Numerical Analysis
Assignment 13

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Problem 1. Problem 5.14

Sol. The result is as follows. We can see the result is better than the simple trapezoidal and simpson rule when $n = 2$ and $n = 3$.

```
function res = gauss_legendre(f, n)
if n == 2
    x = [-0.5773502692, 0.5773502692];
    w = [1.0, 1.0];
elseif n == 3
    x = [-0.7745966692, 0, 0.7745966692];
    w = [5/9, 8/9, 5/9];
elseif n == 4
    x = [-0.8611363116, 0.8611363116, -0.3399810436, 0.3399810436];
    w = [0.3478546451, 0.3478546451, 0.6521451549, 0.6521451549];
elseif n == 5
    x = [-0.9061798459, 0.9061798459, -0.5384693101, 0.5384693101, 0];
    w = [0.2369268851, 0.2369268851, 0.4786286705, 0.4786286705,
        0.5688888889];
elseif n == 6
    x = [-0.9324695142, 0.9324695142, -0.6612093865, 0.6612093865,
        -0.2386191861, 0.2386191861];
    w = [0.1713244924, 0.1713244924, 0.3607615730, 0.3607615730,
        0.4679139346, 0.4679139346];
elseif n == 7
    x = [-0.9491079123, 0.9491079123, -0.7415311856, 0.7415311856,
        -0.4058451514, 0.4058451514, 0];
    w = [0.1294849662, 0.1294849662, 0.2797053915, 0.2797053915,
        0.3818300505, 0.3818300505, 0.4179591837];
elseif n == 8
    x = [-0.9602898565, 0.9602898565, -0.7966664774, 0.7966664774,
        -0.5255324099, 0.5255324099, -0.1834346425, 0.1834346425];
    w = [0.1012285363, 0.1012285363, 0.2223810345, 0.2223810345,
        0.3137066459, 0.3137066459, 0.3626837834, 0.3626837834];
end

res = sum(f(x).*w);
```

```
f1 = @(x) 0.5*exp(-1/4*(x+1).^2);
f2 = @(x) 0.5^3.5*(x+1).^2.5;
f3 = @(x) 4./(1+16*x.^2);
f4 = @(x) pi./(2-cos(pi*x));
f5 = @(x) pi/2*exp(pi/2*(x+1)).*cos(2*pi*x);

N = 7;
res = zeros(N, 5);
for i = 1:N
    res(i, 1) = gauss_legendre(f1, i+1);
    res(i, 2) = gauss_legendre(f2, i+1);
```

```
    res(i, 3) = gauss_legendre(f3, i+1);
    res(i, 4) = gauss_legendre(f4, i+1);
    res(i, 5) = gauss_legendre(f5, i+1);
end
err = abs(res(2:N, :)-res(1:N-1, :))/res(1:N-1, :);
dlmwrite('prob1_res.m', res, 'delimiter', ' ', 'precision', '%2.10f');
```

```
n = 2 0.7465946883 0.2864594283 1.2631578947 2.8042191325 -19.2448713255
n = 3 0.7468145842 0.2857539965 3.9748427673 4.0574495156 9.0897287043
n = 4 0.7468243265 0.2857198254 2.0472848847 3.4509852799 0.9220485107
n = 5 0.7468241268 0.2857155443 3.0886190193 3.7088410791 1.1462063445
n = 6 0.7468241329 0.2857146571 2.4116889285 3.5921379211 1.3298098528
n = 7 0.7468241328 0.2857144179 2.8076823087 3.6434274583 1.3003438200
n = 8 0.7468241329 0.2857143396 2.5600801699 3.6206081804 1.3024781038
```

Problem 2. Problem 5.15

Sol. In fact, this problem is just (3) in Problem 1, the result of which is the third column listed above. When compared to Newton-Cotes, we can see that Newton-Cotes formula does not converge when n gets larger, but Gauss-Lengendre quadrature seems to converge when n becomes larger.

Problem 3. Problem 5.17

Sol. From Problem 4.20 we know, with weight function $w(x) = -\ln(x)$,

$$\varphi_0(x) = 1,\ \varphi_1(x) = \frac{12}{\sqrt{7}}(x - \frac{1}{4}),\ \varphi_2(x) = \frac{\sqrt{647}}{180\sqrt{7}}(x^2 - \frac{5}{7}(x - \frac{1}{4}) - \frac{1}{9}) = \frac{\sqrt{647}}{180\sqrt{7}}(x^2 - \frac{5}{7}x + \frac{17}{252})$$

Thus the roots of $\varphi_2(x)$ are

$$x_{1,2} = \frac{15 \pm \sqrt{106}}{42}$$

and by (5.3.7),

$$w_1 + w_2 = 1,$$
$$w_1x_1 + w_2x_2 = \frac{1}{2}$$

we get

$$w_1 = \frac{1}{2} + \frac{3}{\sqrt{106}},\ w_2 = \frac{1}{2} - \frac{3}{\sqrt{106}}.$$

Hence,

$$I(f) = (\frac{1}{2} + \frac{3}{\sqrt{106}})f(\frac{15+\sqrt{106}}{42}) + (\frac{1}{2} - \frac{3}{\sqrt{106}})f(\frac{15-\sqrt{106}}{42}).$$

Problem 4. Problem 5.19

Sol. From Problem 4.24 we know,

$$S_n(x) = \frac{1}{n+1}T'_{n+1}(x) = \frac{1}{\sqrt{1-x^2}}\sin((n+1)\cos^{-1}x).$$

Then the roots of $S_n(x)$ are

$$x_{n,j} = \cos(\frac{j\pi}{n+1}),\ j = 1, 2, \cdots, n.$$

From the recursion relation we know $a_n = 2$, $c_n = \frac{\gamma_n}{\gamma_{n-1}} = 1$. Since

$$\gamma_0 = (S_0, S_0) = \int_{-1}^{1}\sqrt{1-x^2}dx = \frac{\pi}{2},$$

we know $\gamma_n = \frac{\pi}{2}$. Thus

$$w_j = \frac{-a_n\gamma_n}{S'_n(x_j)S_{n+1}(x_j)} = -\frac{\pi\sin^2\frac{j\pi}{n+1}}{(n+1)\cos(j\pi)} = \begin{cases} -\dfrac{\pi}{n+1}\sin^2\dfrac{j\pi}{n+1}, & n = 2k \\ \dfrac{\pi}{n+1}\sin^2\dfrac{j\pi}{n+1}, & n = 2k+1 \end{cases}$$

From (5.3.10) we also know the error

$$E_n = \frac{\gamma_n}{A_n^2 (2n)!} f^{(2n)}(\eta) = \frac{\pi}{2^{n+1}(2n)!} f^{(2n)}(\eta),$$

for some $\eta \in [-1, 1]$.