Instructions: In problems the problems below, references such as III.2.7 refer to Problem 7 in Section 2 of Chapter III in Conway's book.

If you use results from books including Conway's, please be explicit about what results you are using.

Homework 4 is due in class at Midnight March 9.

Do the following problems:

1. IV.7.1

   Sol. (Discussed with a college classmate) In fact, I don't think that this proposition is correct. For example, pick G the unit disk $B(0,1)$, and $\gamma = \gamma(t) : [0,1] \to B$, s.t. $\gamma(t) = t$ for $0 \le t < 1$, and $\gamma(1) = 0$. Then γ is closed, and by simple calculation we know $V(\gamma) = 2$, which shows γ is rectifiable.

   Let $f = \frac{1}{z-1}$, then f is analytic in $B(0,1)$. But when $t \to 1$, $f \circ \gamma(t) \to \infty$, hence it is not rectifiable.

2. IV.7.2

   (a) Let $f(z) = z$, pick any $z_0 \in \{z \mid d(z, \partial G) < \frac{1}{2}r\}$, then since there is only one point $z = z_0$ satisfies $f(z) = z_0$, by Thm 7.2,

   $$n(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} dz$$

   Since $\frac{1}{z-z_0}$ is analytic on $\{z \mid d(z, \partial G) < \frac{1}{2}r\}$, by Prop 2.15, we know the integral is 0. Hence $\{z \mid d(z, \partial G) < \frac{1}{2}r\} \subset H$.

3. V.1.1

   (a) Around $z = 0$,

   $$\lim_{z \to 0} |zf(z)| = \lim_{z \to 0} |\sin(z)| = \frac{1}{2} \lim_{z \to 0} |e^{iz} - e^{-iz}| \le \lim_{z \to 0} |z| = 0.$$

   Hence by Thm 1.2, $z = 0$ is removable, and $f(0) = 1$ by power series expansion.

   (b) At $z = 0$, $g(z) = \cos(z)$ is analytic, and $\cos(0) = 1$. Thus by Prop 1.4, $z = 0$ is a pole, and the singular part is $\frac{1}{z}$.

   (c) At $z = 0$, $\lim_{z \to 0} zf(z) = \lim_{z \to 0} \cos z - 1 = 0$, then by Thm 1.2, 0 is removable, and $f(0) = 0$ by power series expansion.

   (d) At $z = 0$,

   $$f(z) = \sum_{n=0}^{-\infty} \frac{1}{(-n)!} z^n,$$

   hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \{z \mid |z| > exp(\frac{1}{\delta})\}$.

   (e) At $z = 0$,

   $$f(z) = \frac{1}{z^2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+2} z^n.$$

   Hence 0 is a pole, and the singularity part is $\frac{1}{z}$.

   (f) At $z = 0$,

   $$f(z) = z \sum_{n=0}^{\infty} (-1)^n \frac{z^{-2n}}{n!} = z + \sum_{n=-1}^{-\infty} (-1)^{-n} \frac{z^{2n+1}}{(-n)!}$$

   Hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \mathbb{C}$.

(g) Around $z = 0$, notice $\frac{z^2+1}{z-1}$ is analytic, hence 0 is a pole. Since $|z| < 1$,

$$f(z) = 1 - \frac{1}{z} + \frac{2}{z-1} = 1 - \frac{1}{z} - 2\sum_{n=0}^{\infty} z^n,$$

we know the singular part is $-\frac{1}{z}$.

(h) For any $n > 0$,

$$\lim_{z\to 0} z^n f(z) = \lim_{z\to 0} z^n \frac{1}{\sum_{n=1}^{\infty} \frac{1}{n!} z^n} = \infty,$$

hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \{z \mid |z| > \frac{1}{1-e^\delta}\}$.

