Exercise 1

4a) To find the mass of the sun in meters, we will be using the conversion factor G/c^2
Where G is the gravitational constant and c is the speed of light.

$$M_{sun\_kg} = 1.989 \cdot 10^{30}\ kg$$

$$M_{sun\_m} = M_{sun\_kg} \cdot \frac{G}{c^2}$$

$$M_{sun\_m} = 1.989 \cdot 10^{30} \cdot 7{,}42 \cdot 10^{-28}$$

$$\underline{\underline{M_{sun\_m} = 1475{,}838\ m}}$$

$$r_{sun} = 695'508\ km$$

$$\frac{M_{sun\_m}}{r_{sun}} = \frac{1475{,}838}{695'508'000}$$

$$\underline{\underline{\frac{M_{sun\_m}}{r_{sun}} = 2{,}122 \cdot 10^{-6}}}$$

b) We can use the approximation found in question 3.

$$\frac{\Delta\lambda}{\lambda_{shell}} = \frac{M_{sun\_m}}{r_{sun}} = 2{,}122 \cdot 10^{-6}$$

$$\Delta\lambda = \frac{M_{sun\_m}}{r_{sun}} \cdot \lambda_{shell} = 1{,}061 \cdot 10^{-3}\ nm$$

The color of the sun will not change at all. The light is only redshifted by approximately a thousandth of a nanometer, which does not affect the colour at all.

c) $$M_{Earth} = M_{Earth\_kg} \cdot \frac{G}{c^2}$$

$$M_{Earth} = 4{,}431 \cdot 10^{-3}$$

$$\underline{\underline{\frac{M_{Earth}}{r_{Earth}} = 6{,}955 \cdot 10^{-10}}}$$

d) $$\frac{\Delta\lambda}{\lambda_{shell}} = \frac{M_{Earth}}{r_{Earth}}$$

$$\frac{\Delta\lambda}{\lambda_{shell}} = 6{,}955 \cdot 10^{-10}$$

$$\Delta\lambda = 6{,}955 \cdot 10^{-10} \cdot 500$$

$$\underline{\underline{\Delta\lambda = 3{,}4775 \cdot 10^{-7}}}\ nm$$

The gravitational blueshift of the earth only decreases the wavelength by a fraction of a nanometer.
It will therefore not change the perceived color of the light at all.

5) $\Delta\lambda = 2150 - 600 \text{ nm}$

$\Delta\lambda = 1550 \text{ nm}$

$\lambda_{shell} = 600 \text{ nm}$

$$\frac{\Delta\lambda}{\lambda_{shell}} = \frac{1}{\sqrt{1-\frac{2M}{r}}} - 1$$

$$\sqrt{1-\frac{2M}{r}} = \frac{1}{\frac{\Delta\lambda}{\lambda_{shell}} - 1}$$

$$\frac{2M}{r} = 1 - \frac{1}{\left(\frac{\Delta\lambda}{\lambda_{shell}} - 1\right)^2}$$

$$r = \frac{2M}{1 - \frac{1}{\left(\frac{\Delta\lambda}{\lambda_{shell}} - 1\right)^2}}$$

$$r = \frac{2M}{1 - \frac{1}{\left(\frac{1550}{600} - 1\right)^2}}$$

$$\underline{\underline{r = 3{,}327 M}}$$

6) $$\frac{\Delta\lambda}{\lambda_{shell}} = \frac{1}{\sqrt{1-\frac{2M}{2{,}01M}}} - 1$$

$$\frac{\lambda - \lambda_{shell}}{\lambda_{shell}} = 13{,}177$$

$$\lambda_{shell} = \frac{\lambda}{12{,}177}$$

As the blueshift is extremely strong, we will most likely not be able to observe any stars with the naked eye. Almost all of the light will be blueshifted into the UV spectrum.

Exercise 2

1) As we are looking at such small intervals, we can use Lorentz-transformation and therefore the Schwarzschild line element.
We divide up in two sections. The first one from point 1 to point 2 and the second one from point 2 to point 3.
As we use such small intervals, the distance r is constant for each interval and we can approximate

$$\Delta S_{12} = \Delta \tau_{12}$$
$$\Delta S_{23} = \Delta \tau_{23}$$

$$\Delta S^2 = \left(1-\frac{2M}{r}\right)\Delta t^2 - \frac{\Delta r^2}{\left(1-\frac{2M}{r}\right)} - r^2 \Delta\phi^2$$

$$\Delta \tau_{12} = \sqrt{\left(1-\frac{2M}{r_A}\right)\Delta t_{12}^2 - \frac{\Delta r_{12}}{1-\frac{2M}{r_A}} - r_A^2 \Delta\phi_{12}^2}$$

$$\Delta \tau_{23} = \sqrt{\left(1-\frac{2M}{r_B}\right)\Delta t_{23}^2 - \frac{\Delta r_{23}}{1-\frac{2M}{r_B}} - r_B^2 \Delta\phi_{23}^2}$$

