

B-Series and Split-Explicit Methods

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1 Introduction

MISB-Series is a FORTRAN program to compute order conditions for a class of one step methods where in each stage again a differential equation has to be solved. We start with the following differential equation

$$y' = F(y, y)$$

where the right hand side depends twice on the unknown y . This form of the right hand side involves a large number of ODE's for which special splitting typed integrator are proposed. We list types of equations which appear in the literature.

Other types of partitioning

Additive splitting:

$$y' = F(y, y) = f(y) + g(y)$$

Additive nonlinear-linear splitting:

$$y' = F(y, y) = f(y) + Ny$$

Vector fields on manifolds in frame representation

$$y' = F(y, y) = \sum_{i=1}^N f_i(y) E_i(y)$$

Multiplicative splitting:

$$y' = F(y, y) = A(y)y$$

In all examples the first y argument is the first one in the general framework, and so on.

For this type of equations exponential Runge-Kutta type methods will be analyzed. The methods have the following structure.

$$\begin{aligned}
Y_1 &= y_n \\
Z'_i &= \sum_{j=1}^{i-1} \sum_{p=0}^{\rho_i} a_{ij}^p \left(\frac{\tau}{h}\right)^p F(Y_j, Z_i), \quad Z_i(0) = y_n + \sum_j d_{ij}(Y_j - y_n) \\
Y_i &= Z_i(h),
\end{aligned}$$

where in addition the consistency condition is required

$$\sum_{j=1}^{i-1} a_{ij}^p = 0, \quad p = 1, \dots, \rho_i.$$

We will derive order conditions with the help of B-series. Let us start with the elementary differentials. We will use the abbreviation $F_{(i,j)}^{(n)}$, $i + j = n$ to denote the i -th derivative of f with respect to the first argument and the j -th derivative of f with respect to the second argument.

$$\begin{aligned}
y' &= F \\
y'' &= F_{(1,0)}^{(1)} F + F_{(1,0)}^{(1)} F \\
y^{(3)} &= F_{(2,0)}^{(2)} FF + 2F_{(1,1)}^{(2)} FF + F_{(0,2)}^{(2)} FF \\
&\quad + F_{(1,0)}^{(1)} F_{(1,0)}^{(1)} F + F_{(1,0)}^{(1)} F_{(0,1)}^{(1)} F + F_{(0,1)}^{(1)} F_{(1,0)}^{(1)} F + F_{(0,1)}^{(1)} F_{(0,1)}^{(1)} F
\end{aligned}$$

For a graphical representation we use bi-coloured trees. Black vertices means computation of F at the position (y, y) and white vertices means computation at the position (y, z) . Upwards pointing branches represent partial derivatives with respect to the first argument if the branch leads to a black vertex, and with respect to the second argument if it leads to a white vertex.

The trees are divided in two sets T^b and T^w where T^b contain all trees with a black root and T^w with a white root. Hence, the trees T^b are computed at (y, y) in the leading derivative whereas those of T^w are computed at the mixed argument (y, z) . The trees are defined recursively through:

$$\begin{aligned}
t &= [t_{b1} \dots t_{bk}, t_{w1} \dots t_{wl}]_b, \quad \forall t \in T^b \\
t &= [t_{b1} \dots t_{bk}, t_{w1} \dots t_{wl}]_w, \quad \forall t \in T^w
\end{aligned}$$

Furthermore we introduce the black operator b which replaces a white root node by a black root node.

$$t^b = [t_{b1} \dots t_{bk}, t_{w1} \dots t_{wl}]_w^b = [t_{b1} \dots t_{bk}, t_{w1} \dots t_{wl}]_b$$

$$\begin{aligned}
& G_F([t_{b,1} \dots t_{b,k}, t_{w1} \dots t_{wl}]_b)(y_0, y_0) = \\
& F_{(k,l)}^{(k+l)}(y_0, y_0)(G(t_{b1})(y_0, y_0), \dots, G(t_{bk})(y_0, y_0), G(t_{w1}^b)(y_0, y_0), \dots, G(t_{wl}^b)(y_0, y_0)) \\
& G_F([t_{b,1} \dots t_{b,k}, t_{w1} \dots t_{wl}]_w)(y_0, z_0) = \\
& F_{(k,l)}^{(k+l)}(y_0, y_0)(G(t_{b1})(y_0, y_0), \dots, G(t_{bk})(y_0, y_0), G(t_{w1})(y_0, z_0), \dots, G(t_{wl})(y_0, z_0))
\end{aligned}$$

Two types of formal B-series are introduced

$$\begin{aligned}
B(a, hF, y, y) &= a(\emptyset)y + \sum_{t \in T^b} \frac{a(t)}{\sigma(t)} G_F(t)(y, y) h^{r(t)} \\
C(b, hF, \lambda, y, z) &= b(\emptyset)z + \sum_{t \in T^w} \frac{b^\lambda(t)}{\sigma(t)} G_F(t)(y, z) h^{r(t)}
\end{aligned}$$

where the coefficients $b^\lambda(t)$ of the second B-series are polynomials in a further parameter λ

$$b^\lambda(t) = b_0(t) + \lambda b_1(t) + \lambda^2 b_2(t) + \dots$$

The internal variables are expanded into series by

$$Y_i = B(\eta_i, hF, y, y) := \sum_{t \in T^b} \eta_i(t) \frac{h^{\rho(t)}}{\sigma(t)} G_F(t)(y, y) \quad (1)$$

$$Z_i(\lambda) = C(\zeta_i^\lambda, hF, y, z) := \sum_{t \in T^w} \zeta_i^\lambda(t) \frac{h^{\rho(t)}}{\sigma(t)} G_F(t)(y, z) \quad (2)$$

$$(3)$$

The series coefficients η_i map T^b to \mathbb{R} , where the ζ_i^λ maps T^b to polynomials over \mathbb{R} . Alternatively, we can interpret the B series defined by ζ_i^λ being dependent on a parameter. Note, that we have $\eta_i(\emptyset) = \zeta_i^\lambda(\emptyset) = 1$.

