Statistics of Solutions to Test Models for SPEKF

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Outline

1. Introduction
   - Stochastic Parameterization Extended Kalman Filter (SPEKF)
   - Itô Integration
2. Statistics of $b(t)$ and $\gamma(t)$
   - Mean
   - Variance
   - Covariance
3. Statistics of $u(t)$
   - Mean
   - Variance
   - Covariance
4. Numerical Simulation
   - Parameters and Algorithms
   - Results

Introduction

Signals from nature can be modeled by Langevin equation:

Langevin Equation

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -\gamma(t)u(t) + \mathrm{i}\omega u(t) + \sigma\dot{W}(t) + f(t),$$

where $\dot{W}(t)$ is a Brownian motion, and $f(t)$ is the external force.

A major difficulty in accurate filtering of noisy signals with many degrees of freedom is model error; signal from nature is processed through incomplete physical models, as well as parameterized to inadequate numerical resolution.

B. Gershgorin, *et, al* proposed the Stochastic Parameterization Extended Kalman Filter (SPEKF) to cope with model errors.

Test Model

SDEs in Test Model

$$\begin{cases} \dfrac{du(t)}{dt} = (-\gamma(t) + \mathrm{i}\omega)u(t) + b(t) + f(t) + \sigma W(t), \\ \dfrac{db(t)}{dt} = (-\gamma_b + \mathrm{i}\omega_b)(b(t) - \hat{b}) + \sigma_b W_b(t), \\ \dfrac{d\gamma(t)}{dt} = -d_\gamma(\gamma(t) - \hat{\gamma}) + \sigma_\gamma W_\gamma(t) \end{cases}$$

The initial values are complex random variables, with their first-order and second-order statistics known.

Solution

With knowledge of ODEs, the solution of the SDE set is

$$\begin{cases} b(t) = \hat{b} + (b_0 - \hat{b})e^{\lambda_b(t-t_0)} + \sigma_b \int_{t_0}^{t} e^{\lambda_b(t-s)} dW_b(s) \\ \gamma(t) = \hat{\gamma} + (\gamma_0 - \hat{\gamma})e^{-d_\gamma(t-t_0)} + \sigma_\gamma \int_{t_0}^{t} e^{-d_\gamma(t-s)} dW_\gamma(s) \\ u(t) = e^{-J(t_0,t)+\hat{\lambda}(t-t_0)} u_0 + \int_{t_0}^{t} (b(s)+f(s))e^{-J(s,t)+\hat{\lambda}(s-t_0)} ds \\ \qquad + \sigma \int_{t_0}^{t} e^{-J(s,t)+\hat{\lambda}(s-t_0)} dW(s) \end{cases}$$

with $\lambda_b = -\gamma_b + \mathrm{i}\omega_b$, $\hat{\lambda} = -\hat{\gamma} + \mathrm{i}\omega$, $J(s,t) = \int_s^t (\gamma(s') - \hat{\gamma})ds'$.

Introduction

Itô Isometry

$\forall f \in \mathscr{V}(S,T)$, B_t is a standard Brownian motion,

$$\mathrm{E}\left[\left(\int_S^T f(t,\omega)dB_t\right)^2\right] = \mathrm{E}\left[\int_S^T f^2(t,\omega)dt\right].$$

Linear property of Itô integration

$$(1) \quad \int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t, \text{ a.e.}$$

$$(2) \quad \int_S^T (cf+g)dB_t = c\int_S^T f dB_t + \int_S^T g dB_t, \text{ a.e.}$$

$$(3) \quad \mathrm{E}\left[\int_S^T f dB_t\right] = 0.$$

Outline

Mean of $b(t)$, $\gamma(t)$

With property of Itô integration (3), it is easy to know

$$\mathrm{E}(b(t)) = \hat{b} + (\mathrm{E}[b_0] - \hat{b})e^{\lambda_b(t-t_0)}$$

$$\mathrm{E}(\gamma(t)) = \hat{\gamma} + (\mathrm{E}[\gamma_0] - \hat{\gamma})e^{-d_\gamma(t-t_0)}$$

