

# FYS3500: Particle Physics

## Lecture Notes

Oskar Idland

### Contents

<b>1</b>	<b>History</b>	<b>3</b>
1.1	Proton Discovery: The Rutherford Scattering Experiment . . . . .	3
1.1.1	Conclusion . . . . .	3
1.2	Discovery of the Neutron . . . . .	3
<b>2</b>	<b>Nucleus</b>	<b>3</b>
2.1	Notation . . . . .	4
2.2	Nuclides . . . . .	4
2.2.1	Stable Numbers . . . . .	4
<b>3</b>	<b>Units and Dimensions in Nuclear Physics</b>	<b>4</b>
3.1	Length . . . . .	4
3.2	Time Scale . . . . .	4
3.3	Energy . . . . .	4
3.4	Mass . . . . .	5
<b>4</b>	<b>Nuclear Properties</b>	<b>5</b>
4.1	Connected Terms . . . . .	5
4.2	Charge Distribution . . . . .	6
4.3	Nuclear Charge Distribution from Electron Scattering . . . . .	6
4.4	Nuclear Size . . . . .	6
4.5	Nuclear Mass Distribution from Hadron Scattering . . . . .	6
4.5.1	Fixed Angle of Observation with Changing Energy . . . . .	7
4.6	Conclusion from Charge Radius Experiments . . . . .	7
4.7	Nuclear Mass . . . . .	8
4.8	Deflection Spectrometer . . . . .	8
4.8.1	Calculating the Mass . . . . .	8
<b>5</b>	<b>Binding Energy</b>	<b>9</b>
5.1	Formulas and Definitions . . . . .	9
5.2	Mass of the Nucleus . . . . .	9
5.3	Nuclear Binding Energy (B.E.) . . . . .	10
5.3.1	. . . . .	10
5.4	Nuclear Separation Energy . . . . .	10
5.4.1	Neutron Separation Energy . . . . .	10

5.4.2	Proton Separation Energy . . . . .	11
5.5	Average Binding Energy . . . . .	11
5.6	Semi-Empirical Mass Formula . . . . .	12
5.6.1	Explanation of the Terms in the Semi-Empirical Mass Formula . . . . .	12
5.6.2	SEMF Conclusion . . . . .	13
5.7	Mass Parabolas of Isobars . . . . .	13
5.7.1	Finding the Minimum of the Mass Parabola . . . . .	13
5.7.2	Valley of (beta) stability . . . . .	14
<b>6</b>	<b>Angular Momentum &amp; Parity</b>	<b>15</b>
6.1	Angular Momentum of the Nucleus . . . . .	15
6.1.1	Orbital Angular Momentum . . . . .	16
6.1.2	Spin . . . . .	16
6.1.3	Total Angular Momentum . . . . .	18
6.1.4	Total Angular Momentum of the Nucleus . . . . .	18
6.2	Parity . . . . .	19
6.2.1	Splitting the Wave Function . . . . .	19
<b>7</b>	<b>Electric and Magnetic Moments</b>	<b>20</b>
7.1	Parity Selection Rules . . . . .	20
7.1.1	. . . . .	20
7.1.2	. . . . .	20
7.2	Bohr Magnetron & Nuclear Magnetron . . . . .	21
7.3	Magnetic Moments of Nuclei . . . . .	21
7.3.1	Nuclear Structure from Magnetic Moments . . . . .	21
7.4	. . . . .	22
7.5	Example: Calculating Parity of State . . . . .	23
7.6	Level Schemes & Excited States . . . . .	23
<b>8</b>	<b>Nuclear Force</b>	<b>23</b>
8.1	Effects of the Short Range of the Strong Force . . . . .	24
8.2	Deuteron . . . . .	24
8.2.1	Deuteron Binding Energy . . . . .	24
8.2.2	Nucleon-Nucleon Potential . . . . .	25
8.2.3	Deuteron Spin and Parity . . . . .	26

# 1 History

- 1896: Henri Becquerel discovered radioactivity
- 1898: Marie and Pierre Curie discovered radium and polonium
- 1903: Alphas charge to mass ratio
- 1909: Alphas are helium nuclei
- 1911: Rutherford discovers the nucleus
- 1913: Bohr model of the atom
- 1917: Rutherford discovers the proton
- 1930: Neutrinos were postulated
- 1932: Chadwick discovers the neutron by shooting alpha particles at beryllium.
- 1938: Discovery of nuclear fission
- 1956: Neutrinos were detected

## 1.1 Proton Discovery: The Rutherford Scattering Experiment

Thomson's model of the atom was a positive sphere with electrons embedded in it. Rutherford wanted to test this model by shooting alpha particles at a thin gold foil surrounded by a detector foil. The alpha particles were shot from a radioactive source and when the alpha particles exited, they hit the foil and emitted light.

### 1.1.1 Conclusion

- Most alpha particles went straight through the foil. This implies the atom is mostly empty space.
- Some alpha particles were deflected by a small angle. This implies the positive charge is concentrated in a small volume.
- Sometimes the particles travel backwards. This implies the positive center has most of the mass of the atom.

## 1.2 Discovery of the Neutron

- Shooting alpha particles on beryllium which is much lighter than gold. This

# 2 Nucleus

- Very dense. Carries all the mass.  $2.7 \cdot 10^{14}$  times denser than water.
- The atom is mostly empty space. If the nucleus was the size of a coin, the atom would be 2-3 km in radius.

## 2.1 Notation

- **Notation:**  ${}^A_ZX_N$
- **Isotope:** Same **proton** number  $Z$
- **Isotone:** Same **neutron** number  $N$
- **Isobar:** Same **atomic** mass number  $A = Z + N$

## 2.2 Nuclides

- 92 stable elements
- 280 stable isotopes
- 3000 unstable isotopes
- 6000 more predicted to exist

### 2.2.1 Stable Numbers

$$N = 2, 8, 20, 28, 50, 82, 126 \quad (1)$$

$$Z = 2, 8, 20, 28, 50, 82, \dots \quad (2)$$

## 3 Units and Dimensions in Nuclear Physics

### 3.1 Length

The order of  $10^{-15}\text{m} = 1\text{fm}$  (fermi/femtometer) meter. This is the distance between nucleons.

### 3.2 Time Scale

- $10^{-20}\text{s}$ : Unbound, in the case of nuclear reactions and decays.
- $10^{-9}/10^{-12}\text{s}$ : lifetimes of excited nuclear states through gamma decays.
- Minutes/hours/millions of years: Alpha and beta decays.

### 3.3 Energy

MeV in nuclear physics.

$$1 \text{ MeV} = 1.6 \times 10^{-13} \text{ J}. \quad (3)$$

$$1 \text{ eV} = 1.6 \times 10^{-19} \text{ J} \quad (4)$$

Particle	Mass (kg)	Mass (u)	Mass (MeV/c <sup>2</sup> )
1 atomic mass unit	1.660540 x 10 <sup>-27</sup> kg	1.000 u	931.5 MeV/c <sup>2</sup>
neutron	1.674929 x 10 <sup>-27</sup> kg	1.008664 u	939.57 MeV/c <sup>2</sup>
proton	1.672623 x 10 <sup>-27</sup> kg	1.007276 u	938.28 MeV/c <sup>2</sup>
electron	9.109390 x 10 <sup>-31</sup> kg	0.00054858 u	0.511 MeV/c <sup>2</sup>

Figure 1: Table of the masses of the nucleons.  $c^2 = 931.5 \text{ MeV/u}$ . In reality, the mass of the proton is slightly less than the mass of the neutron. The proton is 2000 times more massive than the electron.

