Homework 5

Instructions: In problems the problems below, references such as III.2.7 refer to Problem 7 in Section 2 of Chapter III in Conway's book.

If you use results from books including Conway's, please be explicit about what results you are using.

Homework 5 is due in class at Midnight Monday, March 26.

Do the following problems:

1. V.3.1

   Proof From (3.1), (3.2) we know for a zero $z = a$ of f,

   $$\frac{f'(z)}{f(z)}g(z) = \frac{m}{z-a}g(z) + \frac{h'(z)}{h(z)}g(z),$$

   where $f(z) = (z-a)^m h(z)$ and h is analytical around a and $h(a) \neq 0$. Similarly, for a pole $z = a$ of f,

   $$\frac{f'(z)}{f(z)}g(z) = \frac{-m}{z-a}g(z) + \frac{h'(z)}{h(z)}g(z),$$

   where $f(z) = (z-a)^{-m}h(z)$ and h is analytical and $h(a) \neq 0$. In both cases, since g is analytic in G, it has no poles in G, so $\frac{h'(z)}{h(z)}g(z)$ is analytic. Hence,

   $$\frac{f'(z)}{f(z)}g(z) = \sum_{k=1}^{n} \frac{g(a_k)}{z-a_k} - \sum_{j=1}^{m} \frac{g(p_j)}{z-p_j} + \frac{h'(z)}{h(z)},$$

   and hence by Cauchy's theorem,

   $$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{i=1}^{n} g(z_i)n(\gamma; z_i) - \sum_{j=1}^{m} g(p_j)n(\gamma; p_j).$$

2. V.3.2

   Sol. Let $h(z) = f(z) - z^n$, $g(z) = z^n$, then by assumptions, on $\{|z| = 1\}$,

   $$|h(z) + g(z)| = |f(z)| < 1 = |g(z)|,$$

   hence by Rouche's theorem,

   $$Z_h - P_h = Z_g - P_g.$$

   Since h, g are analytic on $\bar{B}(0,1)$, $P_h = P_g = 0$. Hence

   $$Z_h = Z_g = 1,$$

   which means the equation has one solution.

3. V.3.3 Hint: Think about the expansion

   $$\frac{1}{f(z) - w} = \frac{1}{f(z)} + \frac{w}{[f(z)]^2} + \cdots + \frac{w^n}{[f(t)]^{n+1}} + \cdots.$$

   The hypotheses allow you to integrate this series termwise in z.

   Proof. Since f is analytic in $\bar{B}(0,R)$, it has no poles and according to assumptions, one zeros in it. Hence

   $$g(w) = \frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z) - w}dz = \frac{1}{2\pi i}\int_\gamma \sum_{n=1}^{\infty} \frac{w^{n-1}}{f(z)^n} zf'(z)dz.$$

Let $f(z) = zh(z)$, by assumption we know $h(0) \neq 0$, hence

$$\left|\frac{w^n}{f(z)^{n+1}} zf'(z)\right| = \left|\frac{w^n}{z^n h(z)^{n+1}} zf'(z)\right| = \left|\left(\frac{w}{z}\right)^n \frac{f'(z)}{h(z)^n}\right| \leq \frac{|w|^n}{R^n} \frac{R^n |f'(z)|}{|f(z)|^n} = \left|\frac{w}{f(z)}\right|^n |f'(z)|.$$

By assumption we know $|w| \leq |f(z)|$, hence for $f(z) \neq w$ (and since the domain is $B(0;\rho)$ which means the inequality is strict), the series

$$\sum_{n=0}^{\infty} \frac{|w|^n}{|f(z)|^n} f'(z)$$

converges, thus by Weierstrass theorem, the series

$$\sum_{n=1}^{\infty} \frac{w^{n-1}}{f(z)^n} zf'(z)dz$$

converges uniformly, hence we can switch the infinity sum with the integral:

$$g(w) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_\gamma \left(\frac{w}{z}\right)^n \frac{f'(z)}{h(z)^n} dz.$$

For an arbitrary triangle path T in $B(0,\rho)$, since w^n is analytic for every $n \geq 0$,

$$\int_T g(w)dw = \frac{1}{2\pi i} \int_T \sum_{n=0}^{\infty} \int_\gamma \left(\frac{w}{z}\right)^n \frac{f'(z)}{h(z)^n} dz = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_\gamma \frac{1}{z^n} \frac{f'(z)}{h(z)^n} dz \int_T w^n dw = 0.$$

Hence by Morera's theorem, g is analytic.