Variance of $b(t)$, $\gamma(t)$

According to definition,

$$\begin{aligned}
\mathrm{Var}(b(t)) &= \mathrm{E}\left[(b(t)-\mathrm{E}[b(t)])(b(t)-\mathrm{E}[b(t)])^*\right] \\
&= e^{-2\gamma_b(t-t_0)}\mathrm{Var}(b_0) + \mathrm{E}\left[\sigma_b^2 \int_{t_0}^{t} e^{\lambda_b(t-s)}dW_b(s)\left(\int_{t_0}^{t} e^{\lambda_b(t-s)}dW_b(s)\right)^*\right] \\
&= e^{-2\gamma_b(t-t_0)}\mathrm{Var}(b_0) + \sigma_b^2 \mathrm{E}\left[\int_{t_0}^{t} e^{-2\gamma_b(t-s)}ds\right]
\end{aligned}$$

The last step takes advantage of Itô isometry.

$$\mathrm{Var}(b(t)) = e^{-2\gamma_b(t-t_0)}\mathrm{Var}(b_0) + \frac{\sigma_b^2}{2\gamma_b}(1-e^{-2\gamma_b(t-t_0)})$$

$$\mathrm{Var}(\gamma(t)) = e^{-2d_\gamma(t-t_0)}\mathrm{Var}(\gamma_0) + \frac{\sigma_\gamma^2}{2d_\gamma}(1-e^{-2d_\gamma(t-t_0)})$$

Covariance of $b(t)$, $\gamma(t)$

$$\mathrm{Cov}(b(t), b(t)^*) = \mathrm{E}\left[(b(t) - \mathrm{E}[b(t)])(b(t)^* - \mathrm{E}[b(t)^*])\right]$$

$$= \mathrm{E}\left[(b_0 - \mathrm{E}[b_0])(b_0^* - \mathrm{E}[b_0^*])e^{2\lambda_b(t-t_0)}\right] + \sigma_b \mathrm{E}\left[(b_0 - \mathrm{E}[b_0])\int_{t_0}^{t} e^{\lambda_b(t-s)} dW_b(s)\right]$$

$$+ \sigma_b \mathrm{E}\left[(b_0^* - \mathrm{E}[b_0^*])e^{\lambda_b(t-t_0)}\int_{t_0}^{t} e^{\lambda_b(t-s)} dW_b(s)\right] + \sigma_b^2 \mathrm{E}\left[(\int_{t_0}^{t} e^{\lambda_b(t-s)} dW_b(s))^2\right]$$

With property of Itô integration (3), the second and third term are both 0; with Itô isometry we know the last term is also 0.

$$\mathrm{Cov}(b(t), b(t)^*) = \mathrm{E}[(b_0 - \mathrm{E}[b_0])(b_0^* - \mathrm{E}[b_0^*])]e^{2\lambda_b(t-t_0)} = \mathrm{Cov}(b_0, b_0^*)e^{2\lambda_b(t-t_0)}$$

$$\mathrm{Cov}(b(t), \gamma(t)) = \mathrm{E}[(b(t) - \mathrm{E}[b(t)])(\gamma(t) - \mathrm{E}[\gamma(t)])] = \mathrm{Cov}(b_0, \gamma_0)e^{(\lambda_b - d_\gamma)(t-t_0)}$$

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Mean of $u(t)$

Using the same properties, it is obvious to find

$$\mathrm{E}[u(t)] = e^{\hat{\lambda}(t-t_0)}\mathrm{E}\left[e^{-J_0(t_0,t)}u_0\right] + \int_{t_0}^{t} e^{\hat{\lambda}(t-s)}\mathrm{E}\left[b(s)e^{-J(s,t)}\right]ds$$

$$+\sigma\int_{t_0}^{t} e^{\hat{\lambda}(t-s)}f(s)\mathrm{E}\left[e^{-J(s,t)}\right]ds.$$

It is necessary to compute expectations of terms like $\mathrm{E}[ze^{bx}]$, where z is a complex-valued Gaussian random variable and x is a real-valued Gaussian variable. We propose two lemmas here.

Mean of $u(t)$

Lemma

$$E\left[ze^{ibx}\right] = (E[z] + ibCov(z,x))e^{ibE[x]-\frac{1}{2}b^2Var(x)}$$

with z being a complex-valued Gaussian, and x a real-valued Gaussian.

