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This is an abstract Complete this summary at the end of the paper

I. INTRODUCTION

In this text we are going to explore our very own star and compare it to another known star named the Sun. This will teach you how one can extract valuable information about a star's physical properties, its birth, life and death. Lets begin by looking at our star in its current state

II. OUR STAR: ITS CURRENT STATE AND ORIGIN

A. The main sequence

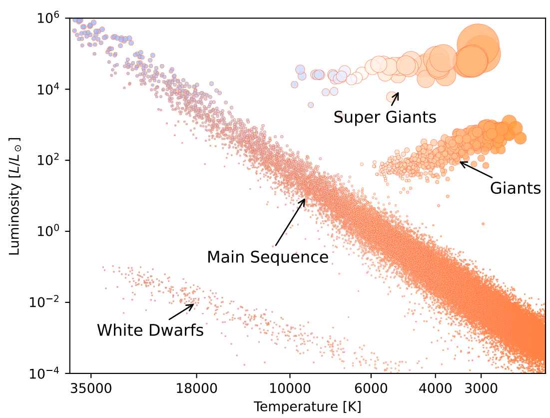

Figure 1. HR-Diagram

In Figure 1 you can see what is called and HR-diagram. This is a way to portray the behavior of most stars in their life cycle. The x-axis is the surface temperature of the star, and the y-axis is the luminosity as fraction of the Sun's luminosity. In Figure 2 you see the Sun's placement in the diagram in its true size relative to other stars. The diagonal strip it occupies is known as the "The Main Sequence" and is where most stars spends most of their life.

The Placement of Our Star

To find our star's position we need both its temperature and luminosity. From earlier measurement we know our star has a surface temperature T of 11733 Kelvin.

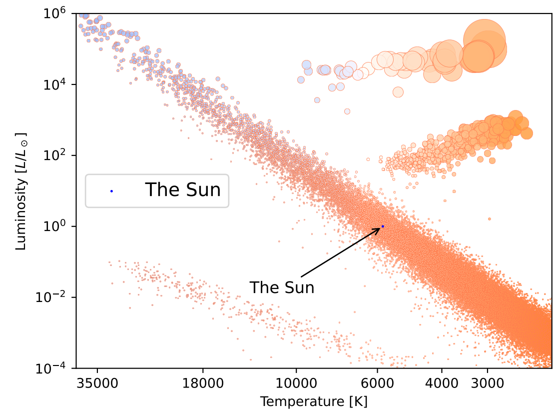

Figure 2. The Sun in the HR-Diagram

Luminosity is defined as energy per unit of time.

$$L = \frac{E}{dt} \tag{1}$$

Flux is defined as energy per unit of area per unit of time.

$$F = \frac{E}{dAdt} \tag{2}$$

We can use this to find a new way to express the Luminosity.

$$L = FdA \tag{3}$$

Stefan-Boltzmann's law Referer til 3d s, 3 dictates that the flux F can be expressed as

$$F = \sigma T^4. \tag{4}$$

Where T is the temperature in Kelvin and σ is Stefan-Boltzmann's constant. We also know the total area of a sphere to be $4\pi r^2$ where r is the radius of the sphere. Combining this and equations 3 & 4 we get the following expression for the luminosity.

$$L = \sigma T^4 4\pi r^2 \tag{5}$$

Using our measured surface temperature of our star we get a luminosity $L = 5.79 \cdot 10^{28} W$ or $1.51 \cdot 10^2 L_\odot$. The use of the " $\odot$ " symbol is a way to express a quantity as a fraction of said quantity in relation to the Sun. An example would be $M_\odot$ and $L_\odot$ being the mass and luminosity of the sun respectively. This kind of notation is useful as it creates an easy baseline of comparison of different stars. Now we have the coordinates needed to plot our star in the HR diagram. It seems our star is a