Properties: first, by Theorem 3.6, notice f has one zero $z = 0$ and no poles,

$$g(0) = \frac{1}{2\pi i} \int_\gamma \frac{zf'(z)}{f(z)} dz = \sum_{i=1}^{n} z_i n(\gamma; z_i) - \sum_{j=1}^{m} p_i n(\gamma, p_j) = 0.$$

4. V.3.5

Proof. First we show it is true for the poles. Since f is meromorphic in G, if z_0 is a limit point of poles, then f is either analytic in a neighbourhood of z_0 or has an isolated singularity at z_0. If f is analytic at z_0, then f is analytic at some $B(z_0, r)$. But since z_0 is a limit point of poles, there must be a pole $z_1 \in B(z_0, r)$, which makes a contradiction. If z_0 is an isolated singularity, then there is some $r > 0$, f is analytic in $B(z_0, r) \setminus \{z_0\}$. By the same reason, there is a pole $z_1 \in B(z_0, r) \setminus \{z_0\}$, which makes a contradiction. Hence it is true for poles.

For the zeros: suppose z_0 is a limit point of zeros. First we claim that z_0 cannot be a pole. Otherwise, since poles cannot have a limit points as we have shown above, f has a Laurent expansion in some $B(z_0, r) \setminus \{z_0\}$:

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n.$$

Then pick $r < 1$, we know

$$\left|\sum_{n=0}^{\infty} a_n (z - z_0)^n\right| < \sum_{n=0}^{\infty} |a_n| r^n < M$$

for some $M > 0$. However, we can pick a $\epsilon > 0$, s.t.

$$\sum_{n=-m}^{-1} a_n \epsilon^n > M.$$

Hence for each $z \in B(z_0, \epsilon)$, $f(z) \neq 0$, which makes a contradiction with that z_0 is a limit point of zeros. Thus, since poles cannot have a limit points, we can pick a $r > 0$, s.t. there is no pole in $B(z_0, r)$, which means f is analytic in $B(z_0, r)$. By Theorem 4.3.7, zeros has no limit points in $B(z_0, r)$, which contradicts with that z_0 is a limit point.

Hence, the proposition holds.

5. V.3.6

6. V.3.7

   Sol. A very trivial version is: change γ to any closed rectifiable curve in G where f, g are meromorphic in a neighbourhood of G with no zeros or poles in γ. In this case we do not need to modify the proof.

7. V.3.10

   Proof. In fact, by problem 2 we know there is an unique z s.t. $|z| < 1$ and $f(z) = z$. When $|f(z)| \leq 1$ on $|z| = 1$, we will show it is not true.

   i) pick $f(z) \equiv 1$, then $f(z) = z$ has no solution in $|z| < 1$.

   ii) pick $|f(z)| < 1$ on $|z| = 1$, then it has one solution in $|z| < 1$.

   iii) pick $f(z) = z$, then is has infinity number of solutions.

   Now we proof the only situations are as above, i.e., if a function f is analytic in the unit dist D, and $f(D) \subset D$, then f has one fixed point in D, except for the identity function.

   In fact, suppose there are two fixed points z_1, z_2. If $z_1 = 0$, then by Schwarz's lemma, there is a constant $|c| = 1$, s.t. $f(z) = cz$ for all z in D. Hence, $f(z_2) = cz_2 = z_2$, which means $c = 1$.

   Now suppose $z_1, z_2 \neq 0$. Let $z_1 = re^{i\theta}$, consider $\varphi(z) = e^{i\theta}\frac{z+r}{rz+1}$, then φ maps D onto itself, and $\varphi(0) = z_1$. Hence the function $g = \varphi^{-1} \circ f \circ \varphi$ maps D to a subset of D with two fixed points $t_1 = \varphi^{-1}(z_1) = 0$, and $t_2 = \varphi^{-1}(z_1) \neq 0$. By the first case, g is the identity function, so f is also the identity function.