

Statistics of Solutions to A Stochastic Differential Equation Set

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- 1 Introduction
 - SDE
 - Itô Integration
- 2 Statistics of $b(t)$ and $\gamma(t)$
 - Mean
 - Variance
 - Covariance
- 3 Statistics of $u(t)$
 - Mean
 - Variance
 - Covariance
- 4 Numerical Simulation

The SDEs

$$\begin{cases} \frac{du(t)}{dt} = (-\gamma(t) + i\omega)u(t) + b(t) + f(t) + \sigma W(t), \\ \frac{db(t)}{dt} = (-\gamma_b + i\omega_b)(b(t) - \hat{b}) + \sigma_b W_b(t), \\ \frac{d\gamma(t)}{dt} = -d_\gamma(\gamma(t) - \hat{\gamma}) + \sigma_\gamma W_\gamma(t) \end{cases}$$

The initial values are complex random variables, with their first-order and second-order statistics known.

Solution

With knowledge of ODEs, the solution of the SDE set is

$$\left\{ \begin{array}{l} b(t) = \hat{b} + (b_0 - \hat{b})e^{\lambda_b(t-t_0)} + \sigma_b \int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s) \\ \gamma(t) = \hat{\gamma} + (\gamma_0 - \hat{\gamma})e^{-d_\gamma(t-t_0)} + \sigma_\gamma \int_{t_0}^t e^{-d_\gamma(t-s)} dW_\gamma(s) \\ u(t) = e^{-J(t_0,t) + \hat{\lambda}(t-t_0)} u_0 + \int_{t_0}^t (b(s) + f(s)) e^{-J(s,t) + \hat{\lambda}(s-t_0)} ds \\ \quad + \sigma \int_{t_0}^t e^{-J(s,t) + \hat{\lambda}(s-t_0)} dW(s) \end{array} \right.$$

with $\lambda_b = -\gamma_b + i\omega_b$, $\hat{\lambda} = -\hat{\gamma} + i\omega$, $J(s,t) = \int_s^t (\gamma(s') - \hat{\gamma}) ds'$.

Itô integration

Itô Isometry

$\forall f \in \mathcal{V}(S, T)$, B_t is a standard Brownian motion,

$$\mathbb{E} \left[\left(\int_S^T f(t, \omega) dB_t \right)^2 \right] = \mathbb{E} \left[\int_S^T f^2(t, \omega) dt \right].$$

Itô Formula

Assume that X_t is a Itô process satisfying $dX_t = udt + vdB_t$, $g(t, x) \in C^2([0, \infty) \times \mathbb{R})$, then $Y_t = g(t, X_t)$ is also a Itô process satisfying

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X_t) \cdot (dX_t)^2,$$

with $dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0$, $dB_t \cdot dB_t = dt$.

Linear property of Itô integration

$$(1) \quad \int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t, \text{ a.e.}$$

$$(2) \quad \int_S^T (cf + g) dB_t = c \int_S^T f dB_t + \int_S^T g dB_t, \text{ a.e.}$$

$$(3) \quad E \left[\int_S^T f dB_t \right] = 0.$$

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With property of Itô integration (3), it is easy to know

$$\mathbb{E}(b(t)) = \hat{b} + (\mathbb{E}[b_0] - \hat{b})e^{\lambda_b(t-t_0)}$$

$$\mathbb{E}(\gamma(t)) = \hat{\gamma} + (\mathbb{E}[\gamma_0] - \hat{\gamma})e^{-d_\gamma(t-t_0)}$$

According to definition,

$$\begin{aligned}\text{Var}(b(t)) &= \mathbb{E}[(b(t) - \mathbb{E}[b(t)])(b(t) - \mathbb{E}[b(t)])^*] \\&= e^{-2\gamma_b(t-t_0)}\text{Var}(b_0) + \mathbb{E}\left[\sigma_b^2 \int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s) \left(\int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s)\right)^*\right] \\&= e^{-2\gamma_b(t-t_0)}\text{Var}(b_0) + \sigma_b^2 \mathbb{E}\left[\int_{t_0}^t e^{-2\gamma_b(t-s)} ds\right]\end{aligned}$$