We consider the expansion of $hF(Y, Z)$ in a B-series when Y, Z are given by B-series with coefficients η, ζ^λ whereas always $\eta(\emptyset) = \zeta(\emptyset) = 1$. The resulting B-series is denoted by $D(\eta, \zeta^\lambda)$.

Theorem 1.1 *For a tree $t = [t_{b1} \dots t_{bk}, t_{w1} \dots t_{wl}]_b$ we have*

$$hF(Y, Z) = C(D(\eta, \zeta^\lambda), hF, y, z) \quad (4)$$

whereas

$$D(\eta, \zeta)(t) = \prod_i \eta(t_{bi}) \prod_j \zeta^\lambda(t_{wj}) \quad (5)$$

The proof requires only an exact calculation of the occurrence of the tree t in the Taylor expansion of hF .

Proof: We denote by $F^{(n)}$ the full tensor representing the n -th derivative of f with respect to a vectorial column variable $(Y; Z)$.

$$\begin{aligned}
hF(Y, Z) &= hF(y, z) + h \sum_{n=1}^{\infty} \frac{1}{n!} F^{(n)} \left[\begin{pmatrix} Y - y \\ Z - z \end{pmatrix}, \dots \right] \\
&= hF(y, z) + h \sum_{n=1}^{\infty} \sum_{k=0}^n \frac{1}{n!} \binom{n}{k} F_{(k, n-k)}^n [Y - y, \dots; Z - z, \dots] \\
&= hF(y, z) + h \sum_{n=1}^{\infty} \sum_{k=0}^n \frac{1}{k!(n-k)!} F_{(k, n-k)}^n [Y - y, \dots; Z - z, \dots]
\end{aligned}$$

Now assume we want to count the number of occurrences of a tree $t = [t_{b1}, \dots, t_{bk}, t_{w1}, \dots, t_{wl}]$ where $t_{b1}, \dots, t_{bk} \in T^b$, $t_{w1}, \dots, t_{wl} \in Tw$. Assume that the multiplicities in which children trees occur are μ_1, \dots, μ_r . Then, such a tree occurs in the Taylor expansion exactly

$$\frac{k!!!}{\mu_1! \cdots \mu_r!}$$

times. If all subtrees are distinct we have $k!!!$ occurrences, but for each multiple occurrence μ_i we have to deduct a factor $1/\mu_i!$. Taken into consideration the recursion formula for the expressions $\sigma(t)$, we end up with the proposition

$$D(\eta, \zeta^\lambda)(t) = \prod_i \eta(t_{bi}) \prod_j \zeta^\lambda(t_{wj}) \quad (6)$$

The algorithm below computes the series coefficients for an algorithm where we have no shift in the initial conditions.

2 Series coefficients

Let us compute the first coefficients of both series and the intermediate series ϕ for all stages. In addition to the standard case we also add the differentials for general additive splitting and linear additive splitting.

For a compact notation we define also

$$\begin{aligned}
a_{ij}^{[0]}[\lambda] &= \sum_{p=0}^{\rho_i} a_{ij}^p \frac{p!}{(p+0)!} \lambda^{p+0} \\
a_{ij}^{[1]}[\lambda] &= \sum_{p=0}^{\rho_i} a_{ij}^p \frac{p!}{(p+1)!} \lambda^{p+1} \\
a_{ij}^{[m]}[\lambda] &= \sum_{p=0}^{\rho_i} a_{ij}^p \frac{p!}{(p+m)!} \lambda^{p+m}
\end{aligned}$$

Algorithm 1 Calculate series coefficient for tree $t = [t_1 t_2 \dots t_m]_w$

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 $t = [t_1 t_2 \dots t_m]_w$ 
for  $i = 1$  to  $ns + 1$  do
   $\phi_i^\lambda = 0$ 
  for  $j = 1$  to  $i - 1$  do
     $r_{ij}^\lambda = 1$ 
    for  $s \in [t_1 t_2 \dots t_m]$  do
      if  $s \in T^w$  then
         $r_{ij}^\lambda = r_{ij}^\lambda * \zeta_{i,\lambda}(s)$  Comment: Polynomial multiplication
      else
         $r_{ij}^\lambda = r_{ij}^\lambda * \eta_j(s)$  Comment: Scalar multiplication
      end if
    end for
  for  $p = 0$  to  $\rho_i$  do
     $\phi_i^\lambda = \phi_i^\lambda + a_{ijp} \lambda^p * r_{ij}^\lambda$ 
  end for
end for
Integrate  $\frac{d}{d\lambda} \zeta_i^\lambda(t) = \phi_i^\lambda$ 
 $\zeta_i^{\lambda^0}(t) = 0$ 
 $\eta_i(t^b) = \zeta_i^1(t)$ 
end for

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A further notation is the product of polynomial

$$\begin{aligned}
(qr)[\lambda] &= q[\lambda]r[\lambda] = \left(\sum_{p=0}^{\rho_1} q^p \lambda^p\right) \left(\sum_{p=0}^{\rho_2} r^p \lambda^p\right) \\
&= q^{\rho_1} r^{\rho_2} \lambda^{\rho_1 + \rho_2} \\
&\quad + (q^{\rho_1} r^{\rho_2 - 1} + q^{\rho_1 - 1} r^{\rho_2}) \lambda^{\rho_1 + \rho_2 - 1} \\
&\quad + (q^{\rho_1} r^{\rho_2 - 2} + q^{\rho_1 - 1} r^{\rho_2 - 1} + q^{\rho_1 - 2} r^{\rho_2}) \lambda^{\rho_1 + \rho_2 - 2} \\
&\quad + \dots \\
&\quad + (q^0 r^1 + q^1 r^0) \lambda^1 \\
&\quad + q^0 r^0 \lambda^0 \\
&= \sum_{p=0}^{\rho_1 + \rho_2} \left(\sum_{s=0}^k q^l r^{p-s} \right) \lambda^p
\end{aligned}$$

where q^p is zero whenever $p \notin [0, \rho_1]$, resp. r^p is zero whenever $p \notin [0, \rho_2]$.

