Numerical Analysis
Assignment 9

Chuan Lu

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Problem 1. Problem 4.16, Page 242

Solution. For each $m < n$, by integration by parts,

$$\begin{aligned}\int_0^\infty e^{-x}x^m\varphi_n(x)dx &= \frac{(-1)^n}{n!}\int_0^\infty x^m\frac{d^n}{dx^n}(x^ne^{-x})dx\\ &= \frac{(-1)^n}{n!}\left(x^m\frac{d^{n-1}}{dx^{n-1}}(x^ne^{-x})\Big|_0^\infty - \int_0^\infty mx^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^ne^{-x})dx\right)\end{aligned}$$

Since

$$x^m\frac{d^{n-1}}{dx^{n-1}}(x^ne^{-x}) = e^{-x}N(x),$$

where $N(x)$ is a polynomial of degree $n-1+m$, by L'Hospital's Rule we know the first term in the integration is 0. Then by induction we know

$$\begin{aligned}\int_0^\infty e^{-x}x^m\varphi_n(x)dx &= \frac{(-1)^{n+1}m}{n!}\int_0^\infty x^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^ne^{-x})dx\\ &= \frac{(-1)^{n+m}m!}{n!}\int_0^\infty \frac{d^{n-m}}{dx^{n-m}}(x^ne^{-x})dx\\ &= \frac{(-1)^{n+m}m!}{n!}\frac{d^{n-m}}{dx^{n-m}}\int_0^\infty x^ne^{-x}dx\\ &= \frac{(-1)^{n+m}m!}{n!}\frac{d^{n-m}}{dx^{n-m}}(n!) = 0.\end{aligned}$$

In the deduction we used the property that $f(x) = x^ne^{-x}$ is absolutely continuous. Since $\varphi_m(x)$ is a polynomial of degree $m < n$, we know

$$(\varphi_n(x), \varphi_m(x)) = 0, \text{ and } (\varphi_n(x), \varphi_n(x)) \neq 0.$$

Hence $\{\varphi_n(x)\}$ is a family of orthogonal polynomials.

Problem 2. Problem 4.18, Page 242

Solution. First, we derive c_n. Multiply both sides of (4.4.21) by $w(x)\varphi_{n-1}(x)$, and then integrate, we get

$$\int w\varphi_{n+1}\varphi_{n-1}dx = \int a_nwx\varphi_n\varphi_{n-1} + \int b_nw\varphi_n\varphi_{n-1} - c_n\int w\varphi_{n-1}^2.$$

Using the orthogonality of φ_n, the left side is 0, and the second term of right side is 0. Then

$$a_n\int wx\varphi_n\varphi_{n-1} = c_n\int w\varphi_{n-1}^2.$$

Since

$$a_n\int wx\varphi_n\varphi_{n-1} = a_n\int w\varphi_n(A_{n-1}x^n + B_{n-1}x^{n-1} + \cdots) = a_n\int w\varphi_nA_{n-1}x^n = a_n\frac{A_{n-1}}{A_n}\int w\varphi_n^2,$$

we have

$$c_n = \frac{a_nA_{n-1}\gamma_n}{A_n\gamma_{n-1}} = \frac{A_{n+1}A_{n-1}\gamma_n}{A_n^2\gamma_{n-1}}.$$

Now we consider b_n. Multiply both sides of (4.4.21) by $w(x)\varphi_n(x)$, then integrate both sides, we get

$$\int w\varphi_{n+1}\varphi_n = \int a_n w x\varphi_n^2 + \int b_n w\varphi_n^2 - \int c_n w\varphi_{n-1}\varphi_n.$$

Using orthogonality, we get

$$\int a_n w x\varphi_n^2 + \int b_n w\varphi_n^2 = 0.$$

The first term can be wrote as

$$\begin{aligned}\int a_n w x\varphi_n^2 &= a_n \int w(A_n x^{n+1} + B_n x + \cdots)\varphi_n = a_n \int w\left(\frac{A_n}{A_{n+1}}\varphi_{n+1} - \frac{A_n B_{n+1} - A_{n+1}B_n}{A_{n+1}}x^n + \cdots\right)\varphi_n \\ &= a_n \int w\left(B_n - \frac{A_n}{A_{n+1}}B_{n+1}\right)x^n\varphi_n = a_n \int w\frac{1}{A_n}\left(B_n - \frac{A_n}{A_{n+1}}B_{n+1}\right)\varphi_n^2.\end{aligned}$$