The last step takes advantage of Itô isometry.

$$\text{Var}(b(t)) = e^{-2\gamma_b(t-t_0)}\text{Var}(b_0) + \frac{\sigma_b^2}{2\gamma_b}(1 - e^{-2\gamma_b(t-t_0)})$$

$$\text{Var}(\gamma(t)) = e^{-2d_\gamma(t-t_0)}\text{Var}(\gamma_0) + \frac{\sigma_\gamma^2}{2d_\gamma}(1 - e^{-2d_\gamma(t-t_0)})$$

Covariance

$$\begin{aligned}\text{Cov}(b(t), b(t)^*) &= \mathbb{E}[(b(t) - \mathbb{E}[b(t)])(b(t)^* - \mathbb{E}[b(t)^*])] \\&= \mathbb{E}\left[(b_0 - \mathbb{E}[b_0])(b_0^* - \mathbb{E}[b_0^*])e^{2\lambda_b(t-t_0)}\right] + \sigma_b \mathbb{E}\left[(b_0 - \mathbb{E}[b_0]) \int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s)\right] \\&\quad + \sigma_b \mathbb{E}\left[(b_0^* - \mathbb{E}[b_0^*])e^{\lambda_b(t-t_0)} \int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s)\right] + \sigma_b^2 \mathbb{E}\left[\left(\int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s)\right)^2\right]\end{aligned}$$

With property of Itô integration (3), the second and third term are both 0; with Itô isometry we know the last term is also 0.

$$\text{Cov}(b(t), b(t)^*) = \mathbb{E}[(b_0 - \mathbb{E}[b_0])(b_0^* - \mathbb{E}[b_0^*])]e^{2\lambda_b(t-t_0)} = \text{Cov}(b_0, b_0^*)e^{2\lambda_b(t-t_0)}$$

$$\text{Cov}(b(t), \gamma(t)) = \mathbb{E}[(b(t) - \mathbb{E}[b(t)])(\gamma(t) - \mathbb{E}[\gamma(t)])] = \text{Cov}(b_0, \gamma_0)e^{(\lambda_b - d_\gamma)(t-t_0)}$$

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Using the same properties, it's obvious to have

$$\begin{aligned} \mathbf{E}[u(t)] &= e^{\hat{\lambda}(t-t_0)} \mathbf{E} \left[e^{-J_0(t_0,t)} u_0 \right] + \int_{t_0}^t e^{\hat{\lambda}(t-s)} \mathbf{E} \left[b(s) e^{-J(s,t)} \right] ds \\ &\quad + \sigma \int_{t_0}^t e^{\hat{\lambda}(t-s)} f(s) \mathbf{E} \left[e^{-J(s,t)} \right] ds. \end{aligned}$$

It is necessary to compute expectations of terms like $\mathbf{E}[ze^{bx}]$, where z is a complex-valued Gaussian random variable and x is a real-valued Gaussian variable. We propose two lemmas here.

Lemma 1

Lemma

$$E\left[ze^{ibx}\right] = (E[z] + ib\text{Cov}(z,x))e^{ibE[x] - \frac{1}{2}b^2\text{Var}(x)}$$

with z being a complex-valued Gaussian, and x a real-valued Gaussian.

Corollary

Under the condition of Lemma 1,

$$E\left[ze^{bx}\right] = (E[z] + b\text{Cov}(z,x))e^{bE[x] + \frac{1}{2}b^2\text{Var}(x)}.$$

Proof of lemma 1 takes advantage of the characteristic function of multivariable Gaussian distribution.