$$(qr)^p = \sum_{s=0}^p q^s q^{p-s}$$

First order:

Differential 1, F , F , N:

$$\begin{aligned}
r_{ij}^\lambda(\tau_w) &= 1 \\
\phi_i^\lambda(\tau_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \\
\zeta_i^\lambda(\tau_w) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \\
\eta_i(\tau_b) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \\
\eta_j(\tau_b) &= \sum_{k=1}^{j-1} a_{jk}^{[1]}[1]
\end{aligned}$$

Second order:

Differential 2, $F_{(1,0)}^{(1)}F$, $f'F$, $N'N$:

$$\begin{aligned}
r_{ij}^\lambda([\tau_b]_w) &= \eta_j^\lambda(\tau_b) = \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \\
\phi_i^\lambda([\tau_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([\tau_b]_w) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \\
\zeta_i^\lambda([\tau_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \\
\eta_i([\tau_b]_b) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \\
\eta_j([\tau_b]_b) &= \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} a_{kl}^{[1]}[1]
\end{aligned}$$

Differential 3, $F_{(0,1)}^{(1)}F$, $g'F$, LN :

$$\begin{aligned}
r_{ij}^\lambda([\tau_w]_w) &= \zeta_i(\tau_w) = \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \\
\phi_i^\lambda([\tau_w]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([\tau_w]_w) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \\
\zeta_i^\lambda([\tau_w]_w) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \right)^{[1]}[\lambda] \\
\eta_i([\tau_w]_b) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \right)^{[1]}[1] \\
\eta_j([\tau_w]_b) &= \left(\sum_{k=1}^{j-1} a_{jk}^{[0]} \sum_{l=1}^{k-1} a_{kl}^{[1]} \right)^{[1]}[1]
\end{aligned}$$

Third order:

Differential 4, $F_{(2,0)}^{(2)} FF, f'' FF, N'' NN :$

$$r_{ij}^\lambda([\tau_b, \tau_b]_w) = (\eta_j(\tau_b))^2 = \left(\sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right)^2$$

$$\begin{aligned} \phi_i^\lambda([\tau_b, \tau_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([\tau_b, \tau_b]_w) \\ &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \left(\sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right)^2 \end{aligned}$$

$$\zeta_i^\lambda([\tau_b, \tau_b]_w) = \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \left(\sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right)^2$$

$$\eta_i([\tau_b, \tau_b]_b) = \sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \left(\sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right)^2$$

$$\eta_j([\tau_b, \tau_b]_b) = \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \left(\sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \right)^2$$

Differential 5, $F_{(1,1)}^{(2)}FF$:

$$\begin{aligned}
r_{ij}^\lambda([\tau_w, \tau_b]_w) &= \zeta_i^\lambda(\tau_w)\eta_j(\tau_b) = \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda]\right) \left(\sum_{k=1}^{j-1} a_{jk}^{[1]}[1]\right) \\
\phi_i^\lambda([\tau_w, \tau_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([\tau_w, \tau_b]_w) \\
&= \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda]\right) \left(\sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1]\right) \\
&= \left(\left(\sum_{j=1}^{i-1} a_{ij}^{[1]}\right) \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1]\right)\right) [\lambda] \\
\zeta_i^\lambda([\tau_w, \tau_b]_w) &= \left(\left(\sum_{j=1}^{i-1} a_{ij}^{[1]}\right) \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1]\right)\right)^{[1]} [\lambda] \\
\eta_i([\tau_w, \tau_b]_b) &= \left(\left(\sum_{j=1}^{i-1} a_{ij}^{[1]}\right) \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1]\right)\right)^{[1]} [1] \\
\eta_j([\tau_w, \tau_b]_b) &= \left(\left(\sum_{k=1}^{j-1} a_{jk}^{[1]}\right) \left(\sum_{k=1}^{j-1} a_{jk}^{[0]} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1]\right)\right)^{[1]} [1]
\end{aligned}$$

Differential 6, $F_{(0,2)}^{(2)}FF, g''FF$:

$$\begin{aligned}
r_{ij}^\lambda([\tau_w, \tau_w]_w) &= (\zeta_i^\lambda(\tau_w))^2 = \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda]\right)^2 \\
\phi_i^\lambda([\tau_w, \tau_w]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] (\tau_w, \tau_w)_w \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda]\right)^2 \\
\zeta_i^\lambda([\tau_w, \tau_w]_w) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[1]\right)^2\right)^{[1]} [\lambda] \\
\eta_i([\tau_w, \tau_w]_b) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[1]\right)^2\right)^{[1]} [1] \\
\eta_j([\tau_w, \tau_w]_b) &= \left(\sum_{k=1}^{j-1} a_{jk}^{[0]} \left(\sum_{k=1}^{j-1} a_{jk}^{[1]}[1]\right)^2\right)^{[1]} [1]
\end{aligned}$$

Differential 7, $F_{(1,0)}^{(1)} F_{(1,0)}^{(1)} F$, $f' f' F$, $N' N' N$:

$$\begin{aligned}
r_{ij}^\lambda([[\tau_b]_b]_w) &= \eta_j([\tau_b]_b) = \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \\
\phi_i^\lambda([[\tau_b]_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([[\tau_b]_b]_w) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \\
\zeta_i^\lambda([[\tau_b]_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \\
\eta_i([[\tau_b]_b]_b) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} a_{kl}^{[1]}[1]
\end{aligned}$$

Differential 8, $F_{(1,0)}^{(1)} F_{(0,1)}^{(1)} F$, $f' g' F$, $N' L N$:

$$\begin{aligned}
r_{ij}^\lambda([[\tau_w]_b]_w) &= \eta_j([\tau_w]_b) = \left(\sum_{k=1}^{j-1} a_{jk}^{[0]} \sum_{l=1}^{k-1} a_{kl}^{[1]} \right)^{[1]}[1] \\
\phi_i^\lambda([[\tau_w]_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([[\tau_w]_b]_w) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \right)^{[1]}[1] \\
\zeta_i^\lambda([[\tau_w]_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \left(\sum_{l=1}^{k-1} a_{kl}^{[1]} \right)^{[1]}[1] \\
\eta_i([[\tau_w]_b]_b) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \left(\sum_{k=1}^{j-1} a_{jk}^{[0]} \sum_{l=1}^{k-1} a_{kl}^{[1]} \right)^{[1]}[1] \\
\eta_j([[\tau_w]_b]_b) &= \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \left(\sum_{l=1}^{k-1} a_{kl}^{[0]} \sum_{r=1}^{l-1} a_{lr}^{[1]} \right)^{[1]}[1]
\end{aligned}$$

