Homework 1

Instructions:
In problems 3. - 5., references such as III.2.7 refer to Problem 7 in Section 2 of Chapter III in Conway's book.

If you use results from books including Conway's, please be explicit about what results you are using.

Homework 1 is due on Dropbox on Monday, February 5.

1. Show that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ without using logarithms.

   Proof. Let $n^{1/n} = 1 + y_n$. First we know $n^{1/n} > 1^{1/n} = 1$, so $y_n > 0$. Then

   $$n = (1+y_n)^n = 1 + ny_n + \frac{n(n-1)}{2}y_n^2 + \cdots + y_n^n > 1 + \frac{n(n-1)}{2}y_n^2.$$

   Thus $\frac{n(n-1)}{2}y_n^2 < n-1$, which means $y_n < \sqrt{2/n}$. Hence $\lim\limits_{n\to\infty} y_n \le \lim\limits_{n\to\infty} \sqrt{2/n} = 0$, which means $y_n \to 0$, and thus $n^{1/n} \to 1$.

2. Given a power series, $\sum_{n=0}^{n=\infty} a_n(z-a)^n$, show that its radius of convergence R satisfies the inequalities

   $$(\limsup |\frac{a_{n+1}}{a_n}|)^{-1} \le R \le \limsup |\frac{a_n}{a_{n+1}}|.$$

   Proof. We only proof the right inequality since it is just the same for the left one. If $R > r > \limsup |\frac{a_n}{a_{n+1}}| = \alpha$, then there is an $N > 0$ s.t. $r > |a_n/a_{n+1}|$ for all $n \ge N$. Let $B = |a_N|r^N$, then $|a_{N+1}|r^{N+1} = |a_{N+1}|rr^N > B$. Hence for all $n > N$ we have $|a_n|r^n > B$, which gives $|a_n z^n| \ge B|z|^n/|r|^n$ when $n > N$. But $|z|/|r| > 1$, which makes $|z|^n/|r|^n \to \infty$ when $n \to \infty$. Hence $\sum a_n z^n$ diverges, so $R \le \alpha$.

3. Problem III.1.6.

   (a). By Theorem 1.3,

   $$\limsup |a_n|^{1/n} = \limsup |a^n|^{1/n} = |a|,$$

   thus $R = \frac{1}{|a|}$.

   (b). By Theorem 1.3,

   $$\limsup |a_n|^{1/n} = \limsup |a^{n^2}|^{1/n} = \limsup |a^n| = \begin{cases} 0, |a| < 1, \\ 1, |a| = 1, \\ \infty, |a| > 1. \end{cases}$$

   Thus

   $$R = \begin{cases} \infty, |a| < 1, \\ 1, |a| = 1, \\ 0, |a| > 1. \end{cases}$$

   (c). By Theorem 1.3,

   $$\limsup |a_n|^{1/n} = \limsup |k^n|^{1/n} = k,$$

   thus $R = \frac{1}{k}$.

   (d). Since

   $$\sum_{n=0}^{\infty} |z|^{n!} < \sum_{n=0}^{\infty} |z|,$$

   and the convergence radius of the latter series is $R' = 1$, we know $R \ge 1$. On the other hand, if $R > 1$, pick $1 < |z| = r < R$, then $|z|^{n!} = r^{n!} \to \infty$ when $n \to \infty$, hence the series diverges. Thus $R = 1$.

4. Problem III.1.7

Proof. On one hand,

$$\sum_{n=1}^{\infty} |a_n||z^{n(n+1)}| \leq \sum_{n=1}^{\infty} |a_n||z^n|,$$

thus $R \geq R' = \lim |a_n/a_{n+1}| = \lim \frac{n+1}{n} = 1$. On the other hand, if $R > 1$, pick $1 < r < R$, then $|a_n|z^{n(n+1)} = \frac{1}{n} r^{n(n+1)} = \frac{1}{n}(1+\delta)^{n(n+1)} > \frac{1}{n}(1+n\delta)^{n+1} > \frac{1}{n}(1+n(n+1)\delta) > (n+1)\delta$. But the last term $\to \infty$ as $n \to \infty$, hence the series diverges. Thus $R = 1$.

When $z = 1$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, it is a Leibniz series, thus converges. When $z = -1$, since $n(n+1)$ is a even number, it is the same with $z = 1$, thus converges. When $z = i$, the series becomes

$$\sum_{n=0}^{\infty} -\left(\frac{(-1)^{4n+1}}{(4n+1)} + \frac{(-1)^{4n+2}}{4n+2}\right) + \left(\frac{(-1)^{4n+3}}{(4n+3)} + \frac{(-1)^{4n+4}}{4n+4}\right) = \sum_{n=0}^{\infty} \frac{1}{4n+1} - \frac{1}{4n+2} - \frac{1}{4n+3} + \frac{1}{4n+4}$$

$$= \sum_{n=0}^{\infty} (-1)^n (\frac{1}{2n+1} - \frac{1}{2n+2}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)}.$$

It is also a Leibniz series, so converges.

5. Problem III.2.6

(i) $z = x + iy$, then $e^z = e^x(\cos y + i \sin y) = i \to x = 0, y = 2k\pi + \frac{\pi}{2}$. Thus $z = i(2k\pi + \frac{\pi}{2}), k \in \mathbb{Z}$.

(ii) $e^x(\cos y + i \sin y) = -1 \to x = 0, y = 2k\pi + \pi$. Thus $z = i(2k\pi + \pi), k \in \mathbb{Z}$.

