Homework 7

Instructions: In problems the problems below, references such as III.2.7 refer to Problem 7 in Section 2 of Chapter III in Conway's book.

If you use results from books including Conway's, please be explicit about what results you are using.

Homework 7 is due at Midnight Wednesday, May 2.

Do the following problems:

1. Let F be the map that takes nonzero vectors in $\mathbb{C}^2$ to vectors in $\mathbb{R}^3$ by the following formula:

$$F(z_1, z_2) := \left(\frac{z_1\bar{z}_2 + \bar{z}_1 z_2}{z_1\bar{z}_1 + \bar{z}_2 z_2}, \frac{z_1\bar{z}_2 - \bar{z}_1 z_2}{i(z_1\bar{z}_1 + \bar{z}_2 z_2)}, \frac{z_1\bar{z}_1 - \bar{z}_2 z_2}{z_1\bar{z}_1 + \bar{z}_2 z_2} \right).$$

Show that:

(a) F defines a bijection between $\mathbb{P}^1(\mathbb{C})$ and the unit sphere in $\mathbb{R}^3$, and

(b) if S denotes stereographic projection from $\mathbb{C}_\infty - \{(0, 0, 1)\}$ to $\mathbb{C}$, then if $[z_1 : z_2] \neq [1 : 0]$,

$$S(F([z_1 : z_2])) = z_1/z_2.$$

Proof. (a) Let $z_1 = r_1 e^{it_1}, z_2 = r_2 e^{it_2}$, then F maps $(r_1 e^{it_1}, r_2 e^{it_2})$ to

$$\left(2\frac{r_1 r_2}{r_1^2 + r_2^2} \cos(t_1 - t_2),\ 2\frac{r_1 r_2}{r_1^2 + r_2^2} \sin(t_1 - t_2),\ \frac{r_1^2 - r_2^2}{r_1^2 + r_2^2} \right)$$

Hence if F maps two vectors $(r_1 e^{it_1}, r_2 e^{it_2})$ and $(r_1' e^{it_1'}, r_2' e^{it_2'})$ to a same point in $\mathbb{R}^3$, then $\tan(t_1 - t_2) = \tan(t_1' - t_2')$, hence $t_1 - t_2 = t_1' - t_2'$. Besides, if we denote $p_1 = \frac{r_1}{r_2}, p_1' = \frac{r_1'}{r_2'}$, then

$$\frac{1 - p_1^2}{p_1} = \frac{1 - p_1'^2}{p_1'},$$

and

$$\frac{1 + p_1^2}{p_1} = \frac{1 + p_1'^2}{p_1'},$$

when we add this two equations we get $p_1 = p_2$. Hence by considering the first term we know $r_1 = r_2$. Hence F is an injection, and since $\|F(z)\| = 1$ we know F is an injection from $\mathbb{P}^1(\mathbb{C})$ to unit sphere.

On the other hand, for each $(\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\theta)$ in unit sphere, we can find r_1, r_2, t_1, t_2, s.t.

$$\frac{r_1 r_2}{r_1^2 + r_2^2} = \frac{1}{2}\sin\varphi,\ t_1 - t_2 = \theta.$$

Hence F is a bijection.

(b)

2. III.3.8

Proof. First, if we choose a, b, c, d to be real, then for each $x \in \mathbb{R}$, we can pick

$$z = \frac{dx - b}{-cx + a} \in \mathbb{R}_\infty$$

then $Tz = x$. For $x = \infty$, if $c = 0$, then by $ad - bc \neq 0$ we know $d \neq 0$, hence we can pick $z = \infty$ and $Tz = x$. If $c \neq 0$, let $z = \frac{a}{c}$ then $Tz = x$.

On the other hand, if $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$, then there are $x_1, x_2, x_3 \in \mathbb{R}_\infty$, s.t. $T(x_1) = 1, T(x_2) = 0, T(x_3) = \infty$. Then by Proposition 3.8,

$$\frac{z - x_2}{z - x_3} / \frac{x_1 - x_2}{x_1 - x_3} = T(z) = \frac{(x_1 - x_3)z - (x_1 - x_3)x_2}{(x_1 - x_2)z - (x_1 - x_2)x_3},$$

which means a, b, c, d are all real.

3. III.3.11

   Proof. Let T_1 be the Mobius transformation which maps (z_1, z_2, z_3) to $(1, 0, \infty)$, and T_2 maps (w_1, w_2, w_3) to $(1, 0, \infty)$. Let $T = T_2 T_1^{-1}$, then T maps $\mathbb{R}_\infty$ onto $\mathbb{R}_\infty$, and hence by Exercise 8,

   $$T(z) = \frac{az+b}{cz+d}$$

   where $a, b, c, d \in \mathbb{R}$. Hence

   $$T(z)^* = T(z).$$

   Then for z, z^* satisfying $T_1(z^*) = T_1(z)^*$, $T_2(z^*) = T_2(T_1^{-1}(T_1(z^*))) = T(T_1(z^*)) = T(T_1(z)^*) = T_2(z)^*$.

4. III.3.13

   Sol. We may consider the function

   $$f(z) = \frac{z^2+1}{2z}$$

   using the same process in Page 45.

   First it is not a bijection, thus is not a conformal map.

5. III.3.14

   Sol. First, consider $T = \frac{1}{z-a}$, it maps G onto a region between two parallel lines, denoted by H. Then we can use a rotation and translation f to map H onto the set $I = \{z : 0 < Im(z) < \frac{\pi}{2}\}$. Then $\exp(I) = \{z : Re(z) > 0\} = J$. At this stage we can use another Mobius transformation $g(z) = \frac{z-1}{z+1}$ to map J to open unit disk. Hence

   $$\varphi(z) = g \circ \exp \circ f \circ T$$

   is what we need, and it is a conformal map.

6. III.3.15

   Sol. In fact, as we have learned in Chapter 6.2, for any $|a| < 1$,

   $$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$$

   is what we want.

7. III.3.16

   Sol. First, $f(z) = \frac{z+1}{z-1}$ maps G to $H = \mathbb{C} - (\{z : z \leq 0\} \cup \{z = 1\})$. Then $g(z) = z^{1/2}$ maps H to $I = \{z : Re(z) > 0\} - \{z = 1\}$. Then $h(z) = \frac{1-z}{1+z}$ maps I to $J = D - \{z = 0\}$ where D is the open unit disk. By Problem 3.3.15 we know $\varphi(z) = \exp(\frac{z+1}{z-1})$ maps D to J conformally, hence φ^{-1} maps J to D conformally. Hence,

   $$\phi = \varphi^{-1} \circ h \circ g \circ f$$

   is what we want, and since the four functions are all analytic and one to one in their domains, ϕ is an analytic function and is also 1-1.

8. III.3.17

   Proof. In fact this is trivial by Open Mapping Theorem, since a subset of a circle cannot be open.