1 Introduction

MISB-Series is a FORTRAN program to compute order conditions for a class of one step methods where in each stage again a differential equation has to be solved. We start with the following differential equation

$$y' = F(y, y)$$

where the right hand depends twice on the unknown y. This form of the right hand side involves a large number of ODE's for which special splitting typed integrator are proposed. We list types of equations which appear in the literature.

Other types of partitioning Additive splitting:

$$y' = F(y, y) = f(y) + f(y)$$

Additive nonlinear-linear splitting:

$$y' = F(y, y) = f(y) + Ny$$

Vector fields on manifolds in frame representation

$$y' = F(y, y) = \sum_{i=1}^{N} f_i(y) E_i(y)$$

Multiplicative splitting:

$$y' = F(y, y) = A(y)y$$

I all examples the first y argument is the first one in the general framework, and so on.

For this type of equations exponential Runge-Kutta type methods will be analyzed. The methods have the following structure.

$$\begin{aligned}
Y_1 &= y_n \\
Z_i' &= \sum_{j=1}^{i-1} \sum_{p=0}^{\rho_i} a_{ij}^p \left(\frac{\tau}{h}\right)^p F(Y_j, Z_i), \quad Z_i(0) = y_n + \sum_j d_{ij}(Y_j - y_n) \\
Y_i &= Z_i(h),
\end{aligned}$$

where in addition the consistency condition is required

$$\sum_{j=1}^{i-1} a_{ij}^p = 0, \; p = 1, \ldots, \rho_i.$$

We will derive order conditions with the help of B-series. Let us start with the elementary differentials. We will use the abbreviation $F^{(n)}_{(i,j)}$, $i+j=n$ to denote the i-th derivative of f with respect to the first argument and the j-th derivative of f with respect to the second argument.

$$\begin{aligned}
y' &= F \\
y'' &= F^{(1)}_{(1,0)}F + F^{(1)}_{(1,0)}F \\
y^{(3)} &= F^{(2)}_{(2,0)}FF + 2F^{(2)}_{(1,1)}FF + F^{(2)}_{(0,2)}FF \\
&+ F^{(1)}_{(1,0)}F^{(1)}_{(1,0)}F + F^{(1)}_{(1,0)}F^{(1)}_{(0,1)}F + F^{(1)}_{(0,1)}F^{(1)}_{(1,0)}F + F^{(1)}_{(0,1)}F^{(1)}_{(0,1)}F
\end{aligned}$$

For a graphical representation we use bi-coloured trees. Black vertices means computation of F at the position (y,y) and and white vertices means computation at the position (y,z). Upwards pointing branches represent partial derivatives with respect to the first argument if the branch leads to a black vertex, and with respect to the second argument if it leads to a white vertex.

The trees are divided in two sets T^b and T^w where T^b contain all trees with a black root and T^w with a white root. Hence, the trees T^b are computed at (y,y) in the leading derivative whereas those of T^w are computed at the mixed argument (y,z). The tress are defined recursively trough:

$$\begin{aligned}
t &= [t_{b1}\dots t_{bk}, t_{w1}\dots t_{wl}]_b, \quad \forall t \in T^b \\
t &= [t_{b1}\dots t_{bk}, t_{w1}\dots t_{wl}]_w, \quad \forall t \in T^w
\end{aligned}$$

Furthermore we introduce the black operator ^{b} which replaces a white root node by a black root node.

$$t^b = [t_{b1}\dots t_{bk}, t_{w1}\dots t_{wl}]_w^b = [t_{b1}\dots t_{bk}, t_{w1}\dots t_{wl}]_b$$

$$G_F([t_{b,1}\dots t_{b,k}, t_{w1}\dots t_{wl}]_b)(y_0,y_0) =$$
$$F^{(k+l)}_{(k,l)}(y_0,y_0)(G(t_{b1})(y_0,y_0),\dots,G(t_{bk})(y_0,y_0),G(t^b_{w1})(y_0,y_0),\dots,G(t^b_{wl})(y_0,y_0))$$
$$G_F([t_{b,1}\dots t_{b,k}, t_{w1}\dots t_{wl}]_w)(y_0,z_0) =$$
$$F^{(k+l)}_{(k,l)}(y_0,y_0)(G(t_{b1})(y_0,y_0),\dots,G(t_{bk})(y_0,y_0),G(t_{w1})(y_0,z_0),\dots,G(t_{wl})(y_0,z_0))$$

Two types of formal B-series are introduced

$$\begin{aligned}
B(a,hF,y,y) &= a(\emptyset)y + \sum_{t\in T^b}\frac{a(t)}{\sigma(t)}G_F(t)(y,y)h^{r(t)} \\
C(b,hF,\lambda,y,z) &= b(\emptyset)z + \sum_{t\in T^w}\frac{b^\lambda(t)}{\sigma(t)}G_F(t)(y,z)h^{r(t)}
\end{aligned}$$

where the coefficients $b^\lambda(t)$ of the second B-series are polynomials in a further parameter λ

$$b^\lambda(t) = b_0(t) + \lambda b_1(t) + \lambda^2 b_2(t) + \dots$$

The internal variables are expanded into series by

$$\begin{aligned} Y_i =& B(\eta_i, hf, y, y) := \sum_{t\in T_y} \eta_i(t)\frac{h^{\rho(t)}}{\sigma(t)} G_F(t)(y,y) && (1)\\ Z_i(\lambda) =& C(\zeta_i^\lambda, hf, y, z) := \sum_{t\in T_w} \zeta_i^\lambda(t)\frac{h^{\rho(t)}}{\sigma(t)} G_F(t)(y,z) && (2)\\ & && (3)\end{aligned}$$

The series coefficients η_i map T_w to $\mathbb{R}$, where the ζ_i^λ maps to polynomials over $\mathbb{R}$. Alternatively, we can interpret the B series defined by ζ_i being dependent on a parameter. Note, that we have $\eta_i(\emptyset) = \zeta_i^\lambda(\emptyset) = 1$.

We consider the expansion of $hf(Y,Z)$ in a B-series when Y, Z are given by B-series with coefficents η, ζ^λ whereas always $\eta(\emptyset) = \zeta(\emptyset) = 1$. The resulting B-series is denoted by $D(\eta, \zeta)$.

Theorem 1.1 *For a tree $t = [t_{b1} \dots t_{bk}, t_{w1} \dots t_{wl}]_b$ we have*

$$hf(Y,Z) = C(D(\eta,\zeta^\lambda), hf, y, z) \tag{4}$$

whereas

$$D(\eta,\zeta))(t) = \prod_i \eta(t_{bi}) \prod_j \zeta^\lambda(t_{wj}) \tag{5}$$

The proof requires only an exact calculation of the occurence of the tree t in the Taylor expansion of hf.