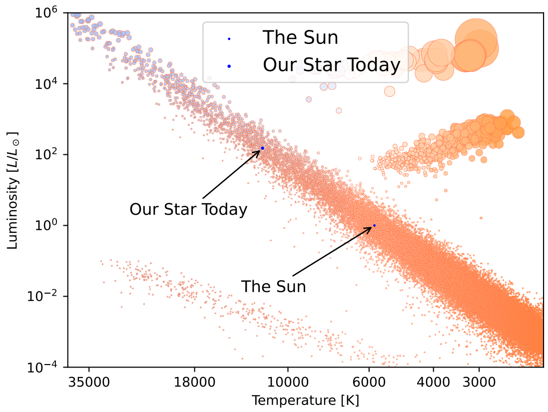

Figure 3. Our Star and the Sun in the HR-Diagram. Both in true size relative to the other stars

main sequence star! This was very likely as this is where most stars spends most their time. Speaking of which, lets see how we can determine the lifespan of our star.

The Lifespan of Our Star

In all stars their is a struggle between two forces. The first being the gravitational force trying to crush all the atoms in the star in on itself. The other being the radiation pressure generated from the fusion of mainly hydrogen inside the star. As more and more hydrogen is turned into helium the mean molecular weight μ of the star increases and as such, its mass increases as well. When the mass increases the gravitational force increases and the star will shrink. The shrinkage will create higher pressure and temperature which will make the fusion more effective, increasing the radiation pressure. The two forces will balance each other until the star has a relatively stable radius until there isn't enough hydrogen to stop the gravitational forces, but we will cover this later. When these two forces are equal, the star has reached hydrostatic equilibrium.

This tells us there is a relationship between the mass of the star and its lifetime t. The equation expressing this relationship (derived in referer til 3d s, 7) is as follows:

$$t = \frac{pMc^2}{L} \tag{6}$$

Where p is the fraction of the total mass being converted into energy, M is the mass of the star, L being the luminosity and c being the speed of light. We will assume this to be 10% as this is true for the Sun. Using equation 6 we get a lifetime of 3.04e+08 years. This is approximately 30 times as long as the Sun!

Relationships between mass, temperature and luminosity

As one can see from the HR-diagram 1, there is a clear relationship between the temperature and luminosity of a star as they sit on the main sequence. We therefore expect to see "well behaved" main sequence stars to obey certain ratios. Derived in Refferer til forelesningsnotat 3b eq 4 we see the two following equations

$$M \propto T_{eff}^2 \tag{7}$$

where the mass of a star is proportional to the surface temperature.

$$L \propto M^4 \tag{8}$$

where the luminosity is proportional with the mass. To see if our star is well behaved we will calculate both ratios and compare them with the Sun.

	M-L Factor	T-M Factor
The Sun	$4.1 \cdot 10^{94}$	$5.9 \cdot 10^{22}$
Our Star	$1.0 \cdot 10^{95}$	$6.4 \cdot 10^{22}$

The T-M factor of both stars are quite close and have the same order of magnitude. With a mass M of about $4.4 M_\odot$ its not surprising to see our M-L factor being a bit higher. Although barely one order of magnitude higher. One could say our star is quite well behaved.

B. Giant Molecular Cloud

Star do not begin their existence as bright, hot spheres creating fusion reaction at a scale we could only dream of on earth. Instead, they start their life as cold Giant Molecular Clouds (GMC) of different gases. For a star to form there need to be a strong enough gravitational pull to overpower the gas pressure. As a consequence, there is a maximum radius this cloud can have for it to collapse in on itself. We cannot go back in time to see what our star looked like before it was born, but we can make an educated guess! For this we will assume our GMC has:

- The same mass as our star today
- Collapsed in on itself without external forced like a supernova
- Is spherically symmetrical
- Has a temperature of 10 K
- Is made up of 75% Hydrogen and 25% Helium

As derived in Referer til part 3b Jeans length, there is a minimum radius requirement radius R_j for a GMC to collapse in on itself.

$$R_j = \left(\frac{15kT}{4\pi G\mu m_h \rho}\right)^{\frac{1}{2}} \tag{9}$$

Where T is the temperature of the cloud, k is Boltzmann's constant, G the gravitational constant and m_h the mass of a hydrogen atom. μ is the mean molecular weight of the cloud which in our case is $\mu = 1.74$. We replace the density ρ with the mass M divided by the volume $V = \frac{4}{3}\pi r^3$. This gives us the new expression we will use to calculate the smallest possible radius our GMC could have had.

$$R = \frac{GM\mu m_h}{5kT} \tag{10}$$

Using the mass of our star we get a Jean's Length R_j of 2.5e+15 meters. To see how big this is in comparison with other stars lets place it into the HR-Diagram! Before we can do this we need to find its luminosity. We already have its radius and temperature so we will use equation 5 again.

$$L_{\mathrm{GMC}} = 1.1 \cdot 10^2 \ L_\odot \tag{11}$$

Now we are finally able to put it in our HR-Diagram! The GMC is at the size of giant stars.