Differential 9, $LN'N$:

$$\begin{aligned}
r_{ij}^\lambda([[\tau_b]_w]_w) &= \zeta_i^\lambda([[\tau_b]_w]) = \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \\
\phi_i^\lambda([[\tau_b]_w]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([[\tau_b]_w]_w) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \\
\zeta_i^\lambda([[\tau_b]_w]_w) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{j=1}^{i-1} a_{ij}^{[1]} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right)^{[1]}[\lambda] \\
\eta_i([[\tau_b]_w]_b) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{j=1}^{i-1} a_{ij}^{[1]} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right)^{[1]}[1] \\
\eta_j([[\tau_b]_w]_b) &= \left(\sum_{k=1}^{j-1} a_{jk}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \right)^{[1]}[1]
\end{aligned}$$

Differential 10, $F_{(0,1)}^{(1)} F_{(0,1)}^{(1)} F$, $g'g'F$, LLN :

$$\begin{aligned}
r_{ij}^\lambda([[\tau_w]_w]_w) &= \zeta_i^\lambda([[\tau_w]_w]) = \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \right)^{[1]}[\lambda] \\
\phi_i^\lambda([[\tau_w]_w]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([[\tau_w]_w]_w) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \right)^{[1]}[\lambda] \\
\zeta_i^\lambda([[\tau_w]_w]_w) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \right)^{[1]} \right)^{[1]}[\lambda] \\
\eta_i([[\tau_w]_w]_b) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \right)^{[1]} \right)^{[1]}[1] \\
\eta_j([[\tau_w]_w]_b) &= \left(\sum_{k=1}^{j-1} a_{jk}^{[0]} \left(\sum_{k=1}^{j-1} a_{jk}^{[0]} \sum_{l=1}^{k-1} a_{kl}^{[1]} \right)^{[1]} \right)^{[1]}[1]
\end{aligned}$$

Fourth order:

Differential 11, $F_{(3,0)}^{(3)} FFF$, $f''' FFF$, $N''' NNN$:

$$\begin{aligned}
r_{ij}^\lambda([\tau_b, \tau_b, \tau_b]_w) &= (\eta_j(\tau_b))^3 = \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \right)^3 \\
\phi_i^\lambda([\tau_b, \tau_b, \tau_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([\tau_b, \tau_b, \tau_b]_w) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \right)^3 \\
\zeta_i^\lambda([\tau_b, \tau_b, \tau_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \right)^3 \\
\eta_i([\tau_b, \tau_b, \tau_b]_b) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \right)^4 \\
\eta_j([\tau_b, \tau_b, \tau_b]_b) &= \left(\sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right)^4
\end{aligned}$$

Differential 12, $F_{(2,1)}^{(3)} FFF$:

$$\begin{aligned}
r_{ij}^\lambda([\tau_b, \tau_b, \tau_w]_w) &= \zeta_i^\lambda(\tau_w)(\eta_j(\tau_b))^2 = \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \right)^2 \\
\phi_i^\lambda([\tau_b, \tau_b, \tau_w]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([\tau_b, \tau_b, \tau_w]_w) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \right)^2 \\
\zeta_i^\lambda([\tau_b, \tau_b, \tau_w]_w) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{j=1}^{i-1} a_{ij}^{[1]} \right)^{[1]} [\lambda] \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \right)^2 \\
\eta_i([\tau_b, \tau_b, \tau_w]_b) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{j=1}^{i-1} a_{ij}^{[1]} \right)^{[1]} [1] \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \right)^2 \\
\eta_j([\tau_b, \tau_b, \tau_w]_b) &= \left(\sum_{k=1}^{j-1} a_{jk}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \right)^{[1]} [1] \left(\sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right)^2
\end{aligned}$$

Differential 13, $F_{(1,2)}^{(3)} FFF$:

$$\begin{aligned}
r_{ij}^\lambda([\tau_b, \tau_w, \tau_w]_w) &= (\zeta_i^\lambda(\tau_w))^2 (\eta_j \tau_b) = \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \right)^2 \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \\
\phi_i^\lambda([\tau_b, \tau_w, \tau_w]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([\tau_b, \tau_w, \tau_w]_w) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \right)^2 \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \\
\zeta_i^\lambda([\tau_b, \tau_w, \tau_w]_w) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \left(\sum_{j=1}^{i-1} a_{ij}^{[1]} \right)^2 \right) \sum_{k=1}^{j-1} a_{jk}^{[1]}[1]^{[1]}[\lambda] \\
\eta_i([\tau_b, \tau_w, \tau_w]_b) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \left(\sum_{j=1}^{i-1} a_{ij}^{[1]} \right)^2 \right) \sum_{k=1}^{j-1} a_{jk}^{[1]}[1]^{[1]}[1] \\
\eta_j([\tau_b, \tau_w, \tau_w]_b) &= \left(\sum_{k=1}^{j-1} a_{jk}^{[0]} \left(\sum_{k=1}^{j-1} a_{jk}^{[1]} \right)^2 \right) \sum_{l=1}^{k-1} a_{kl}^{[1]}[1]^{[1]}[1]
\end{aligned}$$

