Numerical Analysis
Assignment 1

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September 3, 2017

Problem 1. Problem 1.1, Page 43

(a). In fact,

$$\frac{1}{n}n\inf_{a\le x\le b} f(x) \le \frac{1}{n}\sum_{i=1}^{n} f(i) \le \frac{1}{n}n\sup_{a\le x\le b} f(x).$$

So according to intermediate theorem, there exists $\zeta \in [a,b]$, $S = f(\zeta)$.

(b). The proposition now becomes as this:

$$S = \sum_{i=1}^{n} w_i f(x_i) = f(\zeta)\sum_{i=1}^{n} w_i,$$

for some $\zeta \in [a,b]$. The proof is just the same as (a).

Problem 2. Promblem 1.2, Page 43

(a). Denote $f(x) = e^x$, then according to the mean value theorem,

$$|e^x - e^z| = f'(\xi)|x-z|,$$

where $\xi \le 0$, so $|e^x - e^z| \le |x-z|$.

(b). Denote $f(x) = \tan(x)$, then $|\tan(x) - \tan(x)| = f'(\xi)|x-z| \le |x-z|$.

(c). Denote $f(x) = x^p$, then $x^p - y^p = f'(\xi)(x-y),\ \xi \in [x,y]$. Thus $py^{p-1}(x-y) \le x^p - y^p \le px^{p-1}(x-y)$.

Problem 3. Problem 1.4, Page 44

Proof. According to Integral Mean Value theorem,

$$\int_0^h x^2(h-x)^2 g(x)dx = g(\xi)\int_0^h x^2(h-x)^2 dx = \frac{1}{30}h^5 g(\xi),$$

for some $\xi \in [0,h]$.

Problem 4. Problem 1.5, Page 44

(a). Since

$$\left(\int_0^x e^{-t^2}dt\right)' = e^{-x^2},$$

and

$$\begin{aligned} e^{-x^2} &= 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots + \frac{(-x^2)^n}{n!} + \frac{e^{-\theta x^2}}{(n+1)!}(-x^2)^{n+1} \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \frac{(-1)^{n+1}e^{-\theta x^2}}{(n+1)!}x^{2n+2} \end{aligned}$$

We have

$$\frac{1}{x}\int_0^x e^{-t^2}dt = \frac{1}{1\cdot 0!} - \frac{1}{3\cdot 1!}x^2 + \frac{1}{5\cdot 2!}x^4 + \cdots + (-1)^n \frac{1}{(2n+1)\cdot n!}x^{2n} + R_{2n+1},$$

where

$$|R_{2n+1}| \le \frac{x^{2n+1}}{(2n+2)(n+1)!}.$$

(b). Since

$$\left(\sin^{-1}(x)\right)' = \frac{1}{\sqrt{1-x^2}},$$

and

$$\begin{aligned}\frac{1}{\sqrt{1-x^2}} &= 1 - \frac{1}{2}(-x^2) + \frac{1\cdot 3}{2\cdot 4}(-x^2)^2 + \cdots + (-1)^n \frac{(2n-1)!!}{2n!!}(-x^2)^n + (-1)^{n+1}\frac{(2n+1)!!}{(2n+2)!!}\frac{(-x^2)^{n+1}}{(1-\theta x^2)^{n+\frac{3}{2}}} \\ &= 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \cdots + \frac{(2n-1)!!}{2n!!}x^{2n} + \frac{(2n+1)!!}{(2n+2)!!}\frac{x^{2(n+1)}}{(1-\theta x^2)^{n+\frac{3}{2}}}\end{aligned}$$

We have

$$\sin^{-1}(x) = x + \frac{1}{3\cdot 2}x^3 + \frac{1\cdot 3}{5\cdot 2\cdot 4}x^5 + \cdots + \frac{(2n-1)!!}{(2n+1)(2n)!!}x^{2n+1} + R_{2n+2},$$

where

$$|R_{2n+2}| \le \frac{(2n+1)!!}{(2n+2)(2n+2)!!}x^{2n+2}$$

(c). Since

$$\left(\int_0^x \frac{\tan^{-1} t}{t}dt\right)' = \frac{\tan^{-1} x}{x}, \quad \left(\tan^{-1} x\right)' = \frac{1}{1+x^2}$$

and

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + (-1)^{n+1}\frac{x^{2n+2}}{(1+\theta x)^{2n+4}},$$

We have

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + (-1)^{n+1}\frac{x^{2n+3}}{2n+3} + (-1)^{n+2}\frac{x^{2n+5}}{2n+5} + R_{n+3},$$

$$R_{n+3} = (-1)^{n+3}\frac{x^{2n+7}}{(1+\theta x)^{2n+7}}$$

So

$$\frac{1}{x}\int_0^x \frac{\tan^{-1} t}{t}dt = 1 - \frac{1}{3^2}x^2 + \frac{1}{5^2}x^4 - \cdots + (-1)^n \frac{x^{2n}}{(2n+1)^2} + R_{n+1},$$

and

$$|R_{n+1}| \le \frac{x^{2n+2}}{(2n+3)(2n+4)}.$$

(d).

$$\begin{aligned}\cos(x) + \sin(x) &= 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + R_{2n}, \\ R_{2n} &= \frac{x^{2n+2}}{(2n+2)!}cos(\theta_1 x + (n+1)\pi) + \frac{x^{2n+3}}{(2n+3)!}\sin\left(\theta_2 x + \frac{2n+3}{2}\pi\right) \\ &\le \frac{x^{2n+2}}{(2n+2)!} + \frac{x^{2n+3}}{(2n+3)!}.\end{aligned}$$

(e).

$$\log(1-x) = \frac{\ln(1-x)}{\ln 10} = \frac{1}{\ln 10}\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} + R_{n+1}\right),$$

and

$$|R_{n+1}| \le \frac{|x|^{n+2}}{n+2}.$$

(f).

$$\begin{aligned}
\log\left(\frac{1+x}{1-x}\right) &= \frac{1}{\ln 10}(\ln(1+x) - \ln(1-x)) \\
&= \frac{1}{\ln 10}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} - (-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n}) + R_{n+1}\right) \\
&= \frac{2}{\ln 10}\left(x + \frac{x^3}{3} + \cdots + \frac{x^{2n+1}}{2n+1} + R_{n+1}\right),
\end{aligned}$$

and

$$|R_{n+1}| \leq \frac{|x|^{2n+2}}{2n+2}$$