B-Series and Split-Explicit Methods

Oswald Knoth, Jörg Wensch

18. Juli 2020

1 Introduction

MISB-Series is a FORTRAN program to compute order conditions for a class of one step methods where in each stage again a differential equation has to be solved. We start with the following differential equation

$$y' = F(y, y)$$

where the right hand side depends twice on the unknown y. This form of the right hand side involves a large number of ODE's for which special splitting typed integrator are proposed. We list types of equations which appear in the literature.
Other types of partitioning
Additive splitting:

$$y' = F(y, y) = f(y) + g(y)$$

Additive nonlinear-linear splitting:

$$y' = F(y, y) = f(y) + Ny$$

Vector fields on manifolds in frame representation

$$y' = F(y, y) = \sum_{i=1}^{N} f_i(y) E_i(y)$$

Multiplicative splitting:

$$y' = F(y, y) = A(y)y$$

I all examples the first y argument is the first one in the general framework, and so on.

For this type of equations exponential Runge-Kutta type methods will be analyzed. The methods have the following structure.

$$
\begin{aligned}
Y_1 &= y_n \\
Z_i' &= \sum_{j=1}^{i-1} \sum_{p=0}^{\rho_i} a_{ij}^p \left(\frac{\tau}{h}\right)^p F(Y_j, Z_i), \quad Z_i(0) = y_n + \sum_j d_{ij}(Y_j - y_n) \\
Y_i &= Z_i(h),
\end{aligned}
$$

where in addition the consistency condition is required

$$
\sum_{j=1}^{i-1} a_{ij}^p = 0, \; p = 1, \dots, \rho_i.
$$

We will derive order conditions with the help of B-series. Let us start with the elementary differentials. We will use the abbreviation $F_{(i,j)}^{(n)}$, $i + j = n$ to denote the i-th derivative of f with respect to the first argument and the j-th derivative of f with respect to the second argument.

$$
\begin{aligned}
y' &= F \\
y'' &= F_{(1,0)}^{(1)} F + F_{(1,0)}^{(1)} F \\
y^{(3)} &= F_{(2,0)}^{(2)} FF + 2F_{(1,1)}^{(2)} FF + F_{(0,2)}^{(2)} FF \\
&\quad + F_{(1,0)}^{(1)} F_{(1,0)}^{(1)} F + F_{(1,0)}^{(1)} F_{(0,1)}^{(1)} F + F_{(0,1)}^{(1)} F_{(1,0)}^{(1)} F + F_{(0,1)}^{(1)} F_{(0,1)}^{(1)} F
\end{aligned}
$$

For a graphical representation we use bi-coloured trees. Black vertices means computation of F at the position (y, y) and and white vertices means computation at the position (y, z). Upwards pointing branches represent partial derivatives with respect to the first argument if the branch leads to a black vertex, and with respect to the second argument if it leads to a white vertex.

The trees are divided in two sets T^b and T^w where T^b contain all trees with a black root and T^w with a white root. Hence, the trees T^b are computed at (y, y) in the leading derivative whereas those of T^w are computed at the mixed argument (y, z). The tress are defined recursively trough:

$$
\begin{aligned}
t &= [t_{b1} \dots t_{bk}, t_{w1} \dots t_{wl}]_b, \quad \forall t \in T^b \\
t &= [t_{b1} \dots t_{bk}, t_{w1} \dots t_{wl}]_w, \quad \forall t \in T^w
\end{aligned}
$$

Furthermore we introduce the black operator ^{b} which replaces a white root node by a black root node.

$$
t^b = [t_{b1} \dots t_{bk}, t_{w1} \dots t_{wl}]_w^b = [t_{b1} \dots t_{bk}, t_{w1} \dots t_{wl}]_b
$$

$$
\begin{aligned}
&G_F([t_{b,1}\dots t_{b,k},t_{w1}\dots t_{wl}]_b)(y_0,y_0)= \\
&F^{(k+l)}_{(k,l)}(y_0,y_0)(G(t_{b1})(y_0,y_0),\dots,G(t_{bk})(y_0,y_0),G(t^b_{w1})(y_0,y_0),\dots,G(t^b_{wl})(y_0,y_0)) \\
&G_F([t_{b,1}\dots t_{b,k},t_{w1}\dots t_{wl}]_w)(y_0,z_0)= \\
&F^{(k+l)}_{(k,l)}(y_0,y_0)(G(t_{b1})(y_0,y_0),\dots,G(t_{bk})(y_0,y_0),G(t_{w1})(y_0,z_0),\dots,G(t_{wl})(y_0,z_0))
\end{aligned}
$$

Two types of formal B-series are introduced

$$
\begin{aligned}
B(a,hF,y,y) &= a(\emptyset)y+\sum_{t\in T^b}\frac{a(t)}{\sigma(t)}G_F(t)(y,y)h^{r(t)} \\
C(b,hF,\lambda,y,z) &= b(\emptyset)z+\sum_{t\in T^w}\frac{b^\lambda(t)}{\sigma(t)}G_F(t)(y,z)h^{r(t)}
\end{aligned}
$$

where the coefficients $b^\lambda(t)$ of the second B-series are polynomials in a further parameter λ

$$b^\lambda(t)=b_0(t)+\lambda b_1(t)+\lambda^2 b_2(t)+\dots$$

The internal variables are expanded into series by

$$Y_i = B(\eta_i,hF,y,y) := \sum_{t\in T^b}\eta_i(t)\frac{h^{\rho(t)}}{\sigma(t)}G_F(t)(y,y) \tag{1}$$

$$Z_i(\lambda) = C(\zeta_i^\lambda,hF,y,z) := \sum_{t\in T^w}\zeta_i^\lambda(t)\frac{h^{\rho(t)}}{\sigma(t)}G_F(t)(y,z) \tag{2}$$

$$\tag{3}$$

The series coefficients η_i map T^b to $\mathbb{R}$, where the ζ_i^λ maps T^b to polynomials over $\mathbb{R}$. Alternatively, we can interpret the B series defined by ζ_i^λ being dependent on a parameter. Note, that we have $\eta_i(\emptyset)=\zeta_i^\lambda(\emptyset)=1$.

We consider the expansion of $hF(Y,Z)$ in a B-series when Y,Z are given by B-series with coefficents η,ζ^λ whereas always $\eta(\emptyset)=\zeta(\emptyset)=1$. The resulting B-series is denoted by $D(\eta,\zeta^\lambda)$.