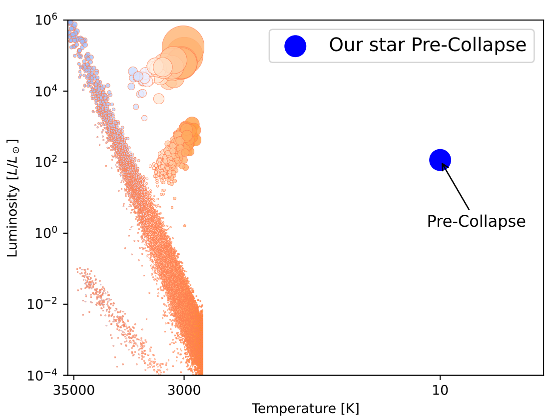

Figure 4. Our star as a GMC plotted true to size

III. NUCLEAR REACTIONS IN OUR STARS CORE

Now we will explore our star internally and take a look at the processes happening on the inside. This will include core temperature, energy production and luminosity. To simplify the problem we will invoke a few assumptions:

- We assume the density of our star is uniform
- We will approximate the pressure inside as an ideal gas, not taking other forces such as radiation pressure into account
- We will assume hydrostatic equilibrium
- We will assume our star consist entirely of protons

A. Core Temparature

To calculate the core temperature we first need an expression of the mass profile of our star. As when we derived equation 10 we can express the mass of our star as the following.

$$M(r) = \rho_0 \frac{4}{3}\pi r^3 \tag{12}$$

As we assume the star to be made up of an ideal gas, the pressure P will obey the ideal gas law.

$$P = \frac{\rho_0 k T(r)}{\mu m_h} \tag{13}$$

Since our star is on the main sequence we know it has reach hydrostatic equilibrium. Which let's us describe the change in pressure $\frac{dP}{dr}$ as a function of radius.

$$\frac{dP}{dr} = -\rho_0 G \frac{M(r)}{r^2} \tag{14}$$

Inserting our equation for the mass profile 12 into our equation for an ideal gas 14, and comparing it with the derivative of our equation for hydrostatic equilibrium we get the following.

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{\rho_0 k T(r)}{\mu m_h} = -\rho_0^2 G \frac{4}{3}\pi r \tag{15}$$

We solve this for $T(r)$.

$$\frac{\mathrm{d}}{\mathrm{d}r}T(r) = -\frac{4\pi G \rho_0 \mu m_h r}{3k} \tag{16}$$

As the core temperature $T_c = T(0)$ and we already know the surface temperature $T(R)$ we want to integrate our equation 16 from the core to the surface.

$$\int_0^R \frac{\mathrm{d}}{\mathrm{d}r}T(r)\mathrm{d}r = T(R) - T(0) = -\frac{4\pi G\rho_0 \mu m_h R}{3k} \tag{17}$$

Replacing $T(0)$ with T_c, $T(R)$ with T_{surface} and solving for this we get the following.

$$T_c = T_{\mathrm{surface}} + \frac{4\pi G \rho_0 \mu m_h R}{3k} \tag{18}$$

As we assumed our star is made op entirely of protons our mean molecular weight $\mu = 1$. Using our final equation 18 we get a core temperature T_c of 17 million Kelvin

$$T_c = 1.72 \cdot 10^7 K \tag{19}$$

B. Energy Production and Luminosity

We are now going to re-calculate the luminosity of our star based on the nuclear reactions happening in its core. To model these reactions we are going to invoke a few assumptions.

