Taylor. $\mathbf{R_{n+1}(x)} = \frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)dt = \frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}(\xi)$. $\ln(\mathbf{1+x}) = \sum_{i=1}^{n}(-1)^{i-1}\frac{x^i}{i}$. $\sin\mathbf{x} = \sum_{i=1}^{n-1}(-1)^{i-1}\frac{x^{2i-1}}{(2i-1)!} + (-1)^n\frac{x^{2n+1}}{(2n+1)}\cos\xi$. $\cos\mathbf{x} = \sum_{i=0}^{n-1}(-1)^i\frac{x^{2i}}{(2i)!} + (-1)^n\frac{x^{2n}}{(2n)!}\cos\xi$. $(\mathbf{1+x})^\alpha = 1 + C_\alpha^1 x + C_\alpha^2 x^2 + \cdots + C_\alpha^n x^n + C_\alpha^{n+1}\frac{x^{n+1}}{(1+\xi)^{n+1-\alpha}}$. $C_\alpha^i = \frac{\alpha(\alpha-1)\cdot(\alpha-i+1)}{i!}$. $\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots + (-1)^n\frac{x^{2n+1}}{2n+1}$, $\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n$. $\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{1*3}{2*4}x^2 + \cdots + (-1)^n\frac{(2n-1)!!}{(2n)!!}x^n$.

Mean value. Diff: f cont and diff; $f(b) - f(a) = f'(\xi)(b-a)$; Int: $w > 0$ int, f cont. $\int_a^b w(x)f(x)dx = f(\xi)\int_a^b w(x)dx$.

float num. $\sigma(0.a_1a_2\cdot a_t)\beta^e$. sig. digits: e.g. $0.02138, 0.02144$: 2; $0.333, 0.33$: 2; (minus 1). **loss of significance:** solve by rationalize; (Or Taylor expansion) **Machine epsilon**: called unit roundoff, chop: β^{-t+1}, round: $\frac{1}{2}\beta^{-t+1}$. the smallest num s.t. $fl(1+\delta) > 1$.

Conv Order: 1: $|x - x_{n+1}| \le c|x - x_n|^p$ for some $c > 0$, $p \ge 1$. 2: $\lim_{n\to\infty}\frac{|x-x_{n+1}|}{|x-x_n|^p} = c$ (2 also called asymptotic rate);

Bisection: cont. (intermediate value thm). Adv: 1: guaranteed to conv. 2: reasonal error bound; Disadv: 1: doesn't take adv of machine eps. 2: conv too slow.

Newton: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, regarded as fixed-point iter $g(x) = x - \frac{f(x)}{f'(x)}$; Conv order and rate: Taylor expansion near α, $\lim_{n\to\infty}\frac{\alpha - x_{n+1}}{(\alpha-x_n)^2} = -\frac{f''(\alpha)}{2f'(\alpha)}$ (Assumption: $f'(\alpha) \neq 0$, f, f', f'' cont, x_0 sufficiently close to α, s.t. $M = \frac{\max|f''|}{\min 2|f'|}$), $M|\alpha - x_0| < 1$). Error estimate: $\alpha - x_n = x_{n+1} - x_n$ (by mean value thm, let $f'(\xi) = f'(x_n)$); Adv: quick. Disadv: 1: doesn't guaranteed to conv. 2: need to know f';

Secant: $x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}f(x_n)$. Error: $\alpha - x_{n+1} = -(\alpha - x_{n-1})(\alpha - x_n)\frac{f[x_{n-1}, x_n, \alpha]}{f[x_{n-1}, x_n]} = -()()\frac{f''(\xi_n)}{2f'(\eta_n)}$. Conv: $p = \frac{\sqrt{5}+1}{2}$, assumption: $f \in C^2, f'(\alpha) \neq 0$, $\delta = \max\{M|e_0|, M|e_1|\} < 1$, let $I = [\alpha - \epsilon, \alpha + \epsilon]$, s.t. $f' \neq 0$ in I. Then $M|e_{n+1}| \le \delta^{q_{n+1}}, q_{n+1} = q_n + q_{n-1}$.

