Homework 1

Instructions:
In problems 3. - 5., references such as III.2.7 refer to Problem 7 in Section 2 of Chapter III in Conway's book.

If you use results from books including Conway's, please be explicit about what results you are using.

Homework 1 is due on Dropbox on Monday, February 5.

1. Show that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ without using logarithms.

   Proof. Let $n^{1/n} = 1 + y_n$. First we know $n^{1/n} > 1^{1/n} = 1$, so $y_n > 0$. Then

$$n = (1+y_n)^n = 1 + ny_n + \frac{n(n-1)}{2}y_n^2 + \cdots + y_n^n > 1 + \frac{n(n-1)}{2}y_n^2.$$

   Thus $\frac{n(n-1)}{2}y_n^2 < n-1$, which means $y_n < \sqrt{2/n}$. Hence $\lim\limits_{n\to\infty} y_n \le \lim\limits_{n\to\infty} \sqrt{2/n} = 0$, which means $y_n \to 0$, and thus $n^{1/n} \to 1$.

2. Given a power series, $\sum_{n=0}^{n=\infty} a_n(z-a)^n$, show that its radius of convergence R satisfies the inequalities

$$(\limsup|\frac{a_{n+1}}{a_n}|)^{-1} \le R \le \limsup|\frac{a_n}{a_{n+1}}|.$$

   Proof. We only proof the right inequality since it is just the same for the left one. If $R > r > \limsup|\frac{a_n}{a_{n+1}}| = \alpha$, then there is an $N > 0$ s.t. $r > |a_n/a_{n+1}|$ for all $n \ge N$. Let $B = |a_N|r^N$, then $|a_{N+1}|r^{N+1} = |a_{N+1}|rr^N > B$. Hence for all $n > N$ we have $|a_n|r^n > B$, which gives $|a_n z^n| \ge B|z|^n/|r|^n$ when $n > N$. But $|z|/|r| > 1$, which makes $|z|^n/|r|^n \to \infty$ when $n \to \infty$. Hence $\sum a_n z^n$ diverges, so $R \le \alpha$.

3. Problem III.1.6.

   (a). By Theorem 1.3,

$$\limsup|a_n|^{1/n} = \limsup|a^n|^{1/n} = |a|,$$

   thus $R = \frac{1}{|a|}$.

   (b). By Theorem 1.3,

$$\limsup|a_n|^{1/n} = \limsup|a^{n^2}|^{1/n} = \limsup|a^n| = \begin{cases} 0, |a| < 1, \\ 1, |a| = 1, \\ \infty, |a| > 1. \end{cases}$$

   Thus

$$R = \begin{cases} \infty, |a| < 1, \\ 1, |a| = 1, \\ 0, |a| > 1. \end{cases}$$

   (c). By Theorem 1.3,

$$\limsup|a_n|^{1/n} = \limsup|k^n|^{1/n} = k,$$

   thus $R = \frac{1}{k}$.

   (d). Since

$$\sum_{n=0}^{\infty}|z|^{n!} < \sum_{n=0}^{\infty}|z|,$$

   and the convergence radius of the latter series is $R' = 1$, we know $R \ge 1$. On the other hand, if $R > 1$, pick $1 < |z| = r < R$, then $|z|^{n!} = r^{n!} \to \infty$ when $n \to \infty$, hence the series diverges. Thus $R = 1$.

4. Problem III.1.7
5. Problem III.2.6
6. Problem III.2.7
7. Problem III.2.9.
8. Problem III.2.13
9. Problem III.2.20