Exercise 1

4a) To find the mass of the sun in meters, we will be using the conversion factor G/c^2
Where G is the gravitational constant and c is the speed of light.

$$M_{sun\_kg} = 1.989 \cdot 10^{30}\ kg$$

$$M_{sun\_m} = M_{sun\_kg} \cdot \frac{G}{c^2}$$

$$M_{sun\_m} = 1.989 \cdot 10^{30} \cdot 7{,}42 \cdot 10^{-28}$$

$$M_{sun\_m} = 1475{,}838\ m$$

$$r_{sun} = 695'508\ km$$

$$\frac{M_{sun\_m}}{r_{sun}} = \frac{1475{,}838}{695'508'000}$$

$$\frac{M_{sun\_m}}{r_{sun}} = 2{,}122 \cdot 10^{-6}$$

b) We can use the approximation found in question 3.

$$\frac{\Delta\lambda}{\lambda_{shell}} = \frac{M_{sun\_m}}{r_{sun}} = 2{,}122 \cdot 10^{-6}$$

$$\Delta\lambda = \frac{M_{sun\_m}}{r_{sun}} \cdot \lambda_{shell} = 1{,}061 \cdot 10^{-3}\ nm$$

The color of the sun will not change at all. The light is only redshifted by approximately a thousandth of a nanometer, which does not affect the colour at all.

c) $M_{Earth} = M_{Earth\_kg} \cdot \frac{G}{c^2}$

$$M_{Earth} = 4{,}431 \cdot 10^{-3}$$

$$\frac{M_{Earth}}{r_{Earth}} = 6{,}955 \cdot 10^{-10}$$

d) $\frac{\Delta\lambda}{\lambda_{shell}} = \frac{M_{Earth}}{r_{Earth}}$

$$\frac{\Delta\lambda}{\lambda_{shell}} = 6{,}955 \cdot 10^{-10}$$

$$\Delta\lambda = 6{,}955 \cdot 10^{-10} \cdot 500$$

$$\Delta\lambda = 3{,}4775 \cdot 10^{-7}\ nm$$

The gravitational blueshift of the earth only decreases the wavelength by a fraction of a nanometer.
It will therefore not change the perceived color of the light at all.

5) $\Delta\lambda = 2150 - 600 \text{ nm}$

$\Delta\lambda = 1550 \text{ nm}$

$\lambda_{shell} = 600 \text{ nm}$

$$\frac{\Delta\lambda}{\lambda_{shell}} = \frac{1}{\sqrt{1-\frac{2M}{r}}} - 1$$

$$\sqrt{1-\frac{2M}{r}} = \frac{1}{\frac{\Delta\lambda}{\lambda_{shell}} - 1}$$

$$\frac{2M}{r} = 1 - \frac{1}{\left(\frac{\Delta\lambda}{\lambda_{shell}} - 1\right)^2}$$

$$r = \frac{2M}{1 - \frac{1}{\left(\frac{\Delta\lambda}{\lambda_{shell}} - 1\right)^2}}$$

$$r = \frac{2M}{1 - \frac{1}{\left(\frac{1550}{600} - 1\right)^2}}$$

$$\underline{\underline{r = 3,327 M}}$$

6) $$\frac{\Delta\lambda}{\lambda_{shell}} = \frac{1}{\sqrt{1-\frac{2M}{2,01M}}} - 1$$

$$\frac{\lambda - \lambda_{shell}}{\lambda_{shell}} = 13,177$$

$$\lambda_{shell} = \frac{\lambda}{12,177}$$

As the blueshift is extremely strong, we will most likely not be able to observe any stars with the naked eye. Almost all of the light will be blueshifted into the UV spectrum.

Exercise 2

1) As we are looking at such small intervals, we can use Lorentz-transformation and therefore the Schwarzschild line element.
We divide up in two sections. The first one from point 1 to point 2 and the second one from point 2 to point 3.
As we use such small intervals, the distance r is constant for each interval and we can approximate

$$\Delta S_{12} = \Delta \tau_{12}$$
$$\Delta S_{23} = \Delta \tau_{23}$$

$$\Delta S^2 = \left(1-\frac{2M}{r}\right)\Delta t^2 - \frac{\Delta r^2}{\left(1-\frac{2M}{r}\right)} - r^2 \Delta\phi^2$$

$$\Delta \tau_{12} = \sqrt{\left(1-\frac{2M}{r_A}\right)\Delta t_{12}^2 - \frac{\Delta r_{12}}{1-\frac{2M}{r_A}} - r_A^2 \Delta\phi_{12}^2}$$

$$\Delta \tau_{23} = \sqrt{\left(1-\frac{2M}{r_B}\right)\Delta t_{23}^2 - \frac{\Delta r_{23}}{1-\frac{2M}{r_B}} - r_B^2 \Delta\phi_{23}^2}$$