Fixed point iter: $x = g(x) \to x_{n+1} = g(x_n)$. **Exist:** assumption: g cont, $g([a,b]) \subset [a,b]$, then $x = g(x)$ has at least one sol in $[a,b]$ (pf: f = g(x)-x, intermediate val thm). **Unique:** (**First case:** exist cond, $|g(x) - g(y)| \le \lambda|x - y|, 0 < \lambda < 1$. $\to$ for any x_0, $x_n \to \alpha$, $|\alpha - x_n| \le \frac{\lambda^n}{1-\lambda}|x_1 - x_0|$) (**2nd case:** exist cond, g diff. $\lambda = \max|g'| < 1 \to$ 1st conclusion, and $\lim\frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha)$). (**3rd case, local ver.** $g \in C^1$ in neighbour of α, $|g(\alpha)| < 1$, x_0 sufficiently close (s.t. $g'(x) < 1$ in this interval) to α. $\to$ 2nd conclusion)

Higher order one-point: $g \in C^p, p \ge 2$ in neighbour of α, $g'(\alpha) = \cdots = g^{(p-1)}(\alpha) = 0, g^{(p)}(\alpha) \neq 0$, x_0 sufficiently close to α; $\to$ order p, $\lim\frac{\alpha - x_{n+1}}{(\alpha - x_n)^p} = (-1)^{p-1}\frac{g^{(p)}(\alpha)}{p!}$. (pf: taylor expansion)

Sys of nonlinear equs: $f_1(x_1, x_2) = 0, f_2(x_1, x_2) = 0$. fixed-point: $\alpha - x_{n+1} = G_n(\alpha - x_n), G_n$ is Jacobian of g, the iteration func (not f). Thm: D closed bounded convex, $g \in C^1$, $g(D) \subset D$, $\lambda = \max\|G\|_\infty < 1$. $\to$ 1. any initial val $\to$ unique sol; 2. $\|\alpha - x_{n+1}\|_\infty \le (\|G\|_\infty + \epsilon_n)\|\alpha - x_n\|_\infty, \epsilon_n \to 0$. **local ver:** in a neighbour of α, if $\|G(\alpha)\|_\infty < 1$, then $x_n \to \alpha$. ($\|G\|_\infty$: maximum of row sums.) **Newton:** $x_{n+1} = x_n - F(x_n)^{-1}f(x_n)$, F is Jacobian of (f_1, f_2).

Poly interpolation: exist & unique: 1. VA = y has unique sol (1.1: $\det V \neq 0$; 1.2: $VA = 0$ only has zero-sol.) 2. Construct (Lagrange basis) - unique: $r(x) = p(x) - q(x)$, $r(x_i) = 0$;

Lagrange: $\Psi(x) = \prod_{i=0}^{n}(x - x_i)$, then $l_j(x) = \frac{\Psi(x)}{(x-x_j)\Psi'(x_j)}$. Pf of res: $G(x) = E(x) - \frac{\Psi(x)}{\Psi(t)}E(t)$, then G has $n+2$ zeros, use Rolle thm. Rounding err: let $f_0 = f(x_0) - \epsilon, \cdots$, bound $E(x) = f(x) - L(f_i)(x)$ (use f_i for interpolation)

Newton: $p_n(x) = f(x_0) + (x - x_0)f[x_0, x_1] + \cdots + (x - x_0)\cdots(x - x_{n-1})f[x_0, \cdots, x_n]$. $f[x_0, \cdots, x_n] = \sum_{j=0}^{n}\frac{f(x_j)}{\Psi'(x_j)}$. Pf of res: construct $p_{n+1}(x)$ by add a point $(t, f(t))$, then $p_{n+1}(t) = f(t)$.

