment 6

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Problem 1. Problem 29, Section 4.4, Page 89

Sol. Both are not true. We can construct a f like this:

$$f(x) = \begin{cases} 1 + \dfrac{1}{n^2},\ n \le x < n + \dfrac{1}{2},\ \forall n \in \mathbb{N} \\ -1,\ n + \dfrac{1}{2} \le x < n + 1,\ \forall n \in \mathbb{N} \end{cases}$$

Then f is measurable, and f is bounded on any bounded set, and

$$a_n = \int_n^{n+1} f = \frac{1}{2n^2}.$$

Clearly the series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{2n^2}$ converges absolutely, but

$$\int_1^{\infty} |f| = \sum_{n=1}^{\infty} (1 + \frac{1}{2n^2}) = \infty,$$

which means f is not integrable on $[1, \infty)$.

Problem 2. Problem 33, Section 4.4, Page 90

Proof. First,

$$|f_n - f| \le |f| + |f_n|,\ \forall n.$$

Then since f is integrable on E, if $\lim\limits_{n\to\infty} \int_E |f_n| = \int_E |f|$, we know $|f_n| + |f|$ converges pointwise a.e. to $2|f|$, and

$$\lim_{n\to\infty} \int_E (|f_n| + |f|) = 2 \int_E |f| < \infty,$$

with General Lebesgue Dominated Convergence Theorem, notice $|f_n - f|$ converges pointwise a.e. to 0,

$$\lim_{n\to\infty} \int_E |f_n - f| = \int_E 0 = 0.$$

On the other hand, notice

$$|f_n| - |f| \le |f_n - f|,\ \forall n.$$

with the same method, since $\int_E |f - f_n| \to 0$, and $|f - f_n|$ converges pointwise to 0,

$$\lim_{n\to\infty} \int_E |f_n - f| = \int_E 0 = 0,$$

we know from $|f_n| - |f|$ converges pointwise a.e. to 0,

$$\lim_{n\to\infty} \int_E |f_n| - |f| = \int_E 0 = 0.$$

Hence

$$\lim_{n\to\infty} \int_E |f_n| = \int_E |f|.$$

Problem 3. Problem 35, Section 4.4, Page 90

Proof. Denote $f_n(x) = f(x, a_n)$, in which $\{a_n\}$ is any series which converges to 0. Then from the condition we know $f_n(x)$ converges pointwise to $f(x)$, and $|f_n(x)| \le g(x)$. Then using Lebesgue Dominated Convergence Theorem, since g is integrable on $[0,1]$, we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx.$$

It shows that

$$\limsup_{y\to 0} \int_0^1 f(x,y)dx = \liminf_{y\to 0} \int_0^1 f(x,y)dx = \int_0^1 f(x)dx,$$

whic means

$$\lim_{y\to 0} \int_0^1 f(x,y)dx = \int_0^1 f(x)dx.$$

For the continuity of h, we need to show that $\forall y_0 \in [0,1]$, $\forall \epsilon > 0$, $\exists \delta > 0$, when $|y - y_0| < \delta$, we have $h(y) - h(y_0) = |\int_0^1 f(x,y)dx - \int_0^1 f(x,y_0)dx| < \epsilon$. Since $f(x,y)$ is continuous in y for each x in a closed set, then $f(x,y)$ is uniformly bounded, then for each x, $\exists \delta_1$, when $|y - y_0| < \delta_1$, $|f(x,y) - f(x,y_0)| < \epsilon$. Then

$$|h(y) - h(y_0)| = \left|\int_0^1 f(x,y)dx - \int_0^1 f(x,y_0)dx\right| \le \int_0^1 |f(x,y) - f(x,y_0)|dx < \epsilon.$$

We know the continuity of h since we can pick $\delta = \delta_1$.

Problem 4. Problem 36, Section 4.4, Page 90

Proof. For any fixed $y \in [0,1]$, suppose $\{h_n\}$ is a sequence with $h_n \to 0$. Let

$$f_n(x) = \frac{f(x, y + h_n) - f(x,y)}{h_n}$$

Since $\partial f / \partial y$ exists,

$$\lim_{n\to\infty} f_n(x) = \lim_{h\to 0} \frac{f(x, y+h) - f(x,y)}{h} = \frac{\partial f}{\partial y}(x,y).$$

It means $f_n(x)$ converges pointwise to $\frac{\partial f}{\partial y}(x,y)$. Thus

$$\exists N > 0,\ \forall n > N, \left| f_n(x) - \frac{\partial f}{\partial y}(x,y)\right| < 1.$$