Proof: We denote by $f^{(n)}$ the full tensor representing the n-th derivative of f with respect to a vectorial column variable $(Y;Z)$, whereas we denote by $f^{(k,l)}$ the tensor denoting a derivative of k-times derivation with respect to Y, and l times derivation with respect to Z, thus excepting exactly k arguments from the Y-part and exactly l arguments with respect to the Z-part.

$$\begin{aligned} hf(Y,Z) =& hf(y,z) + h\sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)} \left[\begin{pmatrix} Y-y \\ Z-z \end{pmatrix}, \dots \right] \\ =& hf(y,z) + h\sum_{n=1}^{\infty}\sum_{k=0}^{n} \frac{1}{n!}\binom{n}{k} f^{(k,n-k)} [Y-y, \dots; Z-z, \dots] \\ =& hf(y,z) + h\sum_{n=1}^{\infty}\sum_{k=0}^{n} \frac{1}{k!(n-k)!} f^{(k,n-k)} [Y-y, \dots; Z-z, \dots] \end{aligned}$$

Now assume we want to count the number of occurrences of a tree $t = [t_{b1}, \dots, t_{bk}, t_{w1}, \dots, t_{wl}]$ where $t_{b1}, \dots, t_{bk} \in T_b$, $t_{w1}, \dots, t_{wl}l \in T_w$. Assume that the multiplicities in which children trees occur are $\mu_1, \dots, \mu_r$. Then, such a tree occurs in the Taylor expansion exactly

$$\frac{k!l!}{\mu_1! \cdot \dots \cdot \mu_r!}$$

times. If all subtrees are distinct we have $k!l!$ occurrences, but for each multiple occurrence μ_i we have to deduct a factor $1/\mu_i!$. Taken into consideration the recursion formula for the expressions $\sigma(t)$, we end up with the proposition

$$D(\eta, \zeta^\lambda))(t) = \prod_i \eta(t_{bi}) \prod_j \zeta^\lambda(t_{wj}) \tag{6}$$

The algorithm below computes the series coefficients for an algorithm where we have no shift in the initial conditions.

Algorithm 1 Calculate series coefficient for tree $t = [t_1 t_2 \dots t_m]_w$

$t = [t_1 t_2 \dots t_m]_w$
for $i = 1$ **to** $ns + 1$ **do**
  $\phi_i^\lambda = 0$
  for $j = 1$ **to** $i - 1$ **do**
    $r_{ij}^\lambda = 1$
    for $s \in [t_1 t_2 \dots t_m]$ **do**
      if $s \in T_w$ **then**
        $r_{ij}^\lambda = r_{ij}^\lambda * \zeta_{i,\lambda}(s)$ Comment: Polynomial multiplication
      else
        $r_{ij}^\lambda = r_{ij}^\lambda * \eta_j(s)$ Comment: Scalar multiplication
      end if
    end for
    for $p = 0$ **to** ρ_i **do**
      $\phi_i^\lambda = \phi_i^\lambda + a_{ijp}\lambda^p * r_{ij}^\lambda$
    end for
  end for
  Integrate $\frac{d}{d\lambda}\zeta_i^\lambda(t)) = \phi_i^\lambda$
  $\zeta_i^{\lambda 0}(t) = 0$
  $\eta_i(t^b) = \zeta_i^1(t)$
end for

2 Series coefficients

Let us compute the first coefficients of both series and the intermediate series ϕ for all stages. In addition to the standard case we also add the differentials for general additive splitting and linear additive splitting.

For a compact notification we define also

$$a_{ij}^{[0]}[\lambda] = \sum_{p=0}^{\rho_i} a_{ij}^p \frac{p!}{(p+0)!} \lambda^{p+0}$$

$$a_{ij}^{[1]}[\lambda] = \sum_{p=0}^{\rho_i} a_{ij}^p \frac{p!}{(p+1)!} \lambda^{p+1}$$

$$a_{ij}^{[m]}[\lambda] = \sum_{p=0}^{\rho_i} a_{ij}^p \frac{p!}{(p+m)!} \lambda^{p+m}$$

A further notation is the product of polynomial

$$\begin{aligned}
(qr)[\lambda] = q[\lambda]r[\lambda] =& (\sum_{p=0}^{\rho_1} q^p \lambda^p)(\sum_{p=0}^{\rho_2} r^p \lambda^p) \\
=& q^{\rho_1} p^{\rho_2} \lambda^{\rho_1+\rho_2} \\
&+ (q^{\rho_1} r^{\rho_2-1} + q^{\rho_1-1} r^{\rho_2})\lambda^{\rho_1+\rho_2-1} \\
&+ (q^{\rho_1} r^{\rho_2-2} + q^{\rho_1-1} r^{\rho_2-1} + q^{\rho_1-1} r^{\rho_2-1})\lambda^{\rho_1+\rho_2-2} \\
&+ \ldots \\
&+ (q^0 r^1 + q^1 r^0)\lambda^1 \\
&+ q^0 r^0 \lambda^0 \\
=& \sum_{p=0}^{\rho_1+\rho_2} (\sum_{s=0}^{k} q^l r^{p-s})\lambda^p
\end{aligned}$$

where q^p is zero whenever $p \notin [0, \rho_1]$, resp. r^p is zero whenever $p \notin [0, \rho_2]$.

$$(qr)^p = \sum_{s=0}^{p} q^s q^{p-s}$$

First order:

Differential 1, F, F , N:

$$
\begin{aligned}
r_{ij}^{\lambda}(\tau_w) &= 1 \\
\phi_i^{\lambda}(\tau_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \\
\zeta_i^{\lambda}(\tau_w) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \\
\eta_i(\tau_b) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[1] \\
\eta_j(\tau_b) &= \sum_{k=1}^{j-1} a_{jk}^{[1]}[1]
\end{aligned}
$$

Second order:
Differential 2, $F^{(1)}_{(1,0)}F$, $f'F$, $N'N$:

$$r^{\lambda}_{ij}([\tau_b]_w) = \eta^{\lambda}_j(\tau_b) = \sum_{k=1}^{j-1} a^{[1]}_{jk}[1]$$

$$\begin{aligned}
\phi^{\lambda}_i([\tau_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_b]_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \\
\zeta^{\lambda}_i([\tau_b]_w) &= \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \\
\eta_i([\tau_b]_b) &= \sum_{j=1}^{i-1} a^{[1]}_{ij}[1] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \\
\eta_j([\tau_b]_b) &= \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]
\end{aligned}$$