Residue $f(t) - \sum_{j=0}^{n}f(x_j)l_j(t) = \frac{(t-x_0)\cdots(t-x_n)}{(n+1)!}f^{(n+1)}(\xi) = f[x_0, \cdots, x_n, x]\prod_{i=0}^{n}(x - x_i)$.

Interpolation basis: 1. Monomial: x^i. adv: easy to write, matrix A easy to compute and evaluate. disadv: $Ax = b$ expensive to solve, sys ill-cond; 2. Lagrange: $l_i(x)$. adv: no need to solve equations, depend on x_i not $y_i \to$ useful when many sets of $\{y_i\}$ on the same $\{x_i\}$. disadv: expensive to compute, hard to add new points; 3. Newton: $\prod(x_i - x_j)$. adv: easy to compute, solve the matrix, bound error, add new points.

Hermite: use primes; $h_i(x) = (1 - 2l_i'(x_i)(x - x_i))l_i^2(x), \hat{h}_i(x) = (x - x_i)l_i^2(x)$, s.t. $h_i(x_j) = \hat{h}_i'(x_j) = \delta_{ij}, h_i'(x_j) = \hat{h}_i(x_j) = 0$. $H(x) = \sum y_ih_i(x) + \sum y_i'\hat{h}_i(x)$. **Res:** $E = f[x_1, x_1, \cdots, x_n, x_n, x]\prod(x - x_i)^2 = \Psi^2(x)\frac{f^{2n}(\xi)}{(2n)!}$, (let p be interpolation on $\{x_i\}_{i=1}^{2n}$, let $x_{2i} \to x_{2i-1}$).

Piecewise: order r: poly order $< r$ in each interval; local cubic: lagrange/hermite;

Spline: Grid $a = x_0 < \cdots < x_n = b$. s is spline order m: 1. s is poly order $< m$ on each $[x_{i-1}, x_i]$; 2. $s^{(r)}$ continuous, for $0 \le r \le m-2$. then s' is a spline of order $m-1$, etc.

cubic spline: order $m = 4$. cond: $s(x_i) = y_i, 0 \le i \le n$. Case 1: $s'(x_0) = y_0', s'(x_n) = y_n'$, complete spline. Case 2: $s''(x_0) = s''(x_n) = 0$, natural spline. **Error:** $\max|f^{(j)}(x) - s^j(x)| \le c_jh^{4-j}\max|f^{(4)}(x)|$, $c_0 = \frac{5}{384}, c_1 = \frac{1}{24}, c_2 = \frac{3}{8}$. $\int_a^b\|s''(x)\|^2dx \le \int_a^b\|g''(x)\|^2dx$, for any g satisfying the conditions as s. In order to solve Runge ($\frac{1}{1+x^2}$), the other way is to interpolate at Chebyshev zeros (see chpt 4).

Trigonometric: for periodic functions. interpolate at $t_j = \frac{2\pi j}{2n+1}, j = 0, \pm1, \cdots$. Use FFT.

Weierstrass appro. thm: f cont on $[a,b]$. for each ϵ, there is a poly p, s.t. $|f - p| \le \epsilon$. (Motivation for best approx: more efficient than interpolation; low degree); Taylor: error increase when $|x|$ get large, and error not distributed evenly. (so could be better).

Minimax: minimax error $\rho_n(f) = \inf_{deg(q)\le n}\|f - q\|_\infty$ (compute: the max error gets at endpoints and some middle points.)

Least square: minimize $M_n(f) = \inf_{deg(r)\le n}\|f - r\|_2$. (compute: derivative of coefficients = 0.)

Weight func: nonnegative, $\int_a^b|x|^nw(x)$ finite for all n; if $\int_a^b w(x)g(x) = 0$ and g nonnegative, then $g = 0$.