Since

$$\left|\frac{\partial f}{\partial y}(x,y)\right| \le g(x)$$

we have

$$|f_n(x)| \le g(x) + 1,$$

and $g(x) + 1$ is integrable on $[0,1]$. By Lebesgue Dominated Convergence Theorem,

$$\int_0^1 f_n(x)dx \to \int_0^1 \frac{\partial f}{\partial y}(x,y)dx.$$

Since $\{h_n\}$ is arbitrary, and f_n is integrable, we know

$$\limsup_{n\to\infty} \int_0^1 f_n(x)dx = \liminf_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{h\to 0} \frac{1}{h}\left(\int_0^1 f(x,y+h)dx - \int_0^1 f(x,y)dx\right) = \frac{d}{dy}\int_0^1 f(x,y)dx.$$

Problem 5. Problem 38, Section 4.5, Page 91

(i).

$$\lim_{n\to\infty} \int_1^n f dx = \lim_{n\to\infty} \sum_{m=1}^{n} \frac{(-1)^m}{m} = -\ln 2,$$

But

$$\int_1^\infty f^+ = \sum_{m=1}^\infty \frac{1}{2m} = \infty,$$

So f is not integrable.

(ii).

$$\lim_{n\to\infty} \int_1^n f = \int_1^\infty \frac{\sin x}{x} dx,$$

with Dirichlet's Criterion, we know this integral converges. But

$$\int_1^\infty |f| \geq \int_1^\infty \frac{1}{2x} dx - \int_1^\infty \frac{\cos 2x}{x} dx,$$

and the second term converges with Dirichlet's Criterion, but the first term $\to \infty$, we know this integral diverges to ∞. Thus f is not integrable.

This two counterexamples do not contradict to the continuity: f is not integrable over the whole set $E = [1, \infty)$.

Problem 6. Problem 39, Section 4.5, Page 91

Proof (i). Denote

$$F_1 = E_1,\ F_n = E_n \setminus \bigcup_{m=1}^{n-1} E_m,\ n \geq 2.$$

Then $\{F_i\}$ is a sequence of disjoint measurable subsets of E. Then using Theorem 20,

$$\int_{\cup_{n=1}^\infty E_n} f = \sum_{n=1}^\infty \int_{F_n} f = \lim_{n\to\infty} \sum_{m=1}^n \int_{F_m} f = \lim_{n\to\infty} \int_{E_n} f.$$

Proof of (ii). Using the same method with (i), only changing F to

$$F_1 = E_1,\ F_n = E_1 \setminus \bigcup_{m=1}^{n-1} E_m,\ n \geq 2.$$

The other parts of proof is just the same.

Problem 7. Problem 44, Section 4.6, Page 95

(i). First, when f is nonnegative, from the definition of integrable functions we know for any $\epsilon > 0$, there is a bounded measurable function with finite support $0 \leq h(x) \leq f(x)$, s.t.

$$\int_{\mathbb{R}} (f - h) = \int_{\mathbb{R}} f - \int_{\mathbb{R}} h \leq \frac{1}{2}\epsilon.$$

Let the support set of h be M with $m(M) < \infty$. Then using Simple Approximation Theorem, there is a simple function η on M, s.t. $0 \leq h - \eta \leq \frac{\epsilon}{2m(M)}$. Let $\eta = 0$ on $\mathbb{R} \setminus M$, then η has finite support, and

$$\int_{\mathbb{R}} |f - \eta| = \int_{\mathbb{R}} (f - h + h - \eta) = \int_{\mathbb{R}} f - h + \int_M h - \eta \leq \frac{1}{2}\epsilon + m(M)\frac{\epsilon}{2m(M)} = \epsilon.$$

When f is an arbitrary integrable function, $f^+,\ f^-$ are nonnegative functions. Let $E_+ = \{x \mid f^+ > 0\},\ E_- = \{x \mid f^- > 0\}$, then there exists nonnegative simple functions $\eta^+,\ \eta^-$, s.t. $\eta^+ = 0$ on $\mathbb{R} \setminus E_+$, and η^+ has finite support on E_+, $\eta^- = 0$ on $\mathbb{R} \setminus E_-$, and η^- has finite support on E_-, and they satisfies

$$\int_{\mathbb{R}} |f^+ - \eta^+| < \epsilon,\ \int_{\mathbb{R}} |f^- - \eta^-| < \epsilon.$$

Let

$$\eta = \begin{cases} \eta^+,\ x \in E_+ \\ \eta^-,\ x \in E_- \end{cases},$$

then $\eta = \eta^+ - \eta^-$ is a simple function with finite support, and

$$\int_{\mathbb{R}} |f - \eta| = \int_{\mathbb{R}} |f^+ - f^- - (\eta^+ - \eta^-)| \leq \int_{\mathbb{R}} |f^+ - \eta^+| + \int_{\mathbb{R}} |f^- - \eta^-| < 2\epsilon.$$