To find the total proper time interval from point 1 to 3, we can sum these intervals together

$$\Delta \tau_{13} = \Delta \tau_{12} + \Delta \tau_{23}$$

$$\Delta \tau_{13} = \sqrt{\left(1-\frac{2M}{r_A}\right)\Delta t_{12}^2 - \frac{\Delta r_{12}}{1-\frac{2M}{r_A}} - r_A^2 \Delta\phi_{12}^2} + \sqrt{\left(1-\frac{2M}{r_B}\right)\Delta t_{23}^2 - \frac{\Delta r_{23}}{1-\frac{2M}{r_B}} - r_B^2 \Delta\phi_{23}^2}$$

2) We use the principle of maximum aging and try to maximize the proper time with respect to t2

$$\frac{\partial}{\partial \phi_2}\Delta \tau_{13} = \frac{\partial}{\partial \phi_2}\sqrt{\underbrace{\left(1-\frac{2M}{r_A}\right)\Delta t_{12}^2 - \frac{\Delta r_{12}}{1-\frac{2M}{r_A}}}_{K_1} - r_A^2 \Delta\phi_{12}^2} + \frac{\partial}{\partial \phi_2}\sqrt{\underbrace{\left(1-\frac{2M}{r_B}\right)\Delta t_{23}^2 - \frac{\Delta r_{23}}{1-\frac{2M}{r_B}}}_{K_2} - r_B^2 \Delta\phi_{23}^2}$$

$$= \frac{\partial}{\partial \phi_2}\sqrt{K_1 - r_A^2 \Delta\phi_{12}^2} + \frac{\partial}{\partial \phi_2}\sqrt{K_2 + r_B \Delta\phi_{23}^2}$$

$$= \frac{1}{\sqrt{K_1 - r_A^2(\phi_2-\phi_1)^2}} \cdot \left(-2r_A^2(\phi_2-\phi_1)^2\right)\cdot(1) + \frac{1}{\sqrt{K_2 - r_B^2(\phi_3-\phi_2)^2}} \cdot \left(-2r_B^2(\phi_3-\phi_2)^2\right)\cdot(-1)$$

$$= -\frac{2r_A^2(\phi_2-\phi_1)^2}{\sqrt{K_1 - r_A^2(\phi_2-\phi_1)^2}} + \frac{2r_B^2(\phi_3-\phi_2)^2}{\sqrt{K_2 - r_B^2(\phi_3-\phi_2)^2}}$$

Setting equal to 0

$$\frac{r_A^2 \Delta\phi_{12}^2}{\sqrt{K_1 - r_A^2\Delta\phi_{12}^2}} = \frac{r_B^2 \Delta\phi_{23}^2}{\sqrt{K_2 - r_B^2\Delta\phi_{23}^2}}$$

$$\frac{r_A^2 \Delta\phi_{12}^2}{\Delta \tau_{12}} = \frac{r_B^2 \Delta\phi_{23}^2}{\Delta \tau_{23}}$$

$$r_A^2 \frac{d\phi_{12}}{d\tau_{12}} = r_B^2 \frac{d\phi_{23}}{d\tau_{23}}$$

This shows that the quantity remains the same even for changing phi and tau.
We can therefore conclude that the quantity is conserved.