Corollary

Under the condition of Lemma 1,

$$E\left[ze^{bx}\right] = (E[z] + bCov(z,x))e^{bE[x]+\frac{1}{2}b^2Var(x)}.$$

Proof of lemma 1 takes advantage of the characteristic function of multivariable Gaussian distribution.

Proof

Let $z = y + \mathrm{i}w, \quad y, w \in \mathbb{R}$. Denote $\mathbf{v} = (x, y, w)$, then $\mathbf{v}$ satisfies the multivariable Gaussian distribution, with its characteristic function

$$\phi_{\mathbf{v}}(\mathbf{s}) = \exp(\mathrm{i}\mathbf{s}^\top \mathrm{E}[\mathbf{v}] - \frac{1}{2}\mathbf{s}^\top \Sigma \mathbf{s}).$$

Let $g(\mathbf{v})$ being the PDF of $\mathbf{v}$, then one knows from that char. func. being Fourier transform of PDF,

$$\phi_{\mathbf{v}}(\mathbf{s}) = \frac{1}{(2\pi)^3} \int e^{\mathrm{i}\mathbf{s}^\top \mathbf{v}} g(\mathbf{v}) d\mathbf{v}$$

According to the differential property of Fourier transform,

$$\frac{\partial \phi_{\mathbf{v}}(\mathbf{s})}{\partial s_2} = \frac{1}{(2\pi)^3} \int \mathrm{i} y_0 e^{\mathrm{i}\mathbf{s}^\top \mathbf{v}} g(\mathbf{v}) d\mathbf{v} = \mathrm{i}\mathrm{E}\left[y_0 e^{\mathrm{i}\mathbf{s}^\top \mathbf{v}} \right].$$

Let $\mathbf{v} = (b, 0, 0)^\top$,

$$\mathrm{E}[y_0 e^{ibx_0}] = -\mathrm{i} \left. \frac{\partial \phi_{\mathbf{v}}(s)}{\partial s_2} \right|_{\mathbf{s}=(b,0,0)^\top}$$

$$\mathrm{E}[y_0 e^{ibx_0}] = -\mathrm{i} \left. \frac{\partial \phi_{\mathbf{v}}(s)}{\partial s_2} \right|_{\mathbf{s}=(b,0,0)^\top}$$

Proof

From PDF of multivariable Gaussian distribution, one knows

$$\frac{\partial \phi_{\mathbf{v}}(\mathbf{s})}{\partial s_2} = (\mathrm{i}\mathrm{E}[y_0] - \mathrm{Var}(y_0)s_2 - \mathrm{Cov}(x_0, y_0)s_1 - \mathrm{Cov}(y_0, w_0)s_3)\phi_{\mathbf{v}}(s)$$

$$\frac{\partial \phi_{\mathbf{v}}(\mathbf{s})}{\partial s_3} = (\mathrm{i}\mathrm{E}[w_0] - \mathrm{Var}(w_0)s_3 - \mathrm{Cov}(x_0, w_0)s_1 - \mathrm{Cov}(y_0, w_0)s_2)\phi_{\mathbf{v}}(s)$$

Compute the partial derivatives at $\mathbf{s} = (b, 0, 0)^\top$,

$$\mathrm{E}\left[y_0 e^{ibx_0}\right] = (\mathrm{E}[y_0] + \mathrm{i}\mathrm{Cov}(x_0, y_0)b)\exp(\mathrm{i}b\mathrm{E}[x_0] - \frac{1}{2}\mathrm{Var}(x_0)b^2)$$

$$\mathrm{E}\left[w_0 e^{ibx_0}\right] = (\mathrm{E}[w_0] + \mathrm{i}\mathrm{Cov}(x_0, w_0)b)\exp(\mathrm{i}b\mathrm{E}[x_0] - \frac{1}{2}\mathrm{Var}(x_0)b^2)$$

Then

$$\mathrm{E}\left[ze^{ibx}\right] = (\mathrm{E}[z] + \mathrm{i}b\mathrm{Cov}(z, x))e^{\mathrm{i}b\mathrm{E}[x] - \frac{1}{2}b^2\mathrm{Var}(x)}.$$

$\square$

Mean of $u(t)$

Lemma

$$E\left[zwe^{bx}\right] = [E[z]E[w] + Cov(z,w^*) + b(E[z]Cov(w,x)) + E[w]Cov(z,x) + b^2 Cov(z,x)Cov(w,x)]\, e^{bE[x]+\frac{b^2}{2}Var(x)}.$$

with z, w being complex-valued Gaussian, and x real-valued Gaussian.

