Introduction to Analysis I
Homework 3
Monday, September 11, 2017

Instructions: This and all subsequent homeworks must be submitted written in LaTeX.
If you use results from books, Royden or others, please be explicit about what results you are using.

Homework 3 is due by midnight, Friday, September 22.

1. (Problem 34, Page 53) Show that there is a continuous, strictly increasing function on the interval $[0,1]$ that maps a set of positive measure onto a set of measure zero.

   Collaborators:

   Solution: Let C be the Cantor set on $[0,1]$, and $\varphi(x)$ be the Cantor function. Define
   $$\phi(x) = \varphi(x) + x,\ x \in C.$$
   We now show $\phi^{-1}(x)$, the inverse of $\phi(x)$, satisfies the properties in the problem. First, since $\phi(x)$ is a continuous, strictly increasing function on $[0,1]$, thus is continuous and strictly increasing on $C \subset [0,1]$.

   Denote $D = \phi(C)$, then according to theorems in the book, $m(D) = 1$. $\forall x_1 < x_2 \in D$, if $\phi^{-1}(x_1) \geq \phi^{-1}(x_2)$, then since ϕ is strictly increasing, $\phi(\phi^{-1}(x_1)) = x_1 + \phi^{-1}(x_1) \geq \phi(\phi^{-1}(x_2)) = x_2 + \phi^{-1}(x_2)$, which means $\phi^{-1}(x_1) < \phi^{-1}(x_2)$, leading to a contradictory. Thus ϕ^{-1} is strictly increasing.

   On the other hand, $\forall x_0 \in D, \forall \epsilon > 0$, we need to show that $\exists \delta > 0,\ \forall x_1 \in D,\ |x_1 - x_0| < \delta$, then $|\phi^{-1}(x_1) - \phi^{-1}(x)| < \epsilon$. Denote $y_1 = \phi(\min\{(\phi^{-1}(x_0) - \epsilon,\ \phi^{-1}(x_0) + \epsilon) \cap C\}),\ y_2 = \phi(\max\{(\phi^{-1}(x_0) - \epsilon,\ \phi^{-1}(x_0) + \epsilon) \cap C\})$, then let $\delta = \min y_1 - x_0, y_2 - x_0$, we have $\forall |x - x_0| < \delta$, $|\phi^{-1}(x) - \phi^{-1}(x_0)| < \epsilon$. Thus ϕ^{-1} is continuous on D.

   Since C is measure zero, we get a function satisfying the properties in the problem.

2. (Problem 37, Page 53) Let f be a continuous function defined on E. Is it true that $f^{-1}(A)$ is always measurable if A is measurable?

   Collaborators:

   Solution: It is not true.

   We may consider the function $\psi(x)$ defined by Proposition 21 on Page 52. It is a strictly increasing continuous function, and it maps a measurable set $A \subset C$, onto a nonmeasurable set. Thus if we consider $\psi^{-1}(x)$, it has been proved as a strictly increasing continuous function in Problem 1. Then $\psi^{-1}(A)$ is nonmeasurable.

3. (Problem 39, Page 53) Let F be the subset of $[0,1]$ constructed in the same manner as the Cantor set except that each of the intervals removed at the nth deletion stage has length $\alpha 3^{-n}$ with $0 < a < 1$. Show that F is a closed set, $[0,1] \sim F$ dense in $[0,1]$, and $m(F) = 1 - a$. Such a set F is called a *generalized Cantor set*.

   Collaborators:

Solution: First, we denote $F_1 = \left[0, \frac{3-\alpha}{6}\right] \cup \left[1 - \frac{3-\alpha}{6}, 1\right]$. With the same process of constructing the Cantor set, we have a collection of F_n. We define the generalized Cantor set as

$$F = \bigcap_{k=1}^{\infty} F_n.$$

Since each F_n is a closed set, we have F as well closed. Then F_n is the disjoint of union of 2^n intervals, each of length $(1-\alpha)\frac{1}{2^n} + \frac{\alpha}{3^n}$. By the finite additivity of Lebesgue measure,

$$m(F_k) = 1 - \alpha + \left(\frac{2}{3}\right)^n \alpha.$$

According to the continuity of measure, we have $m(F) = \lim_{k\to\infty} m(F_k) = 1 - \alpha$.

