Assignment 9

Chuan Lu, 13300180056, chuanlu13@fudan.edu.cn

Problem 1. Problem 2, Page 151

Proof. First, one notices that $e_0 = 0$, thus

$$e_i = e_0 + h\sum_{j=0}^{i-1}\delta_x^+ = h\sum_{j=0}^{i-1}\delta_x^+ e_j.$$

Using Schwarz inequality, one finds

$$\begin{aligned}\|\mathbf{e}\|_{l^2}^2 &= \left\|\left(h\sum_{j=0}^{i-1}\delta_x^+ e_j\right)_i\right\|_{l^2}^2 = \sum_{i=1}^{N}\left(h\sum_{j=0}^{i-1}\delta_x^+ e_j\right)^2 \leqslant \sum_{i=1}^{N} ih^2\sum_{j=0}^{i-1}\left(\delta_x^+ e_j\right)^2 \\ &\leqslant N^2h^2\sum_{i=1}^{N-1}(\delta_x^+ e_j)^2 = \left\|\delta_x^+\mathbf{e}\right\|_{l^2}.\end{aligned}$$

So we have proved (3.1.54). Next we use this inequality to proof the convergence. The truncation error of the differenced system is

$$R_i = -\delta_x^2 u(x_i) - f(x_i) = -\delta_x^2 u(x_i) + \frac{d^2u}{dx^2} = -\frac{h^2}{12} + O(h^4).$$

Define $e_i = u(x_i) - u_i$, then e_i satisfies

$$-\delta_x^2 e_i = R_i.$$

So we have

$$\|\mathbf{e}\| \leqslant \left\|\sigma_x^+ e\right\| \leqslant \|R\| \sim O(h^2).$$

It means the function value is of order-2 convergent. When it comes to the derivatives,

$$\frac{du}{dx}(x_i) = \frac{u(x_{i+1}) - u(x_{i-1})}{2h} + \frac{h^2}{12}u^{(3)}(x_i) + O(h^3).$$

Using the same methods we can get that the values of derivatives are of order-2 convergent. □

Problem 2. Problem 3, Page 151

Proof.

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) = -(a'(x)u'(x) + a(x)u''(x)) = f$$

Thus the three-point-difference scheme is

$$-a_i'\delta_x u_i - a_i\delta_x^2 u_i = f_i.$$

When $f_i \geqslant 0$,

$$\begin{aligned}u_{i+1} &= 2u_i - u_{i-1} - \frac{h^2}{a_i}f_i - \frac{h_i a_i'}{a_i}\delta_x u_i \\ &\leqslant (2 - \frac{h_i a_i'}{a_i})u_i - (1 - \frac{h_i a_i'}{a_i})u_{i-1},\end{aligned}$$

where we used the backward scheme in the last term. According to the same process as Lemma 3.1.12, we finds that either u_i being a constant, or the minimum is reached on the boarder. □

Max-module estimation. We construct a problem:

$$\begin{cases} v_0 = v_n = 0, \\ -\delta_x(a_i\delta_x u_i) = \|\mathbf{R}\|_{l^\infty}, \end{cases}$$

Then $v_i = \frac{1}{N} \sum_{j=1}^{i} \frac{\frac{i}{N} \|\mathbf{R}\|_{l^\infty} + \frac{\sum_{j=1}^{N} \frac{i \|\mathbf{R}\|_{l^\infty}}{a_i}}{\sum_{j=1}^{N} \frac{1}{a_i}}}{a_i}$. Thus

$$\|e_i\| \leqslant v_i \leqslant C \|\mathbf{R}\|_{l^\infty} \leqslant C_2 h^2.$$