Numerical Analysis
Assignment 13

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Problem 1. Problem 5.23

(a). Integrate (5.4.2) with respect to x on $[0,1]$, we get from the convergence of the function series,

$$\int_0^1 (\frac{t}{e^t-1}e^{tx} - \frac{t}{e^t-1})dx = \sum_{j=1}^{\infty}\frac{t^j}{j!}\int_0^1 B_j(x)dx,$$

thus

$$1 - \frac{t}{e^t-1} = \sum_{j=1}^{\infty}\frac{t^j}{j!}\int_0^1 B_j(x)dx,$$

with Taylor expansion of the left term of (5.4.5), we have $B_0 = 1$, thus

$$\sum_{j=1}^{\infty}(B_j + \int_0^1 B_j(x)dx)\frac{t^j}{j!} = 0.$$

The left term can be regarded as a Taylor series of a function f at $t = 0$ which satisfies $f(0) = 0$. By the uniqueness of Taylor expansion of a analytic function, we know $f \equiv 0$. Then for all $j > 0$,

$$B_j + \int_0^1 B_j(x)dx = 0.$$

Now we know $B_0 = 1,\ B_1 = -\frac{1}{2}$, thus from definition (5.4.5),

$$\frac{t}{e^t-1} + \frac{1}{2}t = 1 + \sum_{j=2}^{\infty}B_j\frac{t^j}{j!} = \frac{t}{2}\frac{e^t+1}{e^t-1} \equiv g(t).$$

We have

$$g(-t) = -\frac{t}{2}\frac{e^{-t}+1}{e^{-t}-1} = -\frac{t}{2}\frac{1+e^t}{1-e^t} = g(t),$$

hence $g(t)$ is an even function. Substitute $t = -t$ into last term,

$$1 + \sum_{j=2}^{\infty}(-1)^j B_j\frac{t^j}{j!} = g(-t) = g(t) = 1 + \sum_{j=2}^{\infty}B_j\frac{t^j}{j!}.$$

By simplification,

$$\sum_{j=1}^{\infty}B_{2j+1}\frac{t^{2j+1}}{(2j+1)!} = 0.$$

With the same arguments and the uniqueness of Taylor expansion, we have

$$B_{2j+1} = 0, \text{ for all } j \geq 1.$$

(b). Take derivatives respective to x on both sides of (5.4.2),

$$\frac{t^2}{e^t-1}e^{tx} = \sum_{j=1}^{\infty}B_j'(x)\frac{t^j}{j!}.$$

Hence,

$$\sum_{j=1}^{\infty} B_j'(x)\frac{t^j}{j!} = t\frac{t(e^{tx}-1)}{e^t-1} + \frac{t^2}{e^t-1} = \sum_{j=1}^{\infty} B_j(x)\frac{t^{j+1}}{j!} + \sum_{j=0}^{\infty} B_j\frac{t^{j+1}}{j!} = \sum_{j=2}^{\infty} jB_{j-1}(x)\frac{t^j}{j!} + \sum_{j=1}^{\infty} jB_{j-1}\frac{t^j}{j!}$$

we have

$$tB_1'(x) + \sum_{j=2}^{\infty} B_j'(x)\frac{t^j}{j!} = B_0 t + \sum_{j=2}^{\infty} j(B_{j-1}(x) + B_{j-1})\frac{t^j}{j!}.$$

Since $tB_1'(x) = B_0 t$, with the uniqueness of Taylor expansion,

$$B_j'(x) = j(B_{j-1}'(x) + B_{j-1}), \text{ for all } j \geq 2.$$

As proved in (a), $B_{2j+1} = 0$ for all $j \geq 1$, the conclusion holds.

Problem 2. Problem 5.31

```
function T = romberg(f, a, b, n)
% Romberg extrapolation
% f: function
% x: interval: [a, b]
% n: order
T = zeros(n, n);
T(1, 1) = (feval(f, a)+feval(f, b))/2*(b-a);

for i = 2:n
    k = 2^(i-1);
    h = (b-a)/k;
    eval_points = a + (2*linspace(1, k, k)-1)*h/2;
    sums = sum(feval(f, eval_points));
    T(i, 1) = 0.5*(h*sums + T(i-1, 1));
    m = 1;
    for j = 2:i
        m = m*4;
        T(i, j) = (m*T(i, j-1)-T(i-1, j-1))/(m-1);
    end
end

T = diag(T);
```

```
format long g;

f1 = @(x) exp(-x.^2);
f2 = @(x) x.^2.5;
f3 = @(x) 1./(1+x.^2);
f4 = @(x) 1./(2+cos(x));
f5 = @(x) exp(x).*cos(4*x);

a1 = 0; b1 = 1;
a2 = 0; b2 = 1;
a3 = -4; b3 = 4;
a4 = 0; b4 = 2*pi;
a5 = 0; b5 = pi;

N = 20;
```

```
T1 = romberg(f1, a1, b1, N);
T2 = romberg(f2, a2, b2, N);
T3 = romberg(f3, a3, b3, N);
T4 = romberg(f4, a4, b4, N);
T5 = romberg(f5, a5, b5, N);

res = [T1 T2 T3 T4 T5];

dlmwrite('res2.m', res, 'delimiter', ' ', 'precision', '%2.9f');
```

```
0.683939721 0.500000000 0.470588235 2.094395102 37.920111314
0.731045203 0.339463097 1.223529412 2.792526803 -0.705452807
0.739446087 0.310849407 1.993202614 3.297841177 3.229586000
0.743193648 0.298084229 2.365757648 3.468111523 2.177291245
0.745016009 0.291875140 2.508348805 3.547746390 1.737097813
0.745920955 0.288791713 2.579942644 3.587716961 1.519545872
0.746372654 0.287252625 2.615800086 3.607662729 1.410943245
0.746598407 0.286483408 2.633718795 3.617631337 1.356665152
0.746711272 0.286098841 2.642677198 3.622615109 1.329529004
0.746767702 0.285906563 2.647156280 3.625106928 1.315961292
0.746795918 0.285810424 2.649395806 3.626352829 1.309177482
0.746810025 0.285762355 2.650515567 3.626975779 1.305785582
0.746817079 0.285738320 2.651075447 3.627287254 1.304089633
0.746820606 0.285726303 2.651355387 3.627442991 1.303241659
0.746822369 0.285720294 2.651495357 3.627520860 1.302817671
0.746823251 0.285717290 2.651565342 3.627559794 1.302605678
0.746823692 0.285715788 2.651600335 3.627579261 1.302499681
0.746823912 0.285715037 2.651617831 3.627588995 1.302446683
0.746824023 0.285714661 2.651626579 3.627593862 1.302420183
0.746824078 0.285714473 2.651630953 3.627596295 1.302406934
```

Cmp. When comparing with Simpson's rule and trapezoidal rule, we can find that romberg extrapolation converges quicker than the corresponding order of the two rules, but slower than the 2^N order of the two rules.

Problem 3. Problem 5.37

Problem 4. Problem 5.40