Theorem 1.1 *For a tree $t=[t_{b1}\dots t_{bk},t_{w1}\dots t_{wl}]_b$ we have*

$$hF(Y,Z) = C(D(\eta,\zeta^\lambda),hF,y,z) \tag{4}$$

whereas

$$D(\eta,\zeta))(t)=\prod_i \eta(t_{bi})\prod_j \zeta^\lambda(t_{wj}) \tag{5}$$

The proof requires only an exact calculation of the occurence of the tree t in the Taylor expansion of hF.

Proof: We denote by $F^{(n)}$ the full tensor representing the n-th derivative of f with respect to a vectorial column variable $(Y;Z)$.

$$
\begin{aligned}
hF(Y,Z) =& hF(y,z) + h\sum_{n=1}^{\infty}\frac{1}{n!}F^{(n)}\left[\begin{pmatrix} Y-y \\ Z-z \end{pmatrix}, \dots\right] \\
=& hF(y,z) + h\sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{1}{n!}\binom{n}{k}F^{n}_{(k,n-k)}\left[Y-y,\dots;Z-z,\dots\right] \\
=& hF(y,z) + h\sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}F^{n}_{(k,n-k)}\left[Y-y,\dots;Z-z,\dots\right]
\end{aligned}
$$

Now assume we want to count the number of occurrences of a tree $t = [t_{b1},\dots,t_{bk},t_{w1},\dots,t_{wl}]$ where $t_{b1},\dots,t_{bk} \in T^b$, $t_{w1},\dots,t_{wl} \in Tw$. Assume that the multiplicities in which children trees occur are $\mu_1,\dots,\mu_r$. Then, such a tree occurs in the Taylor expansion exactly

$$
\frac{k!l!}{\mu_1!\cdot\dots\cdot\mu_r!}
$$

times. If all subtrees are distinct we have $k!l!$ occurrences, but for each multiple occurrence μ_i we have to deduct a factor $1/\mu_i!$. Taken into consideration the recursion formula for the expressions $\sigma(t)$, we end up with the proposition

$$
D(\eta,\zeta^{\lambda}))(t) = \prod_i \eta(t_{bi})\prod_j \zeta^{\lambda}(t_{wj}) \tag{6}
$$

The algorithm below computes the series coefficients for an algorithm where we have no shift in the initial conditions.

2 Series coefficients

Let us compute the first coefficients of both series and the intermediate series ϕ for all stages. In addition to the standard case we also add the differentials for general additive splitting and linear additive splitting.

For a compact notification we define also

$$
a_{ij}^{[0]}[\lambda] = \sum_{p=0}^{\rho_i} a_{ij}^{p}\frac{p!}{(p+0)!}\lambda^{p+0}
$$

$$
a_{ij}^{[1]}[\lambda] = \sum_{p=0}^{\rho_i} a_{ij}^{p}\frac{p!}{(p+1)!}\lambda^{p+1}
$$

$$
a_{ij}^{[m]}[\lambda] = \sum_{p=0}^{\rho_i} a_{ij}^{p}\frac{p!}{(p+m)!}\lambda^{p+m}
$$

Algorithm 1 Calculate series coefficient for tree $t = [t_1 t_2 \dots t_m]_w$

$t = [t_1 t_2 \dots t_m]_w$
for $i = 1$ **to** $ns + 1$ **do**
  $\phi_i^\lambda = 0$
  for $j = 1$ **to** $i - 1$ **do**
    $r_{ij}^\lambda = 1$
    for $s \in [t_1 t_2 \dots t_m]$ **do**
      if $s \in T^w$ **then**
        $r_{ij}^\lambda = r_{ij}^\lambda * \zeta_{i,\lambda}(s)$ Comment: Polynomial multiplication
      else
        $r_{ij}^\lambda = r_{ij}^\lambda * \eta_j(s)$ Comment: Scalar multiplication
      end if
    end for
    for $p = 0$ **to** ρ_i **do**
      $\phi_i^\lambda = \phi_i^\lambda + a_{ijp}\lambda^p * r_{ij}^\lambda$
    end for
  end for
  Integrate $\frac{d}{d\lambda}\zeta_i^\lambda(t)) = \phi_i^\lambda$
  $\zeta_i^{\lambda 0}(t) = 0$
  $\eta_i(t^b) = \zeta_i^1(t)$
end for

A further notation is the product of polynomial

$$\begin{aligned}
(qr)[\lambda] = q[\lambda]r[\lambda] =& (\sum_{p=0}^{\rho_1} q^p \lambda^p)(\sum_{p=0}^{\rho_2} r^p \lambda^p) \\
=& q^{\rho_1} p^{\rho_2} \lambda^{\rho_1+\rho_2} \\
&+ (q^{\rho_1} r^{\rho_2-1} + q^{\rho_1-1} r^{\rho_2})\lambda^{\rho_1+\rho_2-1} \\
&+ (q^{\rho_1} r^{\rho_2-2} + q^{\rho_1-1} r^{\rho_2-1} + q^{\rho_1-1} r^{\rho_2-1})\lambda^{\rho_1+\rho_2-2} \\
&+ \dots \\
&+ (q^0 r^1 + q^1 r^0)\lambda^1 \\
&+ q^0 r^0 \lambda^0 \\
=& \sum_{p=0}^{\rho_1+\rho_2} (\sum_{s=0}^{k} q^l r^{p-s})\lambda^p
\end{aligned}$$

where q^p is zero whenever $p \notin [0, \rho_1]$, resp. r^p is zero whenever $p \notin [0, \rho_2]$.

$$(qr)^p = \sum_{s=0}^{p} q^s q^{p-s}$$

First order:

Differential 1, F, F , N:

$$
\begin{aligned}
r_{ij}^{\lambda}(\tau_w) &= 1 \\
\phi_i^{\lambda}(\tau_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \\
\zeta_i^{\lambda}(\tau_w) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \\
\eta_i(\tau_b) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \\
\eta_j(\tau_b) &= \sum_{k=1}^{j-1} a_{jk}^{[1]}[1]
\end{aligned}
$$