### 3.4 Mass

u = unified atomic mass unit. 1 u is defined as 1/12 of the mass of an unbound <sup>12</sup>C atom. Mass is equivalent with energy. Therefore:

$$u = 931.5 \text{ MeV}/c^2 = 1.66 \times 10^{-27} \text{ kg} \quad (5)$$

## 4 Nuclear Properties

The parameters which describe the nucleus are. There are two types of nuclear properties: static and dynamic.

- Static: Charge, Radius, mass, Binding energy, Angular momentum, Parity, Magnetic dipole moment, Electric quadrupole moments, Excited states and their energies.
- Dynamic: Shape, Decay

### 4.1 Connected Terms

- Charge/Charge Distribution: Protons. Found via electron scattering [section 4.3](#) by the Coulomb interaction.
- Matter/Mass Distribution: Nucleons. Found via hadron scattering [section 4.5](#), alpha particles (Rutherford), protons and neutrons by using the strong force.
- Radius: Size of the nucleus (nucleons)

## 4.2 Charge Distribution

To probe the charge distribution of the nucleus, we use charged particles. We also need the following:

- A beam of charged particles (often protons)
- Wavelength should be similar or smaller than the nucleus (about 10fm in diameter).
- Electrons were popular in the 50's.
- An energy of 100 MeV to 1 GeV is needed.
- Calculating the energy needed is done by using the de Broglie wavelength where  $\lambda = h/p$  with  $\lambda \leq 10\text{fm}$ .

## 4.3 Nuclear Charge Distribution from Electron Scattering

- Radius increases with mass number  $A$
- The central nuclear charge density is nearly the same for all nuclei. Nucleons do not seem to concentrate near the center of the nucleus, but instead have a constant distribution along the surface.
- The number of nucleons per unit volume is roughly constant:

$$\frac{A}{\frac{4}{3}\pi R^3} \approx \text{const} \quad (6)$$

- The radius of the nucleus is proportional to  $A^{1/3}$ .

$$R = R_0 A^{1/3} \quad , \quad R_0 \approx 1.2 \text{ fm} \quad (7)$$

## 4.4 Nuclear Size

We can find the radius of a nucleus by using the scattering angle of the local minimum of the Rutherford cross-section, see [figure 2](#). The diffraction pattern is not exactly that of a circular disk, as the nucleus does not have a well-defined surface.

$$\sin \theta = \frac{1.22\lambda}{d} \Rightarrow R = \frac{d}{2} = \frac{1.22\lambda}{2 \sin \theta} \quad (8)$$

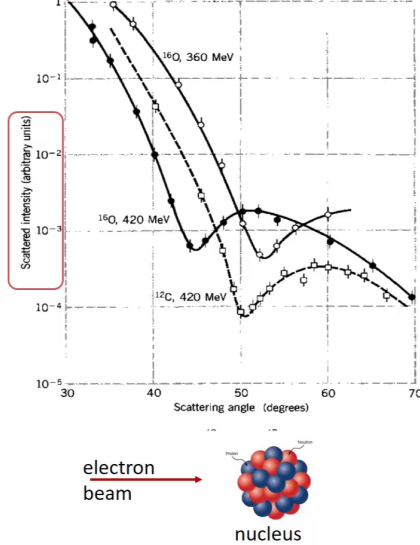
This is only a rough estimate as the angle is calculated in two dimensions, instead of three.

## 4.5 Nuclear Mass Distribution from Hadron Scattering

- Electrons only mostly interact with protons. We therefore use hadrons to study the mass distribution of the nucleus.
- The radius is proportional to the nuclear rather than the Coulomb force.
- The Rutherford experiment showed that the nucleus is a point-like object.

### Electron scattering on nuclei

Examples:  $^{16}\text{O}$  and  $^{12}\text{C}$  & measured cross sections



### Light scattering on a circular disk

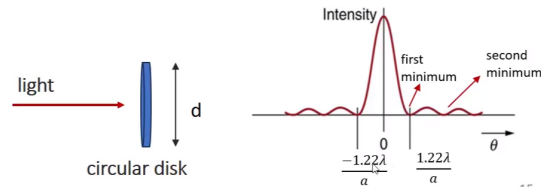
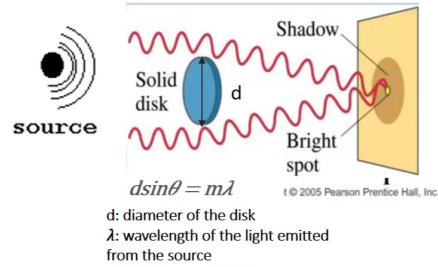


Figure 2: Example of the local minimum of the Rutherford cross-section. The angle is used to calculate the radius of the nucleus.

#### 4.5.1 Fixed Angle of Observation with Changing Energy

- At low energies the alpha particles and the  $^{208}\text{Pb}$  nucleus interact with the Coulomb force as with Rutherford scattering.
- With increasing energy, the repulsion from the Coulomb force is overcome, and the strong force becomes the dominant force. The Rutherford formula no longer holds.
- The alpha particles become absorbed by the nucleus and only a small fraction is scattered.
- When energy is high enough, we get the diffraction pattern.

#### 4.6 Conclusion from Charge Radius Experiments

- The charge and mass radii of nuclei are nearly equal to within about 0.1 fm.
- Both show the same  $A^{1/3}$  dependence with  $R_0 = 1.2 \text{ fm}$ .
- As heavy nuclei have about 50 % more neutrons than protons, we might expect the neutron distribution to be more extended than the proton distribution. This is not the case as the neutrons pull inwards, and the protons push outwards, until they are mixed such that the radius is the same.

## 4.7 Nuclear Mass

## 4.8 Deflection Spectrometer

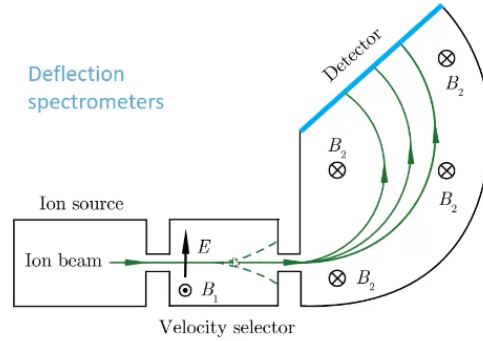


Figure 3: Experimental setup for measuring the mass of a particle.

