Introduction to Analysis
Assignment 8

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Problem 1. Problem 37, Page 123

Sol. (i) Let

$$f(x)=\begin{cases} x\sin(\frac{1}{x}),\ 0<x\leq 1,\\ 0,\ x=0\end{cases}$$

Then f is continuous on $(0,1]$. Since $\lim\limits_{x\to 0^+} f(x)=0=f(0)$, f is continuous on $[0,1]$. For $x_1,x_2\in[\epsilon,1]$ where $\epsilon>0$,

$$\begin{aligned}|f(x_1)-f(x_2)| &= |x_1\sin\frac{1}{x_1}-x_2\sin\frac{1}{x_2}| = |(x_1-x_2)\sin\frac{1}{x_1}+x_2(\sin\frac{1}{x_1}-\sin\frac{1}{x_2})|\\ &\leq |x_1-x_2|\sin\frac{1}{x_1}+2x_2|\cos\frac{1}{2}(\frac{1}{x_1}+\frac{1}{x_2})\sin\frac{1}{2}(\frac{1}{x_1}-\frac{1}{x_2})|\\ &\leq |x_1-x_2|\sin\frac{1}{x_1}+2x_2|\sin\frac{x_2-x_1}{2x_1x_2}| \leq |x_1-x_2|\sin\frac{1}{x_1}+2x_2|\frac{x_2-x_1}{2x_1x_2}|\\ &= |x_1-x_2|(\sin\frac{1}{x_1}+\frac{1}{x_1}) \leq |x_1-x_2|(1+\frac{1}{\epsilon}).\end{aligned}$$

Thus f is Lipschitz, with Proposition 7 we know f is absolutely continuous.
However, with Problem 35 we know f is not of bounded variation on $[0,1]$, and with Remark on Page 122 we know f is not absolutely continuous on $[0,1]$.

Problem 2. Problem 39, Page 123

Problem 3. Problem 41, Page 123

Problem 4. Problem 49, Page 128

Problem 5. Problem 56, Page 129

Problem 6. Problem 59, Page 129