Second order:
Differential 2, $F^{(1)}_{(1,0)}F$, $f'F$, $N'N$:

$$r^{\lambda}_{ij}([\tau_b]_w) = \eta^{\lambda}_j(\tau_b) = \sum_{k=1}^{j-1} a^{[1]}_{jk}[1]$$

$$\phi^{\lambda}_i([\tau_b]_w) = \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_b]_w)$$

$$= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1]$$

$$\zeta^{\lambda}_i([\tau_b]_w) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1]$$

$$\eta_i([\tau_b]_b) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[1] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1]$$

$$\eta_j([\tau_b]_b) = \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]$$

Differential 3, $F^{(1)}_{(0,1)}F$, $g'F$, LN:

$$r^{\lambda}_{ij}([\tau_w]_w) = \zeta_i(\tau_w) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda]$$

$$\phi^{\lambda}_i([\tau_w]_w) = \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_w]_w)$$

$$= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda]$$

$$\zeta^{\lambda}_i([\tau_w]_w) = (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{k=1}^{j-1} a^{[1]}_{jk})^{[1]}[\lambda]$$

$$\eta_i([\tau_w]_b) = (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{k=1}^{j-1} a^{[1]}_{jk})^{[1]}[1]$$

$$\eta_j([\tau_w]_b) = (\sum_{k=1}^{j-1} a^{[0]}_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl})^{[1]}[1]$$

Third order:
Differential 4, $F^{(2)}_{(2,0)}FF$, $f''FF$, $N''NN$:

$$\begin{aligned}
r^{\lambda}_{ij}([\tau_b,\tau_b]_w) &= (\eta_j(\tau_b))^2 = (\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2 \\
\phi^{\lambda}_{i}([\tau_b,\tau_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_b,\tau_b]_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] (\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2 \\
\zeta^{\lambda}_{i}([\tau_b,\tau_b]_w) &= \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] (\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2 \\
\eta_i([\tau_b,\tau_b]_b)) &= \sum_{j=1}^{i-1} a^{[1]}_{ij}[1] (\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2 \\
\eta_j([\tau_b,\tau_b]_b)) &= \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] (\sum_{l=1}^{k-1} a^{[1]}_{kl}[1])^2
\end{aligned}$$

Differential 5, $F^{(2)}_{(1,1)}FF$:

$$
\begin{aligned}
r^\lambda_{ij}([\tau_w,\tau_b]_w &= \zeta^\lambda_i(\tau_w)\eta_j(\tau_b) = (\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda])(\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]) \\
\phi^\lambda_i([\tau_w,\tau_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^\lambda_{ij}([\tau_w,\tau_b]_w) \\
&= (\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda])(\sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda]\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]) \\
&= ((\sum_{j=1}^{i-1} a^{[1]}_{ij})(\sum_{j=1}^{i-1} a^{[0]}_{ij}\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]))[\lambda] \\
\zeta^\lambda_i([\tau_w,\tau_b]_w) &= ((\sum_{j=1}^{i-1} a^{[1]}_{ij})(\sum_{j=1}^{i-1} a^{[0]}_{ij}\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]))^{[1]}[\lambda] \\
\eta_i([\tau_w,\tau_b]_b)) &= ((\sum_{j=1}^{i-1} a^{[1]}_{ij})(\sum_{j=1}^{i-1} a^{[0]}_{ij}\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]))^{[1]}[1] \\
\eta_j([\tau_w,\tau_b]_b)) &= ((\sum_{k=1}^{j-1} a^{[1]}_{jk})(\sum_{k=1}^{j-1} a^{[0]}_{jk}\sum_{l=1}^{k-1} a^{[1]}_{kl}[1]))^{[1]}[1]
\end{aligned}
$$

Differential 6, $F^{(2)}_{(0,2)}FF$, $g''FF$:

$$
\begin{aligned}
r^\lambda_{ij}([\tau_w,\tau_w]_w) &= (\zeta^\lambda_i(\tau_w))^2 = (\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda])^2 \\
\phi^\lambda_i([\tau_w,\tau_w]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda]([\tau_w,\tau_w]_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda](\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda])^2 \\
\zeta^\lambda_i([\tau_w,\tau_w]_w) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij}(\sum_{j=1}^{i-1} a^{[1]}_{ij})^2)^{[1]}[\lambda] \\
\eta_i([\tau_w,\tau_w]_b)) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij}(\sum_{j=1}^{i-1} a^{[1]}_{ij})^2)^{[1]}[1] \\
\eta_j([\tau_w,\tau_w]_b)) &= (\sum_{k=1}^{j-1} a^{[0]}_{jk}(\sum_{k=1}^{j-1} a^{[1]}_{jk})^2)^{[1]}[1]
\end{aligned}
$$

Differential 7, $F^{(1)}_{(1,0)}F^{(1)}_{(1,0)}F$, $f'f'F$, $N'N'N$:

$$\begin{aligned}
r^{\lambda}_{ij}([[\tau_b]_b]_w) &= \eta_j([\tau_b]_b) = \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] \\
\phi^{\lambda}_{i}([[\tau_b]_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([[\tau_b]_b]_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] \\
\zeta^{\lambda}_{i}([[\tau_b]_b]_w) &= \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] \\
\eta_{i}([[\tau_b]_b]_b)) &= \sum_{j=1}^{i-1} a^{[1]}_{ij}[1] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]
\end{aligned}$$

Differential 8, $F^{(1)}_{(1,0)}F^{(1)}_{(0,1)}F$, $f'g'F$, $N'LN$:

$$\begin{aligned}
r^{\lambda}_{ij}([[\tau_w]_b]_w) &= \eta_j([\tau_w]_b) = (\sum_{k=1}^{j-1} a^{[0]}_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl})^{[1]}[1] \\
\phi^{\lambda}_{i}([[\tau_w]_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([[\tau_w]_b]_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{k=1}^{j-1} a^{[1]}_{jk})^{[1]}[1] \\
\zeta^{\lambda}_{i}([[\tau_w]_b]_w) &= \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] (\S \sum_{l=1}^{k-1} a^{[1]}_{kl})^{[1]}[1] \\
\eta_{i}([[\tau_w]_b]_b) &= \sum_{j=1}^{i-1} a^{[1]}_{ij}[1] (\sum_{k=1}^{j-1} a^{[0]}_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl})^{[1]}[1] \\
\eta_{j}([[\tau_w]_b]_b) &= \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] (\sum_{l=1}^{k-1} a^{[0]}_{kl} \sum_{r=1}^{l-1} a^{[1]}_{lr})^{[1]}[1]
\end{aligned}$$