Let $x < y \in [0,1]$. If $y \notin F_k$ for one of the k, then since $[0,1] \setminus F$ is open, there exists $t < y$ in $N(y)$, s.t. $t \in [0,1] \setminus F$. If $x \notin F_k$, it is the same case. Now we assume that $x,\ y \in F$. If $x,\ y$ are not in the same subset of F_k, then $\exists t \in [0,1] \setminus F$. Then we can assume that $x,\ y$ always belong to the same subset. However, the subsets make a nested set with the limit of lower bound and the upper bound be the same. According to Nested Set Theorem, we have $x = y$. Then we have proved that $[0,1] \setminus F$ is dense.

4. Let C be the Cantor set and let φ be the Cantor-Lebesgue function.
   (a) Show that C consists of all $x \in [0,1]$ whose ternary expansion has coefficients equal to 0 or 2, i.e., if $x = \sum_{k\geq 1} c_k 3^{-k}$, where each $c_k = 0, 1,$ or 2, then $x \in C$ if and only if $c_k = 0$ or 2.
   (b) Show that if $x \in C$ and $x = \sum_{k\geq 1} c_k 3^{-k}$, where each $c_k = 0$ or 2, then $\varphi(x) = \sum_{k\geq 1}(\frac{1}{2}c_k)2^{-k}$.

Collaborators:

Solution: (a). We show by induction that a number belongs to the intervals we removed in each step iff its ternary expasion has a coefficient 1. First, the interval removed at the first step can be represented as $(0.1,\ 0.2)$. Thus $0.1 \notin F$, and $0.0, 0.2 \in F$. Assume it holds in the first n steps, then the intervals removed at the $n+1$ step has the representation $(0.a_1a_2\cdots a_n1,\ 0.a_1a_2\cdots a_n2)$, in which $a_i \in \{0,2\}$. Then each number in this interval has an expansion like

$$0.a_1a_2\cdots a_n1a_{n+1}\cdots.$$

Thus each number in F has a coefficient 1 in its expansion. According to the construction process, we can also know that the reverse also holds. Thus the proposition holds.

(b). First, assume $x \in O$. According to the construction of Cantor function, if $x = \sum_k c_k 3^{-k}$ is the ternary expansion of x, and the first 1 in the ternary expansion is in the n_{th} place, then $\varphi(x) = \frac{1}{2^n} + \sum_{k<n} \frac{1}{2}c_k 2^{-k}$.

If $x = \sum_k c_k 3^{-k} \in C$, according to the extension of $\varphi(x)$, $\varphi(x) = \sup\{\varphi(t) \mid t \in O \cap [0,x)\}$. We can construct a sequence $\{t_i\} = \{\sum_k d_k 3^{-k}\}$, s.t. $d_i = c_i, \forall i < m_{i+1}$, where m_i is the position of i_{th} 2 in c_k, and $d_{m_{i+1}} = 1,\ d_k = 0,\ \forall k > m_{i+1}$. Then $\{t_i\}$ is a strictly increasing sequence, and $\lim_{i\to\infty} t_i = x$.

Since $\varphi(x)$ is a strictly increasing function in O, we know that

$$\varphi(x) = \sup\{\varphi(t) \mid t \in O \cap [0,x)\} \geq \varphi(t_i) = \frac{1}{2^i} + \sum_{k<i} \frac{1}{2} d_k 2^{-k}.$$

Thus

$$\varphi(x) \geq \sum_k \frac{1}{2} c_k 2^{-k}$$

when we let $i \to \infty$. On the other hand, $\forall \epsilon > 0$, $\exists i > 0$, $\frac{1}{2^i} < \epsilon$. Thus

$$\varphi(t_i) > \sum_k \frac{1}{2} c_k 2^{-k} - \epsilon.$$

So

$$\varphi(x) = \sup\{\varphi(t) \mid t \in O \cap [0, x)\} = \sum_k \frac{1}{2} c_k 2^{-k}.$$

5. Construct a Cantor-type subset of $[0, 1]$ by removing from each interval remaining at the k^{th} stage, a subinterval of relative length θ_k, $0 < \theta_k < 1$. Show that the remainder has measure zero if and only if $\sum_{k \geq 1} \theta_k = \infty$. (Use the fact that for $a_k > 0$, the product $\Pi_{k=1}^{\infty} a_k$ converges, in the sense that $\lim_{n\to\infty} \Pi_{k=1}^{N} a_k$ exists and is not zero, if and only if $\sum_{k=1}^{\infty} \ln a_k$ converges.)