To find the total proper time interval from point 1 to 3, we can sum these intervals together

$$\Delta \tau_{13} = \Delta \tau_{12} + \Delta \tau_{23}$$

$$\Delta \tau_{13} = \sqrt{\left(1-\frac{2M}{r_A}\right)\Delta t_{12}^2 - \frac{\Delta r_{12}}{1-\frac{2M}{r_A}} - r_A^2 \Delta\phi_{12}^2} + \sqrt{\left(1-\frac{2M}{r_B}\right)\Delta t_{23}^2 - \frac{\Delta r_{23}}{1-\frac{2M}{r_B}} - r_B^2 \Delta\phi_{23}^2}$$

2) We use the principle of maximum aging and try to maximize the proper time with respect to t2

$$\frac{\partial}{\partial \phi_2}\Delta \tau_{13} = \frac{\partial}{\partial \phi_2}\sqrt{\underbrace{\left(1-\frac{2M}{r_A}\right)\Delta t_{12}^2 - \frac{\Delta r_{12}}{1-\frac{2M}{r_A}}}_{K_1} - r_A^2 \Delta\phi_{12}^2} + \frac{\partial}{\partial \phi_2}\sqrt{\underbrace{\left(1-\frac{2M}{r_B}\right)\Delta t_{23}^2 - \frac{\Delta r_{23}}{1-\frac{2M}{r_B}}}_{K_2} - r_B^2 \Delta\phi_{23}^2}$$

$$= \frac{\partial}{\partial \phi_2}\sqrt{K_1 - r_A^2 \Delta\phi_{12}^2} + \frac{\partial}{\partial \phi_2}\sqrt{K_2 + r_B \Delta\phi_{23}^2}$$

$$= \frac{1}{\sqrt{K_1 - r_A^2(\phi_2-\phi_1)^2}} \cdot \left(-2r_A^2(\phi_2-\phi_1)^2\right)\cdot(1) + \frac{1}{\sqrt{K_2 - r_B^2(\phi_3-\phi_2)^2}} \cdot \left(-2r_B^2(\phi_3-\phi_2)^2\right)\cdot(-1)$$

$$= -\frac{2r_A^2(\phi_2-\phi_1)^2}{\sqrt{K_1 - r_A^2(\phi_2-\phi_1)^2}} + \frac{2r_B^2(\phi_3-\phi_2)^2}{\sqrt{K_2 - r_B^2(\phi_3-\phi_2)^2}}$$

Setting equal to 0

$$\frac{r_A^2 \Delta\phi_{12}^2}{\sqrt{K_1 - r_A^2 \Delta\phi_{12}^2}} = \frac{r_B^2 \Delta\phi_{23}^2}{\sqrt{K_2 - r_B^2 \Delta\phi_{23}^2}}$$

$$\frac{r_A^2 \Delta\phi_{12}^2}{\Delta \tau_{12}} = \frac{r_B^2 \Delta\phi_{23}^2}{\Delta \tau_{23}}$$

$$r_A^2 \frac{\partial \phi_{12}}{\partial \tau_{12}} = r_B^2 \frac{\partial \phi_{23}}{\partial \tau_{23}}$$

This shows that the quantity remains the same even for changing phi and tau.
We can therefore conclude that the quantity is conserved.

3) $\gamma = \frac{dt}{d\tau}$

we know that $\frac{d\phi}{dt} = \frac{v_\phi}{r}$

$$r^2 \frac{d\phi}{d\tau} = r^2 \frac{d\phi}{d\tau}\frac{dt}{dt}$$
$$= r^2 \frac{d\phi}{dt}\frac{dt}{d\tau}$$
$$= r^2 \frac{v_\phi}{r}\gamma$$
$$= \gamma r v_\phi$$

For $v \lll c \rightarrow dt \approx d\tau \rightarrow$

This means that $\gamma_{shell} \approx 1$

$$\gamma r v_\phi \approx r v_\phi$$

Classic spin:

$$\frac{L}{m} = r v_\phi$$

We see that these two are now identical, which means the quantity we derived corresponds to spin in classical mechanics.