Differential 14, $F_{(0,3)}^{(3)} FF, g''' FFF$:

$$\begin{aligned}
r_{ij}^\lambda([\tau_w, \tau_w, \tau_w]_w) &= (\zeta_i^\lambda(\tau_w))^3 = \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \right)^3 \\
\phi_i^\lambda([\tau_w, \tau_w, \tau_w]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([\tau_w, \tau_w, \tau_w]_w) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \right)^3 \\
\zeta_i^\lambda([\tau_w, \tau_w, \tau_w]_w) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \left(\sum_{j=1}^{i-1} a_{ij}^{[1]} \right)^3 \right)^{[1]}[\lambda] \\
\eta_i([\tau_w, \tau_w, \tau_w]_b) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \left(\sum_{j=1}^{i-1} a_{ij}^{[1]} \right)^3 \right)^{[1]}[1] \\
\eta_j([\tau_w, \tau_w, \tau_w]_b) &= \left(\sum_{k=1}^{j-1} a_{jk}^{[0]} \left(\sum_{k=1}^{j-1} a_{jk}^{[1]} \right)^3 \right)^{[1]}[1]
\end{aligned}$$

Differential 15, $F_{(2,0)}^{(2)} F_{(1,0)}^{(1)} FF, f'' f' FF, N'' N' NN$

$$\begin{aligned}
r_{ij}^\lambda([\tau_b, [\tau_b]_b]_w) &= \eta_j(\tau_b) \eta_j([\tau_b]_b) = \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \left(\sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \right) \\
\phi_i^\lambda([\tau_b, [\tau_b]_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([\tau_w, \tau_w, \tau_w]_w) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \left(\sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \right) \\
\zeta_i^\lambda([\tau_b, [\tau_b]_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \left(\sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \right) \\
\eta_i([\tau_b, [\tau_b]_b]_b) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \left(\sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \right) \\
\eta_j([\tau_b, [\tau_b]_b]_b) &= \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \left(\sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \sum_{r=1}^{l-1} a_{lr}^{[1]}[1] \right)
\end{aligned}$$

Differential 16, $F_{(1,1)}^{(2)} F_{(1,0)}^{(1)} FF$

$$\begin{aligned}
r_{ij}^\lambda([\tau_b, [\tau_b]_w]_w) &= \eta_j(\tau_b) \zeta_i^\lambda([\tau_b]_w) = \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right) \\
\phi_i^\lambda([\tau_b, [\tau_b]_w]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([\tau_b, [\tau_b]_w]_w) \\
&= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right) \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right) \\
\zeta_i^\lambda([\tau_b, [\tau_b]_w]_w) &= \left(\left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right) \left(\sum_{j=1}^{i-1} a_{ij}^{[1]} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right) \right)^{[1]}[\lambda] \\
\eta_i([\tau_b, [\tau_b]_w]_b) &= \left(\left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right) \left(\sum_{j=1}^{i-1} a_{ij}^{[1]} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right) \right)^{[1]}[1] \\
\eta_j([\tau_b, [\tau_b]_w]_b) &= \left(\left(\sum_{k=1}^{j-1} a_{jk}^{[0]} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \right) \left(\sum_{k=1}^{j-1} a_{jk}^{[1]} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \right) \right)^{[1]}[1]
\end{aligned}$$

Differential 17, $F_{(2,0)}^{(2)} F_{(0,1)}^{(1)} FF$

$$\begin{aligned}
r_{ij}^\lambda([\tau_w, [\tau_b]_b]_w) &= \eta_j([\tau_w]_w) \zeta_i^\lambda(\tau_w) = \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \\
\phi_i^\lambda([\tau_w, [\tau_b]_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([\tau_w, [\tau_b]_b]_w) \\
&= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \right) \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \\
\zeta_i^\lambda([\tau_w, [\tau_b]_b]_w) &= \left(\left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \right) \sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \right) [\lambda] \\
\eta_i([\tau_w, [\tau_b]_b]_b) &= \left(\left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \right) \sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \right) [1]
\end{aligned}$$

Differential 18, $F_{(2,0)}^{(2)} F_{(0,1)}^{(1)} FF$, $f''g'FF$, $N''LNN$

$$\begin{aligned}
r_{ij}^\lambda([\tau_b, [\tau_w]_b]_w) &= \eta_j^\lambda(\tau_b) \eta_j^\lambda([\tau_w]_b) = \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \left(\sum_{k=1}^{j-1} a_{jk}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \right) [1] \\
\phi_i^\lambda([\tau_b, [\tau_w]_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([\tau_w]_w) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \left(\sum_{k=1}^{j-1} a_{jk}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \right) [1] \\
\zeta_i^\lambda([\tau_b, [\tau_w]_b]_w) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \left(\sum_{k=1}^{j-1} a_{jk}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \right) [1] \right) [\lambda] \\
\eta_i([\tau_b, [\tau_w]_b]_b) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \left(\sum_{k=1}^{j-1} a_{jk}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \right) [1] \right) [1]
\end{aligned}$$

Differential 24, $F_{(0,1)}^{(1)} F_{(2,0)}^{(2)} FF, g' f'' FF, LN'' NN$:

$$\begin{aligned}
r_{ij}^\lambda([[\tau_b, \tau_b]_w]_w) &= \zeta_i^\lambda([\tau_b, \tau_b]_w) = \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \left(\sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right)^2 \\
\phi_i^\lambda([[\tau_b, \tau_b]_w]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([\tau_b, \tau_b]_w) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \left(\sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right)^2 \\
\zeta_i^\lambda([[\tau_b, \tau_b]_w]_w) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{j=1}^{i-1} a_{ij}^{[1]} \right)^{[1]} [\lambda] \left(\sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right)^2 \\
\eta_i([[\tau_b, \tau_b]_w]_b) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{j=1}^{i-1} a_{ij}^{[1]} \right)^{[1]} [1] \left(\sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right)^2
\end{aligned}$$

Differential 30, $F_{(0,1)}^{(1)} F_{(1,0)}^{(1)} F_{(1,0)}^{(1)} F, g' f' f' F, LN' N' N$

$$\begin{aligned}
r_{ij}^\lambda([[[\tau_b]_b]_w]_w) &= \zeta_i^\lambda([\tau_b]_b) = \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \\
\phi_i^\lambda([[[\tau_b]_b]_w]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([\tau_b]_b) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \\
\zeta_i^\lambda([[[\tau_b]_b]_w]_w) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{j=1}^{i-1} a_{ij}^{[1]} \right)^{[1]} [\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \\
\eta_i([[[\tau_b]_b]_w]_b) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{j=1}^{i-1} a_{ij}^{[1]} \right)^{[1]} [1] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} a_{kl}^{[1]}[1]
\end{aligned}$$

