Numerical Analysis
Assignment 9

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Problem 1. Problem 4.1, Page 239.

Solution. When $n = 4$,

$$\begin{aligned} p_4(x) &= \sum_{k=0}^{4} C_4^k f(\frac{k}{4})x^k(1-x)^{4-k} = f(0)(1-x)^4 + 4f(\frac{1}{4})x(1-x)^3 + 6f(\frac{1}{2})x^2(1-x)^2 + 4f(\frac{3}{4})x^3(1-x) + f(1)x^4 \\ &= (6-4\sqrt{2})x^4 + 8\sqrt{2}x^3 - (12+6\sqrt{2})x^2 + 2\sqrt{2}x + 6 \\ &= (6-4\sqrt{2})(x-\frac{1}{2})^4 - 3(x-\frac{1}{2})^2 + 3 + \frac{\sqrt{2}}{4} \end{aligned}$$

And the fourth degree Taylor polynomial expanded about $\frac{1}{2}$ is

$$\begin{aligned} q_4(x) &= f(\frac{1}{2}) + f'(\frac{1}{2})(x-\frac{1}{2}) + \frac{1}{2}f''(\frac{1}{2})(x-\frac{1}{2})^2 + \frac{1}{6}f^{(3)}(\frac{1}{2})(x-\frac{1}{2})^3 + \frac{1}{24}f^{(4)}(\frac{1}{2})(x-\frac{1}{2})^4 \\ &= 1 - \frac{1}{2}\pi^2(x-\frac{1}{2})^2 + \frac{1}{24}\pi^4(x-\frac{1}{2})^4 \end{aligned}$$

When $x \to \frac{1}{2}$, $q_4(x) \to 1 = f(\frac{1}{2})$, while $p_4(x) \to 3 + \frac{\sqrt{2}}{4}$. Thus Bernstein polynomials are poor approximations.

Problem 2. Problem 4.5, Page 240

Solution. First we have

$$R(0) = a,\ R'(x) = \frac{b-ac}{(1+cx)^2},\ R'(0) = b - ac,$$

$$R''(x) = -2c(b-ac)(1+cx)^{-3},\ R''(0) = -2c(b-ac).$$

This kind of approximation does not necessarily exists. For example, if we choose f, s.t. $f'(x) = xe^x$. Then $f''(x) = (1+x)e^x$, and $f'(x) = 0, f''(x) = 1$. But since $R''(0) = -2cR'(0) = 0$, it makes a contradiction.

Problem 3. Problem 4.6, Page 240

Solution.

$$R(0) = a = f(0) = 1,$$

$$R'(0) = b - ac = f'(0) = 1,$$

$$R''(0) = -2c(b-ac) = f''(0) = 1.$$

Then $a = 1, b = \frac{1}{2}, c = -\frac{1}{2}$. Thus

$$R(x) = \frac{1+\frac{1}{2}x}{1-\frac{1}{2}x}.$$

And

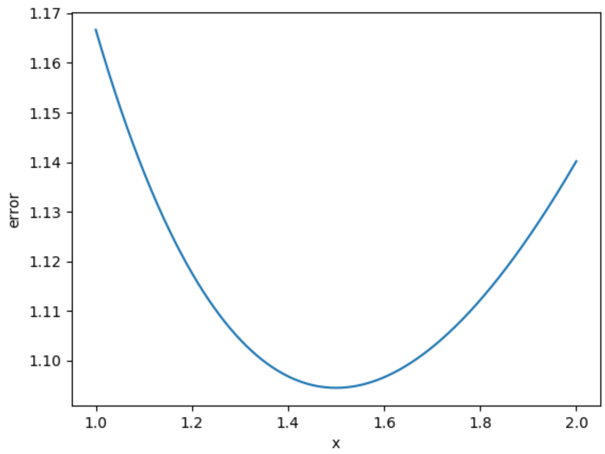
$$f(x) - R(x) = e^x - \frac{e^{\frac{1}{2}x} - \frac{1}{2}f''(\xi)(\frac{1}{2}x)^2}{e^{-\frac{1}{2}x} - \frac{1}{2}f''(\eta)(-\frac{1}{2}x)^2} = \frac{\frac{1}{8}(e^\eta + e^\xi)x^2}{e^{-\frac{1}{2}x} - \frac{1}{8}e^\eta x^2} = \frac{(e^\eta + e^\xi)x^2}{8-4x},$$

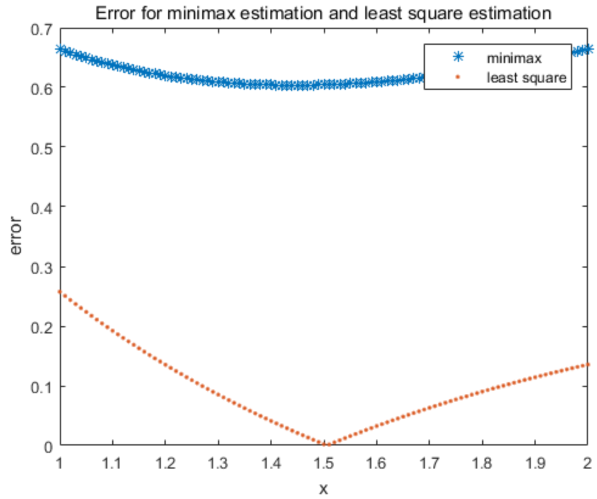
Then

$$\max_{x\in[-1,1]} |f - R| \le \frac{e}{2}.$$

Since

$$|f - p_2| = |\frac{1}{6}e^\eta x^3| \le \frac{e}{6},$$





So the supremum of error of Pade approximation could be larger than Taylor polynomial.

Problem 4. Problem 4.10, Page 241

(a). The linear Taylor polynomial to $f = \ln(x)$ expanding about $\frac{3}{2}$ is

$$p_1(x) = \frac{3}{2} + \frac{2}{3}(x - \frac{3}{2}) = \frac{2}{3}x + \frac{1}{2}.$$

The error of is as follows, the left is error graph for (a) and the right is for (b).
(b). The linear minimax approximation to f is

$$p_2(x) = ax + b.$$

Then there exists x_0, s.t.

$$\ln(1) - (a + b) = \ln(2) - (2a + b) = -(\ln(x_0) - (ax_0 + b)) = \rho.$$

and

$$(\ln(x) - (ax + b))'|_{x_0} = 0$$

Then we have

$$a = \ln(2),\ x_0 = \frac{1}{\ln(2)}, b = \frac{1}{2}(\ln(\frac{1}{2}\ln(2)) + 1),$$

thus

$$p_2(x) = \ln(2)x + \frac{1}{2}(\ln(\frac{1}{2}\ln(2)) + 1)$$

Problem 5. Problem 4.12, Page 241

Solution. The linear least square approximation to f is

$$q(x) = ax + b.$$

Then

$$E = \int_1^2 (\ln(x) - ax - b)^2 dx,$$

and

$$\frac{\partial E}{\partial a} = \int_1^2 (-2x)(\ln(x) - ax - b)dx = 0,$$
$$\frac{\partial E}{\partial b} = \int_1^2 (-2)(\ln(x) - ax - b)dx = 0.$$

Then

$$\begin{cases} \frac{3}{2}a + \frac{3}{2}b = 2\ln(2) - 1 \\ \frac{14}{3}a + 3b = 4\ln(2) - 2 \end{cases}$$

solving this equations, we get

$$a = 0.3,\ b = \frac{4}{3}\ln(2) - \frac{29}{30}.$$

The error graph is showed in (b) of Problem 4.