Numerical Analysis
Assignment 5

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Problem 1. Problem 2.21, Page 121

Solution. According to Theorem 2.7, we should choose c so as to have $|g'(\alpha)| < 1$. Thus

$$|g'(\alpha)| = |1 + cf'(\alpha)| < 1.$$

So c satisfies $-2 < cf'(\alpha) < 0$. For a good rate of convergence, pick c s.t. $cf'(\alpha) \sim 0$.

Problem 2. Problem 2.24, Page 121

Solution. **(a)**

$$g(x) = -16 + 6x + \frac{12}{x}.$$

Thus $g'(2) = 3 > 1$. So the iteration does not converge.
(b)

$$g(x) = \frac{2}{3}x + \frac{1}{x^2}.$$

Thus $g'(3^{\frac{1}{3}}) = \frac{2}{3} - 2\alpha^{-3} = 0$. So the iteration converges. Since

$$\begin{aligned}\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{(\alpha - x_n)^2} &= \lim_{n\to\infty} \frac{-2x_n^3 + 3\alpha x_n^2 - 3}{3x_n^2(\alpha - x_n)^2} = \lim_{x_n\to\alpha} \frac{-2x_n^3 + 3\alpha x_n^2 - 3}{3x_n^2(\alpha - x_n)^2} \\ &= \lim_{x_n\to\alpha} \frac{-6x_n^2 + 6\alpha x_n}{6x_n(\alpha - x_n)^2 - 6x_n^2(\alpha - x_n)} = \lim_{x_n\to\alpha} \frac{1}{\alpha - 2x_n} \\ &= -\frac{1}{\alpha}.\end{aligned}$$

We know the iteration is second-order convergent.
(c)

$$g(x) = \frac{12}{1+x}.$$

Thus $g'(\alpha) = -\frac{3}{4}$, so the iteration converges. Since

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = \frac{(1 + x_n)\alpha - 12}{(1 + x_n)(\alpha - x_n)} = \frac{\alpha}{\alpha - 2x_n - 1} = -\frac{3}{4},$$

We know the iteration is linear convergence, and the rate is $\frac{3}{4}$.

Problem 3. Problem 2.28, Page 122

Proof. Assume $f(x) = (x - \alpha)h(x),\ h(\alpha) \neq 0$. Then

$$\begin{aligned}g(x) = x - \frac{f(x)}{D(x)} &= x - \frac{(x-\alpha)^2h^2(x)}{(x-\alpha)(h(x)+1)h((x-\alpha)h(x)+x) - (x-\alpha)h(x)} \\ &= x - \frac{(x-\alpha)h^2(x)}{(h(x)+1)h((x-\alpha)h(x)+x) - h(x)}.\end{aligned}$$

Thus

$$\begin{aligned}g'(\alpha) &= 1 - \frac{(h^2(x) + 2(x-\alpha)h(x)h'(x))((h(x)+1)h((x-\alpha)h(x)+x) - h(x)) - (x-\alpha)M}{((h(x)+1)h((x-\alpha)h(x)+x) - h(x))^2}\Bigg|_{x=\alpha} \\ &= 1 - \frac{h^4(\alpha)}{h^4(\alpha)} = 0.\end{aligned}$$

And we can know that $g''(\alpha) \neq 0$. Accoding to Theorem 2.8, this iteration is a second-order method.

Problem 4. Problem 2.48, Page 126

Proof. According to the continuity of $\|\cdot\|_\infty$, we can find an ϵ and a closed set $B(\alpha, \delta)$, s.t. $\forall x \in B(\alpha, \delta)$, $\|G(x)\|_\infty \leq 1 - \epsilon$. Then $\forall x_0 \in B = B(\alpha, \delta)$,

$$\|\alpha - x_{n+1}\| = \|g(\alpha) - g(x_n)\| = \|G(\xi)\| \cdot \|\alpha - x_n\| \leq (1-\epsilon)\|\alpha - x_n\| \leq (1-\epsilon)^n \|\alpha - x_0\|.$$

Thus $\|\alpha - x_{n+1}\| \to 0,\ t \to \infty$. So the iteration converges to α. And since $g(B) \subset B$ according to that $\|\alpha - g(x)\| < \|\alpha - x\|$, and $\max\|G(x)\| < 1$, we know that it satisfies the condition of Theorem 2.9.