Differential 9, $LN'N$:

$$r_{ij}^{\lambda}([[\tau_b]_w]_w) = \zeta_i^{\lambda}([[\tau_b]_w]) = \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1]$$

$$\phi_i^{\lambda}([[\tau_b]_w]_w)) = \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^{\lambda}([[\tau_b]_w]_w))$$

$$= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1]$$

$$\zeta_i^{\lambda}([[\tau_b]_w]_w) = (\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{j=1}^{i-1} a_{ij}^{[1]} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1])^{[1]}[\lambda]$$

$$\eta_i([[\tau_b]_w]_b) = (\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{j=1}^{i-1} a_{ij}^{[1]} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1])^{[1]}[1]$$

$$\eta_j([[\tau_b]_w]_b) = (\sum_{k=1}^{j-1} a_{jk}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1])^{[1]}[1]$$

Differential 10, $F_{(0,1)}^{(1)} F_{(0,1)}^{(1)} F$, $g'g'F$, LLN:

$$r_{ij}^{\lambda}([[\tau_w]_w]_w) = \zeta_i^{\lambda}([\tau_w]_w) = (\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]})^{[1]}[\lambda]$$

$$\phi_i^{\lambda}([[\tau_w]_w]_w) = \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^{\lambda}([[\tau_w]_w]_w))$$

$$= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] (\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]})^{[1]}[\lambda]$$

$$\zeta_i^{\lambda}([[\tau_w]_w]_w) = (\sum_{j=1}^{i-1} a_{ij}^{[0]} (\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]})^{[1]})^{[1]}[\lambda]$$

$$\eta_i([[\tau_w]_w]_b) = (\sum_{j=1}^{i-1} a_{ij}^{[0]} (\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]})^{[1]})^{[1]}[1]$$

$$\eta_j([[\tau_w]_w]_b) = (\sum_{k=1}^{j-1} a_{jk}^{[0]} (\sum_{k=1}^{j-1} a_{jk}^{[0]} \sum_{l=1}^{k-1} a_{kl}^{[1]})^{[1]})^{[1]}[1]$$

Fourth order:

Differential 11, $F^{(3)}_{(3,0)}FFF$, $f'''FFF$, $N'''NNN$:

$$r^{\lambda}_{ij}([\tau_b,\tau_b,\tau_b]_w) = (\eta_j(\tau_b))^3 = (\sum_{j=1}^{i-1} a^{[1]}_{ij}[1])^3$$

$$\phi^{\lambda}_i([\tau_b,\tau_b,\tau_b]_w) = \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_b,\tau_b,\tau_b]_w))$$

$$= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda](\sum_{j=1}^{i-1} a^{[1]}_{ij}[1])^3$$

$$\zeta^{\lambda}_i([\tau_b,\tau_b,\tau_b]_w) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda](\sum_{j=1}^{i-1} a^{[1]}_{ij}[1])^3$$

$$\eta_i([\tau_b,\tau_b,\tau_b]_b) = (\sum_{j=1}^{i-1} a^{[1]}_{ij}[1])^4$$

$$\eta_j([\tau_b,\tau_b,\tau_b]_b) = (\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^4$$

Differential 12, $F^{(3)}_{(2,1)}FFF$:

$$r^{\lambda}_{ij}([\tau_b,\tau_b,\tau_w]_w) = \zeta^{\lambda}_i(\tau_w)(\eta_j(\tau_b))^2 = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda](\sum_{j=1}^{i-1} a^{[1]}_{ij}[1])^2$$

$$\phi^{\lambda}_i([\tau_b,\tau_b,\tau_w]_w) = \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_b,\tau_b,\tau_w]_w))$$

$$= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda]\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda](\sum_{j=1}^{i-1} a^{[1]}_{ij}[1])^2$$

$$\zeta^{\lambda}_i([\tau_b,\tau_b,\tau_w]_w) = (\sum_{j=1}^{i-1} a^{[0]}_{ij}\sum_{j=1}^{i-1} a^{[1]}_{ij})^{[1]}[\lambda](\sum_{j=1}^{i-1} a^{[1]}_{ij}[1])^2$$

$$\eta_i([\tau_b,\tau_b,\tau_w]_b) = (\sum_{j=1}^{i-1} a^{[0]}_{ij}\sum_{j=1}^{i-1} a^{[1]}_{ij})^{[1]}[1](\sum_{j=1}^{i-1} a^{[1]}_{ij}[1])^2$$

$$\eta_j([\tau_b,\tau_b,\tau_w]_b) = (\sum_{k=1}^{j-1} a^{[0]}_{jk}\sum_{k=1}^{j-1} a^{[1]}_{jk})^{[1]}[1](\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2$$

Differential 13, $F^{(3)}_{(1,2)}FFF$:

$$\begin{aligned}
r^{\lambda}_{ij}([\tau_b,\tau_w,\tau_w]_w) &= (\zeta^{\lambda}_i(\tau_w))^2(\eta_j\tau_b) = (\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda])^2 \sum_{k=1}^{j-1} a^{[1]}_{jk}[1]) \\
\phi^{\lambda}_i([\tau_b,\tau_w,\tau_w]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_b,\tau_w,\tau_w]_w)) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda](\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda])^2 \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \\
\zeta^{\lambda}_i([\tau_b,\tau_w,\tau_w]_w) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij}(\sum_{j=1}^{i-1} a^{[1]}_{ij})^2)\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^{[1]}[\lambda] \\
\eta_i([\tau_b,\tau_w,\tau_w]_b) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij}(\sum_{j=1}^{i-1} a^{[1]}_{ij})^2)\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^{[1]}[1] \\
\eta_j([\tau_b,\tau_w,\tau_w]_b) &= (\sum_{k=1}^{j-1} a^{[0]}_{jk}(\sum_{k=1}^{j-1} a^{[1]}_{jk})^2)\sum_{l=1}^{k-1} a^{[1]}_{kl}[1])^{[1]}[1]
\end{aligned}$$