Orthogonal poly: $(f, g) = \int_a^b w(x)f(x)g(x)dx$; Gram-Schmidt: $|\varphi_0| = 1$, $\psi(x) = x^n + a_{n,n-1}\varphi_{n-1}(x) + \cdots + a_{n,0}\varphi_0(x)$, $a_{n,j} = -(x^n, \varphi_j)$, $\varphi_n(x) = \frac{\psi_n(x)}{|\psi_n(x)|}$. **Legendre:** $w(x) = 1, \in [-1, 1]$, $P_n(x) = \frac{(-1)^n}{2^nn!}\frac{d^n}{dx^n}[(1 - x^2)^n], P_0 = 1$; $(P_n, P_n) = \frac{2}{2n+1}$. **Chebyshev:** $w(x) = \frac{1}{\sqrt{1-x^2}}, \in [-1, 1]$, $T_n(x) = \cos(n\cos^{-1}x), T_{n+1} = 2xT_n(x) - T_{n-1}(x)$, $(T_n, T_m) = \pi(n = m = 0), 0(n \neq m), \pi/2(n = m > 0)$. If $deg(f) = m$, $f = \sum_{n=0}^{m}\frac{(f, \varphi_n)}{(\varphi_n, \varphi_n)}\varphi_n(x)$. **TRR:** $\varphi_{n+1}(x) = (a_nx + b_n)\varphi_n(x) - c_n\varphi_{n-1}(x)$, where $\varphi_n = A_nx^n + B_nx^{n-1} + \cdots$, $a_n = \frac{A_{n+1}}{A_n}, \gamma_n = (\varphi_n, \varphi_n), b_n = a_n(\frac{B_{n+1}}{A_{n+1}} - \frac{B_n}{A_n}), c_n = \frac{A_{n+1}A_{n-1}}{A_n^2}\frac{\gamma_n}{\gamma_{n-1}}$. **Laguerre:** $w(x) = e^{-x}, [0, \infty), L_n(x) = \frac{1}{n!e^{-x}}\frac{d^n}{dx^n}(x^ne^{-x})$, $|L_n|_2 = 1$ for all n.

General least square sol: φ be (normalized) ort. poly with $w(x)$. $r(x) = \sum_{k=0}^{n} b_k \varphi_k(x)$, where $b_j = (f, \varphi_j)$. **Bessel & Parseval:** $|r_n^*|_2^2 = \sum_{k=0}^{n} (f, \varphi_k)^2 \le |f|_2^2 = \sum_{k=0}^{\infty} (f, \varphi_k)^2$; **Lagendre:** $\varphi_0(x) = \frac{1}{\sqrt{2}}$, $\varphi_n(x) = \sqrt{\frac{2n+1}{2}} \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n}[(1-x^2)^n], (f, \varphi_j) = \int_{-1}^{1} f(x)\varphi_j(x)dx, r_n^*(x) = \sum_{j=0}^{n} (f, \varphi_j)\varphi_j(x)$. **Chebyshev:** $\varphi_0(x) = \frac{1}{\sqrt{\pi}}, \varphi_n(x) = \sqrt{\frac{2}{\pi}} T_n(x), (f, \varphi_j) = \int_{-1}^{1} \frac{f(x)\varphi_j(x)}{\sqrt{1-x^2}} dx$.

Minimax: de la Vallee-Poussin: f cont, $deg(Q) \le n$, $f(x_j) - Q(x_j) = (-1)^j e_j, 0 \le j \le n+1, e_j \ne 0$ of same sign, $a \le x_0 < \cdots < x_{n+1} \le b$, then $\min |e_j| \le \rho_n(f) \le |f - Q|_\infty$. **Equioscillation:** there is a unique poly $deg(q^*) \le n$, s.t. $\rho_n(f) = |f - q^*|_\infty$. and q^* satisfy that there exists $a \le x_0 < \cdots < x_{n+1} \le b$, s.t. $f(x_j) - q^*(x_j) = \sigma(-1)^j \rho_n(f), \sigma = \pm 1$.

