Homework 2 **16.10.9**

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Problem 1. Explain the relationship between spectral clustering, normalized spectral clustering and graph cut.

Solution. We assume $k = 2$ in the following explanation.

For spectral clustering, we already know that the eigenvector u_2 corresponding to the second smallest eigenvalue λ_2 of the Laplacian matrix of the graph is the solution for the minimization

$$\min_{\mathbf{f}} \mathbf{f}^\top \mathbf{L}\mathbf{f},$$
$$s.t. \quad \mathbf{f}^\top \mathbf{f} = 1, \mathbf{f}^\top \mathbf{1} = 0$$

In fact, if we define

$$\begin{aligned} G(f) = G(f_1, \cdots, f_n) &= f^\top \mathbf{L} f - \lambda_1(f^\top f - 1) - \lambda_2 f^\top \mathbf{1} \\ &= \frac{1}{2}\sum_{i,j=1}^{n} w_{ij}(f_i - f_j)^2 - \lambda_1(\sum_{i=1}^{n} f_i^2 - 1) - \lambda_2 \sum_{i=1}^{n} f_i \end{aligned}$$

Then according to Lagrange Theorm, we let

$$\frac{\partial G}{\partial f_i} = \sum_{j=1}^{n}(f_i - f_j) - 2\lambda_1 f_i - \lambda_2 = 0, \quad i = 1 : n$$

If we add the n equations, we get

$$2\lambda_1 \sum_{i,j=1}^{n} f_i + n\lambda_2 = 0$$

Since $f^\top \mathbf{1} = 0$, then $\lambda_2 = 0$. So

$$\sum_{j=1}^{n} w_{ij} f_j = \lambda f_i, \quad i = 1 : n,$$

which means f is an eigenvector of $\mathbf{L}$.

For Min Cut, if we define $\mathbf{f} = (f_1, f_2 \cdots f_n)^\top$