The proof of this lemma is the same as Lemma 1.

Mean of $u(t)$

We now make use of Lemma 1 to obtain the mean of $u(t)$.

$$
\begin{aligned}
\mathrm{E}[u(t)] &= e^{\hat{\lambda}(t-t_0)}(\mathrm{E}[u_0] - \mathrm{Cov}(u_0, J(t_0,t)))e^{-\mathrm{E}[J(t_0,t)]+\frac{1}{2}\mathrm{Var}(J(t_0,t))} \\
&\quad + \int_{t_0}^{t} e^{\hat{\lambda}(t-s)}(\hat{b} + e^{\lambda_b(s-t_0)}(\mathrm{E}[b_0] - \hat{b} - \mathrm{Cov}(b_0, J(s,t))))e^{-\mathrm{E}[J(s,t)]+\frac{1}{2}\mathrm{Var}(J(s,t))}ds \\
&\quad + \int_{t_0}^{t} e^{\hat{\lambda}(t-s)} f(s) e^{-\mathrm{E}[J(s,t)]+\frac{1}{2}\mathrm{Var}(J(s,t))}ds
\end{aligned}
$$

The terms $\mathrm{Cov}(u_0, J(s,t))$, $\mathrm{Cov}(b_0, J(s,t))$, $\mathrm{E}[J(s,t)]$ and $\mathrm{Var}(J(s,t))$ can be found using Itô isometry.

Variance of $u(t)$

Denote $u(t) = A + B + C$,

$$\begin{cases} A = e^{-J(t_0,t)+\hat{\lambda}(t-t_0)}u_0, \\ B = \displaystyle\int_{t_0}^{t}(b(s)+f(s))e^{-J(s,t)+\hat{\lambda}(t-s)}ds, \\ C = \sigma \displaystyle\int_{t_0}^{t} e^{-J(s,t)+\hat{\lambda}(t-s)}dW(s). \end{cases}$$

By definition we find $\mathrm{Var}(u(t)) = \mathrm{E}\left[|u(t)|^2\right] - |\mathrm{E}[u(t)]|^2$, with

$$\mathrm{E}\left[|u(t)|^2\right] = \mathrm{E}\left[|A|^2\right] + \mathrm{E}\left[|B|^2\right] + \mathrm{E}\left[|C|^2\right] + 2\mathrm{Re}\{\mathrm{E}[A^*B]\}.$$

We can obtain $\mathrm{E}\left[|A|^2\right]$ by Lemma 2, and $\mathrm{E}\left[|B|^2\right]$ by Itô isometry and noticing that

$$\mathrm{Cov}(J(s,t),J(r,t)) = \mathrm{Var}(J(s,t)) + \mathrm{Cov}(J(s,t),J(r,s)).$$

$\mathrm{E}\left[|C|^2\right]$ and $\mathrm{Re}\{\mathrm{E}[A^*B]\}$ can also be computed by Itô isometry and property of Itô integration.

Covariances

By definition,

$$\mathrm{Cov}(u(t), u^*(t)) = \mathrm{E}\left[u(t)^2\right] - \mathrm{E}[u(t)]^2$$

$$\mathrm{Cov}(u(t), \gamma(t)) = \mathrm{E}[u(t)(\gamma(t) - \hat{\gamma})] + \mathrm{E}[u(t)](\hat{\gamma} - \mathrm{E}[\gamma(t)])$$

$$\mathrm{Cov}(u(t), b(t)) = \mathrm{E}[u(t)b^*(t)] - \mathrm{E}[u(t)]\mathrm{E}[b(t)]^*$$

$$\mathrm{Cov}(u(t), b^*(t)) = \mathrm{E}[u(t)b(t)] - \mathrm{E}[u(t)]\mathrm{E}[b(t)].$$

Each term can be obtained by Lemma 3 and Itô isometry.

