Introduction to Analysis
Assignment 6

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Problem 1. Problem 29, Section 4.4, Page 89

Sol. Both are not true. We can construct a f like this:

$$f(x) = \begin{cases} 1 + \dfrac{1}{n^2},\ n \leq x < n + \dfrac{1}{2},\ \forall n \in \mathbb{N} \\ -1,\ n + \dfrac{1}{2} \leq x < n + 1,\ \forall n \in \mathbb{N} \end{cases}$$

Then f is measurable, and f is bounded on any bounded set, and

$$a_n = \int_n^{n+1} f = \frac{1}{2n^2}.$$

Clearly the series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{2n^2}$ converges absolutely, but

$$\int_1^{\infty} |f| = \sum_{n=1}^{\infty} (1 + \frac{1}{2n^2}) = \infty,$$

which means f is not integratable on $[1, \infty)$.

Problem 2. Problem 33, Section 4.4, Page 90

Proof. First,

$$|f_n - f| \leq |f| + |f_n|,\ \forall n.$$

Then since f is integratable on E, if $\lim\limits_{n\to\infty} \int_E |f_n| = \int_E |f|$, we know $|f_n| + |f|$ converges pointwise a.e. to $2|f|$, and

$$\lim_{n\to\infty} \int_E (|f_n| + |f|) = 2\int_E |f| < \infty,$$

with General Lebesgue Dominated Convergence Theorem, notice $|f_n - f|$ converges pointwise a.e. to 0,

$$\lim_{n\to\infty} \int_E |f_n - f| = \int_E 0 = 0.$$

On the other hand, notice

$$|f_n| - |f| \leq |f_n - f|,\ \forall n.$$

with the same method, since $\int_E |f - f_n| \to 0$, and $|f - f_n|$ converges pointwise to 0,

$$\lim_{n\to\infty} \int_E |f_n - f| = \int_E 0 = 0,$$

we know from $|f_n| - |f|$ converges pointwise a.e. to 0,

$$\lim_{n\to\infty} \int_E |f_n| - |f| = \int_E 0 = 0.$$

Hence

$$\lim_{n\to\infty} \int_E |f_n| = \int_E |f|.$$