Differential 31, $F_{(1,0)}^{(1)} F_{(0,1)}^{(1)} F_{(1,0)}^{(1)} F, f' g' f' F, N' L N' N$

$$\begin{aligned}
r_{ij}^\lambda([[[\tau_b]_w]_b]_w) &= \eta_j([[\tau_b]_w]_b) = \left(\sum_{k=1}^{j-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \sum_{l=1}^{k-1} a_{kl}^{[1]} [1] \right)^{[1]} [1] \\
\phi_i^\lambda([[[\tau_b]_w]_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]} [\lambda] r_{ij}^\lambda([[[\tau_b]_w]_b]_w) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]} [\lambda] \left(\sum_{k=1}^{j-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \sum_{l=1}^{k-1} a_{kl}^{[1]} [1] \right)^{[1]} [1] \\
\zeta_i^\lambda([[[\tau_b]_w]_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[1]} [\lambda] \left(\sum_{k=1}^{j-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \sum_{l=1}^{k-1} a_{kl}^{[1]} [1] \right)^{[1]} [1] \\
\eta_i([[[\tau_b]_w]_b]_b) &= \sum_{j=1}^{i-1} a_{ij}^{[1]} [1] \left(\sum_{k=1}^{j-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \sum_{l=1}^{k-1} a_{kl}^{[1]} [1] \right)^{[1]} [1]
\end{aligned}$$

Differential 32, $F_{(1,0)}^{(1)} F_{(1,0)}^{(1)} F_{(0,1)}^{(1)} F, f' f' g' F, N' N' L N$:

$$\begin{aligned}
r_{ij}^\lambda([[[\tau_b]_b]_w]_w) &= \zeta_i^\lambda([[\tau_b]_b]_w) = \sum_{j=1}^{i-1} a_{ij}^{[1]} [\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]} [1] \sum_{l=1}^{k-1} a_{kl}^{[1]} [1] \\
\phi_i^\lambda([[[\tau_w]_b]_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]} [\lambda] r_{ij}^\lambda([[[\tau_b]_b]_w]_w) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]} [\lambda] \sum_{j=1}^{i-1} a_{ij}^{[1]} [\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]} [1] \sum_{l=1}^{k-1} a_{kl}^{[1]} [1] \\
\zeta_i^\lambda([[[\tau_b]_b]_w]_w) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{j=1}^{i-1} a_{ij}^{[1]} \right)^{[1]} [\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]} [1] \sum_{l=1}^{k-1} a_{kl}^{[1]} [1] \\
\eta_i([[[\tau_b]_b]_w]_b) &= \left(\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{j=1}^{i-1} a_{ij}^{[1]} \right)^{[1]} [1] \sum_{k=1}^{j-1} a_{jk}^{[1]} [1] \sum_{l=1}^{k-1} a_{kl}^{[1]} [1]
\end{aligned}$$

3 Adding shifted initial conditions

Algorithm:

An identity

$$DR = D(I - D)^{-1} = (D - I + I)(I - D)^{-1} = -I + (I - D)^{-1} = -I + R$$

First order:

Algorithm 2 Calculate series coefficient for tree t

```

 $t = [t_1 t_2 \dots t_m]$ 
for  $i = 1$  to  $ns + 1$  do
   $\phi_i^\lambda = 0$ 
  for  $j = 1$  to  $i - 1$  do
     $r_{ij}^\lambda = 1$ 
    for  $s \in [t_1 t_2 \dots t_m]$  do
      if  $s \in T^w$  then
         $r_{ij}^\lambda = r_{ij}^\lambda * \zeta_{i,\lambda}(s)$  Comment: Polynomial multiplication
      else
         $r_{ij}^\lambda = r_{ij}^\lambda * \eta_j(s)$  Comment: Scalar multiplication
      end if
    end for
    for  $p = 0$  to  $\rho_i$  do
       $\phi_i^\lambda = \phi_i^\lambda + a_{ijp} \lambda^p * r_{ij}^\lambda$ 
    end for
  end for
  Integrate  $\frac{d}{d\lambda} \zeta_i^\lambda(t) = \phi_i^\lambda$ 
   $\zeta_i^0(t) = 0$ 
  for  $j = 0$  to  $i - 1$  do
     $\zeta_i^0(t) = \zeta_i^0(t) + d_{ij} \eta_j(t)$ 
  end for
   $\eta_i(t) = \zeta_i^1(t)$ 
end for

```

Differential 1, F , F' , N :

$$\begin{aligned}
r_{ij}^\lambda(\tau_w) &= 1 \\
\phi_i^\lambda(\tau_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \\
\zeta_i^\lambda(\tau_w) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] + \sum_{j=1}^{i-1} d_{ij} \eta_j(\tau_b) \\
\eta_i(\tau_b) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[1] + \sum_{j=1}^{i-1} d_{ij} \eta_j(\tau_b) \\
\eta_i(\tau_b) &= \sum_{j=1}^{i-1} r_{ij} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \\
\eta_j(\tau_b) &= \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \\
\zeta_i^\lambda(\tau_w) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] + \sum_{j=1}^{i-1} d_{ij} \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1]
\end{aligned}$$

Second order:

Differential 2, $F_{(1,0)}^{(1)} F$, $f' F$, $N' N$:

$$\begin{aligned}
r_{ij}^\lambda([\tau_b]_w) &= \eta_j^\lambda(\tau_b) = \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \\
\phi_i^\lambda([\tau_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([\tau_b]_w) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \\
\zeta_i^\lambda([\tau_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] + \sum_{j=1}^{i-1} d_{ij} \eta_j([\tau_b]_b) \\
\eta_i([\tau_b]_b) &= \sum_{j=1}^{i-1} r_{ij} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} r_{kl} \sum_{r=1}^{l-1} a_{lr}^{[1]}[1] \\
\eta_j([\tau_b]_b) &= \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \sum_{r=1}^{l-1} r_{lr} \sum_{s=1}^{r-1} a_{rs}^{[1]}[1]
\end{aligned}$$