Differential 14, $F^{(3)}_{(0,3)}FF$, $g'''FFF$:

$$\begin{aligned}
r^{\lambda}_{ij}([\tau_w,\tau_w,\tau_w]_w) &= (\zeta^{\lambda}_i(\tau_w))^3 = (\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda])^3 \\
\phi^{\lambda}_i([\tau_w,\tau_w,\tau_w]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_w,\tau_w,\tau_w]_w)) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda](\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda])^3 \\
\zeta^{\lambda}_i([\tau_w,\tau_w,\tau_w]_w) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij}(\sum_{j=1}^{i-1} a^{[1]}_{ij})^3)^{[1]}[\lambda] \\
\eta_i([\tau_w,\tau_w,\tau_w]_b) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij}(\sum_{j=1}^{i-1} a^{[1]}_{ij})^3)^{[1]}[1] \\
\eta_j([\tau_w,\tau_w,\tau_w]_b) &= (\sum_{k=1}^{j-1} a^{[0]}_{jk}(\sum_{k=1}^{j-1} a^{[1]}_{jk})^3)^{[1]}[1]
\end{aligned}$$

Differential 15, $F^{(2)}_{(2,0)}F^{(1)}_{(1,0)}FF$, $f''f'FF$, $N''N'NN$

$$r^{\lambda}_{ij}([\tau_b,[\tau_b]_b]_w) = \eta_j(\tau_b)\eta_j([\tau_b]_b) = \sum_{k=1}^{j-1} a^{[1]}_{jk}[1](\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]\sum_{l=1}^{k-1} a^{[1]}_{kl}[1])$$

$$\begin{aligned}\phi^{\lambda}_i([\tau_b,[\tau_b]_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_w,\tau_w,\tau_w]_w)) \\ &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda]\sum_{k=1}^{j-1} a^{[1]}_{jk}[1](\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]\sum_{l=1}^{k-1} a^{[1]}_{kl}[1])\end{aligned}$$

$$\zeta^{\lambda}_i([\tau_b,[\tau_b]_b]_w) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda]\sum_{k=1}^{j-1} a^{[1]}_{jk}[1](\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]\sum_{l=1}^{k-1} a^{[1]}_{kl}[1])$$

$$\eta_i([\tau_b,[\tau_b]_b]_b) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[1]\sum_{k=1}^{j-1} a^{[1]}_{jk}[1](\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]\sum_{l=1}^{k-1} a^{[1]}_{kl}[1])$$

$$\eta_j([\tau_b,[\tau_b]_b]_b) = \sum_{k=1}^{j-1} a^{[1]}_{jk}[1]\sum_{l=1}^{k-1} a^{[1]}_{kl}[1](\sum_{l=1}^{k-1} a^{[1]}_{kl}[1]\sum_{r=1}^{l-1} a^{[1]}_{lr}[1])$$

Differential 16, $F^{(2)}_{(1,1)}F^{(1)}_{(1,0)}FF$

$$r^{\lambda}_{ij}([\tau_b,[\tau_b]_w]_w) = \eta_j(\tau_b)\zeta^{\lambda}_i([\tau_b]_w) = \sum_{k=1}^{j-1} a^{[1]}_{jk}[1](\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda]\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])$$

$$\begin{aligned}\phi^{\lambda}_i([\tau_b,[\tau_b]_w]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_b,[\tau_b]_w]_w) \\ &= (\sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda]\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])(\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda]\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])\end{aligned}$$

$$\zeta^{\lambda}_i([\tau_b,[\tau_b]_w]_w) = ((\sum_{j=1}^{i-1} a^{[0]}_{ij}\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])(\sum_{j=1}^{i-1} a^{[1]}_{ij}\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]))^{[1]}[\lambda]$$

$$\eta_i([\tau_b,[\tau_b]_w]_b) = ((\sum_{j=1}^{i-1} a^{[0]}_{ij}\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])(\sum_{j=1}^{i-1} a^{[1]}_{ij}\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]))^{[1]}[1]$$

$$\eta_j([\tau_b,[\tau_b]_w]_b) = ((\sum_{k=1}^{j-1} a^{[0]}_{jk}\sum_{l=1}^{k-1} a^{[1]}_{kl}[1])(\sum_{k=1}^{j-1} a^{[1]}_{jk}\sum_{l=1}^{k-1} a^{[1]}_{kl}[1]))^{[1]}[1]$$

Differential 17, $F^{(2)}_{(2,0)}F^{(1)}_{(0,1)}FF$

$$r^{\lambda}_{ij}([\tau_w,[\tau_b]_b]_w) = \zeta^{\lambda}_i(\tau_w)\eta_j([\tau_w]_w) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]$$

$$\phi^{\lambda}_i([\tau_w,[\tau_b]_b]_w) = \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_w,[\tau_b]_b]_w)$$

Differential 18, $F^{(2)}_{(2,0)}F^{(1)}_{(0,1)}FF$, $f''g'FF$, $N''LNN$

$$r^{\lambda}_{ij}([\tau_b,[\tau_w]_b]_w) = \eta^{\lambda}_j(\tau_b)\eta^{\lambda}_j([\tau_w]_b) = \sum_{k=1}^{j-1} a^{[1]}_{jk}[1](\sum_{k=1}^{j-1} a^{[0]}_{jk} \sum_{k=1}^{j-1} a^{[1]}_{jk})^{[1]}[1]$$

$$\begin{aligned}
\phi^{\lambda}_i([\tau_b,[\tau_w]_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([[\tau_w]_w]_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1](\sum_{k=1}^{j-1} a^{[0]}_{jk} \sum_{k=1}^{j-1} a^{[1]}_{jk})^{[1]}[1]
\end{aligned}$$

$$\zeta^{\lambda}_i([\tau_b,[\tau_w]_b]_w) = (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{k=1}^{j-1} a^{[1]}_{jk}[1](\sum_{k=1}^{j-1} a^{[0]}_{jk} \sum_{k=1}^{j-1} a^{[1]}_{jk})^{[1]}[1])^{[1]}[\lambda]$$

$$\eta_i([\tau_b,[\tau_w]_b]_w) = (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{k=1}^{j-1} a^{[1]}_{jk}[1](\sum_{k=1}^{j-1} a^{[0]}_{jk} \sum_{k=1}^{j-1} a^{[1]}_{jk})^{[1]}[1])^{[1]}[1]$$