Near Minimax: use chebyshev least square $C_n(x) = \sum_{j=0}^{n} c_j T_j(x), c_j = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_j(x)}{\sqrt{1-x^2}} dx, c_0 = c_0/2$ as an approximation, then since $f = \sum_{j=0}^{\infty}{}' c_j T_j(x) (f \in C[-1,1])$, the error $f(x) - C_n(x)$ is nearly $c_{n+1}T_{n+1}(x)$. Since $T_{n+1}(x_j) = (-1)^j, x_j = \cos \frac{j\pi}{n+1}, 0 \le j \le n+1$, with Chebyshev Equio. thm, C_n should be nearly equal to the minimax approx. $|f - I_n|_\infty \le \frac{1}{(n+1)!2^n} |f^{(n+1)}|_\infty$

Chebyshev zeros interpolation: from Near minimax, the error is nearly $c_{n+1}T_{n+1}(x)$, thus err be nearly 0 at roots of $T_{n+1}(x) \to x_j = \cos \frac{2j+1}{2n+2}\pi, 0 \le j \le n$. Use interpolation of $f(x)$ on these nodes $I_n(x)$ as approx of C_n and $q_n^*(x)$ $(I_n(x) = \sum f(x_j) l_j(x) = \sum C_n(x_j) l_j(x) + \sum (f(x_j) - C_n(x_j)) l_j(x) = C_n(x))$

Cheb. Poly: $r_n = \inf_{deg(q) \le n-1} (\max_{-1 \le x \le 1} |x^n + q(x)|)$, the minimum attained at $x^n + q(x) = \frac{1}{2^{n-1}} T_n(x), r_n = \frac{1}{2^{n-1}}$.

Cheb. Poly of 2nd kind: $S_n(x) = \frac{1}{n+1} T'_{n+1}(x), [-1,1], w(x) = \sqrt{1-x^2}$, satisfy same TRR with $\{T_n\}$.

Near minimax err: $x_i = \cos \frac{i\pi}{n+1}, 0 \le i \le n+1$, $\sum_{k=0}^{n} c_{n,k} T_k(x_i) + (-1)^i E_n = f(x_i), \sum = \sum' \to c_{n,j} = \frac{2}{n+1} \sum_{i=0}^{n+1} f(x_i) \cos(\frac{ij\pi}{n+1}), \sum = \sum'', E_n = \frac{1}{n+1} \sum_{i=0}^{n+1} (-1)^i f(x_i), \sum = \sum''$. **Notice:** $\sum_{k=1}^{n} kz^k = \frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2} z$

Integration: $I_n(f) = \int_a^b f_n(x)dx = I(f_n) = \sum_{j=1}^{n} w_{j,n} f(x_{j,n})$, use $\{f_n\}$ to approx. f, s.t. $|f - f_n|_\infty \to 0$.

Ways of deriving w_i: 1. compute the integral of interpolation; 2. write down a series of equations.

Asy. err: E_n exact; $\hat{E}_n$ estimate; $\lim_{n\to\infty} \frac{\hat{E}_n}{E_n} = 1$. Corrected: $CT = I + \hat{E}$.

trapezoidal: $E_1(f) = \int_a^b (x-a)(x-b) f[a,b,x] dx = -\frac{(b-a)^3}{12} f''(\eta)$ (use mean value thm). **composite:** $E_n(f) = -\frac{h^3 n}{12} \frac{1}{n} \sum_{j=1}^{n} f''(\eta_j) = -\frac{(b-a)h^2}{12} f''(\eta)$. **Asy err:** $\lim_{n\to\infty} \frac{E_n(f)}{h^2} = -\frac{1}{12} \int_a^b f''(x)dx$.

