Introduction to Analysis
Assignment 8

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Problem 1. Problem 37, Page 123

Sol. (i) Let

$$f(x) = \begin{cases} x\sin(\dfrac{1}{x}), \ 0 < x \le 1, \\ 0, \ x = 0 \end{cases}$$

Then f is continuous on $(0,1]$. Since $\lim\limits_{x\to 0^+} f(x) = 0 = f(0)$, f is continuous on $[0,1]$. For $x_1, x_2 \in [\epsilon, 1]$ where $\epsilon > 0$,

$$\begin{aligned} |f(x_1) - f(x_2)| &= |x_1 \sin\frac{1}{x_1} - x_2 \sin\frac{1}{x_2}| = |(x_1 - x_2)\sin\frac{1}{x_1} + x_2(\sin\frac{1}{x_1} - \sin\frac{1}{x_2})| \\ &\le |x_1 - x_2| \sin\frac{1}{x_1} + 2x_2|\cos\frac{1}{2}(\frac{1}{x_1} + \frac{1}{x_2})\sin\frac{1}{2}(\frac{1}{x_1} - \frac{1}{x_2})| \\ &\le |x_1 - x_2|\sin\frac{1}{x_1} + 2x_2|\sin\frac{x_2 - x_1}{2x_1x_2}| \le |x_1 - x_2|\sin\frac{1}{x_1} + 2x_2|\frac{x_2 - x_1}{2x_1x_2}| \\ &= |x_1 - x_2|(\sin\frac{1}{x_1} + \frac{1}{x_1}) \le |x_1 - x_2|(1 + \frac{1}{\epsilon}). \end{aligned}$$

Thus f is Lipschitz, with Proposition 7 we know f is absolutely continuous.
However, with Problem 35 we know f is not of bounded variation on $[0,1]$, and with Remark on Page 122 we know f is not absolutely continuous on $[0,1]$.
(ii) If not, then there is a $\epsilon > 0$, s.t. for each $\delta > 0$, there is a finite disjoint collection $\{(a_k, b_k)\}$ of open intervals satisfying $\sum\limits_{k=1}^{n}(b_k - a_k) < \delta$, s.t. $\sum\limits_{k=1}^{n}|f(b_k) - f(a_k)| \ge \epsilon$. As suggested in conditions, for each $c > 0$, f is absolutely continuous on $[c,1]$. Then these open intervals must lie in $[0,c]$ for every c. With the continuity of f on $[0,1]$, there exists $c > 0$, s.t. $0 < f(c) - f(0) < \epsilon$. If we take $\delta = c$, because f is increasing, $\sum\limits_{k=1}^{n}(b_k - a_k) < f(c) - f(0) < \epsilon$. It contradicts with our assumption. Hence f is absolutely continuous on $[0,1]$.
(iii) First we show f is absolutely continuous by showing that it satisfies the condition in (ii). For each $c > 0$, since on $[c,1]$ we have

$$\sqrt{x_1} - \sqrt{x_2}| = \frac{|x_1 - x_2|}{\sqrt{x_1} + \sqrt{x_2}} \le \frac{|x_1 - x_2|}{2\sqrt{c}},$$

we know for each $\epsilon > 0$, pick $\delta = 2\sqrt{c}\epsilon$, then for each collection $\{(a_k, b_k)\}$ satisfying $\sum\limits_{k=0}^{n}|b_k - a_k| < \delta$,

$$\sum_{k=1}^{n}|f(b_k) - f(a_k)| < \sum_{k=1}^{n}\frac{|x_1 - x_2|}{2\sqrt{c}} < \epsilon.$$

Hence f is absolutely continuous on $[c,1]$. Since f is increasing, we know f is absolutely continuous on $[0,1]$.
On the other hand, if there exists $\lambda > 0$, s.t.

$$|f(x_1) - f(x_2)| < \lambda|x_1 - x_2|$$

for each x_1, x_2, then if we pick $\max(x_1, x_2) < \frac{1}{4\lambda^2}$, from the argument above we know

$$|f(x_1) - f(x_2)| = \frac{|x_1 - x_2|}{\sqrt{x_1} + \sqrt{x_2}} > \lambda|x_1 - x_2|.$$

Hence f is not Lipschitz.

Problem 2. Problem 39, Page 123

Sol. Suppose E is a measurable set, then for each $\epsilon' > 0$, there exists an open set $O \supset E$, and $m(O \setminus E) < \epsilon'$. Let $O = \bigcup_{k=1}^{\infty} (a_k, b_k)$ be the open decomposition of O, then (a_i, b_i) are pairwise disjoint.

If f is absolutely continuous, by Problem 38, for each $\epsilon > 0$, there is a $\delta > 0$, for each $\{(a_k, b_k)\}$ satisfying $\sum_{k=1}^{\infty} (b_k - a_k) < \delta$, $\sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \epsilon$. In fact, if we take $\delta_1 < \delta - \epsilon'$, then since f is increasing, $f(E) \in f(\cup\{(a_k, b_k)\})$ when $E \in \cup\{(a_k, b_k)\}$, thus

$$m^*(f(E)) < m(f(\cup\{(a_k, b_k)\})) < \delta.$$

On the other hand, since open sets are measurable, with Problem 39 we know the reverse holds.

Problem 3. Problem 41, Page 123

Sol. (i) With the continuity of f and the compactness of $[a, b]$, we know the maximum and minimum of f on $[a, b]$ exists, thus f maps $[a, b]$ to a closed set. Thus f maps a F_σ set to a F_σ set.
(ii)

Problem 4. Problem 49, Page 128

Problem 5. Problem 56, Page 129

Problem 6. Problem 59, Page 129