3) $\gamma_{shell} = \frac{\partial t_{shell}}{\partial \tau}$

we know that $\frac{\partial \phi}{\partial t_{shell}} = \frac{v_{\phi,shell}}{r}$

$$r^2 \frac{\partial \phi}{\partial \tau} = r^2 \frac{\partial \phi}{\partial \tau} \frac{\partial t_{shell}}{\partial t_{shell}}$$
$$= r^2 \frac{\partial \phi}{\partial t_{shell}} \frac{\partial t_{shell}}{\partial \tau}$$
$$= r^2 \frac{v_{\phi,shell}}{r} \gamma_{shell}$$
$$= \underline{\underline{\gamma_{shell}\, r\, v_{\phi,shell}}}$$

4) For $v \ll c \rightarrow \partial t \approx \partial \tau \rightarrow$

This means that $\gamma_{shell} \approx 1$

$$\gamma_{shell}\, r\, v_{\phi,shell} \approx r\, v_{\phi,shell}$$

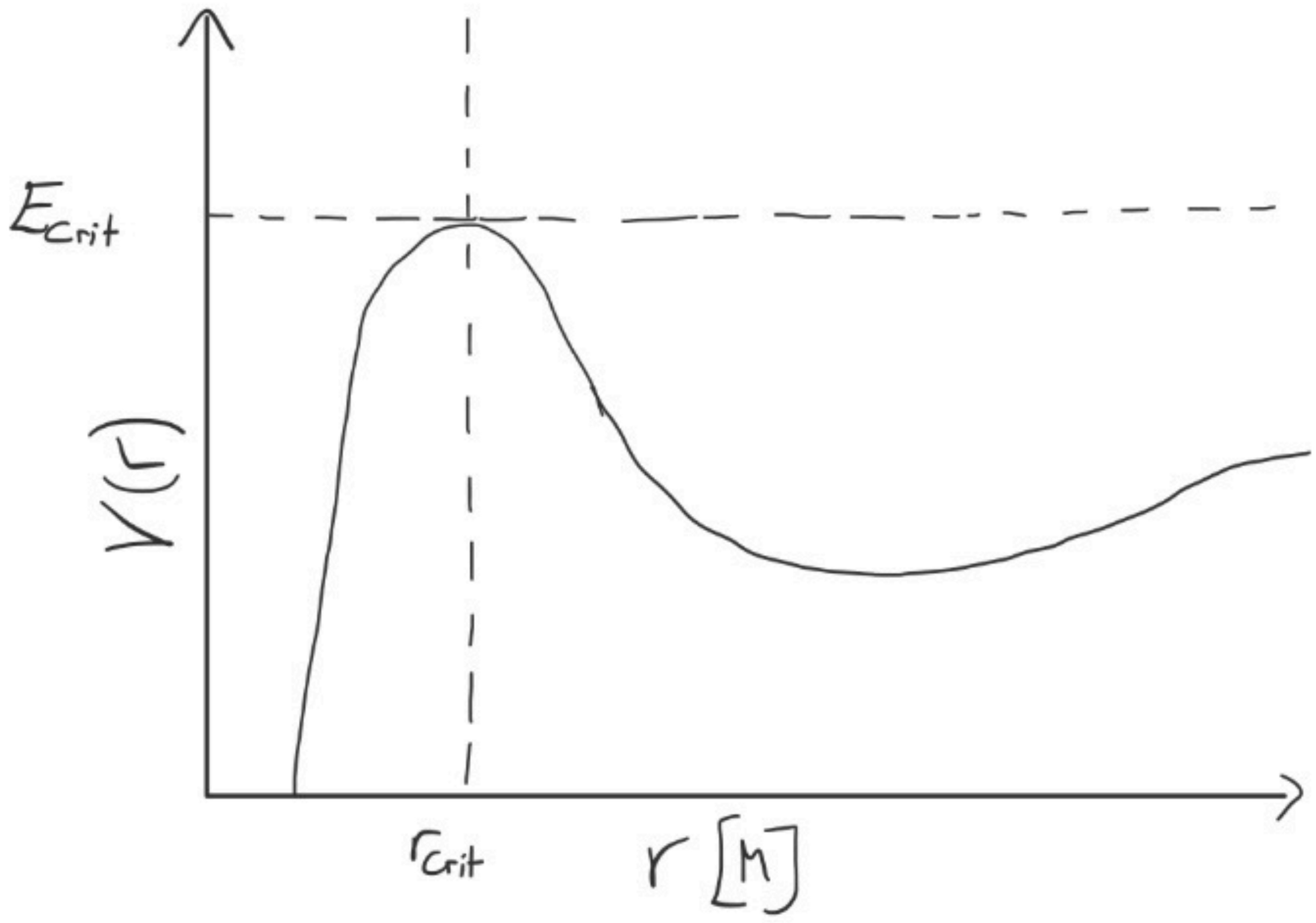
Classic spin:

$$\frac{L}{m} = r v_\phi$$

We see that these two are now identical, which means the quantity we derived corresponds to spin in classical mechanics.