- All nuclear reactions take place within a sphere of 0.2R
- Uniform density
- The core temperature temperature is the same throughout the whole sphere
- Assume energy production occurs via the pp-chain and CNO-cycle
- The core consist of 76.5% Hydrogen, 25.3% Helium and 0.2% Carbon, Oxygen and Nitrogen

As our star has a core temperature T_c of about 17 million Kelvin the most dominant chain reactions in the core is the pp-chain and the CNO-cycle. The basic equation for calculating energy released per kg per second we get from Referer til 3c s, 4.

$$\varepsilon_{\text{AB}} = \varepsilon_{0,\text{ reac}} X_S X_B \rho^\alpha T^\beta \tag{20}$$

In this equation, ρ is the density, α and β are indices which depend on the temperature T and X_A and X_B is the mass fraction of the two nuclei in the reaction which follow the following definition.

$$X_A = \frac{n_A m_A}{nm} = \frac{\text{total mass in type A nuclei}}{\text{total mass}}$$

The pp-chain is actually multiple chain reactions. We will focus on the the most important one, namely the pp-I chain. This chain reaction converts four ${}^1_1$H to ${}^4_2$He and is the most efficient at 15 million Kelvin. We will now introduce a new way of writing temperature. With this notation $T_n = T \cdot 10^n K$. This is used in the equation for the pp-chain as follows, derived in Referer til 3c, s, 6.

$$\varepsilon_{\text{pp}} \approx \varepsilon_{0,\text{ pp}} X_H^2 \rho T_6^4 \tag{21}$$

In this equation, $\varepsilon_{0,\text{ pp}} = 1.08 \cdot 10^{-12} Wm^3/kg^2$, the efficiency of the pp-chain is 0.007 meaning only 0.7% of the mass in each reaction is converted into evergy, ρ the density of our star, $X_H = 0.745$ and $T = T_c$ being our star's core temperature calculated in equation 19. Using all this we get a result of

$$\varepsilon_{\text{pp}} \approx 1.25 \cdot 10^{-5} \tag{22}$$

The CNO-cycle also converts four ${}^1_1$H to ${}^4_2$He. We will use the expression derived in Referer til 3c s, 6 which is valid for temperature around 20 million Kelvin.

$$\varepsilon_{\text{CNO}} = \varepsilon_{0,\text{ CNO}} X_H X_{\text{CNO}} \rho T_6^{20} \tag{23}$$

In this equation, $\varepsilon_{\text{CNO}} = 8.24 \cdot 10^{-31} Wm^3/kg^2$ and $X_{\text{CNO}} = 0.002$. Using all this we get a result of

$$\varepsilon_{\text{CNO}} = 1.58 \cdot 10^{-6} \tag{24}$$

It is worth noticing how much more effective this reaction becomes at higher temperatures as the temperature is to the power of 20! Now that we have calculated both reactions we also notice the fact that the pp-chain is an order of magnitude greater than the CNO-cycle.

We already know the luminosity is measured in energy per second and the energy released form the nuclear reactions is per second per kg. This gives the following relation.

$$\frac{\mathrm{d}}{\mathrm{d}m} L = \varepsilon_{\text{total}} \tag{25}$$

Where $\varepsilon_{\text{total}} = \varepsilon_{\text{pp}} + \varepsilon_{\text{CNO}}$ is the sum of all the reactions. We also know the mass of this sphere of gas is given by

$$m = \rho V(r) = \rho 4\pi r^2$$

meaning

$$\mathrm{d}m = \rho 4\pi r^2 \mathrm{d}r.$$

This gives us a new expression of luminosity

$$\frac{\mathrm{d}}{\mathrm{d}r} L = \rho 4\pi r^2 \varepsilon_{\text{total}} \tag{26}$$

As we assumed the reactions is only happening inside a radius of 0.2 R we can integrate the previous expression.

$$L = \int_0^{0.2R} \rho 4\pi r^2 \varepsilon_{\text{total}} \, \mathrm{d}r \tag{27}$$

Doing the integral in the previous equation 27 we get the following

$$L = \rho \frac{4}{3} \varepsilon_{\text{total}} 0.2R^3 = 2.5 \cdot 10^{25} \tag{28}$$