(i)

$$f(z) = z\sum_{n=0}^{\infty}(-1)^n \frac{z^{-(2n+1)}}{(2n+1)!} = 1 + \sum_{n=-1}^{-\infty}(-1)^{-n}\frac{z^{2n-1}}{(-2n+1)!},$$

hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \{z \mid |z| < \delta\}$.

(j) Same with (i), 0 is an essential singularity, and $f(0 < |z| < \delta) = \{z \mid |z| < \delta^n\}$.

4. V.1.4

   (a)

   $$f(z) = \frac{1}{z}\left(\frac{1}{1-z} - \frac{1}{2(1-z/2)}\right) = \frac{1}{z}\left(\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}(\frac{z}{2})^n\right) = \frac{1}{2z} + \sum_{n=0}^{\infty}(1 - \frac{1}{2^{n+2}})z^n$$

   (b)

   $$f(z) = \frac{1}{z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right) = \frac{1}{z}\left(-\frac{1}{2}\frac{1}{1-\frac{z}{2}} - \frac{1}{z}\frac{1}{1-\frac{1}{z}}\right) = \frac{1}{z}\left(-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty} z^{-n-1}\right) = -\sum_{n=-1}^{\infty}\frac{z^n}{2^{n+2}} - \sum_{n=-\infty}^{-2} z^n$$

   (c)

   $$f(z) = \frac{1}{z}\left(\frac{\frac{1}{z}}{1-\frac{2}{z}} - \frac{\frac{1}{z}}{1-\frac{1}{z}}\right) = \frac{1}{z}\left(\sum_{n=0}^{\infty}(\frac{2}{z})^n - \sum_{n=0}^{\infty}\frac{1}{z^n}\right) = \sum_{n=-\infty}^{-1}(2^{-(n+1)} - 1)z^n.$$

5. V.1.12

   Proof. By (1.11), since f is analytic on $0 < |z| < \infty$,

   $$a_n = \frac{1}{2\pi i}\int_\gamma \frac{\exp(\frac{1}{2}\lambda(z + \frac{1}{z}))}{z^{n+1}}dz$$

   pick $\gamma = \exp(it)$, the unit circle, then

   $$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{\lambda\cos t}e^{-int}dt = \frac{1}{2\pi}\int_0^{2\pi} e^{\lambda\cos t}(\cos nt - i\sin nt)dt,$$

   the real part is

   $$\frac{1}{2\pi}\int_0^{2\pi} e^{\lambda\cos t}\cos nt\,dt = \frac{1}{2\pi}\left(\int_0^{\pi} e^{\lambda\cos t}\cos nt\,dt - \int_\pi^0 e^{\lambda\cos(2\pi-s)}\cos(2\pi - s)ds\right) = \frac{1}{\pi}\int_0^{\pi} e^{\lambda\cos t}\cos nt\,dt.$$

   and the imaginary part is

   $$-\frac{i}{2\pi}\int_0^{2\pi} e^{\lambda\cos t}\sin nt\,dt = -\frac{i}{2\pi}\left(\int_0^{\pi} e^{\lambda\cos t}\sin nt\,dt + \int_\pi^0 e^{\lambda\cos(2\pi-s)}\sin(2\pi - s)ds\right) = 0.$$

   Hence $a_n = \frac{1}{\pi}\int_0^{\pi} e^{\lambda\cos t}\cos nt\,dt$.

2

$$b_n = \frac{1}{2\pi i} \int_\gamma \frac{\exp(\frac{1}{2}\lambda(z - \frac{1}{z}))}{z^{n+1}} dz$$

pick $\gamma = \exp(it)$,

$$b_n = \frac{1}{2\pi} \int_0^{2\pi} e^{\lambda i \sin t} e^{-int} dt = \frac{1}{2\pi} \int_0^{2\pi} \cos(nt - \lambda \sin t) - i \sin(nt - \lambda \sin t) dt.$$