Thus

$$a_n\left(\frac{B_n}{A_n} - \frac{B_{n+1}}{A_{n+1}}\right)\gamma_n + b_n\gamma_n = 0,$$

we know

$$b_n = a_n\left(\frac{B_{n+1}}{A_{n+1}} - \frac{B_n}{A_n}\right).$$

Problem 3. Problem 4.21, Page 243

Proof. Denote $\varphi_n(x) = A_n x^n + B_n x^{n-1} + \cdots$, and $A_n > 0$. Then by (4.4.21),

$$\varphi_{n+1}(x) = (a_n x + b_n)\varphi_n(x) - c_n\varphi_{n-1}(x).$$

We add a $\varphi_0(x)$ to this series, and $\varphi_0(x) = A_0 > 0$. First, when $n = 1$, since

$$\int_a^b w(x)\varphi_1(x)\varphi_0(x)dx = \int_a^b A_0 w(x)\varphi_1(x)dx = 0 = A_0\varphi_1(\xi)\int_a^b w(x)dx,$$

we know $\varphi_1(\xi) = 0,\ \xi \in (a,b)$. Then we show $\varphi_2(x)$ has two different roots in (a,b). First,

$$\varphi_2(\xi) = (a_1\xi + b_1)\varphi_1(\xi) - c_1\varphi_0(\xi) = -c_1\varphi_0(\xi) < 0.$$

Suppose $\varphi_2(x)$ does not change sign in (a,b), then

$$\int w(x)\varphi_2(x)\varphi_0(x) = A_0\int w(x)\varphi_2(x) < 0.$$

It makes a contradiction with orthogonality. Then there $\exists x_1 \in (a,b)$, s.t. $\varphi_2(x_1) = 0$. Since $A_2 > 0$, x_1 cannot be a double root. If $\varphi_2(x)$ has only one root in (a,b), then

$$\varphi_2(x)(x - x_1) = q(x)(x - x_1)^2,$$

integrate by $w(x)$, then since $(x - x_1)$ is of degree 1, left side is 0. But we know $q(x)$ has no root in (a,b), it does not change sign in (a,b), thus the integration is not 0. It makes a contradiction. Thus $\varphi_2(x)$ has two different roots in (a,b), and with $\varphi_2(\xi) < 0$ and $A_2 > 0$, we know the two roots are in (a,ξ) and (ξ,b) seperately. Using the same method, we know that $\varphi_n(x)$ has n different roots in (a,b).
Now we assume this proposition holds for $\varphi_m(x)$, $m \le n$. Denote roots of $\varphi_n(x)$ to be x_i, then since

$$\varphi_{n+1}(x_i) = (a_n x_i + b_n)\varphi_n(x_i) - c_n\varphi_{n-1}(x_i) = -c_n\varphi_{n-1}(x_i),$$

$$\varphi_{n+1}(x_{i+1}) = (a_n x_{i+1} + b_n)\varphi_n(x_{i+1}) - c_n\varphi_{n-1}(x_{i+1}) = -c_n\varphi_{n-1}(x_{i+1}),$$

from the assumption we know $\varphi_{n-1}(x_i)$ and $\varphi_{n-1}(x_{i+1})$ has different signs, which means $\varphi_n(x)$ has a root in each of this intervals. Then from φ_n has n different roots in (a,b), from induction we get the result.

Problem 4. Problem 4.23, Page 243

Solution. We know the orthogonal functions on $[-1,1]$ with weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ are Chebyshev polynomials (normalized):

$$T_0(x) = \frac{1}{\sqrt{\pi}},\ T_1(x) = \sqrt{\frac{2}{\pi}}x,\ T_2(x) = \sqrt{\frac{2}{\pi}}(2x^2 - 1).$$

For any $p(x) = a_0 + a_1 x + a_2 x^2$ of degree 2 that minimizes the distance, we have

$$a_j = (f, T_j), \;\; j = 0, 1, 2.$$

Thus

$$a_0 = (f, T_0) = \frac{1}{\sqrt{\pi}} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \cos^{-1} x dx = \frac{1}{\sqrt{\pi}} \int_{\pi}^{0} -y dy = \frac{(\pi)^{\frac{3}{2}}}{2},$$

$$a_1 = (f, T_1) = \sqrt{\frac{2}{\pi}} \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} \cos^{-1} x dx = \sqrt{\frac{2}{\pi}} \int_{0}^{\pi} y \cos y dy = \sqrt{\frac{2}{\pi}} (y \sin y \Big|_0^{\pi} - \int_0^{\pi} \sin y dy) = -\sqrt{\frac{8}{\pi}},$$

$$a_2 = (f, T_2) = \sqrt{\frac{2}{\pi}} \int_{-1}^{1} \frac{2x^2 - 1}{\sqrt{1-x^2}} \cos^{-1} x dx = \sqrt{\frac{2}{\pi}} \int_{0}^{\pi} y \cos 2y dy = \sqrt{\frac{2}{\pi}} (\frac{y \sin 2y}{2} \Big|_0^{\pi} - \frac{1}{2} \int_0^{\pi} \sin 2y dy) = 0,$$

Thus

$$p_2(x) = -\frac{4}{\pi} x + \frac{\pi}{2}.$$