Exercise 6

1) The gravitational potential per mass



The kinetic potential needs to be higher than the gravitational potential. In other words, the total effective potential needs to be higher than 0.

We need to calculate the energy per mass and the value of the effective potential at its maximum.

2) We use the equation for Energy per mass in the lecture notes 2d

$$\frac{E}{m} = \left(1 - \frac{2M}{R}\right)\frac{dt}{d\tau}$$

Since the shell observer can use special relativity for short time intervals, we can use some of its relations.

$$d\tau = \frac{dt_{shell}}{\gamma_{shell}}$$

$$dt = \frac{dt_{shell}}{\sqrt{1 - \frac{2M}{R}}}$$

$$\frac{dt}{d\tau} = \frac{\gamma_{shell}}{\sqrt{1 - \frac{2M}{R}}}$$

We insert this into the equation for energy per mass

$$\frac{E}{m} = \left(1 - \frac{2M}{R}\right)\frac{\gamma_{shell}}{\sqrt{1 - \frac{2M}{R}}}$$

$$\frac{E}{m} = \sqrt{1 - \frac{2M}{R}} \cdot \gamma_{shell}$$

3) We use the formula for effective potential found in lecture notes 2d. We first find the derivative and set it equal to 0.

$$V_{eff}(r) = \sqrt{\left(1-\frac{2M}{r}\right)\left[1+\frac{\left(\frac{L}{m}\right)^2}{r^2}\right]}$$

$$\frac{dV}{dr} = \frac{1}{2}\cdot\left(\left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{m^2r^2}\right)\right)^{-\frac{1}{2}}\cdot\left(\left(1+\frac{L^2}{m^2r^2}\right)\frac{2M}{r^2}-\left(1-\frac{2M}{r}\right)\frac{2L^2}{m^2r^3}\right)$$

$$\frac{dV}{dr} = \frac{\left(\left(1+\frac{L^2}{m^2r^2}\right)\frac{M}{r^2}-\left(1-\frac{2M}{r}\right)\frac{L^2}{m^2r^3}\right)}{\sqrt{\left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{m^2r^2}\right)}}$$

$$\frac{dV}{dr} = 0$$

$$\left(\left(1+\frac{L^2}{m^2r^2}\right)\frac{M}{r^2}-\left(1-\frac{2M}{r}\right)\frac{L^2}{m^2r^3}\right) = 0$$

$$\frac{M}{r^2}+\frac{ML^2}{m^2r^4}-\frac{L^2}{m^2r^3}+\frac{2ML^2}{m^2r^4} = 0$$

$$\frac{Mm^2r^2}{m^2r^4}+\frac{ML^2}{m^2r^4}-\frac{L^2r}{m^2r^4}+\frac{2ML^2}{m^2r^4} = 0$$

$$Mm^2r^2+ML^2-L^2r+2ML^2 = 0$$

$$Mm^2r^2-L^2r+3ML^2 = 0$$

$$r = \frac{L^2\pm\sqrt{L^4-4\cdot Mm^2\cdot 3ML^2}}{2Mm^2}$$

$$r = \frac{L^2\pm\sqrt{L^4-12M^2m^2L^2}}{2Mm^2}$$

$$r = \frac{L^2\pm L^2\sqrt{1-12M^2m^2\cdot L^{-2}}}{2Mm^2}$$

$$r = \frac{L^2}{2Mm^2}\left(1\pm\sqrt{1-\frac{12M^2m^2}{L^2}}\right)$$

Maximum is at $r = \frac{L^2}{2Mm^2}\left(1-\sqrt{1-\frac{12M^2m^2}{L^2}}\right)$

(the smaller radius)

4) From Classical mechanics, we know the definition of spin to be

$$\vec{L} = \vec{r} \times \vec{p}$$
$$= \vec{r} \times m\vec{v}$$

We can divide the velocity in its components and insert it into the equation for the spin

$$\vec{v} = \vec{v}_\phi + \vec{v}_r$$

$$\vec{L} = m\vec{r} \times (\vec{v}_\phi + \vec{v}_r)$$
$$\vec{L} = m\vec{r} \times \vec{v}_\phi + \vec{r} \times \vec{v}_r \qquad \vec{r} \text{ is perellell to } \vec{v}_r \rightarrow \vec{r} \times \vec{v}_r = 0$$
$$\vec{L} = m\vec{r} \times \vec{v}_\phi$$
$$\vec{L} = m r v_\phi \cdot \sin\phi$$