Proof

Let $z = y + iw$, $y, w \in \mathbb{R}$. Denote $\mathbf{v} = (x, y, w)$, then \mathbf{v} satisfies the multivariable Gaussian distribution, with its characteristic function

$$\phi_{\mathbf{v}}(\mathbf{s}) = \exp(is^{\top} \mathbb{E}[\mathbf{v}] - \frac{1}{2} \mathbf{s}^{\top} \Sigma \mathbf{s}).$$

Let $g(\mathbf{v})$ being the PDF of \mathbf{v} , then one knows from that char. func. being Fourier transform of PDF,

$$\phi_{\mathbf{v}}(\mathbf{s}) = \frac{1}{(2\pi)^3} \int e^{is^{\top} \mathbf{v}} g(\mathbf{v}) d\mathbf{v}$$

According to the differential property of Fourier transform,

$$\frac{\partial \phi_{\mathbf{v}}(\mathbf{s})}{\partial s_2} = \frac{1}{(2\pi)^3} \int i y_0 e^{is^{\top} \mathbf{v}} g(\mathbf{v}) d\mathbf{v} = i \mathbb{E} \left[y_0 e^{is^{\top} \mathbf{v}} \right].$$

Let $\mathbf{v} = (b, 0, 0)^{\top}$,

$$\mathbb{E}[y_0 e^{ibx_0}] = -i \frac{\partial \phi_{\mathbf{v}}(s)}{\partial s_2} \Big|_{\mathbf{s}=(b,0,0)^{\top}}$$

$$\mathbb{E}[y_0 e^{ibx_0}] = -i \frac{\partial \phi_{\mathbf{v}}(s)}{\partial s_2} \Big|_{\mathbf{s}=(b,0,0)^{\top}}$$

From PDF of multivariable Gaussian distribution, one knows

$$\frac{\partial \phi_{\mathbf{v}}(\mathbf{s})}{\partial s_2} = (i\mathbb{E}[y_0] - \text{Var}(y_0)s_2 - \text{Cov}(x_0, y_0)s_1 - \text{Cov}(y_0, w_0)s_3)\phi_{\mathbf{v}}(\mathbf{s})$$

$$\frac{\partial \phi_{\mathbf{v}}(\mathbf{s})}{\partial s_3} = (i\mathbb{E}[w_0] - \text{Var}(w_0)s_3 - \text{Cov}(x_0, w_0)s_1 - \text{Cov}(y_0, w_0)s_2)\phi_{\mathbf{v}}(\mathbf{s})$$

Compute the partial derivatives at $\mathbf{s} = (b, 0, 0)^\top$,

$$\mathbb{E}\left[y_0 e^{ibx_0}\right] = (\mathbb{E}[y_0] + i\text{Cov}(x_0, y_0)b) \exp(ib\mathbb{E}[x_0] - \frac{1}{2}\text{Var}(x_0)b^2)$$

$$\mathbb{E}\left[w_0 e^{ibx_0}\right] = (\mathbb{E}[w_0] + i\text{Cov}(x_0, w_0)b) \exp(ib\mathbb{E}[x_0] - \frac{1}{2}\text{Var}(x_0)b^2)$$

Then

$$\mathbb{E}\left[ze^{ibx}\right] = (\mathbb{E}[z] + ib\text{Cov}(z, x))e^{ib\mathbb{E}[x] - \frac{1}{2}b^2\text{Var}(x)}.$$



Lemma 2

Lemma

$$E\left[zw e^{bx}\right] = [E[z]E[w] + \text{Cov}(z, w^*) + b(E[z]\text{Cov}(w, x)) + E[w]\text{Cov}(z, x) + b^2\text{Cov}(z, x)\text{Cov}(w, x)] e^{bE[x] + \frac{b^2}{2}\text{Var}(x)}.$$

with z, w being complex-valued Gaussian, and x real-valued Gaussian.

The proof of this lemma is the same as Lemma 1.