Simpson: $I_2 = \frac{h}{3}(1,4,1), h = \frac{b-a}{2}$. $E_2 = \int_a^b (x-a)(x-b)(x-c) f[a,b,c,x] dx = \int_a^b w'(x) f[a,b,c,x] dx = -\frac{h^5}{90} f^{(4)}(\eta)$. (integrate by parts; $w(x) = \int_a^x (t-a)(t-b)(t-c)dt, w(a) = w(b) = 0, w(x) > 0 \in (a,b)$.) **composite:** $h = (b-a)/n, I_n = \frac{h}{3}(1,4,2,4,2\cdots,2,4,1)$. $E_n = -\frac{h^4(b-a)}{180} f^{(4)}(\eta)$.

Peano Kernel: $f = p_1 + R_2$, $R_2 = \int_a^x (x-t) f''(t)dt$. compute $E_1(R_2)$ use $\int_a^b \int_a^x G(x,t)dtdx = \int_a^b \int_t^b G(x,t)dxdt$, get $E_n(f) = \int_a^b K(t) f''(t)dt$.

N-C: use lag. poly to approx f. **Err:** 1. n even, $f \in C^{n+2}$. $E_n(f) = C_n h^{n+3} f^{(n+2)}(\eta)$, $C_n = \frac{1}{(n+2)!} \int_0^n u^2(u-1)\cdots(u-n)du$; 2. n odd, $f \in C^{n+1}$. $E_n(f) = C_n h^{n+2} f^{(n+1)}(\eta)$, $C_n = \frac{1}{(n+1)!} \int_0^n u(u-1)\cdots(u-n)du$. **Precision:** m if $I = \hat{I}$ for all poly deg $\le m$.

Conv NC: $I_n(f) \to I(f) \iff$ 1. $I_n(f) \to I(f)$ for all $f \in \mathcal{F}$ dense in poly; 2. $B = \sup_n \sum_{j=0}^{n} |w_{j,n}| < \infty$.

Midpoint: $E_n(f) = \frac{h^2(b-a)}{24} f''(\eta)$.

Gaussian: Regard $I_n(f) = \sum_{j=1}^{n} w_{j,n} f(x_{j,n}) = \int_a^b w(x) f(x) dx$. Use Orthogonal poly: (construct with Hermite). $w_i = \int_a^b w(x) l_i(x) dx$, $E_n(f) = \frac{f^{(2n)}(\eta)}{(2n)!} \int_a^b w(x) \frac{\varphi_n(x)^2}{A_n^2} dx$, x_i be roots of $\varphi_n(x)$ w.r.t. $w(x)$. **Err:** $|E_n(f)| \le 2\rho_{2n-1}(f) \int_a^b w(x)dx$.

G-Legendre: $w(x) = 1, \in [-1,1]$, $w_i = -\frac{2}{(n+1)P'_n(x_i)P_{n+1}(x_i)}$, $E_n(f) = \frac{2^{2n+1}(n!)^4}{(2n+1)((2n)!)^2} \frac{f^{2n}(\eta)}{(2n)!}$.

Comments: roots of I_n are diff. from I_m; can handle near-singular integrals.

Asymptotic Err: Euler-MacLaurin: err in trapezoidal: $E_n(f) = \int_a^b f(x)dx - h \sum_{j=0}^{n}{}'' f(x_j) = -\sum_{i=1}^{m} \frac{B_{2i}}{(2i)!} h^{2i} (f^{(2i-1)}(b) - f^{(2i-1)}(a)) + \frac{h^{2m+2}}{(2m+2)!} \int_a^b \bar{B}_{2m+2}(\frac{x-a}{h}) f^{(2m+2)}(x) dx$, where $\frac{t(e^{xt}-1)}{e^t - 1} = \sum_{j=1}^{\infty} B_j(x) \frac{t^j}{j!}$, $\frac{t}{e^t-1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!}, \bar{B}_j(x)$ be periodic extension of $B_j(x)$ w./ pero. 1. (PF: consider $n = 1$, then $n \ge 1$ be composite form of $n = 1$). **Trapezoidal:** for periodic functions, order of conv. of tra. is greater than any power of h.