Collaborators:

Solution: Denote F_n as the remained set after n steps. Then according to the construction, we have

$$m(F_n) = \prod_{i=1}^{n} (1 - \theta_i).$$

First we have $\sum_{n=1}^{\infty} -\theta_n$ and $\sum_{n=1}^{\infty} \ln(1 - \theta_n)$ be both negative-term series, and the necessity of their convergence is $\lim_{n\to\infty} \theta_n = 0$. On the other hand, when $\lim_{n\to\infty} \theta_n = 0$,

$$\lim_{n\to\infty} \frac{\ln(1 - \theta_n)}{-\theta_n} = 1,$$

thus that $\sum \ln(1 - \theta_n)$ converges is equivalent to the convergence of $\sum -\theta_n$.

Then according to the continuity of measure,

$$\begin{aligned} m(F) > 0 \Leftrightarrow m(\lim_{n\to\infty} F_n) > 0 \Leftrightarrow \prod_{i=1}^{\infty} (1 - \theta_i) \text{ converges} \Leftrightarrow \sum_{i=1}^{\infty} \ln(1 - \theta_i) \text{ converges} \\ \Leftrightarrow \sum_{i=1}^{\infty} \theta_i \text{ converges.} \end{aligned}$$

Thus $m(F) = 0$ iff $\sum \theta_i = \infty$.

6. Let Z be a set of measure zero in $\mathbb{R}$. What is the measure of $\{x^2 \mid x \in Z\}$?

Collaborators:

Solution: $X = \{x^2 \mid x \in Z\}$ is also measure 0.

First, if the set Z is bounded, which means $Z \in [0, M]$. Since $m(Z) = 0$, $\forall \epsilon > 0$, there exists an open cover $\{O_i\} = \{(a_i, b_i)\}$, s.t. $m^*(\cup\{O_i\} \setminus Z) < \epsilon$. Thus $\{(a_i^2, b_i^2)\}$ is an open cover of X, and $m^*(\cup(a_i^2, b_i^2) \setminus X) \leq M \sum(b_i - a_i) < M\epsilon$. Thus now X is a measure-zero set. When $Z \in [-M, 0]$, with the same process above, we can know that X is a measure-zero set. Then when $Z \in [-M, M]$, we know X is also measure zero.

Since Z is measure zero, for $\forall \epsilon > 0$ and integer $n > 0$, we have $m(Z \cap [-n, n]) < 2^{-n}\epsilon$. Thus from the discussion above we have $m(X \cap [-n^2, n^2]) < 2^{-n}\epsilon$. Since

$$X = \bigcup_{n=1}^{\infty} X \cap [-n, n],$$

with the addivity of measure,

$$m(X) < \sum_{n=1}^{\infty} 2^{-n}\epsilon = \epsilon.$$

With the arbitrariness of ϵ, we have $m(X) = 0$.

7. Let $0.\alpha_1\alpha_2\cdots$ be the dyadic development of any $x \in [0,1]$. Let $k_1, k_2, k_3, \dots$ be a fixed permutation of the positive integers $1, 2, \dots$, and consider the transformation T which sends $x = \alpha_1\alpha_2\alpha_3\cdots$ to $Tx := \alpha_{k_1}\alpha_{k_2}\alpha_{k_3}\cdots$. Show that if E is a measurable subset of $[0,1]$ then its image under T, $T(E)$, is also measurable and that $m(T(E)) = m(E)$. That is, show that T is a measure preserving transformation of $[0,1]$. [Consider first the special case where E is a dyadic interval of the form $(s2^{-k}, (s+1)2^{-k})$ and $s = 0, 1, \dots, 2k-1$. Then think about open sets and note that each open set can be written as a countable union of non-overlapping half-open dyadic intervals.]

Collaborators:

Solution: Consider the intervals $E_k^i = [i \cdot 2^-k,\ (i+1)2^{-k})$. Then

$$T[E_k] = \bigcap_{i=1}^{2^k} F_{k_i},$$

where

$$F_{k_i} = \begin{cases} [F_{k_{i-1}}.\text{left},\ F_{k_{i-1}}.\text{mid}), & \text{if } k_i = 0 \\ [F_{k_{i-1}}.\text{mid},\ F_{k_{i-1}}.\text{right}), & \text{if } k_i = 1. \end{cases}$$

Then since the finite intersection of closed sets is measurable, we know that $T[E_k]$ is measurable, and we have $m(T(E_k)) = m(E_k)$ since the probability of the numbers with infinite length be seperated into the intervals. Then since each open interval can be written as a countable union of non-overlapping half-open dyadic intervals, we know from the discussion above that $T(E)$ is measurable, and $m(T(E)) = m(E)$.