Our attempt at calculating the luminosity using chain reactions in the core of the star fell quite short. Our new luminosity is 3 orders of magnitude smaller than the one calculated in equation 5. This could be for many reasons. Its not likely the density being equal inside the entire star, nor the fact that the reactions only occurs in a radius of 0.2R. Both reactions, but especially the CNO-cycle, are highly temperature sensitive. As mentioned earlier, both equations 21 & 23 have their preferred temperature range where both are the most accurate. Our core temperature T_c of 17.2 million Kelvin falls almost dead center between the 15 million Kelvin preferred by the pp-chain and the 20 million preferred by the CNO-cycle. This could be the reason for our failed calculations.

IV. THE DEATH OF OUR STAR

As our star runs out of fuel for its nuclear reactions its inevitable death will come to pass. This will happen in a few different stages we will elaborate on below.

A. Leaving the Main Sequence

Emptying Hydrogen stores in the core

As the hydrogen in the core (the main fuel of our star), is all gone there is too little pressure fighting against gravity and our star. This means we have lost our hydrostatic equilibrium and the core will start to shrink. This shrinkage causes the temperature in and around the core to increase. This is not enough for the fusion of heavier atoms, but the extra heat lets the star "burn" the hydrogen outside the core. This has now the opposite effect where the burning of hydrogen outside the core increases the pressure so much the radius of the star increases a lot. Now our star will become a sub giant as seen in figure 6. The star now has a lot lower surface temperature as a result of it larger size and has therefore moved to the right in the HR-Diagram.

Sub Giant

As a sub giant, our star's radius is so large the temperature reaches a lower limit of 2500 K. The main mechanism of energy transport in the star goes from being radiation to convection. This is a more efficient process meaning its luminosity increases and the star moves up in the HR-Diagram to the "giant" stage group as seen in figure 1. Now its radius lies between $10 \rightarrow 100\ R_\odot$. As a red giant the convection carries material from the core to the surface. This means we can actually observe what materials are in the core by looking at the surface!

Luminosity classification

Luminosity classes are a way to differentiate differentiate how bright a star is. This scale has 6 classes numbered with Roman numerals from I (brightest) to VI (least bright). When on the main sequence stars have a luminosity class V. As our star became a sub giant it received a higher luminosity and a higher class of IV. Now that our star is a giant it has a luminosity class of III.

Electron degeneracy

As our star has a mass $M > 2M_\odot$ it's core will eventually become hot enough to convert helium to carbon and helium to oxygen known as the triple-alpha process. As the core is contracting we discover a new quantum mechanical effect. There us no space left in the core to store electrons at a higher density. There is an upper limit for number of electrons within a certain volume with a certain momentum. This is known as electron degeneracy. Now the star has a new force acting outwards known as the degeneration pressure. This pressure is independent of temperature which means the core does not expand even as the temperature increases. When the core finally reaches temperatures hot enough to burn helium it burns a large portion of it in the core almost simultaneously. This creates a helium flash which manifest itself as an enormous release of energy in an explosion. The luminosity quickly jumps up to the level of bright giants as seen in Figure 6. This explosion makes the core non electron degenerate and the gases acts normally again.

Giant

To regain hydrostatic equilibrium our star's core will begin to expand a new. As it expands there will be parts of the core previously burning hydrogen not being able to as the temperature has become too low. The luminosity decreases and the star enters the *horizontal branch* as seen in Figure 5. After quickly expanding from the burning of helium the star begins to contract again to reach hydrostatic equilibrium. As a consequence the surface temperature increases and it moves to horizontally on the HR-Diagram. There is little to no hydrogen left to burn and the core is mainly made up of carbon and oxygen. As the contraction of the core continues a shell around it becomes hot enough to burn helium. This again introduces new forces pushing outwards meaning the radius of the star increases again. Convection takes over again as the main method of energy transportation and the luminosity reaches class II or in some cases class I. Our star will now have reached the level of bright giant or super giant with a radius of up to 1000 times its original radius.