Differential 24, $F^{(1)}_{(0,1)}F^{(2)}_{(2,0)}FF$, $g'f''FF$, $LN''NN$:

$$\begin{aligned}
r^{\lambda}_{ij}([[\tau_b,\tau_b]_w]_w) &= \zeta^{\lambda}_i([\tau_b,\tau_b]_w)) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda](\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2 \\
\phi^{\lambda}_i([[\tau_b,\tau_b]_w]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_b,\tau_b]_b)_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda](\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2 \\
\zeta^{\lambda}_i([[\tau_b,\tau_b]_w]_w) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{j=1}^{i-1} a^{[1]}_{ij})^{[1]}[\lambda](\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2 \\
\eta_i([[\tau_b,\tau_b]_w]_b) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{j=1}^{i-1} a^{[1]}_{ij})^{[1]}[1](\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2
\end{aligned}$$

Differential 30, $F^{(1)}_{(0,1)}F^{(1)}_{(1,0)}F^{(1)}_{(1,0)}F$, $g'f'f'F$, $LN'N'N$

$$\begin{aligned}
r^{\lambda}_{ij}([[[\tau_b]_b]_w]_w) &= \zeta^{\lambda}_i([[\tau_b]_b]_w]) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] \\
\phi^{\lambda}_i([[[\tau_w]_b]_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([[[\tau_b]_b]_w]_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] \\
\zeta^{\lambda}_i([[[\tau_b]_b]_w]_w) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{j=1}^{i-1} a^{[1]}_{ij})^{[1]})[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] \\
\eta_i([[[\tau_b]_b]_w]_b) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{j=1}^{i-1} a^{[1]}_{ij})^{[1]})[1] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]
\end{aligned}$$

Differential 31, $F^{(1)}_{(1,0)}F^{(1)}_{(0,1)}F^{(1)}_{(1,0)}F$, $f'g'f'F$, $N'LN'N$

$$
\begin{aligned}
r^{\lambda}_{ij}([[[\tau_b]_w]_b]_w) &= \eta_j([[\tau_b]_w]_b) = (\sum_{k=1}^{j-1} a^{[0]}_{ij} \sum_{k=1}^{j-1} a^{[1]}_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1])^{[1]}[1] \\
\phi^{\lambda}_i([[[\tau_b]_w]_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([[[\tau_b]_w]_b]_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda](\sum_{k=1}^{j-1} a^{[0]}_{ij} \sum_{k=1}^{j-1} a^{[1]}_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1])^{[1]}[1] \\
\zeta^{\lambda}_i([[[\tau_b]_w]_b]_w) &= \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda](\sum_{k=1}^{j-1} a^{[0]}_{ij} \sum_{k=1}^{j-1} a^{[1]}_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1])^{[1]}[1] \\
\eta_i([[[\tau_b]_w]_b]_b) &= \sum_{j=1}^{i-1} a^{[1]}_{ij}[1](\sum_{k=1}^{j-1} a^{[0]}_{ij} \sum_{k=1}^{j-1} a^{[1]}_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1])^{[1]}[1]
\end{aligned}
$$

Differential 32, $F^{(1)}_{(1,0)}F^{(1)}_{(1,0)}F^{(1)}_{(0,1)}F$, $f'f'g'F$, $N'N'LN$:

$$
\begin{aligned}
r^{\lambda}_{ij}([[[\tau_b]_b]_w]_w) &= \zeta^{\lambda}_i([[\tau_b]_b]_w]) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] \\
\phi^{\lambda}_i([[[\tau_w]_b]_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([[[\tau_b]_b]_w]_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] \\
\zeta^{\lambda}_i([[[\tau_b]_b]_w]_w) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{j=1}^{i-1} a^{[1]}_{ij})^{[1]})[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] \\
\eta_i([[[\tau_b]_b]_w]_b) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{j=1}^{i-1} a^{[1]}_{ij})^{[1]})[1] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]
\end{aligned}
$$

3 Adding shifted initial conditions

Algorithm:

An identity

$$DR = D(I-D)^{-1} = (D-I+I)(I-D)^{-1} = -I + (I-D)^{-1} = -I + R$$

First order:

Algorithm 2 Calculate series coefficient for tree t

```
t = [t_1 t_2 ... t_m]
for i = 1 to ns + 1 do
  φ_i^λ = 0
  for j = 1 to i − 1 do
    r_ij^λ = 1
    for s ∈ [t_1 t_2 ... t_m] do
      if s ∈ T^w then
        r_ij^λ = r_ij^λ ∗ ζ_{i,λ}(s) Comment: Polynomial multiplication
      else
        r_ij^λ = r_ij^λ ∗ η_j(s) Comment: Scalar multiplication
      end if
    end for
    for p = 0 to ρ_i do
      φ_i^λ = φ_i^λ + a_ijp λ^p ∗ r_ij^λ
    end for
  end for
  Integrate d/dλ ζ_i^λ(t)) = φ_i^λ
  ζ_i^0(t) = 0
  for j = 0 to i − 1 do
    ζ_i^0(t) = ζ_i^0(t) + d_ij η_j(t)
  end for
  η_i(t) = ζ_i^1(t)
end for
```

Differential 1, F, F , N:

$$r^{\lambda}_{ij}(\tau_w) = 1$$

$$\phi^{\lambda}_i(\tau_w) = \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda]$$

$$\zeta^{\lambda}_i(\tau_w) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] + \sum_{j=1}^{i-1} d_{ij}\eta_j(\tau_b)$$

$$\eta_i(\tau_b) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[1] + \sum_{j=1}^{i-1} d_{ij}\eta_j(\tau_b)$$

$$\eta_i(\tau_b) = \sum_{j=1}^{i-1} r_{ij} \sum_{k=1}^{j-1} a^{[1]}_{jk}[1]$$

$$\eta_j(\tau_b) = \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]$$

$$\zeta^{\lambda}_i(\tau_w) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] + \sum_{j=1}^{i-1} d_{ij} \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]$$

Second order:
Differential 2, $F^{(1)}_{(1,0)}F$, $f'F$, $N'N$:

$$r^{\lambda}_{ij}([\tau_b]_w) = \eta^{\lambda}_j(\tau_b) = \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]$$

$$\begin{aligned}
\phi^{\lambda}_i([\tau_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_b]_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]
\end{aligned}$$