The real part is

$$\frac{1}{2\pi} \int_0^{2\pi} \cos(nt - \lambda \sin t) dt = \frac{1}{2\pi} \left(\int_0^\pi \cos(nt - \lambda \sin t) dt - \int_\pi^0 \cos(n(2\pi - s) - \lambda \sin(2\pi - s)) ds \right)$$

$$= \frac{1}{2\pi} \left(\int_0^\pi \cos(nt - \lambda \sin t) dt + \int_0^\pi \cos(-ns + \lambda \sin s) ds \right) = \frac{1}{\pi} \int_0^\pi \cos(nt - \lambda \sin t) dt.$$

and the imaginary part is

$$-\frac{i}{2\pi} \int_0^{2\pi} \sin(nt - \lambda \sin t) dt = -\frac{i}{2\pi} \left(\int_0^\pi \sin(nt - \lambda \sin t) dt - \int_\pi^0 \sin(n(2\pi - s) - \lambda \sin(2\pi - s)) ds \right)$$

$$= -\frac{i}{2\pi} \left(\int_0^\pi \sin(nt - \lambda \sin t) dt - \int_0^\pi \sin(-ns + \lambda \sin s) ds \right) = 0.$$

Hence $b_n = \frac{1}{\pi} \int_0^\pi \cos(nt - \lambda \sin t) dt$.

6. V.1.13

(a) Suppose f is entire and has a removable singularity at ∞, then $g(z) = f(\frac{1}{z})$ has a removable singularity at $z = 0$, thus we can define $g(0) = a < \infty$, which means f is bounded in the neighbourhood of ∞. Hence by Liouville Thm, f is constant.

(b) By assumption, $g(z) = f(\frac{1}{z}) = \frac{1}{z^m} h(z)$, where $h(z)$ is analytic at $z = 0$. Since $f(z) = z^m h(\frac{1}{z})$ is entire, it means at $z = 0$, h has a definition or has a removable singularity, and h is entire. Hence, by (1) we know h is a constant, which means f is a polynomial of degree m.

(c) Let $f(z) = \frac{\prod(z - u_i)}{\prod(z - v_i)}$, then

$$g(z) = f(\frac{1}{z}) = \frac{\prod_{i=1}^n (\frac{1}{z} - u_i)}{\prod_{i=1}^m (\frac{1}{z} - v_i)}$$

has a removable singularity at $z = 0$, which means $\lim_{z \to 0} g(z)$ exists and is not ∞. Notice

$$\lim_{z \to 0} g(z) = \lim_{z \to 0} \frac{z^m \prod_{i=1}^n (\frac{1}{z} - u_i)}{\prod_{i=1}^m (1 - v_i z)},$$

if $m < n$, then $\lim_{z \to 0} g(z) = \infty$, which makes a contradiction. If $m \geq n$, then the limit is well-defined. Hence $f = \frac{p(z)}{q(z)}$, where p, q are polynomials, and $\deg(p) \leq \deg(q)$.

(d) Let $f(z) = \frac{\prod(z - u_i)}{\prod(z - v_i)}$, then

$$g(z) = f(\frac{1}{z}) = \frac{\prod_{i=1}^n (\frac{1}{z} - u_i)}{\prod_{i=1}^k (\frac{1}{z} - v_i)}$$

has a pole of order m at $z = 0$, which means $g(z) = \frac{h(z)}{z^m}$, where $h(z)$ is analytic at $z = 0$. Then

$$z^m g(z) = \frac{z^m \prod_{i=1}^n (\frac{1}{z} - u_i)}{\prod_{i=1}^k (\frac{1}{z} - v_i)} = \frac{z^{m+k} \prod_{i=1}^n (\frac{1}{z} - u_i)}{\prod_{i=1}^m (1 - v_i z)},$$

and we have $m + k = n$, otherwise the order of pole is not m. Hence $f = \frac{p}{q}$, and $\deg(p) - \deg(q) = m$.