(iii) $e^x(\cos y + i \sin y) = -i \to x = 0, y = 2k\pi + \frac{3\pi}{2}$. Thus $z = i(2k\pi + \frac{3\pi}{2}), k \in \mathbb{Z}$.

(iv) $\frac{1}{2}(e^{iz} + e^{-iz}) = 0 \to e^{-y}(\cos x + i \sin x) + e^y(\cos x - i \sin x) = 0 \to \cos x = 0, e^{-y} = e^y \to y = 0, x = k\pi + \frac{\pi}{2}$. Thus $z = k\pi + \frac{\pi}{2}, k \in \mathbb{Z}$.

(v) $\frac{1}{2i}(e^{iz} - e^{-iz}) = 0 \to e^{-y}(\cos x + i \sin x) - e^y(\cos x - i \sin x) = 0 \to -y = y, \sin x = 0 \to y = 0, x = k\pi$. Thus $z = k\pi, k \in \mathbb{Z}$.

6. Problem III.2.7

$$\cos z \cos w = \frac{1}{4}(e^{iz} + e^{-iz})(e^{iw} + e^{-iw}) = \frac{1}{4}(e^{i(z+w)} + e^{-i(z+w)} + e^{i(z-w)} + e^{i(w-z)}),$$

$$\sin z \sin w = -\frac{1}{4}(e^{iz} - e^{-iz})(e^{iw} - e^{-iw}) = \frac{1}{4}(-e^{i(z+w)} - e^{-i(z+w)} + e^{i(z-w)} + e^{i(w-z)})$$

$$\cos z \sin w = \frac{1}{4i}(e^{iz} + e^{-iz})(e^{iw} - e^{-iw}) = \frac{1}{4i}(e^{i(z+w)} - e^{i(z-w)} + e^{i(w-z)} - e^{-i(z+w)})$$

$$\sin z \cos w = \frac{1}{4i}(e^{iz} - e^{-iz})(e^{iw} + e^{-iw}) = \frac{1}{4i}(e^{i(z+w)} + e^{i(z-w)} - e^{i(w-z)} - e^{i(z+w)})$$

Hence,

$$\cos(z+w) = \frac{1}{2}(e^{i(z+w)} + e^{-i(z+w)}) = \cos z \cos w - \sin z \sin w.$$

$$\sin(z+w) = \frac{1}{2i}(e^{i(z+w)} - e^{-i(z+w)}) = \cos z \sin w + \sin z \cos w.$$

7. Problem III.2.9.

Proof.

$$\begin{aligned} |z_n - z| &= |r_n e^{i\theta_n} - re^{i\theta}| = |r_n \cos(\theta_n) - r\cos(\theta) + i(r_n \sin(\theta_n) - r\sin(\theta))| \\ &= \sqrt{(r_n \cos(\theta_n) - r\cos(\theta))^2 + (r_n \sin(\theta_n) - r\sin(\theta))^2} \\ &= \sqrt{r_n^2 + r^2 - 2r_n r \cos(\theta_n - \theta)}. \end{aligned}$$

If $\theta_n \nrightarrow \theta$, then there exists $\epsilon > 0$

$$|z_n - z| \geq \sqrt{2r_n r(1 - \cos(\theta_n - \theta))} > \epsilon,$$

which contradicts with $z_n \to z$. Hence $\theta_n \to \theta$. So

$$|z_n - z| \to |r_n - r|.$$

Thus $r_n \to r$.

8. Problem III.2.13

   $z = f(z)^n \to f(z) = z^{1/n}$, take a branch. Thus

   $$f(z) = z^{1/n} = e^{\frac{1}{n}\log z} = e^{\frac{1}{n}(\log|z| + iArgz + i2k\pi)} = |z|^{1/n} e^{\frac{i}{n}(Argz + 2k\pi)}.$$

   Pick any $k \in \mathbb{Z}$ and we can get a function satisfying the conditions.

9. Problem III.2.20

   $$\log(z_1 \cdots z_n) = \log|z_1 \cdots z_n| + Arg(z_1 \cdots z_n).$$

   We need to show that (Arg is the principle branch.)

   $$Arg(z_1 \cdots z_n) = \sum Arg(z_i).$$

   and

   $$\log|z_1 \cdots z_n| = \sum \log|z_i|.$$

   The second term is trivial from logarithm of real numbers. For the first term, we use induction. Let $z_j = r_j e^{i\theta_j}$, then from the condition we know $-\frac{\pi}{2} < \theta_j < \frac{\pi}{2}$ (*).

   First, when $k = 2$, from the condition we know $2k\pi - \frac{\pi}{2} < \theta_1 + \theta_2 < 2k\pi + \frac{\pi}{2}$. But from (*) we know $-\pi < \theta_1 + \theta_2 < \pi$, thus $-\frac{\pi}{2} < \theta_1 + \theta_2 < \frac{\pi}{2}$. Hence $Arg(z_1 z_2) = Arg(z_1) + Arg(z_2)$.

   Suppose it holds when $\leq n - 1$. Then from induction we know $-\frac{\pi}{2} < \sum_{j=1}^{n-1} \theta_j < \frac{\pi}{2}$, and using the same deduction with $k = 2$, we can know $-\frac{\pi}{2} < \sum_{j=1}^{n} \theta_j < \frac{\pi}{2}$. Hence $Arg(\prod z_i) = \sum Arg(z_i)$.

   If the restrictions are removed, the formula is not valid. For example, $z_1 = e^{i\pi}, z_2 = e^{i\frac{3\pi}{2}}$, then $\log(z_1 z_2) = i\frac{\pi}{2}$, while $\log(z_1) + \log(z_2) = i\frac{5\pi}{2}$.