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Parameters

External forcing $f(t) = \frac{3}{2}e^{0.1it}$, and parameters of the equation set are given

$$\begin{cases} d = 1.5, & d_\gamma = 0.01d \\ \sigma = 0.1549, & \omega = 1.78 \\ \sigma_\gamma = 5\sigma, & \gamma_b = 0.1d \\ \sigma_b = 5\sigma, & \omega_b = \omega \\ \hat{b} = 0, & \hat{\gamma} = 0 \end{cases}$$

We assume the initial values satisfying

$$\begin{cases} \mathrm{Re}(u_0), \mathrm{Im}(u_0) \sim \mathcal{N}(0, \frac{1}{\sqrt{2}}), \text{ i.i.d.} \\ \mathrm{Re}(b_0), \mathrm{Im}(b_0) \sim \mathcal{N}(0, \frac{1}{\sqrt{2}}), \text{ i.i.d.} \\ \gamma_0 \sim \mathcal{N}(0, 1) \end{cases}$$

Parameters

The statistics between initial values are

$$\begin{cases} \mathrm{Cov}(u_0, u_0^*) = 0 \\ \mathrm{Cov}(u_0, \gamma_0) = 0 \\ \mathrm{Cov}(u_0, b_0) = 0 \\ \mathrm{Cov}(u_0, b_0^*) = 0 \end{cases}$$

Euler-Maruyama Scheme

Itô integration can be simulated by E-M scheme:

$$X_j = X_{j-1} + f(X_{j-1})\Delta t + g(X_{t-1})(W(\tau_j) - W(\tau_{j-1})),$$

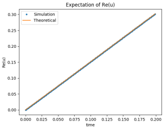
with

$$W(\tau_j) - W(\tau_{j-1}) = \sum_{k=jR-R+1}^{jR} \mathrm{d}W_k,$$

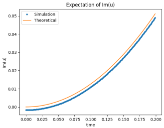
and R being the step length of E-M scheme,

$$\mathrm{d}W = \sqrt{\Delta t} \times \texttt{randn}().$$

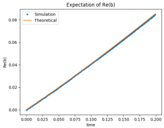
Simulate 10^6 times with $R = 1$, the results are as follows:



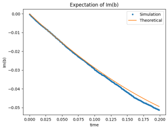
(a) E(Re(u))



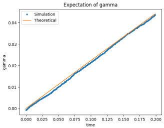
(b) E(Im(u))



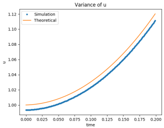
(c) E(Re(b))



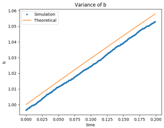
(d) E(Im(b))



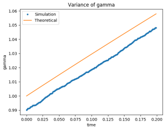
(e) E(γ)



(f) Var(u)



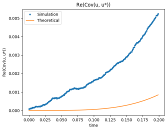
(g) Var(b)



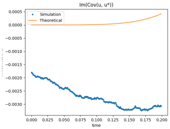
(h) Var(γ)

Figure: Simulation of Expectations and Variances, $n = 10^6$

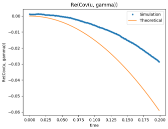
Result



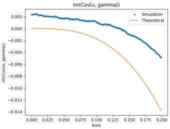
(a) $\mathrm{Re}(\mathrm{Cov}(u,u^*))$



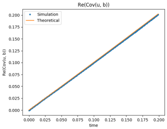
(b) $\mathrm{Im}(\mathrm{Cov}(u,u^*))$



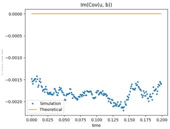
(c) $\mathrm{Re}(\mathrm{Cov}(u,\gamma))$



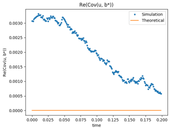
(d) $\mathrm{Im}(\mathrm{Cov}(u,\gamma))$