We now make use of Lemma 1 to obtain the mean of $u(t)$.

$$\begin{aligned} \mathbb{E}[u(t)] &= e^{\hat{\lambda}(t-t_0)} (\mathbb{E}[u_0] - \text{Cov}(u_0, J(t_0, t))) e^{-\mathbb{E}[J(t_0, t)] + \frac{1}{2} \text{Var}(J(t_0, t))} \\ &\quad + \int_{t_0}^t e^{\hat{\lambda}(t-s)} (\hat{b} + e^{\lambda_b(s-t_0)} (\mathbb{E}[b_0] - \hat{b} - \text{Cov}(b_0, J(s, t)))) e^{-\mathbb{E}[J(s, t)] + \frac{1}{2} \text{Var}(J(s, t))} ds \\ &\quad + \int_{t_0}^t e^{\hat{\lambda}(t-s)} f(s) e^{-\mathbb{E}[J(s, t)] + \frac{1}{2} \text{Var}(J(s, t))} ds \end{aligned}$$

The terms $\text{Cov}(u_0, J(s, t))$, $\text{Cov}(b_0, J(s, t))$, $\mathbb{E}[J(s, t)]$ and $\text{Var}(J(s, t))$ can be found using Itô isometry.

Variance

Denote $u(t) = A + B + C$,

$$\begin{cases} A = e^{-J(t_0,t) + \hat{\lambda}(t-t_0)} u_0, \\ B = \int_{t_0}^t (b(s) + f(s)) e^{-J(s,t) + \hat{\lambda}(t-s)} ds, \\ C = \sigma \int_{t_0}^t e^{-J(s,t) + \hat{\lambda}(t-s)} dW(s). \end{cases}$$

By definition we find $\text{Var}(u(t)) = \mathbb{E}[|u(t)|^2] - |\mathbb{E}[u(t)]|^2$, with

$$\mathbb{E}[|u(t)|^2] = \mathbb{E}[|A|^2] + \mathbb{E}[|B|^2] + \mathbb{E}[|C|^2] + 2\text{Re}\{\mathbb{E}[A^*B]\}.$$

We can obtain $\mathbb{E}[|A|^2]$ by Lemma 2, and $\mathbb{E}[|B|^2]$ by Itô isometry and noticing that

$$\text{Cov}(J(s,t), J(r,t)) = \text{Var}(J(s,t)) + \text{Cov}(J(s,t), J(r,s)).$$

$\mathbb{E}[|C|^2]$ and $\text{Re}\{\mathbb{E}[A^*B]\}$ can also be computed by Itô isometry and property of Itô integration.

By definition,

$$\text{Cov}(u(t), u^*(t)) = \mathbb{E}[u(t)^2] - \mathbb{E}[u(t)]^2$$

$$\text{Cov}(u(t), \gamma(t)) = \mathbb{E}[u(t)(\gamma(t) - \hat{\gamma})] + \mathbb{E}[u(t)](\hat{\gamma} - \mathbb{E}[\gamma(t)])$$

$$\text{Cov}(u(t), b(t)) = \mathbb{E}[u(t)b^*(t)] - \mathbb{E}[u(t)]\mathbb{E}[b(t)]^*$$

$$\text{Cov}(u(t), b^*(t)) = \mathbb{E}[u(t)b(t)] - \mathbb{E}[u(t)]\mathbb{E}[b(t)].$$

Each term can be obtained by Lemma 3 and Itô isometry.

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Parameters

External forcing $f(t) = \frac{3}{2}e^{0.1it}$, and parameters of the equation set are given

$$\begin{cases} d = 1.5, & d_\gamma = 0.01d \\ \sigma = 0.1549, & \omega = 1.78 \\ \sigma_\gamma = 5\sigma, & \gamma_b = 0.1d \\ \sigma_b = 5\sigma, & \omega_b = \omega \\ \hat{b} = 0, & \hat{\gamma} = 0 \end{cases}$$