Differential 3, $F_{(0,1)}^{(1)}F$, $g'F$, LN :

$$\begin{aligned}
r_{ij}^\lambda([\tau_w]_w) &= \zeta_i(\tau_w) = \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] + \sum_{j=1}^{i-1} d_{ij} \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \\
\phi_i^\lambda([\tau_w]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^\lambda([\tau_w]_w) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] + \sum_{j=1}^{i-1} d_{ij} \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \right) \\
\zeta_i^\lambda([\tau_w]_w) &= \frac{1}{2} \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \right)^2 + \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \sum_{j=1}^{i-1} d_{ij} \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] + \sum_{j=1}^{i-1} d_{ij} \eta_j([\tau_w]_b) \\
\eta_i([\tau_w]_b) &= \frac{1}{2} \left(\sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \right)^2 + \sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \sum_{j=1}^{i-1} d_{ij} \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] + \sum_{j=1}^{i-1} d_{ij} \eta_j([\tau_w]_b) \\
\eta_i([\tau_w]_b) &= \sum_{j=1}^{i-1} r_{ij} \left(\frac{1}{2} \left(\sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right)^2 + \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{k=1}^{j-1} d_{jk} \sum_{l=1}^{k-1} r_{kl} \sum_{r=1}^{l-1} a_{lr}^{[1]}[1] \right) \\
&= \sum_{j=1}^{i-1} r_{ij} \left(\frac{1}{2} \left(\sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \right)^2 - \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] - \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \right) \\
\eta_j([\tau_w]_b) &= \left(\sum_{k=1}^{j-1} a_{jk}^{[0]} \sum_{l=1}^{k-1} a_{kl}^{[1]} \right)^{[1]}[1]
\end{aligned}$$

	El. Diff.	El. Diff.	El. Diff.	t	Notation	Notation	$\sigma(t)$	Order Conditions	Order Owren
1	F	F	N	•	τ_b	.	1	$\sum a_{ij}^{[1]} [1]$	
2	$F_{(1,0)}^{(1)} F$	$f' F$	$N' N$	•	$[\tau_b]_b$	[.]	1	$\sum a_{ij}^{[1]} [1] \sum a_{jk}^{[1]} [1]$	
3	$F_{(0,1)}^{(1)} F$	$g' F$	LN	•	$[\tau_w]_b$	[o]	1	$(\sum a_{ij}^{[0]} \sum a_{jk}^{[1]})^{[1]} [1]$	
4	$F_{(2,0)}^{(2)} FF$	$f'' FF$	$N'' NN$	•	$[\tau_b, \tau_b]_b$	[.,.]	1	$\sum a_{ij}^{[1]} [1] (\sum a_{jk}^{[1]} [1])^2$	
5	$F_{(1,1)}^{(2)} FF$			•	$[\tau_w, \tau_b]_b$	[o,.]	2	$((\sum a_{ij}^{[1]}) (\sum a_{ij}^{[0]} \sum a_{jk}^{[1]} [1]))^{[1]} [1]$	
6	$F_{(0,2)}^{(2)} FF$	$g'' FF$		•	$[\tau_w, \tau_w]_b$	[o,o]	1	$(\sum a_{ij}^{[0]} (\sum a_{ij}^{[1]})^2)^{[1]} [1]$	
7	$F_{(1,0)}^{(1)} F_{(1,0)}^{(1)} F$	$f' f' F$	$N' N' N$	•	$[[\tau_b]_b]_b$	[[.]]	1	$\sum a_{ij}^{[1]} [1] \sum a_{jk}^{[1]} [1] \sum a_{kl}^{[1]} [1]$	
8	$F_{(1,0)}^{(1)} F_{(0,1)}^{(1)} F$	$f' g' F$	$N' LN$	•	$[[\tau_w]_b]_b$	[[o]]	1	$\sum a_{ij}^{[1]} [1] (\sum a_{jk}^{[0]} \sum a_{kl}^{[1]})^{[1]} [1]$	
9	$F_{(0,1)}^{(1)} F_{(1,0)}^{(1)} F$	$g' f' F$	$LN' N$	•	$[[\tau_b]_w]_b$	[(.)]	1	$(\sum a_{ij}^{[0]} \sum a_{ij}^{[1]} \sum a_{jk}^{[1]} [1])^{[1]} [1]$	
10	$F_{(0,1)}^{(1)} F_{(0,1)}^{(1)} F$	$g' g' F$	LLN	•	$[[\tau_w]_w]_b$	[(o)]	1	$(\sum a_{ij}^{[0]} (\sum a_{ij}^{[0]} \sum a_{jk}^{[1]})^{[1]})^{[1]} [1]$	

El. Diff.	El. Diff.	El. Diff.	t	Notation	Notation	$\sigma(t)$	Order Conditions	Order Owren
11 $F_{(3,0)}^{(3)} FFF$	$f''' FFF$	$N''' NNN$		$[\tau_b, \tau_b, \tau_b]_b$	$[., ., .]$	1	$(\sum a_{ij}^{[1]} [1])^4$	
12 $F_{(2,1)}^{(3)} FFF$				$[\tau_b, \tau_b, \tau_w]_b$	$[., ., o]$	1	$(\sum a_{ij}^{[0]} \sum a_{ij}^{[1]})^{[1]} [1] (\sum a_{ij}^{[1]} [1])^2$	
13 $F_{(1,2)}^{(3)} FFF$				$[\tau_b, \tau_w, \tau_w]_b$	$[., o, o]$	1	$(\sum a_{ij}^{[0]} (\sum a_{ij}^{[1]})^2)^{[1]} [1] \sum a_{ij}^{[1]} [1]$	
14 $F_{(0,3)}^{(3)} FFF$	$g''' FFF$			$[\tau_w, \tau_w, \tau_w]_b$	$[o, o, o]$	1	$(\sum a_{ij}^{[0]} (\sum a_{ij}^{[1]})^3)^{[1]} [1]$	