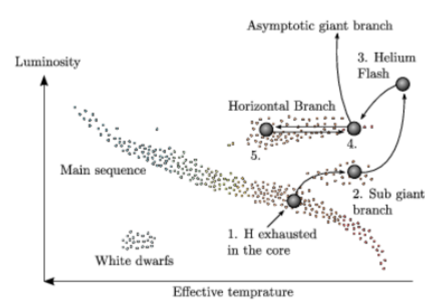

Figure 5. HR-Diagram showing evolution of a star moving from main sequence to the horizontal branch

B. The End

As our star experiences expansion and contraction multiple times as it oscillates between helium flashes it will lose more and more of its helium. In the end, the only thing left is an electron degenerate core of carbon and oxygen. It has now entered its final stage of life. It will live and die as a white dwarf becoming colder and colder through radiation as there is no more energy production. While its temperature is still relatively high, its luminosity is now very low and it will live the rest of its life in the bottom left corner of the HR-Diagram as seen in Figure 1.

Mass

Our star has a mass $M < 8M_\odot$. We will therefore assume its mass as a white dwarf M_{WD} will obey the following equation.

$$M_{\text{WD}} = \frac{M}{8M_\odot} M_{\text{Chandresekhar}} \tag{29}$$

where $M_{\text{Chandresekhar}} = 1.4M_\odot$. Using our current mass we get a mass M_{WD} of

$$M_{\text{WD}} = 1.55 \cdot 10_{30} \text{ kg} = 0.78M_\odot \tag{30}$$

Radius

We want to find an expression of how large our star will become as a white dwarf. To do this we are going to assume the following.

- Hydrostatic equilibrium
- Uniform density

We can use the equation for hydrostatic equilibrium as derived in referer til 3e eq, 1.

$$\frac{P}{R} \approx \frac{GM}{R^2}\frac{4M}{3\pi R^3} = \frac{3GM^2}{4\pi R^5} \tag{31}$$

Replacing the pressure P with the degeneration pressure we get the equation from referer til 3e eq, 2.

$$\left(\frac{3}{\pi}\right)^{\frac{2}{3}} \frac{h^2}{20m_e} n_e^{\frac{5}{3}} = \frac{3GM^2}{4\pi R^5} \tag{32}$$

where n_e is the electron number density which we can be rewritten using the total gas density as.

$$n_e = n_p = \frac{\rho_{\text{p}}}{m_H} = \frac{Z}{A}\frac{\rho}{m_H}$$

where Z is the number of protons per nucleus and A is the number of nucleons per nucleus. We replace the density ρ with $\frac{M}{V}$. As we assume the gas to be neutral there should be one neutron for every proton meaning $\frac{Z}{A} = \frac{1}{2}$. Inserting this to equation 31 and solving for R we get an expression of the radius of our star as a white dwarf R_{WD}.

$$R_{\text{WD}} \approx \left(\frac{3}{2\pi}\right)^{\frac{3}{4}} \frac{h^2}{20m_eG} \left(\frac{1}{2m_H}\right)^{\frac{5}{3}} M_{WD}^{-\frac{1}{3}} \tag{33}$$

where m_e is the mass of an electron, m_h the mass of a proton and M_{WD} is the mass of our white dwarf. Using the mass we calculated in equation 30 we get a radius R_{WD}

$$R_{\text{WD}} = 1.55 \cdot 10^6 \; m \tag{34}$$

which is at the same order of magnitude as the planet Terra.

Fun Facts

We know that 1L of water weighs around 1kg. Lets try and see how much 1L of our white dwarf weighs! As we know its weigh and radius we can find its density $\rho_D = \frac{M}{V} = 9.82 \cdot 10^{10} \frac{\text{kg}}{\text{m}^3}$. As one litre is 1dm^3 we just divide by 1000 and get at mass of 9.82 $\cdot 10^7$kg. Now that is heavy!

Ever wondered what it would be like to walk on a neutron star? Of course you have, but lets see if it is even possible. Using Newtons universal law of gravitation we get the following expression for the acceleration you would feel on its surface.

$$a_D = G\frac{M_{\text{WD}}}{R^2} = 4.27 \cdot 10^7 \; \frac{\text{m}}{\text{s}^2} \tag{35}$$

It does not seem likely there will be any walking on the neutron star after all. That is a lot of force pulling oneself down to the ground, but now you know!