We know that v_ϕ in the proper time of a shell observer is $r\frac{d\phi}{d\tau}$

$$\vec{L} = m r^2 \frac{d\phi}{d\tau} (\vec{e}_r \times e_\phi)$$
$$\frac{L}{m} = r^2 \frac{d\phi}{d\tau}$$

$$\vec{L} = m r V_\phi \sin\phi$$
$$\vec{L} = r p \cdot \sin\phi$$

For a shell observer, measuring velocity and time, we can use special relativity

$$L = r\, p_{shell} \sin\phi \quad , \text{ where } \quad p_{shell} = m\, \gamma_{shell} v_{shell}$$
$$L = r m\, \gamma_{shell} v_{shell} \sin\phi$$

$$\frac{L}{m} = r\, \gamma_{shell} v_{shell} \sin\phi$$

6) We will be using the Schwarzschild geometry and Schwarzschild line element since we are free falling.

$$ds^2 = d\tau^2$$

Since we have assume L = 0, we will not have a tangential component and can simplify the Schwarzschild line element.

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2M}{r}}$$

From the formula for energy per mass, we find that

$$\frac{dt}{d\tau} = \frac{\frac{E}{m}}{1 - \frac{2M}{r}} \qquad d\tau^2 = \left(\frac{1 - \frac{2M}{r}}{\frac{E}{m}}\right)^2 dt^2$$

Inserting this into the Schwarzschild line element formula yields

$$\left(\frac{1 - \frac{2M}{r}}{\frac{E}{m}}\right)^2 dt^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2M}{r}}$$

Solving for $\frac{dr}{dt}$

$$\frac{dr}{dt} = \left(1 - \frac{2M}{r}\right) \sqrt{\left(1 - \frac{1 - \frac{2M}{r}}{\frac{E^2}{m^2}}\right)}$$

However, since we are now inside of the black hole, we cannot use dt anymore and have to use the wrist-watch time instead.

$$\frac{dr}{d\tau} = \frac{dt}{d\tau} \cdot \frac{dr}{dt}$$

$$\frac{dr}{d\tau} = \frac{\frac{E}{m}}{1 - \frac{2M}{r}} \left(1 - \frac{2M}{r}\right) \sqrt{\left(1 - \frac{1 - \frac{2M}{r}}{\frac{E^2}{m^2}}\right)}$$

$$\frac{dr}{d\tau} = \sqrt{\frac{E^2}{m^2} - \left(1 - \frac{2M}{r}\right)}$$

$$d\tau = \frac{1}{\sqrt{\frac{E^2}{m^2} - \left(1 - \frac{2M}{r}\right)}} dr$$

To find the total time from entering the event horizon until reaching the singularity, we can integrate this expression from r = 0 to r = 2M

$$\tau = \int_0^{2M} \frac{1}{\sqrt{\frac{E^2}{m^2} - \left(1 - \frac{2M}{r}\right)}} dr$$

By using the mass of the Black hole Sagittarius A in the Milky Way galaxy we find that

$$M = 4 \cdot 10^6 M_\odot$$

$$\underline{\underline{\tau \approx 4{,}767\ s}}$$

Exercise 7

1) Lorentz-transformation:

$$t' = \gamma (t - vx)$$

$$dt' = \gamma_{shell} \left(-v_{shell}\, dx_{shell} + dt_{shell} \right)$$

Since $dx = dr$ (and $d\phi = 0$)

$$dt' = -v_{shell}\, \gamma_{shell}\, dr_{shell} + \gamma_{shell}\, dt_{shell}$$

2) By using the Schwarzschild line element for an observer at rest we get that

$$\Delta t_{shell} = \sqrt{\left(1 - \frac{2M}{r}\right)}\, \Delta t$$

We now express the line element using Lorentz and Schwarzschild geometry.
Due to invariability of the line element, we can set these equal.
When now measuring a small radial distance at the same time, we can set