Differential 3, $F^{(1)}_{(0,1)}F$, $g'F$, LN:

$$r^{\lambda}_{ij}([\tau_w]_w) = \zeta_i(\tau_w) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda]$$

$$\begin{aligned}
\phi^{\lambda}_i([\tau_w]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_w]_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] \\
\zeta^{\lambda}_i([\tau_w]_w) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{k=1}^{j-1} a^{[1]}_{jk})^{[1]}[\lambda] \\
\eta_i([\tau_w]_b) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{k=1}^{j-1} a^{[1]}_{jk})^{[1]}[1] \\
\eta_j([\tau_w]_b) &= (\sum_{k=1}^{j-1} a^{[0]}_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl})^{[1]}[1]
\end{aligned}$$

Third order:
Differential 4, $F^{(2)}_{(2,0)}FF$, $f''FF$, $N''NN$:

$$
\begin{aligned}
r^{\lambda}_{ij}([\tau_b,\tau_b]_w) &= (\eta_j(\tau_b))^2 = (\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2 \\
\phi^{\lambda}_i([\tau_b,\tau_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_b,\tau_b]_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] (\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2 \\
\zeta^{\lambda}_i([\tau_b,\tau_b]_w) &= \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] (\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2 \\
\eta_i([\tau_b,\tau_b]_b)) &= \sum_{j=1}^{i-1} a^{[1]}_{ij}[1] (\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2 \\
\eta_j([\tau_b,\tau_b]_b)) &= \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] (\sum_{l=1}^{k-1} a^{[1]}_{kl}[1])^2
\end{aligned}
$$

Differential 5, $F^{(2)}_{(1,1)}FF$:

$$\begin{aligned}
r^{\lambda}_{ij}([\tau_w,\tau_b]_w) &= \zeta^{\lambda}_i(\tau_w)\eta_j(\tau_b) = (\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda])(\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]) \\
\phi^{\lambda}_i([\tau_w,\tau_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_w,\tau_b]_w) \\
&= (\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda])(\sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda]\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]) \\
&= ((\sum_{j=1}^{i-1} a^{[1]}_{ij})(\sum_{j=1}^{i-1} a^{[0]}_{ij}\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]))[\lambda] \\
\zeta^{\lambda}_i([\tau_w,\tau_b]_w) &= ((\sum_{j=1}^{i-1} a^{[1]}_{ij})(\sum_{j=1}^{i-1} a^{[0]}_{ij}\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]))^{[1]}[\lambda] \\
\eta_i([\tau_w,\tau_b]_b)) &= ((\sum_{j=1}^{i-1} a^{[1]}_{ij})(\sum_{j=1}^{i-1} a^{[0]}_{ij}\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]))^{[1]}[1] \\
\eta_j([\tau_w,\tau_b]_b)) &= ((\sum_{k=1}^{j-1} a^{[1]}_{jk})(\sum_{k=1}^{j-1} a^{[0]}_{jk}\sum_{l=1}^{k-1} a^{[1]}_{kl}[1]))^{[1]}[1]
\end{aligned}$$

Differential 6, $F^{(2)}_{(0,2)}FF$, $g''FF$:

$$\begin{aligned}
r^{\lambda}_{ij}([\tau_w,\tau_w]_w) &= (\zeta^{\lambda}_i(\tau_w))^2 = (\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda])^2 \\
\phi^{\lambda}_i([\tau_w,\tau_w]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda]([\tau_w,\tau_w]_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda](\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda])^2 \\
\zeta^{\lambda}_i([\tau_w,\tau_w]_w) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij}(\sum_{j=1}^{i-1} a^{[1]}_{ij})^2)^{[1]}[\lambda] \\
\eta_i([\tau_w,\tau_w]_b)) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij}(\sum_{j=1}^{i-1} a^{[1]}_{ij})^2)^{[1]}[1] \\
\eta_j([\tau_w,\tau_w]_b)) &= (\sum_{k=1}^{j-1} a^{[0]}_{jk}(\sum_{k=1}^{j-1} a^{[1]}_{jk})^2)^{[1]}[1]
\end{aligned}$$

Differential 7, $F^{(1)}_{(1,0)}F^{(1)}_{(1,0)}F$, $f'f'F$, $N'N'N$:

$$r^\lambda_{ij}([[\tau_b]_b]_w) = \eta_j([\tau_b]_b) = \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]$$

$$\begin{aligned}\phi^\lambda_i([[\tau_b]_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^\lambda_{ij}([[\tau_b]_b]_w) \\ &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]\end{aligned}$$

$$\zeta^\lambda_i([[\tau_b]_b]_w) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]$$

$$\eta_i([[\tau_b]_b]_b)) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[1] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]$$

Differential 8, $F^{(1)}_{(1,0)}F^{(1)}_{(0,1)}F$, $f'g'F$, $N'LN$:

$$r^\lambda_{ij}([[\tau_w]_b]_w) = \eta_j([\tau_w]_b) = (\sum_{k=1}^{j-1} a^{[0]}_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl})^{[1]}[1]$$

$$\begin{aligned}\phi^\lambda_i([[\tau_w]_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^\lambda_{ij}([[\tau_w]_b]_w) \\ &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{k=1}^{j-1} a^{[1]}_{jk})^{[1]}[1]\end{aligned}$$

$$\zeta^\lambda_i([[\tau_w]_b]_w) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] (\S \sum_{l=1}^{k-1} a^{[1]}_{kl})^{[1]}[1]$$

$$\eta_i([[\tau_w]_b]_b) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[1] (\sum_{k=1}^{j-1} a^{[0]}_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl})^{[1]}[1]$$

$$\eta_j([[\tau_w]_b]_b) = \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] (\sum_{l=1}^{k-1} a^{[0]}_{kl} \sum_{r=1}^{l-1} a^{[1]}_{lr})^{[1]}[1]$$

Differential 9, $LN'N$:

$$
\begin{aligned}
r_{ij}^{\lambda}([[\tau_b]_w]_w) &= \zeta_i^{\lambda}([[\tau_b]_w]) = \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \\
\phi_i^{\lambda}([[\tau_b]_w]_w)) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^{\lambda}([[\tau_b]_w]_w)) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \\
\zeta_i^{\lambda}([[\tau_b]_w]_w) &= (\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{j=1}^{i-1} a_{ij}^{[1]} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1])^{[1]}[\lambda] \\
\eta_i([[\tau_b]_w]_b) &= (\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{j=1}^{i-1} a_{ij}^{[1]} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1])^{[1]}[1] \\
\eta_j([[\tau_b]_w]_b) &= (\sum_{k=1}^{j-1} a_{jk}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1])^{[1]}[1]
\end{aligned}
$$