V. APPENDIX

A. Detailed HR-Diagram

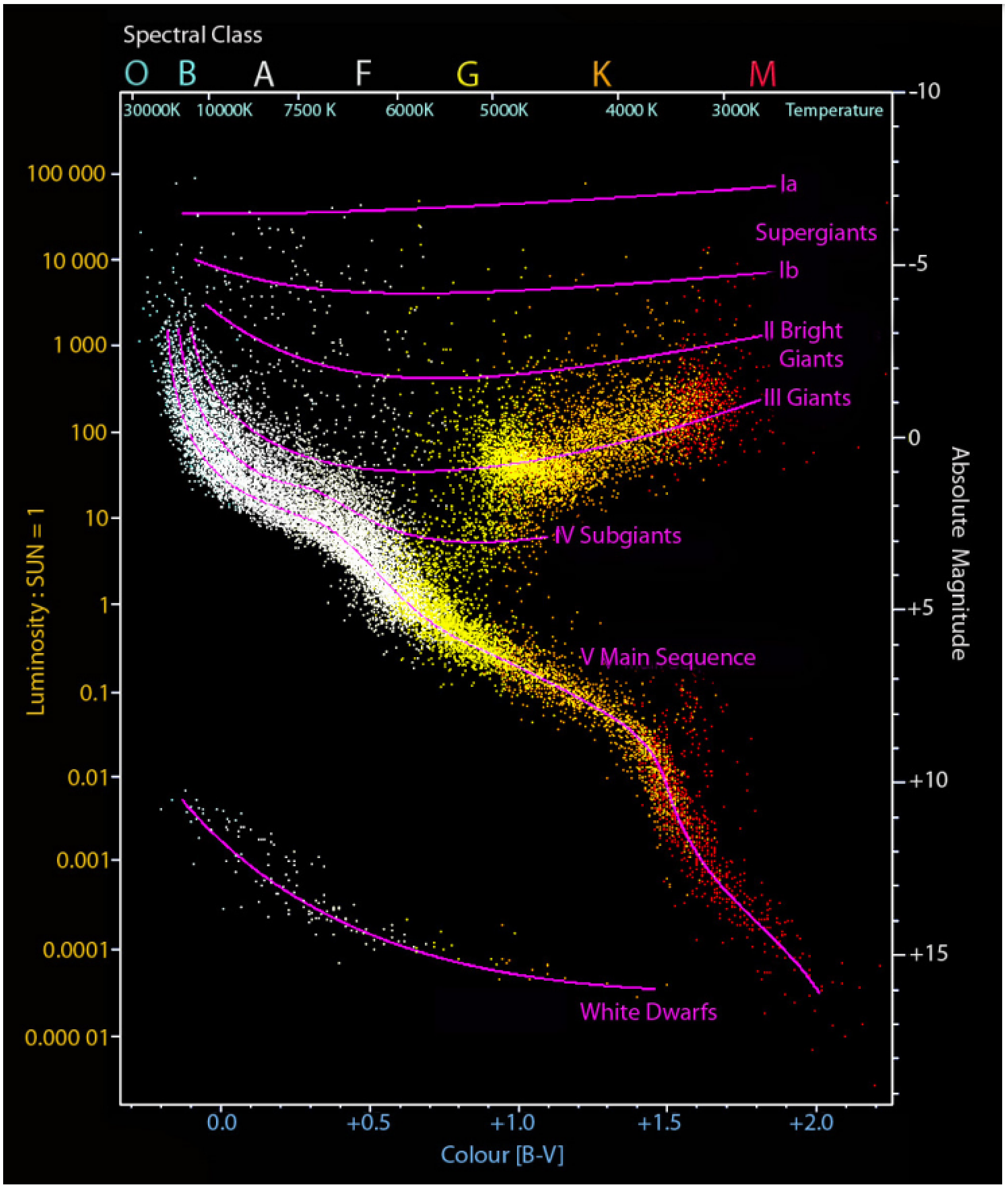

Figure 6. HR-Diagram showing different categorizations of stars Referer til 3d s, 2

ACKNOWLEDGMENTS

REFERENCES