Homework 4

Instructions: In problems the problems below, references such as III.2.7 refer to Problem 7 in Section 2 of Chapter III in Conway's book.

If you use results from books including Conway's, please be explicit about what results you are using.

Homework 4 is due in class at Midnight March 9.

Do the following problems:

1. IV.7.1

   Sol. (Discussed with a college classmate) In fact, I don't think that this proposition is correct. For example, pick G the unit disk $B(0,1)$, and $\gamma = \gamma(t) : [0,1] \to B$, s.t. $\gamma(t) = t$ for $0 \le t < 1$, and $\gamma(1) = 0$. Then γ is closed, and by simple calculation we know $V(\gamma) = 2$, which shows γ is rectifiable. Let $f = \frac{1}{z-1}$, then f is analytic in $B(0,1)$. But when $t \to 1$, $f \circ \gamma(t) \to \infty$, hence it is not rectifiable.

2. IV.7.2

   (a) Let $f(z) = z$, pick any $z_0 \in \{z \mid d(z, \partial G) < \frac{1}{2}r\}$, then since there is only one point $z = z_0$ satisfies $f(z) = z_0$, by Thm 7.2,
$$n(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}dz$$
   Since $\frac{1}{z-z_0}$ is analytic on $\{z \mid d(z, \partial G) < \frac{1}{2}r\}$, by Prop 2.15, we know the integral is 0. Hence $\{z \mid d(z, \partial G) < \frac{1}{2}r\} \subset H$.

3. V.1.1

   (a) Around $z = 0$,
$$\lim_{z\to 0}|zf(z)| = \lim_{z\to 0}|\sin(z)| = \frac{1}{2}\lim_{z\to 0}|e^{iz} - e^{-iz}| \le \lim_{z\to 0}|z| = 0.$$
   Hence by Thm 1.2, $z = 0$ is removable, and $f(0) = 1$ by power series expansion.

   (b) At $z = 0$, $g(z) = \cos(z)$ is analytic, and $\cos(0) = 1$. Thus by Prop 1.4, $z = 0$ is a pole, and the singular part is $\frac{1}{z}$.

   (c) At $z = 0$, $\lim_{z\to 0} zf(z) = \lim_{z\to 0}\cos z - 1 = 0$, then by Thm 1.2, 0 is removable, and $f(0) = 0$ by power series expansion.

   (d) At $z = 0$,
$$f(z) = \sum_{n=0}^{-\infty}\frac{1}{(-n)!}z^n,$$
   hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \{z \mid |z| > exp(\frac{1}{\delta})\}$.

   (e) At $z = 0$,
$$f(z) = \frac{1}{z^2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n} = \frac{1}{z} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+2}z^n.$$
   Hence 0 is a pole, and the singularity part is $\frac{1}{z}$.

   (f) At $z = 0$,
$$f(z) = z\sum_{n=0}^{\infty}(-1)^n\frac{z^{-2n}}{n!} = z + \sum_{n=-1}^{-\infty}(-1)^{-n}\frac{z^{2n+1}}{(-n)!}$$
   Hence 0 is an essential singularity, and $f(0 < |z| < \delta) = \mathbb{C}$.

(g) Around $z=0$, notice $\frac{z^2+1}{z-1}$ is analytic, hence 0 is a pole. Since $|z|<1$,

$$f(z) = 1 - \frac{1}{z} + \frac{2}{z-1} = 1 - \frac{1}{z} - 2\sum_{n=0}^{\infty} z^n,$$

we know the singular part is $-\frac{1}{z}$.

(h) For any $n>0$,

$$\lim_{z\to 0} z^n f(z) = \lim_{z\to 0} z^n \frac{1}{\sum_{n=1}^{\infty}\frac{1}{n!}z^n} = \infty,$$

hence 0 is an essential singularity, and $f(0<|z|<\delta) = \{z \mid |z| > \frac{1}{1-e^{\delta}}\}$.

(i)

$$f(z) = z\sum_{n=0}^{\infty}(-1)^n \frac{z^{-(2n+1)}}{(2n+1)!} = 1 + \sum_{n=-1}^{-\infty}(-1)^{-n}\frac{z^{2n-1}}{(-2n+1)!},$$

hence 0 is an essential singularity, and $f(0<|z|<\delta) = \{z \mid |z| < \delta\}$.

(j) Same with (i), 0 is an essential singularity, and $f(0<|z|<\delta) = \{z \mid |z| < \delta^n\}$.

4. V.1.4

(a)

$$f(z) = \frac{1}{z}\left(\frac{1}{1-z} - \frac{1}{2(1-z/2)}\right) = \frac{1}{z}\left(\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}(\frac{z}{2})^n\right) = \frac{1}{2z} + \sum_{n=0}^{\infty}(1-\frac{1}{2^{n+2}})z^n$$

(b)

$$f(z) = \frac{1}{z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right) = \frac{1}{z}\left(-\frac{1}{2}\frac{1}{1-\frac{z}{2}} - \frac{1}{z}\frac{1}{1-\frac{1}{z}}\right) = \frac{1}{z}\left(-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty} z^{-n-1}\right) = -\sum_{n=-1}^{\infty}\frac{z^n}{2^{n+2}} - \sum_{n=-\infty}^{-2} z^n$$

(c)

$$f(z) = \frac{1}{z}\left(\frac{\frac{1}{z}}{1-\frac{2}{z}} - \frac{\frac{1}{z}}{1-\frac{1}{z}}\right) = \frac{1}{z}\left(\sum_{n=0}^{\infty}(\frac{2}{z})^n - \sum_{n=0}^{\infty}\frac{1}{z^n}\right) = \sum_{n=-\infty}^{-1}(2^{-(n+1)}-1)z^n.$$

5. V.1.12

Proof. By (1.11),

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{\exp(\frac{1}{2}\lambda(z+\frac{1}{z}))}{z^{n+1}}dz$$

pick $\gamma = \exp(it)$, the unit circle, then

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{\lambda\cos t}e^{-int}dt = \frac{1}{2\pi}\int_0^{2\pi} e^{\lambda\cos t}(\cos nt - i\sin nt)dt$$

$$b_n = \frac{1}{2\pi i}\int_\gamma \frac{\exp(\frac{1}{2}\lambda(z-\frac{1}{z}))}{z^{n+1}}dz$$

pick $\gamma = \exp(it)$,

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} e^{\lambda i\sin t}e^{-int}dt = \frac{1}{2\pi}\int_0^{2\pi}\cos(nt-\lambda\sin t) - i\sin(nt-\lambda\sin t)dt.$$

6. V.1.13

7. V.1.17

8. V.2.1

9. V.2.2
10. V.2.3
11. V.2.4
12. V.2.5