Introduction to Analysis
Assignment 8

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Problem 1. Problem 37, Page 123

Sol. (i) Let

$$f(x) = \begin{cases} x\sin(\dfrac{1}{x}),\ 0 < x \le 1, \\ 0,\ x = 0 \end{cases}$$

Then f is continuous on $(0,1]$. Since $\lim\limits_{x\to 0^+} f(x) = 0 = f(0)$, f is continuous on $[0,1]$. For $x_1, x_2 \in [\epsilon, 1]$ where $\epsilon > 0$,

$$\begin{aligned} |f(x_1) - f(x_2)| &= |x_1 \sin\frac{1}{x_1} - x_2 \sin\frac{1}{x_2}| = |(x_1 - x_2)\sin\frac{1}{x_1} + x_2(\sin\frac{1}{x_1} - \sin\frac{1}{x_2})| \\ &\le |x_1 - x_2|\sin\frac{1}{x_1} + 2x_2|\cos\frac{1}{2}(\frac{1}{x_1} + \frac{1}{x_2})\sin\frac{1}{2}(\frac{1}{x_1} - \frac{1}{x_2})| \\ &\le |x_1 - x_2|\sin\frac{1}{x_1} + 2x_2|\sin\frac{x_2 - x_1}{2x_1x_2}| \le |x_1 - x_2|\sin\frac{1}{x_1} + 2x_2|\frac{x_2 - x_1}{2x_1x_2}| \\ &= |x_1 - x_2|(\sin\frac{1}{x_1} + \frac{1}{x_1}) \le |x_1 - x_2|(1 + \frac{1}{\epsilon}). \end{aligned}$$

Thus f is Lipschitz, with Proposition 7 we know f is absolutely continuous.
However, with Problem 35 we know f is not of bounded variation on $[0,1]$, and with Remark on Page 122 we know f is not absolutely continuous on $[0,1]$.
(ii) If not, then there is a $\epsilon > 0$, s.t. for each $\delta > 0$, there is a finite disjoint collection $\{(a_k, b_k)\}$ of open intervals satisfying $\sum\limits_{k=1}^{n}(b_k - a_k) < \delta$, s.t. $\sum\limits_{k=1}^{n}|f(b_k) - f(a_k)| \ge \epsilon$. As suggested in conditions, for each $c > 0$, f is absolutely continuous on $[c,1]$. Then these open intervals must lie in $[0,c]$ for every c. With the continuity of f on $[0,1]$, there exists $c > 0$, s.t. $0 < f(c) - f(0) < \epsilon$. If we take $\delta = c$, because f is increasing, $\sum\limits_{k=1}^{n}(b_k - a_k) < f(c) - f(0) < \epsilon$. It contradicts with our assumption. Hence f is absolutely continuous on $[0,1]$.
(iii) First we show f is absolutely continuous by showing that it satisfies the condition in (ii). For each $c > 0$, since on $[c,1]$ we have

$$\sqrt{x_1} - \sqrt{x_2}| = \frac{|x_1 - x_2|}{\sqrt{x_1} + \sqrt{x_2}} \le \frac{|x_1 - x_2|}{2\sqrt{c}},$$

we know for each $\epsilon > 0$, pick $\delta = 2\sqrt{c}\epsilon$, then for each collection $\{(a_k, b_k)\}$ satisfying $\sum\limits_{k=0}^{n}|b_k - a_k| < \delta$,

$$\sum_{k=1}^{n}|f(b_k) - f(a_k)| < \sum_{k=1}^{n}\frac{|x_1 - x_2|}{2\sqrt{c}} < \epsilon.$$

Hence f is absolutely continuous on $[c,1]$. Since f is increasing, we know f is absolutely continuous on $[0,1]$.
On the other hand, if there exists $\lambda > 0$, s.t.

$$|f(x_1) - f(x_2)| < \lambda|x_1 - x_2|$$

for each x_1, x_2, then if we pick $\max(x_1, x_2) < \frac{1}{4\lambda^2}$, from the argument above we know

$$|f(x_1) - f(x_2)| = \frac{|x_1 - x_2|}{\sqrt{x_1} + \sqrt{x_2}} > \lambda|x_1 - x_2|.$$

Hence f is not Lipschitz.

Problem 2. Problem 39, Page 123

Problem 3. Problem 41, Page 123

Problem 4. Problem 49, Page 128

Problem 5. Problem 56, Page 129

Problem 6. Problem 59, Page 129