Differential 10, $F_{(0,1)}^{(1)} F_{(0,1)}^{(1)} F$, $g'g'F$, LLN:

$$
\begin{aligned}
r_{ij}^{\lambda}([[\tau_w]_w]_w) &= \zeta_i^{\lambda}([\tau_w]_w) = (\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]})^{[1]}[\lambda] \\
\phi_i^{\lambda}([[\tau_w]_w]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^{\lambda}([[\tau_w]_w]_w)) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] (\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]})^{[1]}[\lambda] \\
\zeta_i^{\lambda}([[\tau_w]_w]_w) &= (\sum_{j=1}^{i-1} a_{ij}^{[0]} (\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]})^{[1]})^{[1]}[\lambda] \\
\eta_i([[\tau_w]_w]_b) &= (\sum_{j=1}^{i-1} a_{ij}^{[0]} (\sum_{j=1}^{i-1} a_{ij}^{[0]} \sum_{k=1}^{j-1} a_{jk}^{[1]})^{[1]})^{[1]}[1] \\
\eta_j([[\tau_w]_w]_b) &= (\sum_{k=1}^{j-1} a_{jk}^{[0]} (\sum_{k=1}^{j-1} a_{jk}^{[0]} \sum_{l=1}^{k-1} a_{kl}^{[1]})^{[1]})^{[1]}[1]
\end{aligned}
$$

Fourth order:

Differential 11, $F^{(3)}_{(3,0)}FFF$, $f'''FFF$, $N'''NNN$:

$$\begin{aligned}
r^{\lambda}_{ij}([\tau_b,\tau_b,\tau_b]_w) &= (\eta_j(\tau_b))^3 = (\sum_{j=1}^{i-1} a^{[1]}_{ij}[1])^3 \\
\phi^{\lambda}_i([\tau_b,\tau_b,\tau_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_b,\tau_b,\tau_b]_w)) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda](\sum_{j=1}^{i-1} a^{[1]}_{ij}[1])^3 \\
\zeta^{\lambda}_i([\tau_b,\tau_b,\tau_b]_w) &= \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda](\sum_{j=1}^{i-1} a^{[1]}_{ij}[1])^3 \\
\eta_i([\tau_b,\tau_b,\tau_b]_b) &= (\sum_{j=1}^{i-1} a^{[1]}_{ij}[1])^4 \\
\eta_j([\tau_b,\tau_b,\tau_b]_b) &= (\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^4
\end{aligned}$$

Differential 12, $F^{(3)}_{(2,1)}FFF$:

$$\begin{aligned}
r^{\lambda}_{ij}([\tau_b,\tau_b,\tau_w]_w) &= \zeta^{\lambda}_i(\tau_w)(\eta_j(\tau_b))^2 = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda](\sum_{j=1}^{i-1} a^{[1]}_{ij}[1])^2 \\
\phi^{\lambda}_i([\tau_b,\tau_b,\tau_w]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_b,\tau_b,\tau_w]_w)) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda](\sum_{j=1}^{i-1} a^{[1]}_{ij}[1])^2 \\
\zeta^{\lambda}_i([\tau_b,\tau_b,\tau_w]_w) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{j=1}^{i-1} a^{[1]}_{ij})^{[1]}[\lambda](\sum_{j=1}^{i-1} a^{[1]}_{ij}[1])^2 \\
\eta_i([\tau_b,\tau_b,\tau_w]_b) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{j=1}^{i-1} a^{[1]}_{ij})^{[1]}[1](\sum_{j=1}^{i-1} a^{[1]}_{ij}[1])^2 \\
\eta_j([\tau_b,\tau_b,\tau_w]_b) &= (\sum_{k=1}^{j-1} a^{[0]}_{jk} \sum_{k=1}^{j-1} a^{[1]}_{jk})^{[1]}[1](\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2
\end{aligned}$$

Differential 13, $F^{(3)}_{(1,2)}FFF$:

$$r^{\lambda}_{ij}([\tau_b,\tau_w,\tau_w]_w) = (\zeta^{\lambda}_i(\tau_w))^2(\eta_j\tau_b) = (\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda])^2 \sum_{k=1}^{j-1} a^{[1]}_{jk}[1])$$

$$\phi^{\lambda}_i([\tau_b,\tau_w,\tau_w]_w) = \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_b,\tau_w,\tau_w]_w))$$

$$= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda](\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda])^2 \sum_{k=1}^{j-1} a^{[1]}_{jk}[1]$$

$$\zeta^{\lambda}_i([\tau_b,\tau_w,\tau_w]_w) = (\sum_{j=1}^{i-1} a^{[0]}_{ij}(\sum_{j=1}^{i-1} a^{[1]}_{ij})^2)\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^{[1]}[\lambda]$$

$$\eta_i([\tau_b,\tau_w,\tau_w]_b) = (\sum_{j=1}^{i-1} a^{[0]}_{ij}(\sum_{j=1}^{i-1} a^{[1]}_{ij})^2)\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^{[1]}[1]$$

$$\eta_j([\tau_b,\tau_w,\tau_w]_b) = (\sum_{k=1}^{j-1} a^{[0]}_{jk}(\sum_{k=1}^{j-1} a^{[1]}_{jk})^2)\sum_{l=1}^{k-1} a^{[1]}_{kl}[1])^{[1]}[1]$$

Differential 14, $F^{(3)}_{(0,3)}FF$, $g'''FFF$:

$$r^{\lambda}_{ij}([\tau_w,\tau_w,\tau_w]_w) = (\zeta^{\lambda}_i(\tau_w))^3 = (\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda])^3$$

$$\phi^{\lambda}_i([\tau_w,\tau_w,\tau_w]_w) = \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_w,\tau_w,\tau_w]_w))$$

$$= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda](\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda])^3$$

$$\zeta^{\lambda}_i([\tau_w,\tau_w,\tau_w]_w) = (\sum_{j=1}^{i-1} a^{[0]}_{ij}(\sum_{j=1}^{i-1} a^{[1]}_{ij})^3)^{[1]}[\lambda]$$

$$\eta_i([\tau_w,\tau_w,\tau_w]_b) = (\sum_{j=1}^{i-1} a^{[0]}_{ij}(\sum_{j=1}^{i-1} a^{[1]}_{ij})^3)^{[1]}[1]$$

$$\eta_j([\tau_w,\tau_w,\tau_w]_b) = (\sum_{k=1}^{j-1} a^{[0]}_{jk}(\sum_{k=1}^{j-1} a^{[1]}_{jk})^3)^{[1]}[1]$$