7. **V.1.17**

Proof. If not, first suppose f has a pole of order m at a. Then $f(z) = \frac{g(z)}{(z-a)^m}$, and g is analytic on G.

i) If $g(a) = 0$, then $f(z) = 0$, and we can define $f(a) = 0$, thus a is a removable singularity.

ii) If $g(a) \neq 0$, then according to the isolation of zeros, $\exists r > 0$, s.t. $g(z) \neq 0$ in $B(a, 2r)$. Consider $H = B(a, r)$, by max modulus theorem, $\min_H |g| = \min_{\partial H} |g|$, denote is as $c \neq 0$. Hence

$$\int\int_H |f(x+iy)|^2 dxdy = \int\int_H \frac{|g(x+iy)|^2}{|x+iy-a|^{2m}} dxdy \geq \int\int_H \frac{c^2}{|x+iy-a|^2} dxdy.$$

Let $x = Re(a) + s\cos t, y = Im(a) + s\sin t$, then

$$\int\int_H |f(x+iy)|^2 dxdy \geq \int_0^r \int_0^{2\pi} \frac{c^2}{s^{2m}} s\,ds\,dt = 2\pi c^2 \int_0^r s^{1-2m} ds.$$

If $m = 1$, then the integral becomes

$$\int_0^r s^{-1} ds = \ln(s)\Big|_0^r = \infty.$$

If $m > 1$, then the integral is

$$\int_0^r s^{1-2m} ds = \frac{1}{2-2m} s^{2-2m}\Big|_0^r = \infty.$$

Since the integrand is nonnegative,

$$\int_G |f|^2 \geq \int_H |f|^2 = \infty,$$

which makes a contradiction. With the same method, we know a is not an essential singularity. Hence a is a removable one.

By the deduction we know, if $0 < p < 2$, a could be a removable singularity or a pole of order m, which satisfies $pm < 2$. If $p \geq 2$, then a is a removable singularity.

8. **V.2.1**

(a) Notice $(x^4 + x^2 + 1)(x^2 - 1) = x^6 - 1$, so $f(z) = \frac{z^2}{z^4+z^2+1}$ has four poles of order 1, and they are sixth roots of 1, which are $a_i = e^{i\theta_i}$ where

$$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}.$$

Consider the two poles in the upper-half plane a_1, a_2, then

$$Res(f, a_1) = \lim_{z \to a_1} (z - a_1)f(z) = a_1^2 (a_1 - a_2)^{-1}(a_1 - a_3)^{-1}(a_1 - a_4)^{-1} = \frac{1}{2\sqrt{3}i} e^{\frac{\pi}{3}i},$$

$$Res(f, a_2) = \lim_{z \to a_2} (z - a_2)f(z) = a_2^2 (a_2 - a_1)^{-1}(a_2 - a_3)^{-1}(a_2 - a_4)^{-1} = -\frac{1}{2\sqrt{3}i} e^{\frac{2\pi}{3}i}.$$

Let γ be the closed path which is the boundary of upper half of disk of radius $R > 1$ with center 0, in counter-clockwise direction. Then by Residue Theorem,

$$\frac{1}{2\pi i} \int_\gamma f = Res(f, a_1) + Res(f, a_2) = \frac{1}{2\sqrt{3}i}$$

And

$$\int_\gamma f = \int_{-R}^R \frac{x^2}{x^4 + x^2 + 1} dx + i \int_0^\pi \frac{R^3 e^{i3t}}{R^4 e^{i4t} + R^2 e^{i2t} + 1} dt.$$

4

Notice $|R^4 e^{i4t} + R^2 e^{i2t} + 1| \geq R^4 - R^2 = 1$ when R is sufficiently large, so

$$\left| i \int_0^\pi \frac{R^3 e^{i3t}}{R^4 e^{i4t} + R^2 e^{i2t} + 1} dt \right| \leq \frac{\pi R^3}{R^4 - R^2 - 1} \to 0 \text{ as } R \to \infty.$$

Since $\frac{x^2}{x^4 + x^2 + 1} \geq 0, \forall x \in \mathbb{R}$, we know

$$\int_0^\infty \frac{x^2}{x^4 + x^2 + 1} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x^2}{x^4 + x^2 + 1} = \frac{1}{2} \lim_{R \to \infty} \int_{-R}^R \frac{x^2}{x^4 + x^2 + 1} dx = \frac{\pi}{\sqrt{3}}.$$

For the followsing problems, I will not write all the steps since I don't have enough time to do so. I will only write the key points.