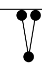
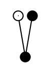
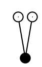
$$\zeta^{\lambda}_i([\tau_b]_w) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] + \sum_{j=1}^{i-1} d_{ij}\eta_j([\tau_b]_b)$$

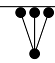
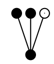
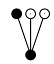
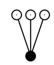
$$\eta_i([\tau_b]_b) = \sum_{j=1}^{i-1} r_{ij} \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} r_{kl} \sum_{r=1}^{l-1} a^{[1]}_{lr}[1]$$

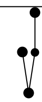
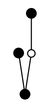
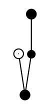
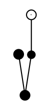
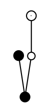
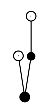
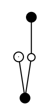
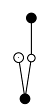
$$\eta_j([\tau_b]_b) = \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] \sum_{r=1}^{l-1} r_{lr} \sum_{s=1}^{r-1} a^{[1]}_{rs}[1]$$

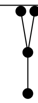
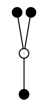
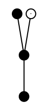
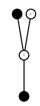
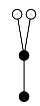
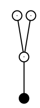
Differential 3, $F^{(1)}_{(0,1)}F$, $g'F$, LN:

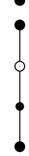
$$
\begin{aligned}
r^{\lambda}_{ij}([\tau_w]_w) &= \zeta_i(\tau_w) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] + \sum_{j=1}^{i-1} d_{ij} \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] \\
\phi^{\lambda}_i([\tau_w]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_w]_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] (\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] + \sum_{j=1}^{i-1} d_{ij} \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]) \\
\zeta^{\lambda}_i([\tau_w]_w) &= \frac{1}{2}(\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda])^2 + \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] \sum_{j=1}^{i-1} d_{ij} \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] + \sum_{j=1}^{i-1} d_{ij} \eta_j([\tau_w]_b)) \\
\eta_i([\tau_w]_b) &= \frac{1}{2}(\sum_{j=1}^{i-1} a^{[1]}_{ij}[1])^2 + \sum_{j=1}^{i-1} a^{[1]}_{ij}[1] \sum_{j=1}^{i-1} d_{ij} \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] + \sum_{j=1}^{i-1} d_{ij} \eta_j([\tau_w]_b)) \\
\eta_i([\tau_w]_b) &= \sum_{j=1}^{i-1} r_{ij} (\frac{1}{2}(\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2 + \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{k=1}^{j-1} d_{jk} \sum_{l=1}^{k-1} r_{kl} \sum_{r=1}^{l-1} a^{[1]}_{lr}[1]) \\
&= \sum_{j=1}^{i-1} r_{ij} (\frac{1}{2}(\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2 - \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] - \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]) \\
\eta_j([\tau_w]_b) &= (\sum_{k=1}^{j-1} a^{[0]}_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl})^{[1]}[1]
\end{aligned}
$$

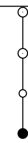
	El. Diff.	El. Diff.	El. Diff.	t	Notation	Notation	$\sigma(t)$	Order Conditions	Order Owren
1	F	F	N		τ_b	$.$	1	$\sum a_{ij}^{[1]}[1]$	
2	$F^{(1)}_{(1,0)}F$	$f'F$	$N'N$		$[\tau_b]_b$	$[.]$	1	$\sum a_{ij}^{[1]}[1]\sum a_{jk}^{[1]}[1]$	
3	$F^{(1)}_{(0,1)}F$	$g'F$	LN		$[\tau_w]_b$	$[o]$	1	$(\sum a_{ij}^{[0]}\sum a_{jk}^{[1]})^{[1]}[1]$	
4	$F^{(2)}_{(2,0)}FF$	$f''FF$	$N''NN$		$[\tau_b,\tau_b]_b$	$[.,.]$	1	$\sum a_{ij}^{[1]}[1](\sum a_{jk}^{[1]}[1])^2$	
5	$F^{(2)}_{(1,1)}FF$				$[\tau_w,\tau_b]_b$	$[o,.]$	2	$((\sum a_{ij}^{[1]})(\sum a_{ij}^{[0]}\sum a_{jk}^{[1]}[1]))^{[1]}[1]$	
6	$F^{(2)}_{(0,2)}FF$	$g''FF$			$[\tau_w,\tau_w]_b$	$[o,o]$	1	$(\sum a_{ij}^{[0]}(\sum a_{ij}^{[1]})^2)^{[1]}[1]$	
7	$F^{(1)}_{(1,0)}F^{(1)}_{(1,0)}F$	$f'f'F$	$N'N'N$		$[[\tau_b]_b]_b$	$[[.]]$	1	$\sum a_{ij}^{[1]}[1]\sum a_{jk}^{[1]}[1]\sum a_{kl}^{[1]}[1]$	
8	$F^{(1)}_{(1,0)}F^{(1)}_{(0,1)}F$	$f'g'F$	$N'LN$		$[[\tau_w]_b]_b$	$[[o]]$	1	$\sum a_{ij}^{[1]}[1](\sum a_{jk}^{[0]}\sum a_{kl}^{[1]})^{[1]}[1]$	
9	$F^{(1)}_{(0,1)}F^{(1)}_{(1,0)}F$	$g'f'F$	$LN'N$		$[[\tau_b]_w]_b$	$[(.)]$	1	$(\sum a_{ij}^{[0]}\sum a_{ij}^{[1]}\sum a_{jk}^{[1]}[1])^{[1]}[1]$	
10	$F^{(1)}_{(0,1)}F^{(1)}_{(0,1)}F$	$g'g'F$	LLN		$[[\tau_w]_w]_b$	$[()]$	1	$(\sum a_{ij}^{[0]}(\sum a_{ij}^{[0]}\sum a_{jk}^{[1]})^{[1]})^{[1]}[1]$	