Differential 15, $F^{(2)}_{(2,0)}F^{(1)}_{(1,0)}FF$, $f''f'FF$, $N''N'NN$

$$r^{\lambda}_{ij}([\tau_b,[\tau_b]_b]_w) = \eta_j(\tau_b)\eta_j([\tau_b]_b) = \sum_{k=1}^{j-1} a^{[1]}_{jk}[1](\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]\sum_{l=1}^{k-1} a^{[1]}_{kl}[1])$$

$$\phi^{\lambda}_{i}([\tau_b,[\tau_b]_b]_w) = \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_w,\tau_w,\tau_w]_w))$$

$$= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda]\sum_{k=1}^{j-1} a^{[1]}_{jk}[1](\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]\sum_{l=1}^{k-1} a^{[1]}_{kl}[1])$$

$$\zeta^{\lambda}_{i}([\tau_b,[\tau_b]_b]_w) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda]\sum_{k=1}^{j-1} a^{[1]}_{jk}[1](\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]\sum_{l=1}^{k-1} a^{[1]}_{kl}[1])$$

$$\eta_{i}([\tau_b,[\tau_b]_b]_b) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[1]\sum_{k=1}^{j-1} a^{[1]}_{jk}[1](\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]\sum_{l=1}^{k-1} a^{[1]}_{kl}[1])$$

$$\eta_{j}([\tau_b,[\tau_b]_b]_b) = \sum_{k=1}^{j-1} a^{[1]}_{jk}[1]\sum_{l=1}^{k-1} a^{[1]}_{kl}[1](\sum_{l=1}^{k-1} a^{[1]}_{kl}[1]\sum_{r=1}^{l-1} a^{[1]}_{lr}[1])$$

Differential 16, $F^{(2)}_{(1,1)}F^{(1)}_{(1,0)}FF$

$$r^{\lambda}_{ij}([\tau_b,[\tau_b]_w]_w) = \eta_j(\tau_b)\zeta^{\lambda}_i([\tau_b]_w) = \sum_{k=1}^{j-1} a^{[1]}_{jk}[1](\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda]\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])$$

$$\phi^{\lambda}_{i}([\tau_b,[\tau_b]_w]_w) = \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_b,[\tau_b]_w]_w)$$

$$= (\sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda]\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])(\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda]\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])$$

$$\zeta^{\lambda}_{i}([\tau_b,[\tau_b]_w]_w) = ((\sum_{j=1}^{i-1} a^{[0]}_{ij}\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])(\sum_{j=1}^{i-1} a^{[1]}_{ij}\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]))^{[1]}[\lambda]$$

$$\eta_{i}([\tau_b,[\tau_b]_w]_b) = ((\sum_{j=1}^{i-1} a^{[0]}_{ij}\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])(\sum_{j=1}^{i-1} a^{[1]}_{ij}\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]))^{[1]}[1]$$

$$\eta_{j}([\tau_b,[\tau_b]_w]_b) = ((\sum_{k=1}^{j-1} a^{[0]}_{jk}\sum_{l=1}^{k-1} a^{[1]}_{kl}[1])(\sum_{k=1}^{j-1} a^{[1]}_{jk}\sum_{l=1}^{k-1} a^{[1]}_{kl}[1]))^{[1]}[1]$$

Differential 17, $F^{(2)}_{(2,0)}F^{(1)}_{(0,1)}FF$

$$r^{\lambda}_{ij}([\tau_w,[\tau_b]_b]_w) = \zeta^{\lambda}_i(\tau_w)\eta_j([\tau_w]_w) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda]\sum_{k=1}^{j-1} a^{[1]}_{jk}[1]\sum_{l=1}^{k-1} a^{[1]}_{kl}[1]$$

$$\phi^{\lambda}_i([\tau_w,[\tau_b]_b]_w) = \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_w,[\tau_b]_b]_w))$$

Differential 18, $F^{(2)}_{(2,0)}F^{(1)}_{(0,1)}FF$, $f''g'FF$, $N''LNN$

$$r^{\lambda}_{ij}([\tau_b,[\tau_w]_b]_w) = \eta^{\lambda}_j(\tau_b)\eta^{\lambda}_j([\tau_w]_b) = \sum_{k=1}^{j-1} a^{[1]}_{jk}[1](\sum_{k=1}^{j-1} a^{[0]}_{jk}\sum_{k=1}^{j-1} a^{[1]}_{jk})^{[1]}[1]$$

$$\begin{aligned}
\phi^{\lambda}_i([\tau_b,[\tau_w]_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([[\tau_w]_w]_w)) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda]\sum_{k=1}^{j-1} a^{[1]}_{jk}[1](\sum_{k=1}^{j-1} a^{[0]}_{jk}\sum_{k=1}^{j-1} a^{[1]}_{jk})^{[1]}[1] \\
\zeta^{\lambda}_i([\tau_b,[\tau_w]_b]_w) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij}\sum_{k=1}^{j-1} a^{[1]}_{jk}[1](\sum_{k=1}^{j-1} a^{[0]}_{jk}\sum_{k=1}^{j-1} a^{[1]}_{jk})^{[1]}[1])^{[1]}[\lambda] \\
\eta_i([\tau_b,[\tau_w]_b]_w) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij}\sum_{k=1}^{j-1} a^{[1]}_{jk}[1](\sum_{k=1}^{j-1} a^{[0]}_{jk}\sum_{k=1}^{j-1} a^{[1]}_{jk})^{[1]}[1])^{[1]}[1]
\end{aligned}$$

Differential 24, $F^{(1)}_{(0,1)}F^{(2)}_{(2,0)}FF$, $g'f''FF$, $LN''NN$:

$$\begin{aligned}
r^{\lambda}_{ij}([[\tau_b,\tau_b]_w]_w) &= \zeta^{\lambda}_i([\tau_b,\tau_b]_w)) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda](\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2 \\
\phi^{\lambda}_i([[\tau_b,\tau_b]_w]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_b,\tau_b]_b)_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda](\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2 \\
\zeta^{\lambda}_i([[\tau_b,\tau_b]_w]_w) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{j=1}^{i-1} a^{[1]}_{ij})^{[1]}[\lambda](\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2 \\
\eta_i([[\tau_b,\tau_b]_w]_b) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{j=1}^{i-1} a^{[1]}_{ij})^{[1]}[1](\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2
\end{aligned}$$