We assume the initial values satisfying

$$\begin{cases} \operatorname{Re}(u_0), \operatorname{Im}(u_0) \sim \mathcal{N}(0, \frac{1}{\sqrt{2}}), & \text{i.i.d.} \\ \operatorname{Re}(b_0), \operatorname{Im}(b_0) \sim \mathcal{N}(0, \frac{1}{\sqrt{2}}), & \text{i.i.d.} \\ \gamma_0 \sim \mathcal{N}(0, 1) & \text{i.i.d.} \end{cases}$$

The statistics between initial values are

$$\begin{cases} \text{Cov}(u_0, u_0^*) = 0 \\ \text{Cov}(u_0, \gamma_0) = 0 \\ \text{Cov}(u_0, b_0) = 0 \\ \text{Cov}(u_0, b_0^*) = 0 \end{cases}$$

Euler-Maruyama Scheme

Itô integration can be simulated by E-M scheme:

$$X_j = X_{j-1} + f(X_{j-1})\Delta t + g(X_{j-1})(W(\tau_j) - W(\tau_{j-1})),$$

with

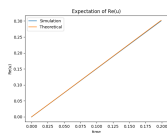
$$W(\tau_j) - W(\tau_{j-1}) = \sum_{k=jR-R+1}^{jR} dW_k,$$

and R being the step length of E-M scheme,

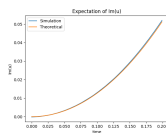
$$dW = \sqrt{\Delta t} * \text{randn}().$$

Simulation

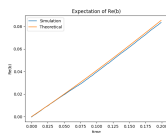
Simulate 100,000 times with $R = 1$, the results are as follows:



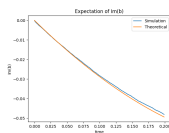
(a) $E(\text{Re}(u))$



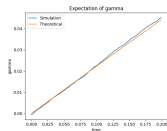
(b) $E(\text{Im}(u))$



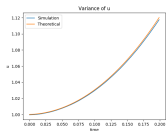
(c) $E(\text{Re}(b))$



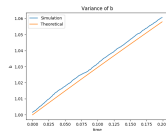
(d) $E(\text{Im}(b))$



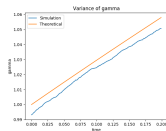
(e) $E(\gamma)$



(f) $\text{Var}(u)$



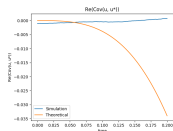
(g) $\text{Var}(b)$



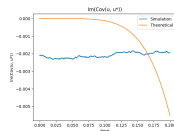
(h) $\text{Var}(\gamma)$

Figure: Simulation of Expectations and Variances

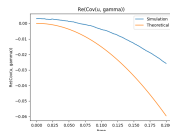
Simulation



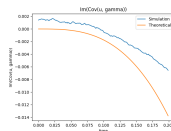
(a) $\text{Re}(\text{Cov}(u, u^*))$



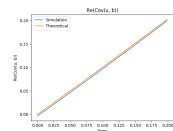
(b) $\text{Im}(\text{Cov}(u, u^*))$



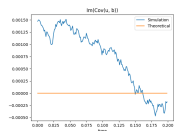
(c) $\text{Re}(\text{Cov}(u, \gamma))$



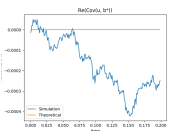
(d) $\text{Im}(\text{Cov}(u, \gamma))$



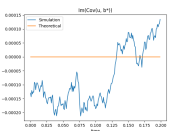
(e) $\text{Re}(\text{Cov}(u, b))$



(f) $\text{Im}(\text{Cov}(u, b))$



(g) $\text{Re}(\text{Cov}(u, b^*))$



(h) $\text{Im}(\text{Cov}(u, b^*))$

Figure: Simulation of Covariances

From the simulation results we find that the simulation of expectations and variances fit the theoretical results properly; but the simulation of covariances have an error of $O(10^{-3})$. It could be caused by the condition number of multiplication of complex numbers.

Thank you!