- Shooting a ray of charge particles affected by a magnetic field and measuring the deflection we can calculate its mass.
- To measure an entire particle they must be ionized. The electrons carry so little mass that they are neglected.
- After ionization, the particles travel through an electric and magnetic field.
- Only the particles with the right velocity will pass through the fields and be subjected to the new magnetic field.
- The new field will deflect the particles according to their  $m/q$  value.

### 4.8.1 Calculating the Mass

$$F_B = q\vec{v} \times \vec{B} \quad (9)$$

The field and velocity are perpendicular.

$$F_B = qvB \quad (10)$$

$$F_E = F_B \Rightarrow qE = qvB \Rightarrow v = \frac{E}{B_1} \quad (11)$$

$B_1$  is the first magnetic field as seen in [figure 3](#). The force from the magnetic field centripetal force.

$$F_B = \frac{mv^2}{r} = qvB_2 \quad (12)$$

$$\frac{mv}{r} = qB_2 \quad (13)$$

$$\frac{m}{q} = \frac{B_2 r}{v} \quad (14)$$



The radius of the circle is given by  $r = \rho$ . Setting  $B_1 = B_2$  gives the following for the mass.

$$m = \frac{B_1 B_2 \rho}{E} = \frac{B^2 \rho q}{E} \quad (15)$$

where  $q$  is the charge of the particle.

### Accuracy

- These measurements are very important for mass models used in other parts of physics.
- The accuracy is about  $\Delta m/m = 10^{-6}$ , but that is not enough.
- The mass doublet technique gives a precision of  $10^{-8} / 10^{-9}$

## 5 Binding Energy

### 5.1 Formulas and Definitions

- Binding Energy: The energy required to keep the nucleus together. The mass of the nucleus is not equal to the sum of its parts. The mass of the individual nucleons is higher than the mass of the nucleus. The difference is the binding energy.

$$Zm_p + Nm_n - M_{\text{Nucleus}} = \text{Binding Energy} \Rightarrow Zm_p + Nm_n > M_{\text{Nucleus}} \quad (16)$$

### 5.2 Mass of the Nucleus

The total mass of the atom is the mass of the nucleus and electrons, minus the binding energy of the electrons.

$$M_{\text{Atom}} = M_{\text{Nucleus}} + Zm_e - \underbrace{\sum_{i=1}^Z B_i/c^2}_{\text{Often negligible}} \quad (17)$$

$$M_{\text{Atom}} = M_{\text{Nucleus}} + Zm_e \quad (18)$$

$M$  usually refers to the mass of the entire atom, and so the subscript "Atom" is often omitted. We usually write the atom using the following notation:

$$M({}_Z^A X_N) = M_{\text{Nucleus}}({}_Z^A X_N) + Zm_e \quad (19)$$

Multiplying by  $c^2$  we get the mass in energy units ( $E = mc^2$ ):

$$M_{\text{Nucleus}}({}_Z^A X_N) = M({}_Z^A X_N) - Zm_e c^2 \quad (20)$$

$$\underline{\underline{M_{\text{Nucleus}}({}_Z^A X_N) c^2 = M({}_Z^A X_N) c^2 - Zm_e c^2}} \quad (21)$$

### 5.3 Nuclear Binding Energy (B.E.)

This energy is very small compared to the mass energy of the nucleus. We can derive this from the mass of the nucleus.

$$B.E. = (Zm_p + Nm_n - M_N({}^A_ZX_N)) c^2 \quad (22)$$

$$= (Zm_p + Nm_n - (M({}^A_ZX_N) - Zm_e)) c^2 \quad (23)$$

$$= \left( \underbrace{Z(m_p + m_e)}_{\text{Hydrogen}} + Nm_n - M({}^A_ZX_N) \right) c^2 \quad (24)$$

$$(25)$$

$$B.E. = \underline{\underline{(Zm({}^1H) + Nm_n - M({}^A_ZX_N)) c^2}} \quad (26)$$

As the units so far has been energy ( $mc^2$ ) we can switch to MeV.

$$B.E. = [mc^2] = [uc^2] = u931.5\text{MeV} / u \Rightarrow c^2 = 931.5\text{MeV}/u \quad (27)$$

$$B.E. = \underline{\underline{(Zm({}^1H) + Nm_n - M({}^A_ZX_N)) 931.5\text{MeV}/u}} \quad (28)$$

#### 5.3.1

Example: Helium  ${}^4_2\text{He}$  We use the formula for binding energy from [equation \(28\)](#) to calculate the binding energy of the helium atom  ${}^4_2\text{He}$ .

$$B.E. = (2m_p + 2m_n - M({}^4_2\text{He})) 931.5\text{MeV}/u \quad (29)$$

$$= (2 \cdot 1.007825u + 2 \cdot 1.008664u - 4.002603u) \cdot 931.5\text{MeV}/u \quad (30)$$

$$= \underline{\underline{0.0304 \cdot 931.5 \text{ MeV} = 28.3 \text{ MeV}}} \quad (31)$$

The ratio between the binding energy and the rest mass of the nucleus is very small. Using the binding energy from [equation \(31\)](#) and the mass of the helium nucleus, we can calculate the ratio:

$$\frac{28.3}{3728} = 0.75\% \quad (32)$$

### 5.4 Nuclear Separation Energy

The energy required to separate a proton  $S_p$  or a neutron  $S_n$  from the nucleus.

#### 5.4.1 Neutron Separation Energy

It requires lower energy to remove a neutron from a nucleus with an odd number of neutrons. This is because one is unpaired.

$$S_n = (M({}^{A-1}_ZX_{N-1}) - M({}^A_ZX_N + m_n)) c^2 \quad (33)$$

This can also be expressed using binding energies as mass and energy are equivalent through  $E = mc^2$ :

$$S_n = B({}^A_ZX_N) - B({}^{A-1}_ZX_{N-1}) \quad (34)$$

### 5.4.2 Proton Separation Energy

Using the same logic as for the neutron separation energy [section 5.4.1](#), we can express the proton separation energy through the binding energies. It's important to keep in mind that after losing a proton, the element changes.

$$S_p = \left( M \left( {}_{Z-1}^{A-1}Y_N \right) - M \left( {}_Z^A X_N + \underbrace{m_p + m_n}_{{}_1^1\text{H}} \right) \right) c^2 \quad (35)$$

$$S_p = B \left( {}_Z^A X_N \right) - B \left( {}_{Z-1}^{A-1}Y_N \right) \quad (36)$$

### 5.5 Average Binding Energy

- Except for very light nuclei, the binding energy per nucleon is linear. It's almost constant at around 8 MeV/nucleon.
- The highest binding energy per nucleon is around  $A = 60$  with the highest binding energy per nucleon at  ${}^{56}\text{Fe}$ .
- When going from heavier elements towards iron we get nuclear fission
- When going from lighter elements towards iron we get nuclear fusion

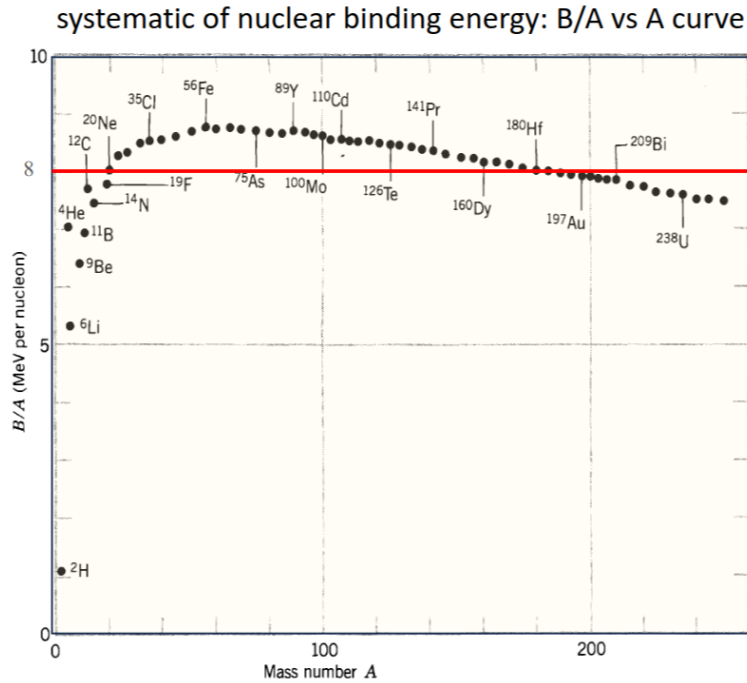


Figure 4

## 5.6 Semi-Empirical Mass Formula

- Sets out to explain the binding energies of nuclei.
- It is semi-empirical as the five of its constant are found by experiment.
- Tries to recreate the binding energy per nucleus graph in [figure 4](#) by using the *liquid drop model*.

$$B = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_{\text{asym}} \frac{(A-2Z)^2}{A} + \delta \quad (37)$$

### 5.6.1 Explanation of the Terms in the Semi-Empirical Mass Formula

- **$a_v A$ : Volume term.** The binding energy is proportional to the volume of the nucleus approximated to a sphere ( $V = 4\pi R^3/3$ ). This dominates the binding energy for large nuclei.

$$a_v \approx 15.8 \text{ MeV} \quad (38)$$

The linear dependence of the binding energy on the number of nucleons tells us that the strong force is short range as each nucleon only interacts with its nearest neighbors.

- **$a_s A^{2/3}$ : Surface term.** The volume term is not quite accurate as the nucleons on the surface have fewer neighbors. This term corrects for that. The binding energy is proportional to  $\pi R^2$

$$a_s \approx 16.8 \text{ MeV} \quad (39)$$

- **$a_c Z(Z-1)A^{-1/3}$ : Coulomb term.** The binding energy is reduced by the repulsion between the protons. It is therefore detracted. The Coulomb force is long range and is therefore proportional to  $Z(Z-1)$  as all protons interact.

$$a_c \approx 0.72 \text{ MeV} \quad (40)$$

- **$a_{\text{asym}}(A-2Z)^2 A^{-1}$ : Asymmetry term.** Stable nuclei have a balance between protons and neutrons. As the ratio of protons to neutrons deviate from 1, the nuclei becomes less stable (lower binding energy). This inhibits Hydrogen or Helium atoms with many neutrons. It is caused by the Pauli exclusion principle as nucleons are fermions and therefore can not occupy the same state at once.

$$a_{\text{asym}} \approx 23 \text{ MeV} \quad (41)$$

Heavier nuclei must have more neutrons to fight the Coulomb repulsion. The term gets relatively small as the number of nucleons increases.

- **$\delta$ : Pairing term.** This term is not included in the original formula, but is added to account for the fact that nuclei with an even number of protons and neutrons are more stable. This is because the nucleons in the same space-state can be coupled to have a total spin of 0. They are therefore closer together and therefore more tightly bound with a higher binding energy. This is called even-even nuclei.

$$\delta = \begin{cases} +a_p S^{-3/4}, & \text{if even(N)-even(Z)} \\ 0, & \text{if odd(A)} \\ -a_p S^{-3/4}, & \text{if odd(N)-odd(Z)} \end{cases} \quad (42)$$

$$a_p \approx 34 \text{ MeV} \quad (43)$$

### 5.6.2 SEMF Conclusion

- The semi-empirical mass formula was a first attempt at understanding how binding energy works.
- It is semi-empirical as the constants are found by experiment.
- A negative binding energy means the nucleus is not bound and is therefore not stable.

$$B = \underbrace{a_v A - a_s A^{2/3} - a_c Z(Z-1)A^{-1/3}}_{\text{Liquid-drop model for energy calculations}} - \underbrace{a_{\text{asym}}(A-2Z)^2 A^{-1} + \delta}_{\text{Interactions between nucleons}} \quad (44)$$

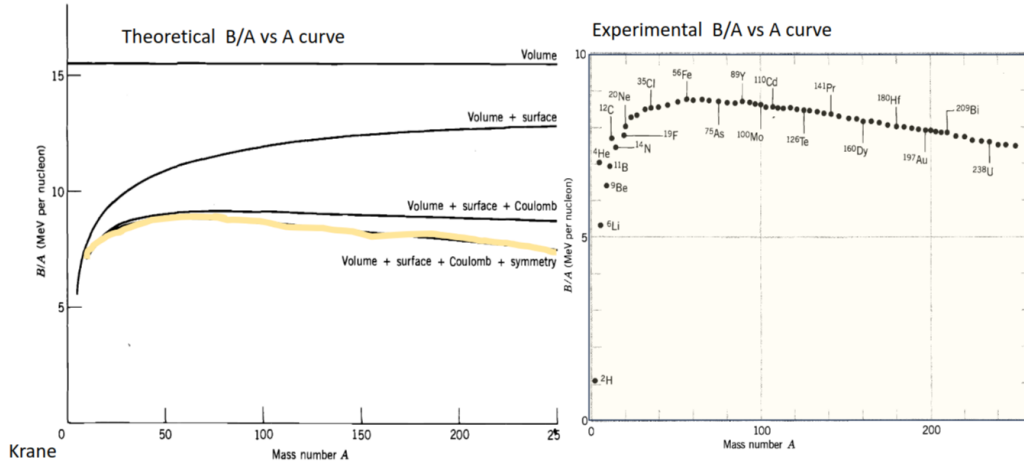


Figure 5: Plot of how the different terms in the semi-empirical mass formula [equation \(37\)](#) gets us closer to the experimental values

## 5.7 Mass Parabolas of Isobars

Isobars have the same number of nucleons ( $A$ ).

$$M(A, Z) = Z(\overbrace{m_p + m_e}^{M(^1H)}) + (\underbrace{A - Z}_{\text{neut. num.}})m_n - B(A, Z)/c^2 \quad (45)$$

$$B(A, Z) = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_a \frac{(A-2Z)^2}{A} + \delta(A, Z) \quad (46)$$

### 5.7.1 Finding the Minimum of the Mass Parabola

As the parabola is mass  $M$  as a function of  $Z$ , we can find the minimum by taking the derivative with respect to  $Z$  and setting it equal to zero.

$$\frac{\partial M}{\partial Z} = 0 \quad (47)$$

$$Z_{\min} = \frac{(m_n - m_p - m_e) + a_c A^{-1/3} + 4a_{\text{sym}}}{2a_c A^{-1/3} + 8a_{\text{sym}} A^{-1}} \quad (48)$$

We can approximate this as the following:

$$Z_{\min} \approx \frac{A}{2} - \frac{1}{1 + \frac{1}{4}A^{2/3}a_c/a_{\text{sym}}} \quad , \quad a_{\text{sym}} \approx 23 \text{ MeV} \quad , \quad a_c \approx 0.72 \text{ MeV} \quad (49)$$

**Example:  $A = 10$**  This is stable for smaller nuclei.

$$Z_{\min} \approx 5 \quad \text{and} \quad \frac{Z_{\min}}{A} \approx 0.5 \quad (50)$$

**Example:  $A = 200$**  A lower ratio is stable for larger nuclei.

$$Z_{\min} \approx 79 \quad \text{and} \quad \frac{Z_{\min}}{A} \approx 0.4 \quad (51)$$

### 5.7.2 Valley of (beta) stability

- As can be seen in [figure 6](#), we have two parabolas for  $A = 128$  as it can be odd-odd or even-even. Higher binding energy is more stable.
- The even-even isobar is more stable as explained in [section 5.6.1](#), because the nucleons can pair up in the same space-state with opposite spins and therefore be closer to each other and thus more stable.
- Only the atom in the bottom of the valley is stable. The others are prompt to beta decay downwards.
- Double beta decay can happen with even numbers of nucleons as can be seen for  $A = 128$  with  $Z = 52$ , as  $Z = 53$  has higher energy, and it is therefore forced to decay all the way up to  $Z = 54$ .

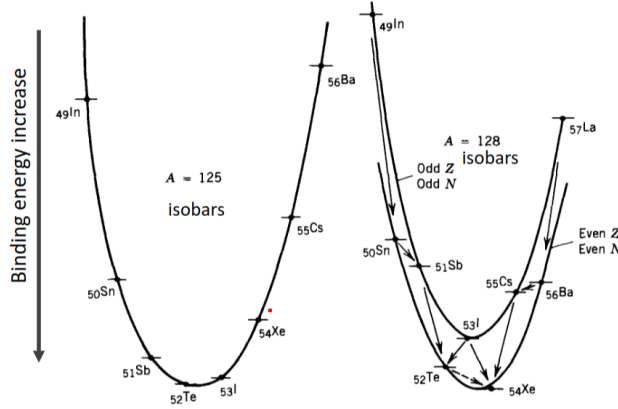


Figure 6: Valley of (beta) stability for different isobars with  $A = 125$  and  $A = 128$ . The higher the binding energy, the more stable the isobar.

## Beta Decay

- $\beta^+$ : Proton rich nuclei decay by converting a proton into a neutron, a positron and a neutrino.
- $\beta^-$ : Neutron rich nuclei decay by converting a neutron into a proton, an electron and an antineutrino.

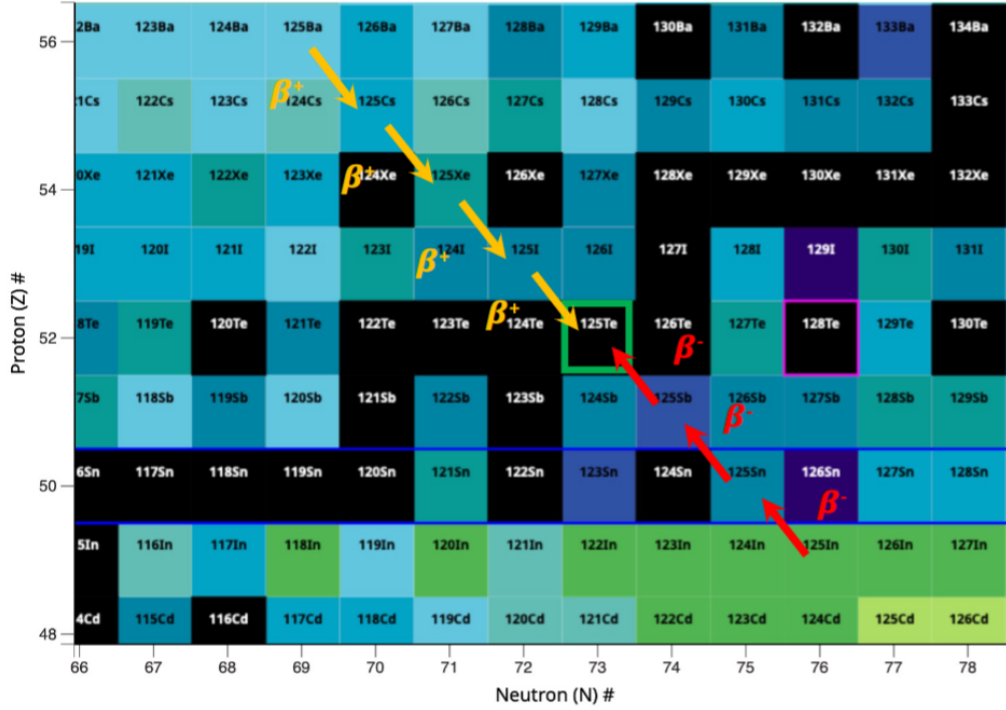


Figure 7: Chart showing different elements and their decays.

## 6 Angular Momentum & Parity

### 6.1 Angular Momentum of the Nucleus

- Total angular momentum  $j = l + s$  is the sum of the orbital angular momentum  $l$  and the spin  $s$ .
- Both  $l$  and  $s$  are quantized, and the total angular momentum  $j$  is also quantized.
- Nucleons are fermions and therefore spin half particles.
- Fermions can't rotate, but still have spin  $s$ . There is no classical analogy for this.  $l$  is the orbital angular momentum and is just like the classical angular momentum.

### 6.1.1 Orbital Angular Momentum

Angular momentum is a vector and thus has both magnitude and direction. As the values are quantized we use the quantum numbers  $l$ ,  $s$  and  $j$  to describe the magnitude and direction.

- $l$  is the orbital angular momentum and can take the values  $0, 1, 2, 3, \dots$
- Magnitude:

$$l = \sqrt{l(l+1)}\hbar \quad (52)$$

$$l_z = m_l \hbar \quad , \quad m_l \in \{-l, -l+1, \dots, l-1, l\} \quad (53)$$

- Direction [figure 8](#):

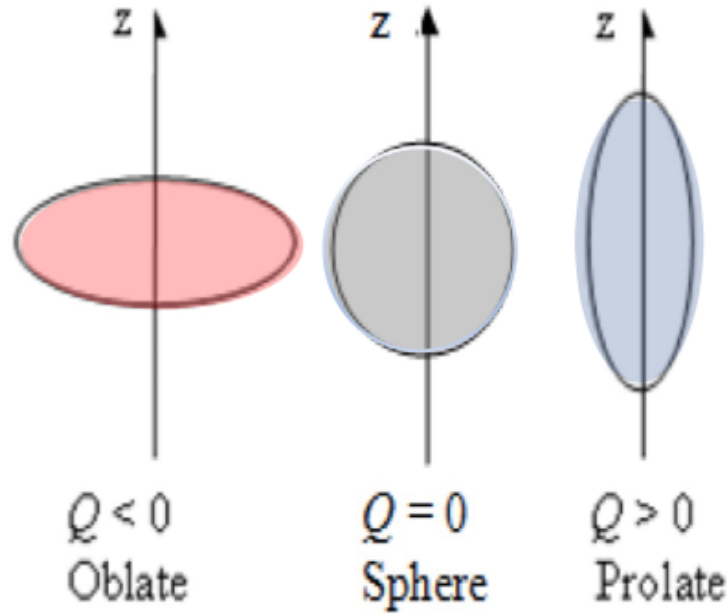


Figure 8: Orbital angular momentum vector visualized on a sphere.

### 6.1.2 Spin

- Spin is a property of particles and is not related to the motion of the particle.
- Spin is quantized and can take the values  $s = \frac{1}{2}$  or  $s = -\frac{1}{2}$ .
- Magnitude:

$$s = \sqrt{s(s+1)}\hbar \quad (54)$$

$$s_z = m_s \hbar \quad , \quad m_s \in \{-s, -s+1, \dots, s-1, s\} \quad (55)$$

- As the spin  $s$  can only be  $1/2$ , the magnetic quantum number  $m_s$  can only be  $\pm 1/2$



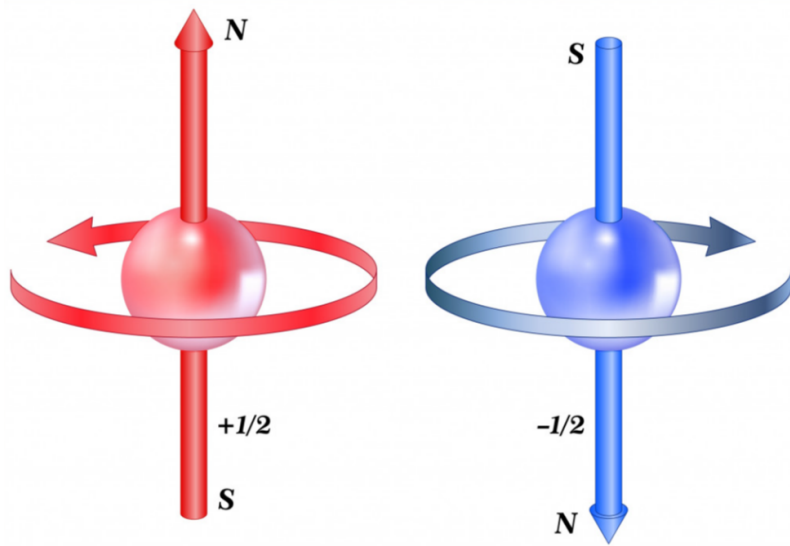


Figure 9: Visual representation of the spin of a nucleon in a magnetic field.

- Direction **figure 9**: There is no classical analogy for direction, but if in a magnetic field, the spin will align with or against the field

### 6.1.3 Total Angular Momentum

- The total angular momentum  $j$  is the sum of the orbital angular momentum  $l$  and the spin  $s$ .
- Magnitude:

$$j = \sqrt{j(j+1)}\hbar \quad (56)$$

$$j_z = m_j \hbar \quad , \quad m_j \in \{-j, -j+1, \dots, j-1, j\} \quad (57)$$

$$m_j = m_l + m_s = m_l \pm \frac{1}{2} \quad (58)$$

- Direction [figure 10](#):

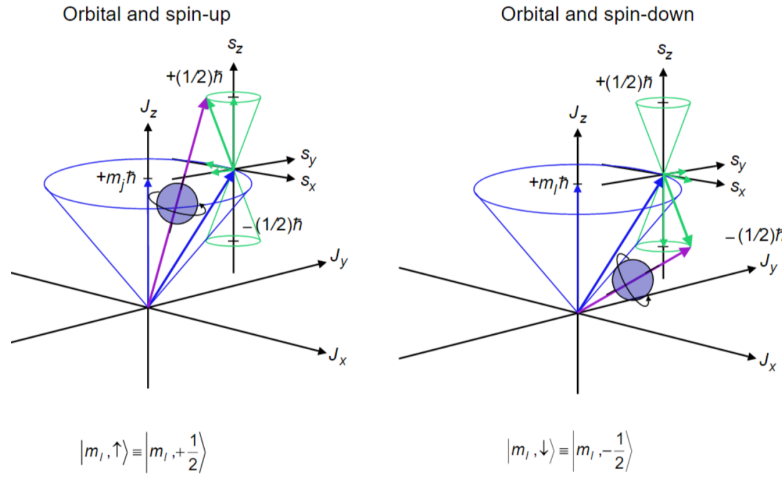


Figure 10: Total angular momentum visualized in 3D.

### 6.1.4 Total Angular Momentum of the Nucleus

- The sum of the angular momentum of all the nucleons in the nucleus.

$$\vec{I} = \sum_{i=1}^A \vec{j}_i \quad , \quad \vec{j}_i = \vec{l}_i + \vec{s}_i \quad (59)$$

$$I = \sqrt{I(I+1)}\hbar \quad (60)$$

$$I_z = m\hbar \quad , \quad m \in \{-I, -I+1, \dots, I-1, I\} \quad (61)$$

- As each nucleus has half-integer total angular momentum, odd number of nucleons  $A$  will have half-integer total angular momentum, and even number of nucleons will have integer total angular momentum.
- All the known even-even nuclei have spin-0 ground states.
- As a result, the ground state of an odd  $A$  nucleus must be the  $j$ -value of the odd proton or neutron.

## 6.2 Parity

Parity is the behavior of a system under the inversion of all spatial coordinates  $\vec{r} \rightarrow -\vec{r}$

- Cartesian coordinates:  $r \rightarrow (-x, -y, -z)$ .
- Spherical coordinates:  $r \rightarrow (-r, \pi - \theta, \varphi + \pi)$ .
- The parity operator is  $\hat{P}$  and has two effects on the wave function:
  - Even parity (+):  $\hat{P}\psi(\vec{r}) = \psi(\vec{r})$ .
  - Odd parity (-):  $\hat{P}\psi(\vec{r}) = -\psi(\vec{r})$ .
  - An even function is symmetric around the origin and an odd function is antisymmetric around the origin. This means  $\psi(-r) = \psi(r)$  or  $\psi(-r) = -\psi(r)$ .
  - Visual representation [figure 11](#):

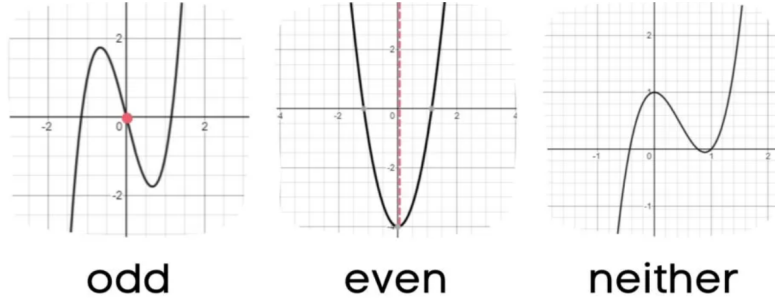


Figure 11: Visual representation of even and odd functions.

### 6.2.1 Splitting the Wave Function

- The wave function can be split into its radial and angular parts.

$$\Psi(\vec{r}) = R(r)Y(\theta, \varphi) \quad (62)$$

$$\hat{P}R(r) = R(r) \quad (63)$$

$$\hat{P}Y(\theta, \varphi) = (-1)^l Y_l^m(\theta, \phi) \quad (64)$$

- Parity of state with orbital angular momentum  $l$

$$\pi(-1)^l \quad (65)$$

- By convention, the intrinsic parity of the nucleon is  $\pi = +1$ , because they are fermions. Anti-fermions (like positron) have  $\pi = -1$ .
- For a composite system, the parity is the product of the intrinsic parities of the constituents.

$$\pi_{\text{total}} = (-1)^L \pi_1 \pi_2 \pi_3 \dots, \quad L = l_1 + l_2 + l_3 + \dots \quad (66)$$

## 7 Electric and Magnetic Moments

- The protons create a magnetic and electric fields.
- A distribution of charge is assigned an electric dipole moment of either monopole, dipole, quadrupole, octopole, etc.
- A spherical charge distribution gives only a monopole.
- A circular current only gives a magnetic dipole.
- Nuclei tend to have as simple of dipole moments as possible.
  - $L = 0$ : Monopole
  - $L = 1$ : Dipole
  - $L = 2$ : Quadrupole
  - $L = 3$ : Octopole

### 7.1 Parity Selection Rules

#### 7.1.1

Electric Dipole Moments  $E_0$

$$L = 0, 2 \tag{67}$$

Allowed values are  $L \in 0, 2$  with a parity of  $(-1)^L$ . A dipole is a measure of the separation of positive and negative charge. In the nucleus there is no separation. The electric monopole moment is just the charge of the nucleus  $Z \cdot e$ .

#### 7.1.2

Magnetic Dipole Moments  $M_1$

$$L = 1 \tag{68}$$

Allowed values are  $L = 1$  with a parity of  $(-1)^{L+1} = 1$ . The magnetic monopole has not been observed.

As the charged particles are moving, they create a magnetic field. For an electron orbiting a nucleus, we get the following:

$$|\vec{\mu}| = \frac{e}{2\pi r/v} \pi r^2 = \frac{e}{2m} |\vec{l}| \tag{69}$$

This connects the magnetic moment to the mass of the particle. The same goes for the protons in the nucleus. We know the  $z$ -component of the orbital angular momentum and can be inserted to the equation:

$$\mu = \frac{e\hbar}{2m} l \tag{70}$$

## 7.2 Bohr Magnetron & Nuclear Magnetron

For atomic motion, the electron mass is used.

$$\mu_B = \frac{e\hbar}{2m_e} = 5.788 \cdot 10^{-5} \text{ eV/T} \quad (71)$$

For nuclear motion, the proton mass is used.

$$\mu_N = \frac{e\hbar}{2m_p} = 3.152 \cdot 10^{-8} \text{ eV/T} \quad (72)$$

As  $\mu_B \gg \mu_N$ , the nuclear magnetic moment plays much smaller role in atomic physics.

## 7.3 Magnetic Moments of Nuclei

- Magnetic Dipole Moment:
  - The magnetic dipole moment of the nucleons is caused by their orbital motion.  $\mu = g_l \mu_N l$ .
  - The g-factor  $g_l$  is a dimensionless quantity characterizing the magnetic moment of the atom, nucleus or other particle in question.
  - Protons have  $g_l = 1$
  - Neutrons have  $g_l = -0.5$ . It was believed to be zero, but it proves it's not a point particle.
- Spin Magnetic Dipole Moment:
  - The magnetic dipole moment of the nucleons is caused by their spin.  $\mu = g_s \mu_N s$ .
  - The g-factor  $g_s$  is a dimensionless quantity characterizing the magnetic moment of the atom, nucleus or other particle in question.
  - Protons have  $g_s = 5.59 \pm 0.0000022$
  - Neutrons have  $g_s = -3.82 \pm 0.0000022$ . This is unexpected as the neutron is a neutral particle. This shows there charge inside the neutron, and it is not a point particle.
  - Electrons have  $g_s = 2$

### 7.3.1 Nuclear Structure from Magnetic Moments

- The pairing force favors the coupling of the nucleons such that the sum of their total angular momentum is zero.
- As a result, the magnetic moment of the nucleus is determined by the unpaired nucleons.
- Example [figure 12](#):

Nuclide	$\mu(\mu_N)$
n	-1.9130418
p	+2.7928456
$^2\text{H (D)}$	+0.8574376
$^{17}\text{O}$	-1.89379
$^{57}\text{Fe}$	+0.09062293
$^{57}\text{Co}$	+4.733
$^{93}\text{Nb}$	+6.1705

Figure 12: Table showing the magnetic dipole moments of different nuclei. The box in red shows how larger atoms have a larger magnetic dipole moment, caused by more unpaired nucleons.

## 7.4

Electric Quadrupole Moments  $E_2$  & Shape of the Nucleus Visual representation of the electric quadrupole moments effect on the shape of the nucleus in [figure 13](#).

$$eQ = e \int \psi^* (3z^2 - r^2) \psi \, dv \quad (73)$$

Experiment shows that large nuclei like Barium, has a pear-like shape.

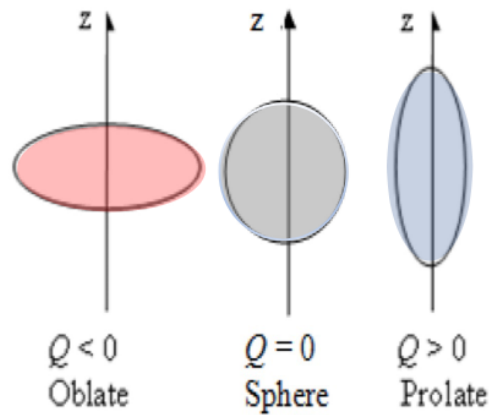


Figure 13: Shape of the nucleus as a function of the electric quadrupole moment.

## 7.5 Example: Calculating Parity of State

**Case 1: Calculate the parity of two nucleons in the  $p_{3/2}$  orbital** In the  $p_{3/2}$  orbital, we know  $l = 1$ . As all nucleons are fermions, the parity  $\pi$  of the orbital is  $(-1)^l = -1$ .

**Case 2: Calculate the parity of two nucleons in the  $g_{9/2}$  orbital** In the  $g_{9/2}$  orbital, we know  $l = 4$ . As all nucleons are fermions, the parity  $\pi$  of the orbital is  $(-1)^l = 1$ .

## 7.6 Level Schemes & Excited States

- Some nuclei have more excited states than others. This is regularly associated with even- $Z$  and even- $N$  nuclei in the interval  $150 \leq A \leq 190$ .
- Comparing the level schemes of different nuclei :

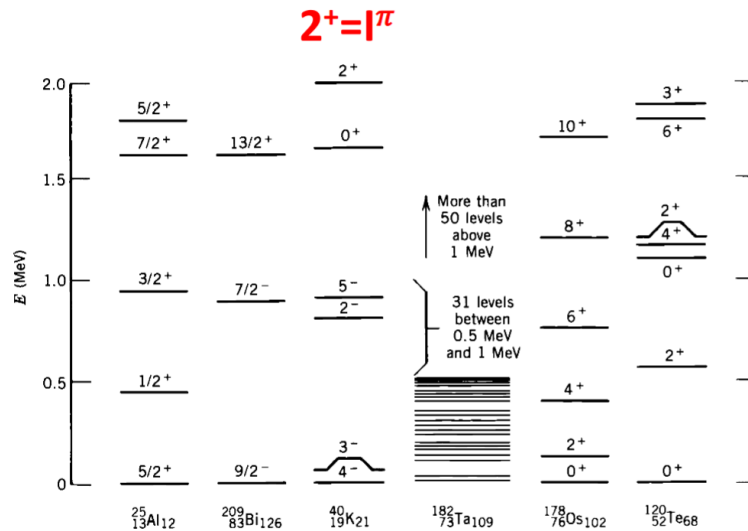


Figure 14: Some nuclei have more complex level schemes than others

## 8 Nuclear Force

- The strong force is very attractive at short distances. Even stronger than the Coulomb force.
- Negligible at greater distances than 1-2 fm.
- Some particles are immune, such as electrons. Electrons are 100,000 fm away from the nucleus.
- The strong force becomes very repelling at distances smaller than 1 fm.
- Nuclear force is nearly charge independent. We know this from experiments on excited states of *mirror nuclei* (same  $A$ , but opposite  $N$  and  $Z$ ) as seen in [figure 15](#).

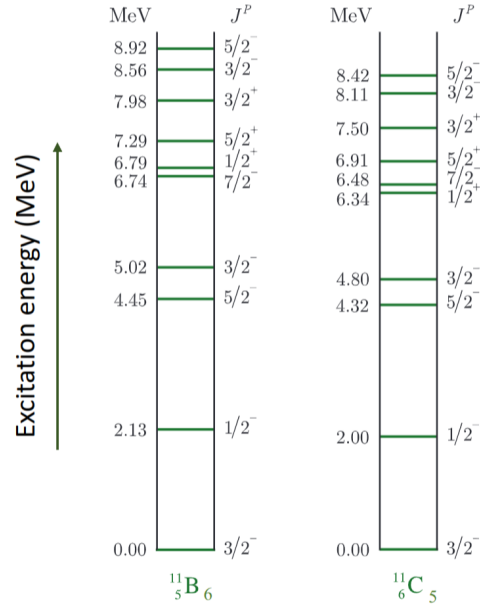


Figure 15: Comparison of the excitation levels of mirror nuclei. In this case we have  $^{11}_5\text{B}_6$  and  $^{11}_6\text{C}_5$

## 8.1 Effects of the Short Range of the Strong Force

- When shooting alpha particles at a nucleus, the alpha particles are repelled by the Coulomb force if they do not have enough energy to get close enough to the nucleus. Then the strong force takes over and the alpha particles are attracted to the nucleus. This is why the Rutherford model does not work at lower energies.
- The linear dependence on the binding energy per nucleon shows that the strong force is short range. If it were long range, each nucleon would attract all the others. Then the term in the binding energy as seen in the first term of [equation \(37\)](#), would be quadratic and not linear ( $\alpha_v A$ )

## 8.2 Deuteron

- Consist of a proton and a neutron (nucleus of deuterium).
- To understand the structure of the atoms we would need to study its excited states. The problem is that deuteron is weakly bound and has no excited states.

### 8.2.1 Deuteron Binding Energy

There are multiple ways of calculating the binding energy of the deuteron.

1. **Mass spectroscopy:** Find the difference in mass between the deuteron and the proton and neutron.

$$B = (M(^1\text{H}) + m_n - m(^2\text{H})) c^2 = 2.225 \text{ MeV} \quad (74)$$



2. **Nuclear reaction:** The gamma ray emitted when a neutron is captured by a proton is almost the binding energy. It has only energy, but can be converted to mass through  $E = mc^2$ .

$${}^1\text{H} + n \rightarrow {}^2\text{H} + \gamma \quad (75)$$

$$E_\gamma \approx B = M_{\text{initial}} - M_{\text{final}} \quad (76)$$

$$B = (M({}^1\text{H}) + m_n - M({}^2\text{H})) c^2 = 2.224 \text{ MeV} \quad (77)$$

### 8.2.2 Nucleon-Nucleon Potential

- We assume that the potential between the nucleons is a finite square well with a potential depth of  $-V_0$
- Solving for the Schrödinger equation for specific energy values and applying the boundary conditions we get the following results:

$$k_1 \cot(k_1 R) = -k_2 \quad (78)$$

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}} \quad k_2 = \sqrt{\frac{2m(V_0 + E)}{\hbar^2}} \quad (79)$$

- The radius  $R$  is now connected to the energy, and we know from scattering experiments that the radius is around 2.1 fm.
- The solution gives a potential of  $V_0 = 35 \text{ MeV}$ .
- The binding energy of the deuteron is just below the potential depth as seen in [figure 16](#).

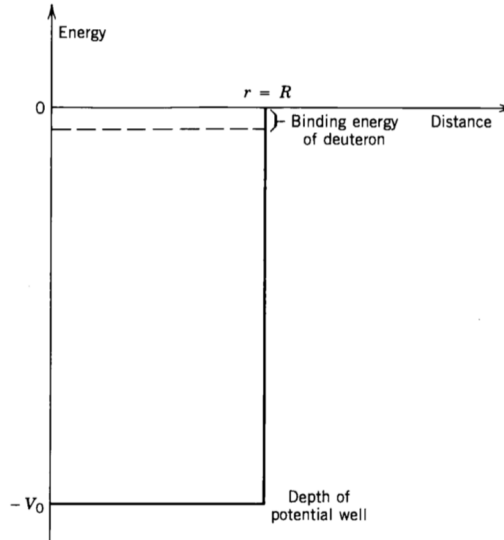


Figure 16: Square well potential for the deuteron.

### 8.2.3 Deuteron Spin and Parity

**Total Spin:**

$$\vec{I} = \vec{S}_p + \vec{S}_n + \vec{l} \quad (80)$$

**Spin Configuration with  $I = 1$**

1. Aligning the spins gives  $I = 1$  and  $S = 1$  with  $l = 0$ . This is a positive parity state with  $\pi = (-1)^l = 1$ . We then get  $I^\pi$
2. Aligning the spins with gives  $I = 3$

**Electric Quadrupole Moment** The deuteron has a small non-zero electric quadrupole moment. This makes it so about 4% of the time the deuteron is in an excited state with  $l = 2$ .

## 9 The Standard Model