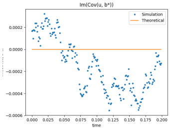
(e) $\mathrm{Re}(\mathrm{Cov}(u,b))$



(f) $\mathrm{Im}(\mathrm{Cov}(u,b))$



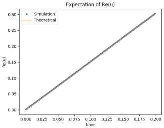
(g) $\mathrm{Re}(\mathrm{Cov}(u,b^*))$



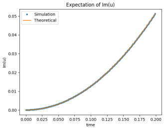
(h) $\mathrm{Im}(\mathrm{Cov}(u,b^*))$

Figure: Simulation of Covariances, $n = 10^7$

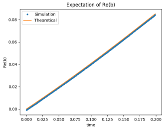
Simulate 10^7 times with $R = 1$, the results are as follows:



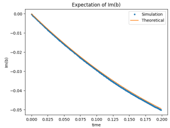
(a) E(Re(u))



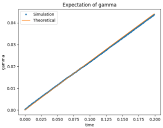
(b) E(Im(u))



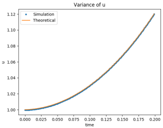
(c) E(Re(b))



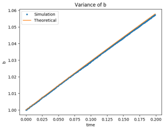
(d) E(Im(b))



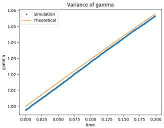
(e) E(γ)



(f) Var(u)



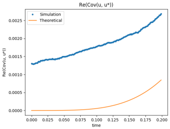
(g) Var(b)



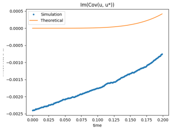
(h) Var(γ)

Figure: Simulation of Expectations and Variances, $n = 10^7$

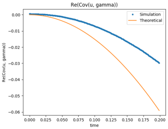
Result



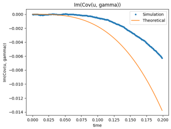
(a) $\mathrm{Re}(\mathrm{Cov}(u, u^*))$



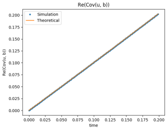
(b) $\mathrm{Im}(\mathrm{Cov}(u, u^*))$



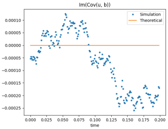
(c) $\mathrm{Re}(\mathrm{Cov}(u, \gamma))$



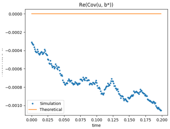
(d) $\mathrm{Im}(\mathrm{Cov}(u, \gamma))$



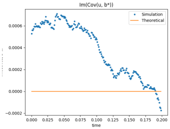
(e) $\mathrm{Re}(\mathrm{Cov}(u, b))$



(f) $\mathrm{Im}(\mathrm{Cov}(u, b))$



(g) $\mathrm{Re}(\mathrm{Cov}(u, b^*))$



(h) $\mathrm{Im}(\mathrm{Cov}(u, b^*))$

Figure: Simulation of Covariances, $n = 10^7$

Discussion

From the simulation results we find that the simulation of expectations fit the theoretical results satisfyingly.
For the results of Variances and Covariances,

- When $n = 10^6$, the error of variances is about $O(10^{-2}) \sim O(10^{-3})$,
- error of covariances is about $O(10^{-4})$.
- When $n = 10^7$, the error of $\mathrm{Var}(u),\ \mathrm{Var}(b)$ is about $O(10^{-6})$, and error of $\mathrm{Var}(\gamma)$ is about $O(10^{-3}) \sim O(10^{-4})$,
- error of $\mathrm{Cov}(u,u^*),\ \mathrm{Cov}(u,\gamma)$ is about $O(10^{-3})$, error of $\mathrm{Cov}(u,b),\ \mathrm{Cov}(u,b^*)$ is $O(10^{-4})$.

Discussion

We can find that with the increase of simulations, errors of all statistics decrease almost linearly. Thus we can believe, when $n \to \infty$, the simulation result would converge to the theoretical result.
In fact, the source of errors is that initial values are all random variables, and according to Law of large numbers, the error would converge to 0 when $n \to \infty$.

We can get third-order and fourth-order statistics of u, b, γ with the same methods, which could be written in a form of multiple integrals of statistics of initial values.

Thank you!