(b) We know

$$\int_r^R \frac{\cos x - 1}{x^2} dx = \frac{1}{2} \left(\int_r^R + \int_{-R}^{-r} \frac{e^{ix} - 1}{x^2} dx \right)$$

Consider $f = \frac{e^{iz} - 1}{z^2}$, then f has a simple pole at $z = 0$. Let γ be the curve in Example 2.7, then

$$0 = \int_r^R + \int_{-R}^{-r} \frac{e^{ix} - 1}{x^2} dx + \int_{\gamma_r} + \int_{\gamma_R} \frac{e^{iz} - 1}{z^2} dz$$

Since

$$\left| \int_{\gamma_R} \frac{e^{iz} - 1}{z^2} dz \right| = \left| \int_0^\pi \frac{\exp(iRe^{it}) - 1}{Re^{it}} dt \right| \leq \frac{1}{R} \int_0^\pi \exp(-R \sin t) dt,$$

and by Example 2.7 we know the integral $\to 0$ as $R \to \infty$. And since $\frac{e^{iz} - 1 - iz}{z^2}$ has a removable singularity at $z = 0$, by Example 2.7 we know

$$\lim_{\gamma \to 0} \left| \int_{\gamma_r} \frac{e^{iz} - 1 - iz}{z^2} dz \right| = 0,$$

but $\int_\gamma \frac{i}{z} dz = \pi$, hence

$$\lim_{\gamma \to 0} \left| \int_{\gamma_r} \frac{e^{iz} - 1}{z^2} dz \right| = \pi.$$

Hence

$$\int_0^\infty \frac{\cos x - 1}{x^2} dx = \lim_{r \to 0, R \to \infty} \int_r^R \frac{\cos x - 1}{x^2} dx = -\frac{\pi}{2}.$$

(c) The same with Example 2.9, let $z = e^{i\theta}$, then

$$\int_0^\pi \frac{\cos 2\theta}{1 - 2a \cos \theta + a^2} d\theta = -\frac{i}{4} \int_\gamma \frac{z^4 + 2z^2 + 1}{-az^2 + (a^2 + 1)z - a} dz,$$

where $\gamma = \{|z| = 1\}$. By residue theorem,

$$\int_\gamma \frac{z^4 + 2z^2 + 1}{-az^2 + (a^2 + 1)z - a} dz = 2\pi i Res(f, a) = 2\pi i \frac{(a^2 + 1)^2}{1 - a^2}.$$

Hence

$$\int_0^\pi \frac{\cos 2\theta}{1 - 2a \cos \theta + a^2} d\theta = \frac{\pi}{2} \frac{(a^2 + 1)^2}{1 - a^2}.$$

(d) Same with (c), let $z = e^{i\theta}$, then

$$\int_0^\pi \frac{d\theta}{(a + \cos \theta)^2} = -2i \int_\gamma \frac{z}{(z^2 + 2az + 1)^2} dz,$$

5

where $\gamma = \{|z| = 1\}$. By residue theorem,

$$\int_\gamma \frac{z}{(z^2 + 2az + 1)^2} dz = \frac{1}{2}\pi i a(a^2 - 1)^{-\frac{3}{2}}.$$

Hence

$$\int_0^\pi \frac{d\theta}{(a + \cos\theta)^2} = \pi a(a^2 - 1)^{-\frac{3}{2}}.$$