Richardson: $I - I_n = \frac{d_2}{n^2} + \frac{d_4}{n^4} + \cdots$, $I - I_{n/2} = \frac{4d_2}{n^2} + \frac{16d_4}{n^4} + \cdots$. Then $I = \frac{4I_n - I_{n/2}}{3} - \frac{4d_4}{n^4} - \cdots$. $I_n^{(k)} = \frac{4^k I_n^{(k-1)} - I_{n/2}^{(k-1)}}{4^k - 1}$. (less computing, low order conv cmp to Gaussian).

Romberg: $I_k(f) = I_{2^k}^{(k)}(f)$.

Singular: In function/in interval; sol: **1. change variable**(singularity happens in func and derivatives/interval): $u = x^\alpha, u = \frac{1}{x^\alpha}$; EXP: $I = \int_1^\infty \frac{f}{x^p} dx, x = \frac{1}{u^\alpha}, I = \alpha \int_0^1 u^{(p-1)\alpha - 1} f(\frac{1}{u^\alpha}) du$, pick $(p-1)\alpha - 1$ larger to max smoothness on $x = 0$. 2. for $I = \int_a^b$ has endpoint singularity, let $\psi(t) = \exp(\frac{-c}{1-t^2}), \varphi(t) = a + \frac{b-a}{\gamma} \int_{-1}^{t} \psi(u)du, -1 \le t \le 1, \gamma = \int_{-1}^{1} \psi(u)du$. Let $x = \varphi(t), I = \int_{-1}^{1} f(\varphi(t))\varphi'(t)dt$ **2. Gaussian Quad:** EXP: $I = \int_0^\infty g(x)dx = \int_0^\infty e^{-x} f(x)dx$. Use Gauss-Laguerre $(w(x) = e^{-x}, L_n(x) = \frac{1}{n! e^{-x}} \frac{d^n}{dx^n} x^n e^{-x})$. EXP: $I = \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx$, use Chebyshev, $x_{j,n} = \cos \frac{2j-1}{2n}\pi, 1 \le j \le n, w_{j,n} = \frac{\pi}{n}$. **Analytic:** $I = \int_0^b f(x)\log(x)dx = \int_0^\epsilon + \int_\epsilon^b = I_1 + I_2$. Take the first terms of Taylor series of f in I_1. **4. Product Integration:** $I(f) = \int_a^b w(x)f(x)$, let all singularity in w. Produce $\{f_n\}$, $|f - f_n|_\infty \to 0$, $I_n(f) = I(f_n)$ easily computed, then $|I(f) - I_n(f)| \le |f - f_n|_\infty \int_a^b |w(x)|dx$. EXP: product trapezoidal for $I(f) = \int_0^b f(x)\log(x)dx, x_j = jh$. For $x_{j-1} \le x \le x_j, f_n(x) = \frac{1}{h}((x_j - x)f(x_{j-1}) + (x - x_{j-1})f(x_j))$, then $|f - f_n| \le \frac{h^2}{8}|f''|$.

Diff: 0. Definition. 1. use poly. interpolation. $f'(x) = p'_n(x) = \sum_{j=0}^{n} f(x_j) l'_j(x)$, error $= \Psi'_n(x) \frac{f^{(n+1)}(\xi_1)}{(n+1)!} + \Psi_n(x) \frac{f^{(n+2)}(\xi_2)}{(n+2)!} = O(h^n)$ (when $\Psi'_n(x) \ne 0$) and $= O(h^{n+1})$ (when $\Psi'_n(x) = 0$). (So choose nodes to have $\Psi'_n(x) = 0$) $\to$ if n odd, nodes symmetrically about x. **2. Method of Undetermined Coeff.** EXP: $f''(x) = Af(x+h) + Bf(x) + Cf(x-h)$, use Taylor expansion of $f(x), f(x \pm h)$. Notice: error dominates the result when h gets very small.

Some Formulas: $\sum_{k=1}^{n} kz^k = \frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2} z$ $\int_0^1 x^n \ln(x)dx = -\frac{1}{(n+1)^2}$.