Differential 30, $F^{(1)}_{(0,1)}F^{(1)}_{(1,0)}F^{(1)}_{(1,0)}F$, $g'f'f'F$, $LN'N'N$

$$\begin{aligned}
r^{\lambda}_{ij}([[[\tau_b]_b]_w]_w) &= \zeta^{\lambda}_i([[\tau_b]_b]_w]) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] \\
\phi^{\lambda}_i([[[\tau_w]_b]_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([[[\tau_b]_b]_w]_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] \\
\zeta^{\lambda}_i([[[\tau_b]_b]_w]_w) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{j=1}^{i-1} a^{[1]}_{ij})^{[1]})[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] \\
\eta_i([[[\tau_b]_b]_w]_b) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{j=1}^{i-1} a^{[1]}_{ij})^{[1]})[1] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]
\end{aligned}$$

Differential 31, $F^{(1)}_{(1,0)}F^{(1)}_{(0,1)}F^{(1)}_{(1,0)}F$, $f'g'f'F$, $N'LN'N$

$$\begin{aligned}
r^{\lambda}_{ij}([[[\tau_b]_w]_b]_w) &= \eta_j([[\tau_b]_w]_b) = (\sum_{k=1}^{j-1} a^{[0]}_{ij} \sum_{k=1}^{j-1} a^{[1]}_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1])^{[1]}[1] \\
\phi^{\lambda}_i([[[\tau_b]_w]_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([[[\tau_b]_w]_b]_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda](\sum_{k=1}^{j-1} a^{[0]}_{ij} \sum_{k=1}^{j-1} a^{[1]}_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1])^{[1]}[1] \\
\zeta^{\lambda}_i([[[\tau_b]_w]_b]_w) &= \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda](\sum_{k=1}^{j-1} a^{[0]}_{ij} \sum_{k=1}^{j-1} a^{[1]}_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1])^{[1]}[1] \\
\eta_i([[[\tau_b]_w]_b]_b) &= \sum_{j=1}^{i-1} a^{[1]}_{ij}[1](\sum_{k=1}^{j-1} a^{[0]}_{ij} \sum_{k=1}^{j-1} a^{[1]}_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1])^{[1]}[1]
\end{aligned}$$

Differential 32, $F^{(1)}_{(1,0)}F^{(1)}_{(1,0)}F^{(1)}_{(0,1)}F$, $f'f'g'F$, $N'N'LN$:

$$\begin{aligned}
r^{\lambda}_{ij}([[[\tau_b]_b]_w]_w) &= \zeta^{\lambda}_i([[\tau_b]_b]_w]) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] \\
\phi^{\lambda}_i([[[\tau_w]_b]_b]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([[[\tau_b]_b]_w]_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] \\
\zeta^{\lambda}_i([[[\tau_b]_b]_w]_w) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{j=1}^{i-1} a^{[1]}_{ij})^{[1]})[\lambda] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] \\
\eta_i([[[\tau_b]_b]_w]_b) &= (\sum_{j=1}^{i-1} a^{[0]}_{ij} \sum_{j=1}^{i-1} a^{[1]}_{ij})^{[1]})[1] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]
\end{aligned}$$

3 Adding shifted initial conditions

Algorithm:

An identity

$$DR = D(I-D)^{-1} = (D-I+I)(I-D)^{-1} = -I + (I-D)^{-1} = -I + R$$

First order:

Algorithm 2 Calculate series coefficient for tree t

$t = [t_1 t_2 \dots t_m]$
for $i = 1$ **to** $ns + 1$ **do**
  $\phi_i^\lambda = 0$
  for $j = 1$ **to** $i - 1$ **do**
    $r_{ij}^\lambda = 1$
    for $s \in [t_1 t_2 \dots t_m]$ **do**
      if $s \in T_w$ **then**
        $r_{ij}^\lambda = r_{ij}^\lambda * \zeta_{i,\lambda}(s)$ Comment: Polynomial multiplication
      else
        $r_{ij}^\lambda = r_{ij}^\lambda * \eta_j(s)$ Comment: Scalar multiplication
      end if
    end for
    for $p = 0$ **to** ρ_i **do**
      $\phi_i^\lambda = \phi_i^\lambda + a_{ijp} \lambda^p * r_{ij}^\lambda$
    end for
  end for
  Integrate $\frac{d}{d\lambda} \zeta_i^\lambda(t)) = \phi_i^\lambda$
  $\zeta_i^0(t) = 0$
  for $j = 0$ **to** $i - 1$ **do**
    $\zeta_i^0(t) = \zeta_i^0(t) + d_{ij} \eta_j(t)$
  end for
  $\eta_i(t) = \zeta_i^1(t)$
end for

Differential 1, F, F , N:

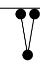
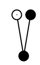
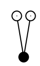
$$
\begin{aligned}
r_{ij}^{\lambda}(\tau_w) &= 1 \\
\phi_i^{\lambda}(\tau_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \\
\zeta_i^{\lambda}(\tau_w) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] + \sum_{j=1}^{i-1} d_{ij}\eta_j(\tau_b) \\
\eta_i(\tau_b) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[1] + \sum_{j=1}^{i-1} d_{ij}\eta_j(\tau_b) \\
\eta_i(\tau_b) &= \sum_{j=1}^{i-1} r_{ij} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \\
\eta_j(\tau_b) &= \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \\
\zeta_i^{\lambda}(\tau_w) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] + \sum_{j=1}^{i-1} d_{ij} \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1]
\end{aligned}
$$

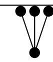
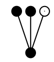
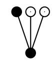
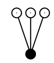
Second order:
Differential 2, $F_{(1,0)}^{(1)}F$, $f'F$, $N'N$:

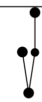
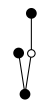
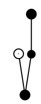
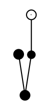
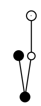
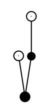
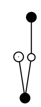
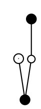
$$
\begin{aligned}
r_{ij}^{\lambda}([\tau_b]_w) &= \eta_j^{\lambda}(\tau_b) = \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \\
\phi_i^{\lambda}([\tau_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] r_{ij}^{\lambda}([\tau_b]_w) \\
&= \sum_{j=1}^{i-1} a_{ij}^{[0]}[\lambda] \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \\
\zeta_i^{\lambda}([\tau_b]_w) &= \sum_{j=1}^{i-1} a_{ij}^{[1]}[\lambda] \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] + \sum_{j=1}^{i-1} d_{ij}\eta_j([\tau_b]_b) \\
\eta_i([\tau_b]_b) &= \sum_{j=1}^{i-1} r_{ij} \sum_{k=1}^{j-1} a_{jk}^{[1]}[1] \sum_{l=1}^{k-1} r_{kl} \sum_{r=1}^{l-1} a_{lr}^{[1]}[1] \\
\eta_j([\tau_b]_b) &= \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a_{kl}^{[1]}[1] \sum_{r=1}^{l-1} r_{lr} \sum_{s=1}^{r-1} a_{rs}^{[1]}[1]
\end{aligned}
$$

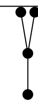
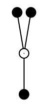
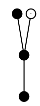
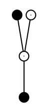
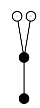
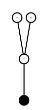
Differential 3, $F^{(1)}_{(0,1)}F$, $g'F$, LN:

$$\begin{aligned}
r^{\lambda}_{ij}([\tau_w]_w) &= \zeta_i(\tau_w) = \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] + \sum_{j=1}^{i-1} d_{ij} \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] \\
\phi^{\lambda}_{i}([\tau_w]_w) &= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda] r^{\lambda}_{ij}([\tau_w]_w) \\
&= \sum_{j=1}^{i-1} a^{[0]}_{ij}[\lambda](\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] + \sum_{j=1}^{i-1} d_{ij} \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]) \\
\zeta^{\lambda}_{i}([\tau_w]_w) &= \frac{1}{2}(\sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda])^2 + \sum_{j=1}^{i-1} a^{[1]}_{ij}[\lambda] \sum_{j=1}^{i-1} d_{ij} \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] + \sum_{j=1}^{i-1} d_{ij}\eta_j([\tau_w]_b)) \\
\eta_i([\tau_w]_b) &= \frac{1}{2}(\sum_{j=1}^{i-1} a^{[1]}_{ij}[1])^2 + \sum_{j=1}^{i-1} a^{[1]}_{ij}[1] \sum_{j=1}^{i-1} d_{ij} \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1] + \sum_{j=1}^{i-1} d_{ij}\eta_j([\tau_w]_b)) \\
\eta_i([\tau_w]_b) &= \sum_{j=1}^{i-1} r_{ij}(\frac{1}{2}(\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2 + \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{k=1}^{j-1} d_{jk} \sum_{l=1}^{k-1} r_{kl} \sum_{r=1}^{l-1} a^{[1]}_{lr}[1]) \\
&= \sum_{j=1}^{i-1} r_{ij}(\frac{1}{2}(\sum_{k=1}^{j-1} a^{[1]}_{jk}[1])^2 - \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] - \sum_{k=1}^{j-1} a^{[1]}_{jk}[1] \sum_{k=1}^{j-1} r_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl}[1]) \\
\eta_j([\tau_w]_b) &= (\sum_{k=1}^{j-1} a^{[0]}_{jk} \sum_{l=1}^{k-1} a^{[1]}_{kl})^{[1]}[1]
\end{aligned}$$

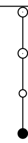
	El. Diff.	El. Diff.	El. Diff.	t	Notation	Notation	$\sigma(t)$	Order Conditions	Order Owren
1	F	F	N		τ_b	$.$	1	$\sum a_{ij}^{[1]}[1]$	
2	$F_{(1,0)}^{(1)}F$	$f'F$	$N'N$		$[\tau_b]_b$	$[.]$	1	$\sum a_{ij}^{[1]}[1]\sum a_{jk}^{[1]}[1]$	
3	$F_{(0,1)}^{(1)}F$	$g'F$	LN		$[\tau_w]_b$	$[o]$	1	$(\sum a_{ij}^{[0]}\sum a_{jk}^{[1]})^{[1]}[1]$	
4	$F_{(2,0)}^{(2)}FF$	$f''FF$	$N''NN$		$[\tau_b,\tau_b]_b$	$[.,.]$	1	$\sum a_{ij}^{[1]}[1](\sum a_{jk}^{[1]}[1])^2$	
5	$F_{(1,1)}^{(2)}FF$				$[\tau_w,\tau_b]_b$	$[o,.]$	2	$((\sum a_{ij}^{[1]})(\sum a_{ij}^{[0]}\sum a_{jk}^{[1]}[1]))^{[1]}[1]$	
6	$F_{(0,2)}^{(2)}FF$	$g''FF$			$[\tau_w,\tau_w]_b$	$[o,o]$	1	$(\sum a_{ij}^{[0]}(\sum a_{ij}^{[1]})^2)^{[1]}[1]$	
7	$F_{(1,0)}^{(1)}F_{(1,0)}^{(1)}F$	$f'f'F$	$N'N'N$		$[[\tau_b]_b]_b$	$[[.]]$	1	$\sum a_{ij}^{[1]}[1]\sum a_{jk}^{[1]}[1]\sum a_{kl}^{[1]}[1]$	
8	$F_{(1,0)}^{(1)}F_{(0,1)}^{(1)}F$	$f'g'F$	$N'LN$		$[[\tau_w]_b]_b$	$[[o]]$	1	$\sum a_{ij}^{[1]}[1](\sum a_{jk}^{[0]}\sum a_{kl}^{[1]})^{[1]}[1]$	
9	$F_{(0,1)}^{(1)}F_{(1,0)}^{(1)}F$	$g'f'F$	$LN'N$		$[[\tau_b]_w]_b$	$[(.)]$	1	$(\sum a_{ij}^{[0]}\sum a_{ij}^{[1]}\sum a_{jk}^{[1]}[1])^{[1]}[1]$	
10	$F_{(0,1)}^{(1)}F_{(0,1)}^{(1)}F$	$g'g'F$	LLN		$[[\tau_w]_w]_b$	$[()]$	1	$(\sum a_{ij}^{[0]}(\sum a_{ij}^{[0]}\sum a_{jk}^{[1]})^{[1]})^{[1]}[1]$	

