# Numerical Analysis
# Assignment 9

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## Problem 1. Problem 4.16, Page 242

***Solution.*** For each $m < n$, by integration by parts,

$$\begin{aligned}\int_0^\infty e^{-x}x^m\varphi_n(x)dx &= \frac{(-1)^n}{n!}\int_0^\infty x^m\frac{d^n}{dx^n}(x^ne^{-x})dx\\ &= \frac{(-1)^n}{n!}\left(x^m\frac{d^{n-1}}{dx^{n-1}}(x^ne^{-x})\Big|_0^\infty - \int_0^\infty mx^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^ne^{-x})dx\right)\end{aligned}$$

Since

$$x^m\frac{d^{n-1}}{dx^{n-1}}(x^ne^{-x}) = e^{-x}N(x),$$

where $N(x)$ is a polynomial of degree $n-1+m$, by L'Hospital's Rule we know the first term in the integration is 0. Then by induction we know

$$\begin{aligned}\int_0^\infty e^{-x}x^m\varphi_n(x)dx &= \frac{(-1)^{n+1}m}{n!}\int_0^\infty x^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^ne^{-x})dx\\ &= \frac{(-1)^{n+m}m!}{n!}\int_0^\infty \frac{d^{n-m}}{dx^{n-m}}(x^ne^{-x})dx\\ &= \frac{(-1)^{n+m}m!}{n!}\frac{d^{n-m}}{dx^{n-m}}\int_0^\infty x^ne^{-x}dx\\ &= \frac{(-1)^{n+m}m!}{n!}\frac{d^{n-m}}{dx^{n-m}}(n!) = 0.\end{aligned}$$

In the deduction we used the property that $f(x) = x^ne^{-x}$ is absolutely continuous. Since $\varphi_m(x)$ is a polynomial of degree $m < n$, we know

$$(\varphi_n(x), \varphi_m(x)) = 0, \text{ and } (\varphi_n(x), \varphi_n(x)) \neq 0.$$

Hence $\{\varphi_n(x)\}$ is a family of orthogonal polynomials.

## Problem 2. Problem 4.18, Page 242

***Solution.*** First, we derive $c_n$. Multiply both sides of (4.4.21) by $w(x)\varphi_{n-1}(x)$, and then integrate, we get

$$\int w\varphi_{n+1}\varphi_{n-1}dx = \int a_nwx\varphi_n\varphi_{n-1} + \int b_nw\varphi_n\varphi_{n-1} - c_n\int w\varphi_{n-1}^2.$$

Using the orthogonality of $\varphi_n$, the left side is 0, and the second term of right side is 0. Then

$$a_n\int wx\varphi_n\varphi_{n-1} = c_n\int w\varphi_{n-1}^2.$$

Since

$$a_n\int wx\varphi_n\varphi_{n-1} = a_n\int w\varphi_n(A_{n-1}x^n + B_{n-1}x^{n-1} + \cdots) = a_n\int w\varphi_nA_{n-1}x^n = a_n\frac{A_{n-1}}{A_n}\int w\varphi_n^2,$$

we have

$$c_n = \frac{a_nA_{n-1}\gamma_n}{A_n\gamma_{n-1}} = \frac{A_{n+1}A_{n-1}\gamma_n}{A_n^2\gamma_{n-1}}.$$

Now we consider $b_n$. Multiply both sides of (4.4.21) by $w(x)\varphi_n(x)$, then integrate both sides, we get

$$\int w\varphi_{n+1}\varphi_n = \int a_n w x \varphi_n^2 + \int b_n w \varphi_n^2 - \int c_n w \varphi_{n-1}\varphi_n.$$

Using orthogonality, we get

$$\int a_n w x \varphi_n^2 + \int b_n w \varphi_n^2 = 0.$$

The first term can be wrote as

$$\begin{aligned}\int a_n w x \varphi_n^2 &= a_n \int w(A_n x^{n+1} + B_n x + \cdots)\varphi_n = a_n \int w(\frac{A_n}{A_{n+1}}\varphi_{n+1} - \frac{A_n B_{n+1} - A_{n+1}B_n}{A_{n+1}} x^n + \cdots)\varphi_n \\ &= a_n \int w(B_n - \frac{A_n}{A_{n+1}} B_{n+1}) x^n \varphi_n = a_n \int w \frac{1}{A_n}(B_n - \frac{A_n}{A_{n+1}} B_{n+1})\varphi_n^2.\end{aligned}$$

Thus

$$a_n(\frac{B_n}{A_n} - \frac{B_{n+1}}{A_{n+1}})\gamma_n + b_n\gamma_n = 0,$$

we know

$$b_n = a_n(\frac{B_{n+1}}{A_{n+1}} - \frac{B_n}{A_n}).$$

**Problem 3.** Problem 4.21, Page 243

**Problem 4.** Problem 4.23, Page 243