$\Delta\phi = 0$, $\Delta t = \Delta t_{shell} = 0$

$$\Delta t^2_{shell} - \Delta r^2_{shell} = \left(1 - \frac{2M}{r}\right)\Delta t^2 - \frac{\Delta r^2}{\left(1 - \frac{2M}{r}\right)}$$

$$\Delta r^2_{shell} = \frac{\Delta r^2}{\left(1 - \frac{2M}{r}\right)}$$

$$\Delta r_{shell} = \frac{\Delta r}{\sqrt{1 - \frac{2M}{r}}}$$

Inserting this into the equation from exercise 1 gives

$$dt' = -\frac{v_{shell}\, \gamma_{shell}\, dr}{\sqrt{\left(1 - \frac{2M}{r}\right)}} + \gamma_{shell} \sqrt{\left(1 - \frac{2M}{r}\right)}\, dt$$

3) $$\frac{dr_{shell}}{dt_{shell}} = -\sqrt{\frac{2M}{r}}$$

$$dt' = -\frac{v_{shell}\,\gamma_{shell}\,dr}{\sqrt{\left(1-\frac{2M}{r}\right)}} + \gamma_{shell}\sqrt{\left(1-\frac{2M}{r}\right)}\,dt$$

$$dt' = \frac{\sqrt{\frac{2M}{r}}\,\gamma_{shell}\,dr}{\sqrt{\left(1-\frac{2M}{r}\right)}} + \gamma_{shell}\sqrt{\left(1-\frac{2M}{r}\right)}\,dt$$

$$\sqrt{\left(1-\frac{2M}{r}\right)}\,dt' = \sqrt{\frac{2M}{r}}\,\gamma_{shell}\,dr + \left(1-\frac{2M}{r}\right)\gamma_{shell}\,dt$$

$$\gamma_{shell}\sqrt{\left(1-\frac{2M}{r}\right)}\,dt = dt' - \frac{\sqrt{\frac{2M}{r}}\,\gamma_{shell}\,dr}{\sqrt{\left(1-\frac{2M}{r}\right)}}$$

We insert for γ_{shell} : $\quad \gamma_{shell} = \frac{1}{\sqrt{1-\frac{2M}{r}}}$

$$dt = dt' - \frac{\sqrt{\frac{2M}{r}}\,dr}{\left(1-\frac{2M}{r}\right)}$$

4) $$ds^2 = \left(1-\frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)} - r^2 d\phi^2$$

$$ds^2 = \left(1-\frac{2M}{r}\right)\left(dt'^2 - \frac{2\sqrt{\frac{2M}{r}}\,dr\,dt'}{\left(1-\frac{2M}{r}\right)} + \frac{\frac{2M}{r}\,dr^2}{\left(1-\frac{2M}{r}\right)^2}\right) - \frac{dr^2}{\left(1-\frac{2M}{r}\right)} - r^2 d\phi^2$$

$$ds^2 = \left(1-\frac{2M}{r}\right)dt'^2 - 2\sqrt{\frac{2M}{r}}\,dr\,dt' + \frac{\frac{2M}{r}\,dr^2}{\left(1-\frac{2M}{r}\right)} - \frac{dr^2}{\left(1-\frac{2M}{r}\right)} - r^2 d\phi^2$$

$$ds^2 = \left(1-\frac{2M}{r}\right)dt'^2 - 2\sqrt{\frac{2M}{r}}\,dr\,dt' - dr^2 - r^2 d\phi^2$$

5) We use the Schwarzschild line element and set

$$d\phi = 0 \qquad d\tau = 0$$

$$ds^2 = d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 d\phi^2$$

$$\left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} = 0$$

$$dr^2 = \left(1 - \frac{2M}{r}\right)^2 dt^2$$

$$\frac{dr}{dt} = \pm \left(1 - \frac{2M}{r}\right)$$

$$dr = \left(1 - \frac{2M}{r}\right) dt$$

$$dr = \pm \left(1 - \frac{2M}{r}\right) dt' \mp \sqrt{\frac{2M}{r}}\, dr$$

$$\frac{dr}{dt'} \left(1 \pm \sqrt{\frac{2M}{r}}\right) = \pm \left(1 - \frac{2M}{r}\right)$$

$$\frac{dr}{dt'} = \frac{\pm \left(1 - \frac{2M}{r}\right)}{\left(1 \pm \sqrt{\frac{2M}{r}}\right)}$$

$$\frac{dr}{dt'} = \pm \left(1 \mp \sqrt{\frac{2M}{r}}\right)$$

$$\frac{dr}{dt'} = \pm 1 - \sqrt{\frac{2M}{r}}$$