9. V.2.2

(a)

$$\int_0^\infty \frac{dx}{(x^2 + a^2)^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(x^2 + a^2)^2}.$$

Consider $R > a$, let $\gamma = \{|z| = R, Im(z) > 0\}$, then by residue thm, since f has a pole of order 2, $z = ai$,

$$\int_\gamma + \int_{-R}^R \frac{dz}{(z^2 + a^2)^2} = 2\pi i Res(f, ai) = \frac{\pi}{2a^3}.$$

But

$$\left|\int_\gamma \frac{dz}{(z^2 + a^2)^2}\right| \leq \int_\gamma \left|\frac{dz}{(z^2 + a^2)^2}\right| \leq \int_0^\pi \frac{1}{|R^2 - a^2|^2} dt \leq \frac{\pi}{|R^2 - a^2|^2} \to 0, \text{ as } R \to \infty.$$

Hence

$$\int_0^\infty \frac{dx}{(x^2 + a^2)^2} = \frac{\pi}{4a^3}.$$

(b) Define $G, l(z), \gamma$ as in Example 2.10, then

$$\int_\gamma \frac{(l(z))^3}{1 + z^2} dz = \int_r^R \frac{(\log x)^3}{1 + x^2} dx + \int_0^\pi \frac{(\log R + i\theta)^3}{1 + R^2 e^{2i\theta}} iRe^{i\theta} d\theta + \int_{-R}^{-r} \frac{(\log|x| + \pi i)^3}{1 + x^2} dx + \int_\pi^0 \frac{(\log r + i\theta)^3}{1 + r^2 e^{2i\theta}} ire^{i\theta} d\theta$$

by residue thm,

$$\int_\gamma \frac{(l(z))^3}{1 + z^2} dz = 2\pi i Res(f, i) = -\frac{1}{8}\pi^4 i.$$

And

$$\int_r^R \frac{(\log x)^3}{1 + x^2} dx + \int_{-R}^{-r} \frac{(\log|x| + \pi i)^3}{1 + x^2} dx = \int_r^R \frac{(\log x)^3 + (\log x + \pi i)^3}{1 + x^2} dx$$

$$= 2\int_r^R \frac{(\log x)^3}{1 + x^2} + 3\pi i \int_r^R \frac{(\log x)^2}{1 + x^2} dx - 3\pi^2 \int_r^R \frac{\log x}{1 + x^2} dx - \pi^3 i \int_r^R \frac{1}{1 + x^2} dx$$

we know from example 2.10, as $r \to 0, R \to \infty$, the third term is 0, and the last term is

$$-\pi^3 i \int_0^\infty \frac{1}{1 + x^2} dx = -\frac{1}{2}\pi^4 i.$$

Now we consider the second term. Using the same notation $G, l(z), \gamma$,

$$\int_\gamma \frac{(l(z))^2}{1 + z^2} dz = \int_r^R \frac{(\log x)^2}{1 + x^2} dx + \int_0^\pi \frac{(\log R + i\theta)^2}{1 + R^2 e^{2i\theta}} iRe^{i\theta} d\theta + \int_{-R}^{-r} \frac{(\log|x| + \pi i)^2}{1 + x^2} dx + \int_\pi^0 \frac{(\log r + i\theta)^2}{1 + r^2 e^{2i\theta}} ire^{i\theta} d\theta$$

by residue theorem,

$$\int_\gamma \frac{(l(z))^2}{1 + z^2} dz = 2\pi i Res(f, i) = -\frac{1}{4}\pi^3.$$

And

$$\int_r^R \frac{(\log x)^2}{1 + x^2} dx + \int_{-R}^{-r} \frac{(\log|x| + \pi i)^2}{1 + x^2} dx = \int_r^R \frac{(\log x)^2 + (\log x + \pi i)^2}{1 + x^2} dx$$

$$= 2\int_r^R \frac{(\log x)^2}{1 + x^2} + 2\pi i \int_r^R \frac{\log x}{1 + x^2} dx - \pi^2 \int_r^R \frac{1}{1 + x^2} dx$$