	El. Diff.	El. Diff.	El. Diff.	t	Notation	Notation	$\sigma(t)$	Order Conditions	Order Owren
15	$F_{(2,0)}^{(2)} F_{(1,0)}^{(1)} FF$	$f'' f' FF$	$N'' N' NN$		$[\tau_b, [\tau_b]_b]_b$	$[\cdot, [\cdot]]$	1	$\sum a_{ij}^{[1]} [1] \sum a_{jk}^{[1]} [1] (\sum a_{jk}^{[1]} [1] \sum a_{kl}^{[1]} [1])$	
16	$F_{(1,1)}^{(2)} F_{(1,0)}^{(1)} FF$				$[\tau_b, [\tau_b]_w]_b$	$[\cdot, (\cdot)]$	1	$(\sum a_{ij}^{[0]} \sum a_{jk}^{[1]} [1]) (\sum a_{ij}^{[1]} \sum a_{jk}^{[1]} [1])^{[1]} [1]$	
17	$F_{(1,1)}^{(2)} F_{(1,0)}^{(1)} FF$				$[\tau_w, [\tau_b]_b]_b$	$[o, [\cdot]]$	1		
18	$F_{(2,0)}^{(2)} F_{(0,1)}^{(1)} FF$	$f'' g' FF$	$N'' L NN$		$[\tau_b, [\tau_w]_b]_b$	$[\cdot, [o]]$	1		
19	$F_{(1,1)}^{(2)} F_{(0,1)}^{(1)} FF$				$[\tau_b, [\tau_w]_w]_b$	$[\cdot, (o)]$	1		
20	$F_{(1,1)}^{(2)} F_{(1,0)}^{(1)} FF$				$[\tau_w, [\tau_b]_b]_b$	$[o, [o]]$	1		
21	$F_{(0,2)}^{(2)} F_{(1,0)}^{(1)} FF$	$g'' f' FF$			$[\tau_w, [\tau_b]_w]_b$	$[o, (\cdot)]$	1		
22	$F_{(0,2)}^{(2)} F_{(0,1)}^{(1)} FF$	$g'' g' FF$			$[\tau_w, [\tau_w]_w]_b$	$[o, (o)]$	1		

	El. Diff.	El. Diff.	El. Diff.	t	Notation	Notation	$\sigma(t)$
23	$F_{(1,0)}^{(1)} F_{(2,0)}^{(2)} FF$	$f' f'' FF$	$N' N'' NN$		$[[\tau_b, \tau_b]_b]_b$	$[[\cdot, \cdot]]$	1
24	$F_{(0,1)}^{(1)} F_{(2,0)}^{(2)} FF$	$g' f'' FF$	$LN'' NN$		$[[\tau_b, \tau_b]_w]_b$	$[(\cdot, \cdot)]$	1
25	$F_{(1,0)}^{(1)} F_{(1,1)}^{(2)} FF$				$[[\tau_b, \tau_b]_b]_b$	$[[\cdot, \cdot]]$	1
26	$F_{(1,0)}^{(1)} F_{(2,0)}^{(2)} FF$				$[[\tau_b, \tau_b]_b]_b$	$[[\cdot, \cdot]]$	1
27	$F_{(1,0)}^{(1)} F_{(0,2)}^{(2)} FF$	$f' g'' FF$			$[[\tau_w, \tau_w]_b]_b$	$[[o, o]]$	1
28	$F_{(1,0)}^{(1)} F_{(2,0)}^{(2)} FF$	$g' g'' FF$			$[[\tau_w, \tau_w]_w]_b$	$[(o, o)]$	1

	El. Diff.	El. Diff.	El. Diff.	t	Notation	Notation	$\sigma(t)$
29	$F_{(1,0)}^{(1)} F_{(1,0)}^{(1)} F_{(1,0)}^{(1)} F$	$f' f' f' F$	$N' N' N' N$		$[[[\tau_b]_b]_b]_b$	$[[[\cdot]]]$	1
30	$F_{(0,1)}^{(1)} F_{(1,0)}^{(1)} F_{(1,0)}^{(1)} F$	$g' f' f' F$	$LN' N' N$		$[[[\tau_w]_b]_b]_b$	$[[([\cdot])]]$	1
31	$F_{(1,0)}^{(1)} F_{(0,1)}^{(1)} F_{(1,0)}^{(1)} F$	$f' g' f' F$	$N' LN' N$		$[[[\tau_b]_w]_b]_b$	$[[(\cdot)]]$	1
32	$F_{(1,0)}^{(1)} F_{(1,0)}^{(1)} F_{(0,1)}^{(1)} F$	$f' f' g' F$	$N' N' LN$		$[[[\tau_b]_b]_w]_b$	$[[[o]]]$	1

	El. Diff.	El. Diff.	El. Diff.	t	Notation	Notation	$\sigma(t)$
				○			
				○			
				○			
33	$F_{(0,1)}^{(1)} F_{(0,1)}^{(1)} F_{(0,1)}^{(1)} F$	$g' g' g' F$	$LLLN$	●	$[[[\tau_w]_w]_w]_b$	$[((o))]$	1
				○			
				○			
				●			
34	$F_{(0,1)}^{(1)} F_{(0,1)}^{(1)} F_{(1,0)}^{(1)} F$	$f' g' g' F$	$N' LLN$	●	$[[[\tau_w]_w]_b]_b$	$[[(o)]]$	1
				○			
				●			
				○			
35	$F_{(1,0)}^{(1)} F_{(0,1)}^{(1)} F_{(1,0)}^{(1)} F$	$g' f' g' F$	$LN' LN$	●	$[[[\tau_w]_b]_w]_b$	$[([o])]$	1
				●			
				○			
				○			
36	$F_{(0,1)}^{(1)} F_{(0,1)}^{(1)} F_{(0,1)}^{(1)} F$	$g' g' f' F$	$LLN' N$	●	$[[[\tau_b]_w]_w]_b$	$[((\cdot))]$	1