FYS3500 Problem set 5

Problem 1

a) Experimental evidences for the nuclear shell model are, among other:

1) Especially tight bound nuclei occuring at specific N and Z, called «magic numbers» (so elements with Z: magic tend to have more stable isotopes)

2) Nuclei with magic neutron number have smaller neutron absorption cross section. Nuclei with one neutron excess with respect to a magic number emits this readily.

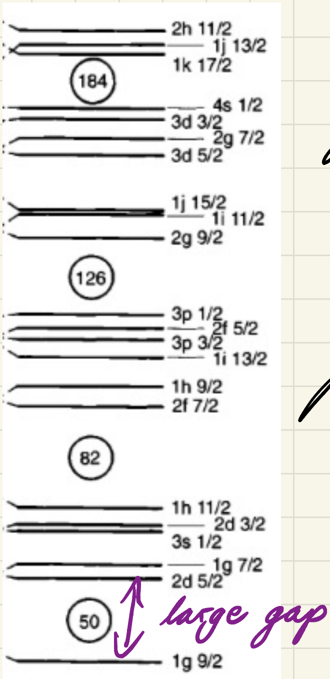
3) At magic numbers for low-A nuclei, spikes occur in the plot showing the average bining energy per nucleon.

4) Doubly-magic nuclei (both N and Z are magic numbers) have higher first excited states compared to neighbours.

Etc :-)

b)

Magic numbers refer to N and Z numbers that corresponds to closed shells where there is an especially large gap to the next shell.



There are
2, 8, 20, 28, 50, 82, 126...

Ex: If we have 50 protons, there is a large energy gap up to the next level.

c) We denote the shells in the shell models as follows: $n l_j$

n : the # level with that l-value (eg $2s_{1/2}$ is the second $s_{1/2}$-level)

l : orbital angular momentum of the state, where we use the historic notation

$s \Rightarrow l = 0$

$p \Rightarrow l = 1$

$d \Rightarrow l = 2$

$f \Rightarrow l = 3$

$g \Rightarrow l = 4$

$h \Rightarrow l = 5$

Norwegian "huskeregel"

se på den fine gutten, hei!

j : total angular momentum of a state, given as $\vec{j} = \vec{l} + \vec{s}$.

$s = 1/2$ for all nucleons, so for each l, $j = |l \pm 1/2|$

e.g. 1d : $l = 2$ and $s = 1/2$, so
$j \in 2 - 1/2\ ,\ 2 + 1/2 \Rightarrow \frac{3}{2}\ \&\ \frac{5}{2}$

The number of nucleons that fits in each shell is given by the mj degeneracy. As two nucleons cannot be in the same state (Pauli), they must have different values of the mj quantum number.

$m_j \in -j, -j+1, \ldots, j$
$\Rightarrow (2j+1)$ different m_j for each j
$\Rightarrow (2j+1)$ nucleons fit in each shell

e.g. $1d_{5/2}$

$m_j : -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$

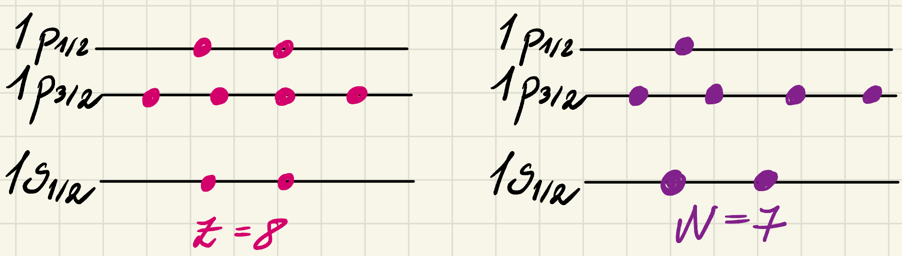
$\Rightarrow 6$ nucleons fit in $1d_{5/2}$

d) Nucleons want to be paired such that the total angular momenta of pairs of nucleons cancel. Since all nucleons are paired in even-even nuclei, all contributions cancel, and the ground state is 0+.

e) In the extreme single-particle model, the configuration of the last unpaired nucleon determines the spin and parity of a nuclear state.

-> We must thus determine in which orbital the unpaired nucleon is.

- $^{15}_{8}O_7$ has $Z=8$ (even) and $N=7$ (odd) and the shell configuration looks like this:



The last unpaired nucleon is in $1p_{1/2} \Rightarrow l=1$, and parity is given by

$$P_{int} \cdot (-1)^{l=1} = (-1)$$

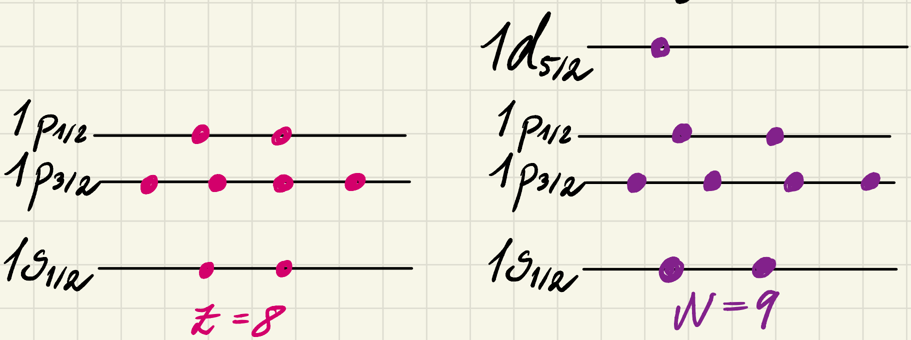
$\uparrow$ intrinsic parity $\Rightarrow +1$ for n & p

$j = 1/2$

$\Rightarrow$ g.s. of ^{15}O is $1/2^-$

- $^{16}_{8}O_8$ is even-even, and has therefore gs 0^+

- $^{17}_{8}O_9$ has nucleon config



$\Rightarrow$ last unpaired nucleon in $1d_{5/2}$, and thus g.s. is $5/2^+$.

$^{30}_{15}P_{15}$

f)

In odd-odd nuclei there are one unpaired neutron and one unpaired proton, so to suggest possible candidates for the ground state, we must combine the neutron and proton states:

$$\vec{J} = \vec{j_1} + \vec{j_2} \Rightarrow J \in |j_1 - j_2|, \ldots, j_1 + j_2$$

$$P = p_1 \cdot p_2$$

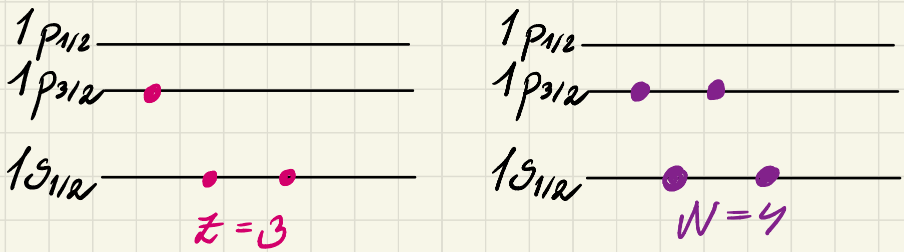
This gives several possible states, and we cannot determine which is at the lowest energies.

Other effects that results in observed nuclear states to deviate from the predictions of the shell model are e.g.

- couplings of more than one valence nucleon
- collective states: vibrations and rotations (FYS4570!)

Problem 2

a) Config of $^7_3Li_4$:



Last unpaired nucleon in $1p_{3/2}$

$\Rightarrow$ ground state is $\underline{\underline{3/2^-}}$.

From eq. 8.28, we know that the nuclear magnetic dipole moment can be written as

$$\mu = g_j \, j \, \mu_N$$

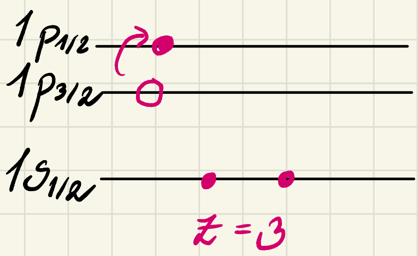
$1p_{3/2} \Rightarrow j = l + s$, so eq 8.31 gives

$$j g_p \,(\text{for } j = l + 1/2) = j + 2.3$$
$$= 3/2 + 2.3 = 3.8$$

$\Rightarrow \underline{\underline{\mu = 3.8 \mu_N}}$

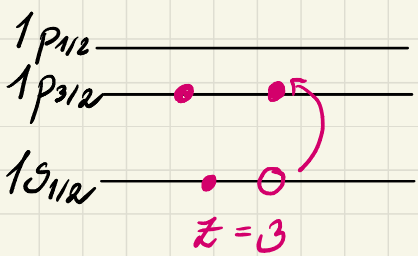
Probable first excited states configs:
(for proton excitations)

1. ex:



Unpaired proton in $1p_{1/2} \Rightarrow$ state is $1/2^-$

2. ex:

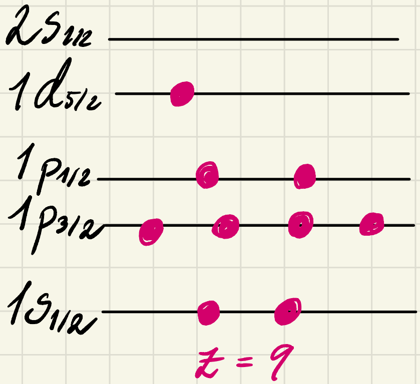


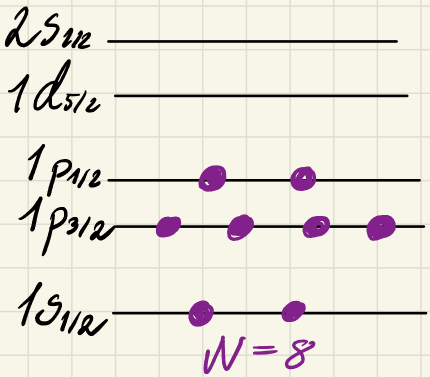
Unpaired proton in $1s_{1/2} \Rightarrow$ state is $1/2^+$.

Experimentally: the first excited state is $1/2^-$.

b) To get $J^P = 5/2^+$ for $^{17}_{9}F_8$

$\Rightarrow$ last unpaired nucleon in $d_{5/2}$





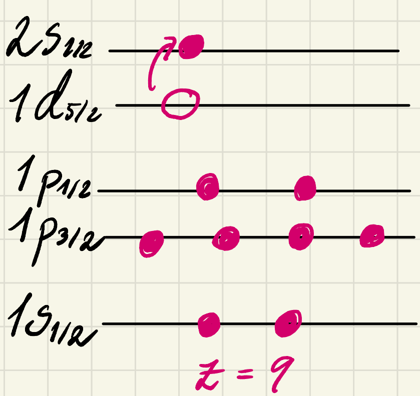
$Z: (1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2(1d_{5/2})^1$

$N: (1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2$

written notation

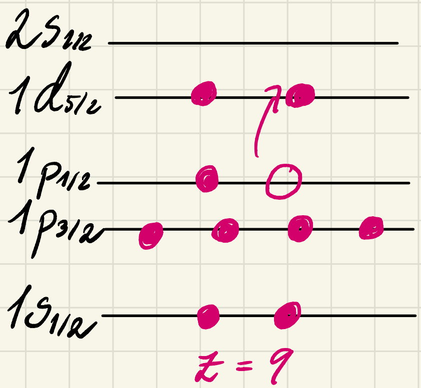
To get first excited state $J^P = 1/2^+$

$\Rightarrow$ last unpaired nuc in $s_{1/2}$



To get second excited state $J^P = 1/2^-$

$\Rightarrow$ unpaired nucleon in $1p_{1/2}$



Problem 3

a) We know that

$$\hat{\vec{j}} = \hat{\vec{s}} + \hat{\vec{l}}$$

$$\Rightarrow \hat{\vec{j}}^2 = (\hat{\vec{s}} + \hat{\vec{l}})^2$$

$$\hat{\vec{j}}^2 = \hat{\vec{s}}^2 + 2\hat{\vec{s}}\cdot\hat{\vec{l}} + \hat{\vec{l}}^2$$

$$2\hat{\vec{s}}\cdot\hat{\vec{l}} = \hat{\vec{j}}^2 - \hat{\vec{s}}^2 - \hat{\vec{l}}^2$$

$$\hat{\vec{s}}\cdot\hat{\vec{l}} = \frac{1}{2}(\hat{\vec{j}}^2 - \hat{\vec{s}}^2 - \hat{\vec{l}}^2)$$

b) We know that the expectation value of $\hat{\vec{s}}$ is $s(s+1)\hbar^2$, and similar for j and l. Thus

$$\langle \hat{\vec{s}}\cdot\hat{\vec{l}} \rangle = \frac{1}{2}[j(j+1) - s(s+1) - l(l+1)]\hbar^2$$

c) $\langle \hat{\vec{l}} \cdot \hat{\vec{s}} \rangle_{j=l+1/2}$

$$= \frac{1}{2}\left[(l+1/2)(l+\underbrace{1/2+1}_{3/2}) - l(l+1) - \frac{1}{2}\underbrace{\left(\frac{1}{2}+1\right)}_{3/2}\right]\hbar^2$$

$$= \frac{1}{2}\left[\cancel{l^2} + \frac{3}{2}l + \frac{1}{2}l + \cancel{\frac{3}{4}} - \cancel{l^2} - l - \cancel{\frac{3}{4}}\right]\hbar^2$$

$$= \frac{l}{2}\hbar^2$$

$\langle \hat{\vec{l}} \cdot \hat{\vec{s}} \rangle_{j=l-1/2}$

$$= \frac{1}{2}\left[(l-1/2)(l\underbrace{-1/2+1}_{+1/2}) - l(l+1) - \frac{1}{2}\underbrace{\left(\frac{1}{2}+1\right)}_{3/2}\right]\hbar^2$$

$$= \frac{1}{2}\left[l^2 + \frac{1}{2}l - \frac{1}{2}l - \frac{1}{4} - l^2 - l - \frac{3}{4}\right]\hbar^2$$

$$= \frac{1}{2}\left[-l - 1\right]\hbar^2 = -\frac{1}{2}(l+1)\hbar^2$$

$$\Delta E = \langle \vec{\hat{l}} \cdot \vec{\hat{s}} \rangle_{j=l+1/2} - \langle \vec{\hat{l}} \cdot \vec{\hat{s}} \rangle_{j=l-1/2}$$

$$= \frac{l}{2} \hbar^2 - \left(-\frac{1}{2} (l+1) \hbar^2 \right)$$

$$= \frac{l}{2} \hbar^2 + \left(\frac{1}{2} l + \frac{1}{2} \right) \hbar^2$$

$$= \underline{\underline{\frac{1}{2} (2l+1) \hbar^2}}$$

We see that the energy splitting increases with larger l.

From fig. 8.4 in M&S2019, we see that the $j = l + 1/2$ lies at the lowest energy, which is because the Hamiltonian $\hat{H}$ includes the term

$-V_{SO} \vec{\hat{l}} \cdot \vec{\hat{s}}$

$\Rightarrow$ negative sign : largest value for $\langle \vec{\hat{l}} \cdot \vec{\hat{s}} \rangle$ gives the smallest energy