	El. Diff.	El. Diff.	El. Diff.	t	Notation	Notation	$\sigma(t)$	Order Conditions	Order Owren
11	$F^{(3)}_{(3,0)}FFF$	$f'''FFF$	$N'''NNN$		$[\tau_b, \tau_b, \tau_b]_b$	$[., ., .]$	1	$(\sum a^{[1]}_{ij}[1])^4$	
12	$F^{(3)}_{(2,1)}FFF$				$[\tau_b, \tau_b, \tau_w]_b$	$[., ., o]$	1	$(\sum a^{[0]}_{ij} \sum a^{[1]}_{ij})^{[1]}[1](\sum a^{[1]}_{ij}[1])^2$	
13	$F^{(3)}_{(1,2)}FFF$				$[\tau_b, \tau_w, \tau_w]_b$	$[., o, o]$	1	$(\sum a^{[0]}_{ij}(\sum a^{[1]}_{ij})^2)^{[1]}[1] \sum a^{[1]}_{ij}[1]$	
14	$F^{(3)}_{(0,3)}FFF$	$g'''FFF$			$[\tau_w, \tau_w, \tau_w]_b$	$[o, o, o]$	1	$(\sum a^{[0]}_{ij}(\sum a^{[1]}_{ij})^3)^{[1]}[1]$	

	El. Diff.	El. Diff.	El. Diff.	t	Notation	Notation	$\sigma(t)$	Order Conditions	Order Owren
15	$F^{(2)}_{(2,0)}F^{(1)}_{(1,0)}FF$	$f''f'FF$	$N''N'NN$		$[\tau_b,[\tau_b]_b]_b$	$[.,[.]]$	1	$\sum a^{[1]}_{ij}[1]\sum a^{[1]}_{jk}[1](\sum a^{[1]}_{jk}[1]\sum a^{[1]}_{kl}[1])$	
16	$F^{(2)}_{(1,1)}F^{(1)}_{(1,0)}FF$				$[\tau_b,[\tau_b]_w]_b$	$[.,(.)]$	1	$((\sum a^{[0]}_{ij}\sum a^{[1]}_{jk}[1])(\sum a^{[1]}_{ij}\sum a^{[1]}_{jk}[1]))^{[1]}[1]$	
17	$F^{(2)}_{(1,1)}F^{(1)}_{(1,0)}FF$				$[\tau_w,[\tau_b]_b]_b$	$[o,[.]]$	1		
18	$F^{(2)}_{(2,0)}F^{(1)}_{(0,1)}FF$	$f''g'FF$	$N''LNN$		$[\tau_b,[\tau_w]_b]_b$	$[.,[o]]$	1		
19	$F^{(2)}_{(1,1)}F^{(1)}_{(0,1)}FF$				$[\tau_b,[\tau_w]_w]_b$	$[.,(o)]$	1		
20	$F^{(2)}_{(1,1)}F^{(1)}_{(1,0)}FF$				$[\tau_w,[\tau_b]_b]_b$	$[o,[o]]$	1		
21	$F^{(2)}_{(0,2)}F^{(1)}_{(1,0)}FF$	$g''f'FF$			$[\tau_w,[\tau_b]_w]_b$	$[o,(.)]$	1		
22	$F^{(2)}_{(0,2)}F^{(1)}_{(0,1)}FF$	$g''g'FF$			$[\tau_w,[\tau_w]_w]_b$	$[o,(o)]$	1		

	El. Diff.	El. Diff.	El. Diff.	t	Notation	Notation	$\sigma(t)$
23	$F^{(1)}_{(1,0)}F^{(2)}_{(2,0)}FF$	$f'f''FF$	$N'N''NN$		$[[\tau_b,\tau_b]_b]_b$	$[[.,.]]$	1
24	$F^{(1)}_{(0,1)}F^{(2)}_{(2,0)}FF$	$g'f''FF$	$LN''NN$		$[[\tau_b,\tau_b]_w]_b$	$[(.,.)]$	1
25	$F^{(1)}_{(1,0)}F^{(2)}_{(1,1)}FF$				$[[\tau_b,\tau_b]_b]_b$	$[[.,.]]$	1
26	$F^{(1)}_{(1,0)}F^{(2)}_{(2,0)}FF$				$[[\tau_b,\tau_b]_b]_b$	$[[.,.]]$	1
27	$F^{(1)}_{(1,0)}F^{(2)}_{(0,2)}FF$	$f'g''FF$			$[[\tau_w,\tau_w]_b]_b$	$[[o,o]]$	1
28	$F^{(1)}_{(1,0)}F^{(2)}_{(2,0)}FF$	$g'g''FF$			$[[\tau_w,\tau_w]_w]_b$	$[(o,o)]$	1

	El. Diff.	El. Diff.	El. Diff.	t	Notation	Notation	$\sigma(t)$
29	$F^{(1)}_{(1,0)}F^{(1)}_{(1,0)}F^{(1)}_{(1,0)}F$	$f'f'f'F$	$N'N'N'N$		$[[[\tau_b]_b]_b]_b$	$[[[.]]]$	1
30	$F^{(1)}_{(0,1)}F^{(1)}_{(1,0)}F^{(1)}_{(1,0)}F$	$g'f'f'F$	$LN'N'N$		$[[[\tau_w]_b]_b]_b$	$[([.])]$	1
31	$F^{(1)}_{(1,0)}F^{(1)}_{(0,1)}F^{(1)}_{(1,0)}F$	$f'g'f'F$	$N'LN'N$		$[[[\tau_b]_w]_b]_b$	$[[(.)]]$	1
32	$F^{(1)}_{(1,0)}F^{(1)}_{(1,0)}F^{(1)}_{(0,1)}F$	$f'f'g'F$	$N'N'LN$		$[[[\tau_b]_b]_w]_b$	$[[[o]]]$	1

	El. Diff.	El. Diff.	El. Diff.	t	Notation	Notation	$\sigma(t)$
33	$F^{(1)}_{(0,1)}F^{(1)}_{(0,1)}F^{(1)}_{(0,1)}F$	$g'g'g'F$	$LLLN$		$[[[\tau_w]_w]_w]_b$	$[((o))]$	1
34	$F^{(1)}_{(0,1)}F^{(1)}_{(0,1)}F^{(1)}_{(1,0)}F$	$f'g'g'F$	$N'LLN$		$[[[\tau_w]_w]_b]_b$	$[[(o)]]$	1
35	$F^{(1)}_{(1,0)}F^{(1)}_{(0,1)}F^{(1)}_{(1,0)}F$	$g'f'g'F$	$LN'LN$		$[[[\tau_w]_b]_w]_b$	$[([o])]$	1
36	$F^{(1)}_{0,1)}F^{(1)}_{(0,1)}F^{(1)}_{(0,1)}F$	$g'g'f'F$	$LLN'N$		$[[[\tau_b]_w]_w]_b$	$[((.))]$	1