(ii). From (i) we know there is a simple function η which has finite support (denoted as E) and $\int_{\mathbb{R}} |f-\eta| < \epsilon$. Since E is a measurable set of finite measure, with Lemma 22, for any $\delta_1 > 0$, there is a $n > 0$,

$$m(E \cap (\mathbb{R} \setminus [-n, n])) < \delta_1.$$

Let $I = [-n, n]$, using the result of Problem 3.18, for any $\delta_2 > 0$, there is a step function s on I, and a close set $F \subset I$, s.t. $|\eta - s| < \delta_2$ on F, and $m(I \setminus F) < \delta_2$. Set $s(x) = 0$ for x outside I.
With Proposition 23, for each $\epsilon > 0$, $\exists \delta > 0$, if $m(A) < \delta$, then $\int_A |f| < \epsilon$. Let $\delta_1 = \delta$, $\delta_2 = \min(\delta, \frac{\epsilon}{2n})$, then

$$\begin{aligned}\int_{\mathbb{R}} |f-s| &\le \int_{\mathbb{R}} |f-\eta| + \int_{\mathbb{R}} |\eta - s| < \epsilon + \int_F |\eta - s| + \int_{I\setminus F} |\eta - s| + \int_{\mathbb{R}\setminus I} |\eta - s| \\ &< \epsilon + 2n\frac{\epsilon}{2n} + \epsilon + \epsilon = 4\epsilon.\end{aligned}$$

(iii). Use Lusin's Theorem, the proof is the same with (ii), and we only need to change the step function to continuous function.

Problem 8. Problem 25, Section 18.2, Page

Solution. With η being the counting measure, suppose nonnegative f on $\mathbb{N}$, and let nonnegative $f_n(x)$ be

$$f_n(k) = \begin{cases} f(k), \ 0 \le k \le n \\ 0, \ k > n \end{cases}$$

then $f_n \to f$ pointwise on $\mathbb{N}$, and $\{f_n\}$ is increasing. So by Monotone Convergence Thm,

$$\lim_{n\to\infty} \int_{\mathbb{N}} f_n = \int_{\mathbb{N}} f.$$

Since

$$\int_{\mathbb{N}} f_n = \sum_{i=1}^{n} \int_{\{i\}} f_n + \int_{\{i \ge n\}} f_n = \sum_{i=1}^{n} \int_{\{i\}} f_n = \sum_{i=1}^{n} f_n(i),$$

we have

$$\int_{\mathbb{N}} f = \sum_{n=0}^{\infty} f(n) < \infty.$$

Problem 9. Problem 26, Section 18.2, Page

Solution. With the definition of Dirac measure, let $g \equiv f(x_0)$, then

$$m(\{g = f\}) = m_{\delta_{x_0}}(\{x \in X \mid f(x) = g(x)\}) = 1 = m(X).$$

Then $f = g$, a.e. on X.

$$\int_X f(x) d\delta_{x_0} = \int_X g d\delta_{x_0} = f(x_0) \int_X 1 d\delta_{x_0} = f(x_0) < \infty.$$

Problem 10. Problem 27, Section 18.3, Page

(i). First we show that if f is integrable on X, then the set

$$F = \{x \mid f(x) \neq 0\}$$

is countable. Otherwise, since f is integrable, it means f^+ and f^- are both integrable. We now assume f is nonnegative, then

$$F = \bigcup_{n=1}^{\infty} \{x \mid f(x) \ge \frac{1}{n}\}.$$

If F is uncountable, then there must exist n, s.t. $F_n = \{x \mid f(x) \ge \frac{1}{n}\}$ is uncountable, and $m(F_n) = \infty$. Thus

$$\int_X f d\eta \ge \int_{F_n} f d\eta \ge \frac{1}{n} \times \infty = \infty,$$

which makes a contradiction. In this case, suppose there is a bijection from F to a subset of $\mathbb{N}$, then this problem is the same with Problem 8.

(ii). Since $x_0 \in X$, and $\mathcal{M}$ is the $\sigma-$algebra of all subsets of X, then $\{x_0\}$ is measurable. For any simple function $h(x) \leq |f(x)|$, suppose

$$h(x) = \sum_{i=1}^{n} c_i 1_{E_i},$$

and $x_0 \in c_1$, then

$$\int_X h(x) d\mu_{x_0} = \sum_{i=1}^{n} c_i m(E_i) = c_1 \leq |f(x_0)|.$$

Then according to the definition of integration,

$$\int_X |f(x)| \leq |f(x_0)|.$$

On the other hand, let

$$h(x) = \begin{cases} |f(x_0)|, \ x = x_0 \\ 0, \ x \neq x_0 \end{cases}$$

then

$$\int_X h(x) = \int_{\{x_0\}} h(x_0) = |f(x_0)|.$$

Hence

$$\int_X |f(x)| d\mu_{x_0} = |f(x_0)|.$$

When f is arbitrary real-valued function, define f^+ and f^- as in the textbook, then $f = f^+ - f^-$, and

$$\int_X f = \int_X f^+ - \int_X f^-.$$