Numerical Analysis
Assignment 6

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Problem 1. Problem 3.1, Page 185

(a). If we expand $V_n(x)$ by the last row, then we can know

$$V_n(x) = A_{n+1,n+1}x^n + A_{n+1,n}x^{n-1} + \cdots + A_{n,1},$$

where $A_{i,j}$ is the cofactors of V. Then $V_n(x)$ is a polynomial of degree n, thus it has n roots. By replacing x with $x_i, i = 0, 1, \cdots, n-1$, we can find $V_n(x_i) = 0$, thus $x = x_i$ are exactly roots of V_n. On the other hand, the coefficient of x^n is just $V_{n-1}(x_{n-1})$, thus

$$V_n(x) = V_{n-1}(x_{n-1}) \prod_{i=0}^{n-1}(x - x_i).$$

(b). With the definition of X we know

$$\begin{aligned}
\det(X) = V_n(x_n) &= V_{n-1}(x_{n-1}) \prod_{i=0}^{n-1}(x_n - x_i) \\
&= V_{n-2}(x_{n-2}) \prod_{i=0}^{n-1}(x_n - x_i) \prod_{i=0}^{n-2}(x_{n-1} - x_i) \\
&= \cdots \\
&= V_0(x_0) \prod_{k=0}^{n-1} \prod_{i=0}^{k-1}(x_{k+1} - x_i) = \prod_{0 \le j < i \le n}(x_i - x_j).
\end{aligned}$$

Problem 2. Problem 3.6, Page 186

Solution. We know from linear interpolation,

$$|E(x)| = \frac{(x - x_0)(x_1 - x)}{2} \cdot \sin(\xi),\ x_0 \le \xi \le x_1.$$

Since $|f''(t)| = |sin(t)| \le 1$, we should take h, s.t. $\frac{h^2}{8} \le 1e^{-6}$. Thus we may take $h = 0.002$.
And we need a table entries of 7 significant digits so as not to let the rounding error dominate the interpolation error.

Problem 3. Problem 3.8, Page 187

Solution. The Lagrange quadratic interpolation of f at x_i is

$$L(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f_2.$$

Then the rounding error of quadratic interpolation is

$$\begin{aligned}
R(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \epsilon_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \epsilon_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \epsilon_2 \\
&= \frac{1}{2h^2}((x - x_1)(x - x_2)\epsilon_0 - 2(x - x_0)(x - x_2)\epsilon_1 + (x - x_0)(x - x_1)\epsilon_2)
\end{aligned}$$

Since the maximum of a quadratic function is at either endpoints or vertex,

$$\max|R(x)| \leq \max\left(|\epsilon_0|,\ |\epsilon_2|, \frac{1}{2h^2}\left(\frac{h^2}{4} + 2h^2 + \frac{h^2}{4}\right)|\epsilon|\ \right) = 1.25|\epsilon|.$$

Problem 4. Problem 3.21, Page 189

Solution. First notice $p(x) - 1$ has three roots. So we may assume

$$p(x) = q(x)x(x-1)(x+1) + 1,\ q(x) = ax^2 + bx + c.$$

Then we replace x with $-2, 2, 3$, we have

$$\begin{cases} q(-2) = 1 = 4a - 2b + c \\ q(2) = 1 = 4a + 2b + c \\ q(3) = 1 = 9a + 3b + c \end{cases}$$

Then $q(x) = 1$ has at least three roots, which means $q(x) = 1$. Thus the degree of $p(x)$ is 3.

Problem 5. Problem 3.24, Page 189

Proof. With (3.2.12) we know

$$|\Psi(x)| = \left|\frac{f^{(n)}(\xi)}{n!}\prod_{i=0}^{n}(x - x_i)\right| < \frac{1}{n!}h^{n+1} \times n! = h^{n+1}.$$

Then $\max|e^x - p_n(x)| \to 0$, when $n \to \infty$.