Introduction to Analysis I
Homework 3
Monday, September 11, 2017

Instructions: This and all subsequent homeworks must be submitted written in LaTeX.
If you use results from books, Royden or others, please be explicit about what results you are using.

Homework 3 is due by midnight, Friday, September 22.

1. (Problem 34, Page 53) Show that there is a continuous, strictly increasing function on the interval $[0, 1]$ that maps a set of positive measure onto a set of measure zero.

   Collaborators:

   Solution: Let C be the Cantor set on $[0, 1]$, and $\varphi(x)$ be the Cantor function. Define

   $$\phi(x) = \varphi(x) + x, \ x \in C.$$

   We now show $\phi^{-1}(x)$, the inverse of $\phi(x)$, satisfies the properties in the problem. First, since $\phi(x)$ is a continuous, strictly increasing function on $[0, 1]$, thus is continuous and strictly increasing on $C \subset [0, 1]$.

   Denote $D = \phi(C)$, then according to theorems in the book, $m(D) = 1$. $\forall x_1 < x_2 \in D$, if $\phi^{-1}(x_1) \geq \phi^{-1}(x_2)$, then since ϕ is strictly increasing, $\phi(\phi^{-1}(x_1)) = x_1 + \phi^{-1}(x_1) \geq \phi(\phi^{-1}(x_2)) = x_2 + \phi^{-1}(x_2)$, which means $\phi^{-1}(x_1) < \phi^{-1}(x_2)$, leading to a contradictory. Thus ϕ^{-1} is strictly increasing.

   On the other hand, $\forall x_0 \in D, \forall \epsilon > 0$, since ϕ is continuous on C, then $\forall \delta > 0$, $\exists \epsilon_1 > 0$, $\forall y \in C$, $|y - \phi^{-1}(x_0)| < \epsilon_1$, $|\phi(y) - x_0| < \delta$. Denote $\epsilon_1 = \min(\epsilon_1, \epsilon)$, then $\forall y \in C$, $|y - \phi^{-1}(x_0)| < \epsilon_1$, $|\phi(y) - x_0| < \delta$. Thus according to properties of strictly increasing bijection, $\forall x \in D$, $|x - x_0| < \delta$, $|\phi^{-1}(x) - \phi^{-1}(x_0)| < \epsilon_1 \leq \epsilon$. It means that ϕ^{-1} is continuous on D.

   Since C is measure zero, we get a function satisfying the properties in the problem.

2. (Problem 37, Page 53) Let f be a continuous function defined on E. Is it true that $f^{-1}(A)$ is always measurable if A is measurable?

   Collaborators:

   Solution:

3. (Problem 39, Page 53) Let F be the subset of $[0, 1]$ constructed in the same manner as the Cantor set except that each of the intervals removed at the nth deletion stage has length $\alpha 3^{-n}$ with $0 < a < 1$. Show that F is a closed set, $[0, 1] \sim F$ dense in $[0, 1]$, and $m(F) = 1 - a$. Such a set F is called a *generalized Cantor set.*

   Collaborators:

   Solution:

4. Let C be the Cantor set and let φ be the Cantor-Lebesgue function.

   (a) Show that C consists of all $x \in [0, 1]$ whose ternary expansion has coefficients equal to 0 or 2, i.e., if $x = \sum_{k \geq 1} c_k 3^{-k}$, where each $c_k = 0, 1,$ or 2, then $x \in C$ if and only if $c_k = 0$ or 2.

(b) Show that if $x \in C$ and $x = \sum_{k\geq 1} c_k 3^{-k}$, where each $c_k = 0$ or 2, then $\varphi(x) = \sum_{k\geq 1}(\frac{1}{2}c_k)2^{-k}$.

Collaborators:

Solution:

5. Construct a Cantor-type subset of $[0, 1]$ by removing from each interval remaining at the k^{th} stage, a subinterval of relative length θ_k, $0 < \theta_k < 1$. Show that the remainder has measure zero if and only if $\sum_{k\geq 1}\theta_k = \infty$. (Use the fact that for $a_k > 0$, the product $\Pi_{k=1}^{\infty} a_k$ converges, in the sense that $\lim_{n\to\infty}\Pi_{k=1}^{N} a_k$ exists and is not zero, if and only if $\sum_{k=1}^{\infty}\ln a_k$ converges.)

Collaborators:

Solution:

6. Let Z be a set of measure zero in $\mathbb{R}$. What is the measure of $\{x^2 \mid x \in Z\}$?

Collaborators:

Solution: $X = \{x^2 \mid x \in Z\}$ is also measure 0.

On one hand, we can define a map

$$\begin{aligned} \phi : X &\to Z, \\ x &\mapsto \sqrt{x}, \quad \text{if } \sqrt{x} \in Z. \\ x &\mapsto -\sqrt{x}, \quad \text{if } \sqrt{x} \notin Z \text{ and } -\sqrt{x} \in Z. \end{aligned}$$

Then it is a bijection from X to a subset of Z.

On the other hand, the map

$$\varphi : Z \to X$$

7. Let $0.\alpha_1\alpha_2\cdots$ be the dyadic development of any $x \in [0, 1]$. Let $k_1, k_2, k_3, \ldots$ be a fixed permutation of the positive integers $1, 2, \ldots$, and consider the transformation T which sends $x = \alpha_1\alpha_2\alpha_3\cdots$ to $Tx := \alpha_{k_1}\alpha_{k_2}\alpha_{k_3}\cdots$. Show that if E is a measurable subset of $[0, 1]$ then its image under T, $T(E)$, is also measurable and that $m(T(E)) = m(E)$. That is, show that T is a measure preserving transformation of $[0, 1]$. [Consider first the special case where E is a dyadic interval of the form $(s2^{-k}, (s + 1)2^{-k})$ and $s = 0, 1, \ldots, 2k - 1$. Then think about open sets and note that each open set can be written as a countable union of non-overlapping half-open dyadic intervals.]

Collaborators:

Solution: