

# Statistics of Solutions to Test Models for SPEKF

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## 1 Introduction

- Stochastic Parameterization Extended Kalman Filter (SPEKF)
- Itô Integration

## 2 Statistics of $b(t)$ and $\gamma(t)$

- Mean
- Variance
- Covariance

## 3 Statistics of $u(t)$

- Mean
- Variance
- Covariance

## 4 Numerical Simulation

- Parameters and Algorithms
- Results

# Introduction

Signals from nature can be modeled by Langevin equation:

## Langevin Equation

$$\frac{du(t)}{dt} = -\gamma(t)u(t) + i\omega u(t) + \sigma \dot{W}(t) + f(t),$$

where  $\dot{W}(t)$  is a Brownian motion, and  $f(t)$  is the external force.

A major difficulty in accurate filtering of noisy signals with many degrees of freedom is model error; signal from nature is processed through incomplete physical models, as well as parameterized to inadequate numerical resolution.

B. Gershgorin, *et, al* proposed the Stochastic Parameterization Extended Kalman Filter (SPEKF) to cope with model errors.

## SDEs in Test Model

$$\begin{cases} \frac{du(t)}{dt} = (-\gamma(t) + i\omega)u(t) + b(t) + f(t) + \sigma W(t), \\ \frac{db(t)}{dt} = (-\gamma_b + i\omega_b)(b(t) - \hat{b}) + \sigma_b W_b(t), \\ \frac{d\gamma(t)}{dt} = -d_\gamma(\gamma(t) - \hat{\gamma}) + \sigma_\gamma W_\gamma(t) \end{cases}$$

The initial values are complex random variables, with their first-order and second-order statistics known.

## Solution

With knowledge of ODEs, the solution of the SDE set is

$$\left\{ \begin{array}{l} b(t) = \hat{b} + (b_0 - \hat{b})e^{\lambda_b(t-t_0)} + \sigma_b \int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s) \\ \gamma(t) = \hat{\gamma} + (\gamma_0 - \hat{\gamma})e^{-d_\gamma(t-t_0)} + \sigma_\gamma \int_{t_0}^t e^{-d_\gamma(t-s)} dW_\gamma(s) \\ u(t) = e^{-J(t_0,t) + \hat{\lambda}(t-t_0)} u_0 + \int_{t_0}^t (b(s) + f(s)) e^{-J(s,t) + \hat{\lambda}(s-t_0)} ds \\ \quad + \sigma \int_{t_0}^t e^{-J(s,t) + \hat{\lambda}(s-t_0)} dW(s) \end{array} \right.$$

with  $\lambda_b = -\gamma_b + i\omega_b$ ,  $\hat{\lambda} = -\hat{\gamma} + i\omega$ ,  $J(s,t) = \int_s^t (\gamma(s') - \hat{\gamma}) ds'$ .

# Introduction

## Itô Isometry

$\forall f \in \mathcal{V}(S, T)$ ,  $B_t$  is a standard Brownian motion,

$$\mathbb{E} \left[ \left( \int_S^T f(t, \omega) dB_t \right)^2 \right] = \mathbb{E} \left[ \int_S^T f^2(t, \omega) dt \right].$$

## Linear property of Itô integration

$$(1) \quad \int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t, \text{ a.e.}$$

$$(2) \quad \int_S^T (cf + g) dB_t = c \int_S^T f dB_t + \int_S^T g dB_t, \text{ a.e.}$$

$$(3) \quad \mathbb{E} \left[ \int_S^T f dB_t \right] = 0.$$

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# Mean of $b(t)$ , $\gamma(t)$

With property of Itô integration (3), it is easy to know

$$E(b(t)) = \hat{b} + (E[b_0] - \hat{b})e^{\lambda_b(t-t_0)}$$

$$E(\gamma(t)) = \hat{\gamma} + (E[\gamma_0] - \hat{\gamma})e^{-d_\gamma(t-t_0)}$$



# Variance of $b(t)$ , $\gamma(t)$

According to definition,

$$\begin{aligned}\text{Var}(b(t)) &= \text{E}[(b(t) - \text{E}[b(t)])(b(t) - \text{E}[b(t)])^*] \\&= e^{-2\gamma_b(t-t_0)} \text{Var}(b_0) + \text{E} \left[ \sigma_b^2 \int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s) \left( \int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s) \right)^* \right] \\&= e^{-2\gamma_b(t-t_0)} \text{Var}(b_0) + \sigma_b^2 \text{E} \left[ \int_{t_0}^t e^{-2\gamma_b(t-s)} ds \right]\end{aligned}$$

The last step takes advantage of Itô isometry.

$$\text{Var}(b(t)) = e^{-2\gamma_b(t-t_0)} \text{Var}(b_0) + \frac{\sigma_b^2}{2\gamma_b} (1 - e^{-2\gamma_b(t-t_0)})$$

$$\text{Var}(\gamma(t)) = e^{-2d_\gamma(t-t_0)} \text{Var}(\gamma_0) + \frac{\sigma_\gamma^2}{2d_\gamma} (1 - e^{-2d_\gamma(t-t_0)})$$

# Covariance of $b(t)$ , $\gamma(t)$

$$\begin{aligned}\text{Cov}(b(t), b(t)^*) &= \text{E}[(b(t) - \text{E}[b(t)])(b(t)^* - \text{E}[b(t)^*])] \\ &= \text{E}\left[(b_0 - \text{E}[b_0])(b_0^* - \text{E}[b_0^*])e^{2\lambda_b(t-t_0)}\right] + \sigma_b \text{E}\left[(b_0 - \text{E}[b_0]) \int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s)\right] \\ &\quad + \sigma_b \text{E}\left[(b_0^* - \text{E}[b_0^*])e^{\lambda_b(t-t_0)} \int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s)\right] + \sigma_b^2 \text{E}\left[\left(\int_{t_0}^t e^{\lambda_b(t-s)} dW_b(s)\right)^2\right]\end{aligned}$$

With property of Itô integration (3), the second and third term are both 0; with Itô isometry we know the last term is also 0.

$$\text{Cov}(b(t), b(t)^*) = \text{E}[(b_0 - \text{E}[b_0])(b_0^* - \text{E}[b_0^*])e^{2\lambda_b(t-t_0)}] = \text{Cov}(b_0, b_0^*)e^{2\lambda_b(t-t_0)}$$

$$\text{Cov}(b(t), \gamma(t)) = \text{E}[(b(t) - \text{E}[b(t)])(\gamma(t) - \text{E}[\gamma(t)])] = \text{Cov}(b_0, \gamma_0)e^{(\lambda_b - d_\gamma)(t-t_0)}$$

# Outline

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# Mean of $u(t)$

Using the same properties, it is obvious to find

$$\begin{aligned} \mathbf{E}[u(t)] &= e^{\hat{\lambda}(t-t_0)} \mathbf{E} \left[ e^{-J_0(t_0,t)} u_0 \right] + \int_{t_0}^t e^{\hat{\lambda}(t-s)} \mathbf{E} \left[ b(s) e^{-J(s,t)} \right] ds \\ &\quad + \sigma \int_{t_0}^t e^{\hat{\lambda}(t-s)} f(s) \mathbf{E} \left[ e^{-J(s,t)} \right] ds. \end{aligned}$$

It is necessary to compute expectations of terms like  $\mathbf{E}[ze^{bx}]$ , where  $z$  is a complex-valued Gaussian random variable and  $x$  is a real-valued Gaussian variable. We propose two lemmas here.

# Mean of $u(t)$

## Lemma

$$E\left[ze^{ibx}\right] = (E[z] + ib\text{Cov}(z,x))e^{ibE[x] - \frac{1}{2}b^2\text{Var}(x)}$$

*with  $z$  being a complex-valued Gaussian, and  $x$  a real-valued Gaussian.*

## Corollary

*Under the condition of Lemma 1,*

$$E\left[ze^{bx}\right] = (E[z] + b\text{Cov}(z,x))e^{bE[x] + \frac{1}{2}b^2\text{Var}(x)}.$$

Proof of lemma 1 takes advantage of the characteristic function of multivariable Gaussian distribution.

# Proof

Let  $z = y + iw$ ,  $y, w \in \mathbb{R}$ . Denote  $\mathbf{v} = (x, y, w)$ , then  $\mathbf{v}$  satisfies the multivariable Gaussian distribution, with its characteristic function

$$\phi_{\mathbf{v}}(\mathbf{s}) = \exp(i\mathbf{s}^\top \mathbf{E}[\mathbf{v}] - \frac{1}{2}\mathbf{s}^\top \Sigma \mathbf{s}).$$

Let  $g(\mathbf{v})$  being the PDF of  $\mathbf{v}$ , then one knows from that char. func. being Fourier transform of PDF,

$$\phi_{\mathbf{v}}(\mathbf{s}) = \frac{1}{(2\pi)^3} \int e^{i\mathbf{s}^\top \mathbf{v}} g(\mathbf{v}) d\mathbf{v}$$

According to the differential property of Fourier transform,

$$\frac{\partial \phi_{\mathbf{v}}(\mathbf{s})}{\partial s_2} = \frac{1}{(2\pi)^3} \int i y_0 e^{i\mathbf{s}^\top \mathbf{v}} g(\mathbf{v}) d\mathbf{v} = i \mathbf{E} \left[ y_0 e^{i\mathbf{s}^\top \mathbf{v}} \right].$$

Let  $\mathbf{v} = (b, 0, 0)^\top$ ,

$$\mathbf{E}[y_0 e^{ibx_0}] = -i \frac{\partial \phi_{\mathbf{v}}(s)}{\partial s_2} \Big|_{\mathbf{s}=(b,0,0)^\top}$$

$$\mathbf{E}[y_0 e^{ibx_0}] = -i \frac{\partial \phi_{\mathbf{v}}(s)}{\partial s_2} \Big|_{\mathbf{s}=(b,0,0)^\top}$$

From PDF of multivariable Gaussian distribution, one knows

$$\frac{\partial \phi_{\mathbf{v}}(\mathbf{s})}{\partial s_2} = (iE[y_0] - \text{Var}(y_0)s_2 - \text{Cov}(x_0, y_0)s_1 - \text{Cov}(y_0, w_0)s_3)\phi_{\mathbf{v}}(\mathbf{s})$$

$$\frac{\partial \phi_{\mathbf{v}}(\mathbf{s})}{\partial s_3} = (iE[w_0] - \text{Var}(w_0)s_3 - \text{Cov}(x_0, w_0)s_1 - \text{Cov}(y_0, w_0)s_2)\phi_{\mathbf{v}}(\mathbf{s})$$

Compute the partial derivatives at  $\mathbf{s} = (b, 0, 0)^\top$ ,

$$E[y_0 e^{ibx_0}] = (E[y_0] + i\text{Cov}(x_0, y_0)b) \exp(ibE[x_0] - \frac{1}{2}\text{Var}(x_0)b^2)$$

$$E[w_0 e^{ibx_0}] = (E[w_0] + i\text{Cov}(x_0, w_0)b) \exp(ibE[x_0] - \frac{1}{2}\text{Var}(x_0)b^2)$$

Then

$$E[z e^{ibx}] = (E[z] + ib\text{Cov}(z, x)) e^{ibE[x] - \frac{1}{2}b^2\text{Var}(x)}.$$



## Lemma

$$E\left[zw e^{bx}\right] = [E[z]E[w] + \text{Cov}(z, w^*) + b(E[z]\text{Cov}(w, x)) + E[w]\text{Cov}(z, x) + b^2\text{Cov}(z, x)\text{Cov}(w, x)] e^{bE[x] + \frac{b^2}{2}\text{Var}(x)}.$$

with  $z, w$  being complex-valued Gaussian, and  $x$  real-valued Gaussian.

The proof of this lemma is the same as Lemma 1.



# Mean of $u(t)$

We now make use of Lemma 1 to obtain the mean of  $u(t)$ .

$$\begin{aligned} \mathbb{E}[u(t)] &= e^{\hat{\lambda}(t-t_0)} (\mathbb{E}[u_0] - \text{Cov}(u_0, J(t_0, t))) e^{-\mathbb{E}[J(t_0, t)] + \frac{1}{2} \text{Var}(J(t_0, t))} \\ &\quad + \int_{t_0}^t e^{\hat{\lambda}(t-s)} (\hat{b} + e^{\lambda_b(s-t_0)} (\mathbb{E}[b_0] - \hat{b} - \text{Cov}(b_0, J(s, t)))) e^{-\mathbb{E}[J(s, t)] + \frac{1}{2} \text{Var}(J(s, t))} ds \\ &\quad + \int_{t_0}^t e^{\hat{\lambda}(t-s)} f(s) e^{-\mathbb{E}[J(s, t)] + \frac{1}{2} \text{Var}(J(s, t))} ds \end{aligned}$$

The terms  $\text{Cov}(u_0, J(s, t))$ ,  $\text{Cov}(b_0, J(s, t))$ ,  $\mathbb{E}[J(s, t)]$  and  $\text{Var}(J(s, t))$  can be found using Itô isometry.

# Variance of $u(t)$

Denote  $u(t) = A + B + C$ ,

$$\begin{cases} A = e^{-J(t_0,t) + \hat{\lambda}(t-t_0)} u_0, \\ B = \int_{t_0}^t (b(s) + f(s)) e^{-J(s,t) + \hat{\lambda}(t-s)} ds, \\ C = \sigma \int_{t_0}^t e^{-J(s,t) + \hat{\lambda}(t-s)} dW(s). \end{cases}$$

By definition we find  $\text{Var}(u(t)) = \mathbb{E}[|u(t)|^2] - |\mathbb{E}[u(t)]|^2$ , with

$$\mathbb{E}[|u(t)|^2] = \mathbb{E}[|A|^2] + \mathbb{E}[|B|^2] + \mathbb{E}[|C|^2] + 2\text{Re}\{\mathbb{E}[A^*B]\}.$$

We can obtain  $\mathbb{E}[|A|^2]$  by Lemma 2, and  $\mathbb{E}[|B|^2]$  by Itô isometry and noticing that

$$\text{Cov}(J(s,t), J(r,t)) = \text{Var}(J(s,t)) + \text{Cov}(J(s,t), J(r,s)).$$

$\mathbb{E}[|C|^2]$  and  $\text{Re}\{\mathbb{E}[A^*B]\}$  can also be computed by Itô isometry and property of Itô integration.

By definition,

$$\text{Cov}(u(t), u^*(t)) = \mathbb{E}[u(t)^2] - \mathbb{E}[u(t)]^2$$

$$\text{Cov}(u(t), \gamma(t)) = \mathbb{E}[u(t)(\gamma(t) - \hat{\gamma})] + \mathbb{E}[u(t)](\hat{\gamma} - \mathbb{E}[\gamma(t)])$$

$$\text{Cov}(u(t), b(t)) = \mathbb{E}[u(t)b^*(t)] - \mathbb{E}[u(t)]\mathbb{E}[b(t)]^*$$

$$\text{Cov}(u(t), b^*(t)) = \mathbb{E}[u(t)b(t)] - \mathbb{E}[u(t)]\mathbb{E}[b(t)].$$

Each term can be obtained by Lemma 3 and Itô isometry.

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# Parameters

External forcing  $f(t) = \frac{3}{2}e^{0.1it}$ , and parameters of the equation set are given

$$\begin{cases} d = 1.5, & d_\gamma = 0.01d \\ \sigma = 0.1549, & \omega = 1.78 \\ \sigma_\gamma = 5\sigma, & \gamma_b = 0.1d \\ \sigma_b = 5\sigma, & \omega_b = \omega \\ \hat{b} = 0, & \hat{\gamma} = 0 \end{cases}$$

We assume the initial values satisfying

$$\begin{cases} \operatorname{Re}(u_0), \operatorname{Im}(u_0) \sim \mathcal{N}(0, \frac{1}{\sqrt{2}}), \text{ i.i.d.} \\ \operatorname{Re}(b_0), \operatorname{Im}(b_0) \sim \mathcal{N}(0, \frac{1}{\sqrt{2}}), \text{ i.i.d.} \\ \gamma_0 \sim \mathcal{N}(0, 1) \end{cases}$$

The statistics between initial values are

$$\begin{cases} \text{Cov}(u_0, u_0^*) = 0 \\ \text{Cov}(u_0, \gamma_0) = 0 \\ \text{Cov}(u_0, b_0) = 0 \\ \text{Cov}(u_0, b_0^*) = 0 \end{cases}$$

# Euler-Maruyama Scheme

Itô integration can be simulated by E-M scheme:

$$X_j = X_{j-1} + f(X_{j-1})\Delta t + g(X_{j-1})(W(\tau_j) - W(\tau_{j-1})),$$

with

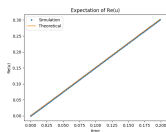
$$W(\tau_j) - W(\tau_{j-1}) = \sum_{k=jR-R+1}^{jR} dW_k,$$

and  $R$  being the step length of E-M scheme,

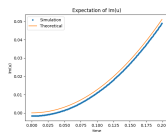
$$dW = \sqrt{\Delta t} \times \text{randn}().$$

# Result

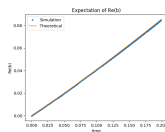
Simulate  $10^6$  times with  $R = 1$ , the results are as follows:



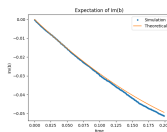
(a)  $E(\text{Re}(u))$



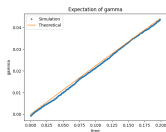
(b)  $E(\text{Im}(u))$



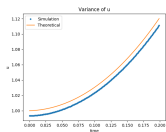
(c)  $E(\text{Re}(b))$



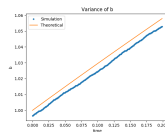
(d)  $E(\text{Im}(b))$



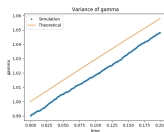
(e)  $E(\gamma)$



(f)  $\text{Var}(u)$



(g)  $\text{Var}(b)$

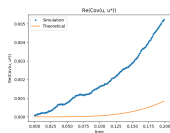


(h)  $\text{Var}(\gamma)$

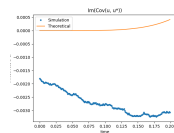
Figure: Simulation of Expectations and Variances,  $n = 10^6$



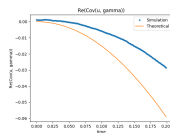
# Result



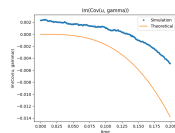
(a)  $\text{Re}(\text{Cov}(u, u^*))$



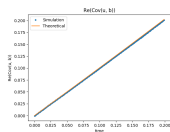
(b)  $\text{Im}(\text{Cov}(u, u^*))$



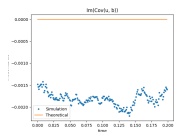
(c)  $\text{Re}(\text{Cov}(u, \gamma))$



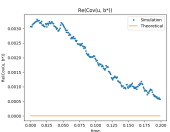
(d)  $\text{Im}(\text{Cov}(u, \gamma))$



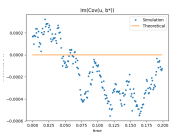
(e)  $\text{Re}(\text{Cov}(u, b))$



(f)  $\text{Im}(\text{Cov}(u, b))$



(g)  $\text{Re}(\text{Cov}(u, b^*))$

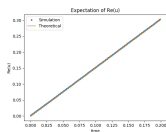


(h)  $\text{Im}(\text{Cov}(u, b^*))$

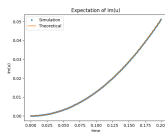
Figure: Simulation of Covariances,  $n = 10^7$

# Results

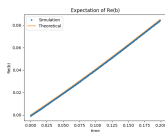
Simulate  $10^7$  times with  $R = 1$ , the results are as follows:



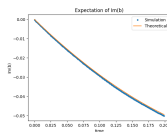
(a)  $E(\text{Re}(u))$



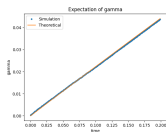
(b)  $E(\text{Im}(u))$



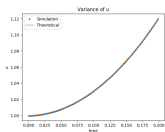
(c)  $E(\text{Re}(b))$



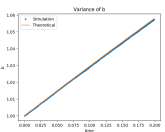
(d)  $E(\text{Im}(b))$



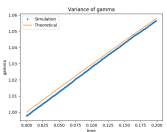
(e)  $E(\gamma)$



(f)  $\text{Var}(u)$



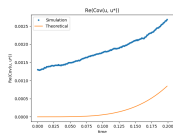
(g)  $\text{Var}(b)$



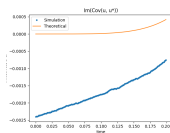
(h)  $\text{Var}(\gamma)$

Figure: Simulation of Expectations and Variances,  $n = 10^7$

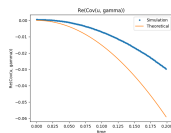
# Result



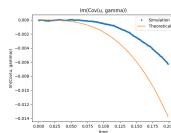
(a)  $\text{Re}(\text{Cov}(u, u^*))$



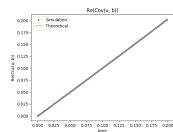
(b)  $\text{Im}(\text{Cov}(u, u^*))$



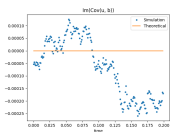
(c)  $\text{Re}(\text{Cov}(u, \gamma))$



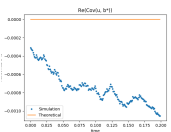
(d)  $\text{Im}(\text{Cov}(u, \gamma))$



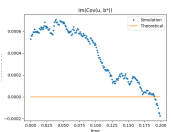
(e)  $\text{Re}(\text{Cov}(u, b))$



(f)  $\text{Im}(\text{Cov}(u, b))$



(g)  $\text{Re}(\text{Cov}(u, b^*))$



(h)  $\text{Im}(\text{Cov}(u, b^*))$

Figure: Simulation of Covariances,  $n = 10^7$

From the simulation results we find that the simulation of expectations fit the theoretical results satisfyingly.

For the results of Variances and Covariances,

- When  $n = 10^6$ , the error of variances is about  $O(10^{-2}) \sim O(10^{-3})$ ,
- error of covariances is about  $O(10^{-4})$ .
- When  $n = 10^7$ , the error of  $\text{Var}(u)$ ,  $\text{Var}(b)$  is about  $O(10^{-6})$ , and error of  $\text{Var}(\gamma)$  is about  $O(10^{-3}) \sim O(10^{-4})$ ,
- error of  $\text{Cov}(u, u^*)$ ,  $\text{Cov}(u, \gamma)$  is about  $O(10^{-3})$ , error of  $\text{Cov}(u, b)$ ,  $\text{Cov}(u, b^*)$  is  $O(10^{-4})$ .

One can find that with the increase of simulations, errors decrease almost linearly. Thus one could believe that when  $n \rightarrow \infty$ , the simulation would converge to the theoretical result. In fact, error of simulation comes from that initial values are all random variables; according to Law of large numbers, the errors would converge to 0 when  $n \rightarrow \infty$ .

We can get third-order and fourth-order statistics of  $u, b, \gamma$  with the same methods, which could be written in a form of multiple integrals of statistics of initial values.

# Thank you!