	El. Diff.	El. Diff.	El. Diff.	t	Notation	Notation	$\sigma(t)$	Order Conditions	Order Owren
11	$F^{(3)}_{(3,0)}FFF$	$f'''FFF$	$N'''NNN$		$[\tau_b, \tau_b, \tau_b]_b$	$[., ., .]$	1	$(\sum a^{[1]}_{ij}[1])^4$	
12	$F^{(3)}_{(2,1)}FFF$				$[\tau_b, \tau_b, \tau_w]_b$	$[., ., o]$	1	$(\sum a^{[0]}_{ij} \sum a^{[1]}_{ij})^{[1]}[1](\sum a^{[1]}_{ij}[1])^2$	
13	$F^{(3)}_{(1,2)}FFF$				$[\tau_b, \tau_w, \tau_w]_b$	$[., o, o]$	1	$(\sum a^{[0]}_{ij}(\sum a^{[1]}_{ij})^2)^{[1]}[1] \sum a^{[1]}_{ij}[1]$	
14	$F^{(3)}_{(0,3)}FFF$	$g'''FFF$			$[\tau_w, \tau_w, \tau_w]_b$	$[o, o, o]$	1	$(\sum a^{[0]}_{ij}(\sum a^{[1]}_{ij})^3)^{[1]}[1]$	

	El. Diff.	El. Diff.	El. Diff.	t	Notation	Notation	$\sigma(t)$	Order Conditions	Order Owren
15	$F^{(2)}_{(2,0)}F^{(1)}_{(1,0)}FF$	$f''f'FF$	$N''N'NN$		$[\tau_b,[\tau_b]_b]_b$	$[.,[.]]$	1	$\sum a^{[1]}_{ij}[1]\sum a^{[1]}_{jk}[1](\sum a^{[1]}_{jk}[1]\sum a^{[1]}_{kl}[1])$	
16	$F^{(2)}_{(1,1)}F^{(1)}_{(1,0)}FF$				$[\tau_b,[\tau_b]_w]_b$	$[.,(.)]$	1	$((\sum a^{[0]}_{ij}\sum a^{[1]}_{jk}[1])(\sum a^{[1]}_{ij}\sum a^{[1]}_{jk}[1]))^{[1]}[1]$	
17	$F^{(2)}_{(1,1)}F^{(1)}_{(1,0)}FF$				$[\tau_w,[\tau_b]_b]_b$	$[o,[.]]$	1		
18	$F^{(2)}_{(2,0)}F^{(1)}_{(0,1)}FF$	$f''g'FF$	$N''LNN$		$[\tau_b,[\tau_w]_b]_b$	$[.,[o]]$	1		
19	$F^{(2)}_{(1,1)}F^{(1)}_{(0,1)}FF$				$[\tau_b,[\tau_w]_w]_b$	$[.,(o)]$	1		
20	$F^{(2)}_{(1,1)}F^{(1)}_{(1,0)}FF$				$[\tau_w,[\tau_b]_b]_b$	$[o,[o]]$	1		
21	$F^{(2)}_{(0,2)}F^{(1)}_{(1,0)}FF$	$g''f'FF$			$[\tau_w,[\tau_b]_w]_b$	$[o,(.)]$	1		
22	$F^{(2)}_{(0,2)}F^{(1)}_{(0,1)}FF$	$g''g'FF$			$[\tau_w,[\tau_w]_w]_b$	$[o,(o)]$	1		

	El. Diff.	El. Diff.	El. Diff.	t	Notation	Notation	$\sigma(t)$
23	$F^{(1)}_{(1,0)}F^{(2)}_{(2,0)}FF$	$f'f''FF$	$N'N''NN$		$[[\tau_b,\tau_b]_b]_b$	$[[.,.]]$	1
24	$F^{(1)}_{(0,1)}F^{(2)}_{(2,0)}FF$	$g'f''FF$	$LN''NN$		$[[\tau_b,\tau_b]_w]_b$	$[(.,.)]$	1
25	$F^{(1)}_{(1,0)}F^{(2)}_{(1,1)}FF$				$[[\tau_b,\tau_b]_b]_b$	$[[.,.]]$	1
26	$F^{(1)}_{(1,0)}F^{(2)}_{(2,0)}FF$				$[[\tau_b,\tau_b]_b]_b$	$[[.,.]]$	1
27	$F^{(1)}_{(1,0)}F^{(2)}_{(0,2)}FF$	$f'g''FF$			$[[\tau_w,\tau_w]_b]_b$	$[[o,o]]$	1
28	$F^{(1)}_{(1,0)}F^{(2)}_{(2,0)}FF$	$g'g''FF$			$[[\tau_w,\tau_w]_w]_b$	$[(o,o)]$	1

	El. Diff.	El. Diff.	El. Diff.	t	Notation	Notation	$\sigma(t)$
29	$F^{(1)}_{(1,0)}F^{(1)}_{(1,0)}F^{(1)}_{(1,0)}F$	$f'f'f'F$	$N'N'N'N$		$[[[\tau_b]_b]_b]_b$	$[[[.]]]$	1
30	$F^{(1)}_{(0,1)}F^{(1)}_{(1,0)}F^{(1)}_{(1,0)}F$	$g'f'f'F$	$LN'N'N$		$[[[\tau_w]_b]_b]_b$	$[([.])]$	1
31	$F^{(1)}_{(1,0)}F^{(1)}_{(0,1)}F^{(1)}_{(1,0)}F$	$f'g'f'F$	$N'LN'N$		$[[[\tau_b]_w]_b]_b$	$[[(.)]]$	1
32	$F^{(1)}_{(1,0)}F^{(1)}_{(1,0)}F^{(1)}_{(0,1)}F$	$f'f'g'F$	$N'N'LN$		$[[[\tau_b]_b]_w]_b$	$[[[o]]]$	1

	El. Diff.	El. Diff.	El. Diff.	t	Notation	Notation	$\sigma(t)$
33	$F^{(1)}_{(0,1)}F^{(1)}_{(0,1)}F^{(1)}_{(0,1)}F$	$g'g'g'F$	$LLLN$		$[[[\tau_w]_w]_w]_b$	$[((o))]$	1
34	$F^{(1)}_{(0,1)}F^{(1)}_{(0,1)}F^{(1)}_{(1,0)}F$	$f'g'g'F$	$N'LLN$		$[[[\tau_w]_w]_b]_b$	$[[(o)]]$	1
35	$F^{(1)}_{(1,0)}F^{(1)}_{(0,1)}F^{(1)}_{(1,0)}F$	$g'f'g'F$	$LN'LN$		$[[[\tau_w]_b]_w]_b$	$[([o])]$	1
36	$F^{(1)}_{0,1)}F^{(1)}_{(0,1)}F^{(1)}_{(0,1)}F$	$g'g'f'F$	$LLN'N$		$[[[\tau_b]_w]_w]_b$	$[((.))]$	1