6

when $r \to 0, R \to \infty$, the second term is 0, and the last term is

$$-\pi^2 \int_0^\infty \frac{1}{1+x^2} dx = -\frac{1}{2}\pi^3.$$

Also, if $\rho > 0$ then

$$\left| \rho \int_0^\pi \frac{(\log \rho + i\theta)^2}{1 + \rho^2 e^{2i\theta}} e^{i\theta} d\theta \right| \leq \rho \frac{|\log \rho|^2}{|1-\rho^2|} \int_0^\pi d\theta + 2\rho \frac{|\log \rho|}{|1-\rho^2|} \int_0^\pi \theta d\theta + \frac{\rho}{|1-\rho^2|} \int_0^\pi \theta^2 d\theta$$

$$= \rho \frac{|\log \rho|^2}{|1-\rho^2|} \pi + \rho \frac{|\log \rho|}{|1-\rho^2|} \pi^2 + \frac{\rho}{|1-\rho^2|} \frac{\pi^3}{3}$$

let $\rho \to 0^+$ or $\rho \to \infty$, these terms will be 0. It is just the same for the three-order case, thus

$$\int_0^\infty \frac{(\log x)^2}{1+x^2} dx = \frac{1}{8}\pi^3,$$

hence

$$\int_0^\infty \frac{(\log x)^3}{1+x^2} dx = 0.$$

(c) First,

$$\int_0^\infty \frac{\cos ax}{(1+x^2)^2} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{e^{iax}}{(1+x^2)^2} dx.$$

Let $\gamma = \{|z| = R, Im(z) > 0\} \cup [-R, R]$, then by residue theorem,

$$\int_\gamma \frac{e^{iaz}}{(1+z^2)^2} dz = 2\pi i Res(f, i) = \frac{(a+1)e^{-a}\pi}{2}.$$

And

$$\left| \int_{\gamma_R} \frac{e^{iaz}}{(1+z^2)^2} dz \right| = \left| \int_0^\pi \frac{e^{-aR\sin\theta}(\cos(aR\cos\theta) + i\sin(aR\cos\theta))}{(1+R^2 e^{2i\theta})^2} iRe^{i\theta} d\theta \right|$$

$$\leq \left| \int_0^\pi \frac{R}{|1-R^2|^2} d\theta \right| \to 0, \text{ when } R \to \infty.$$

Hence

$$\int_0^\infty \frac{\cos ax}{(1+x^2)^2} dx = \frac{(a+1)e^{-a}\pi}{4}.$$

10. V.2.3

Sol. We have proved that f has an essential singularity at $z = 0$. Denote G be the region surrounded by γ.

i) If $0 \notin G$, then by Cauchy's Thm, $\int_\gamma f = 0$.

ii) If $0 \in G$, then by Laurent expansion, $Res(f, 0) = 1$. But by a problem in last homework, we can construct a γ, for each $n \in \mathbb{N}$, $n(\gamma, 0) = n$. Hence $\int_\gamma f = 2\pi ni, n \in \mathbb{N}$.

11. V.2.4

Sol. In fact, in a neighbourhood of a, $f = \frac{c_{-1}}{z-a} + \sum_{n=0}^\infty c_n(z-a)^n$, and $g = \sum_{n=0}^\infty d_n(z-a)^n$. Hence by definition,

$$Res(fg, a) = c_{-1}d_0 = Res(f, a)g(a).$$

12. V.2.5

Sol. Notice in the last problem, each simple pole of f is a simple pole of fg. By residue thm,

$$\frac{1}{2\pi i} \int_\gamma fg = \sum_{k=1}^n n(\gamma, a_k) Res(fg, a_k) = \sum_{k=1}^n n(\gamma, a_k) Res(f, a_k)g(a_k).$$