FYS3500: Particle Physics
Lecture Notes

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1 History

- 1896: Henri Becquerel discovered radioactivity
- 1898: Marie and Pierre Curie discovered radium and polonium
- 1903: Alphas charge to mass ratio
- 1909: Alphas are helium nuclei
- 1911: Rutherford discovers the nucleus
- 1913: Bohr model of the atom
- 1917: Rutherford discovers the proton
- 1930: Neutrinos were postulated
- 1932: Chadwick discovers the neutron by shooting alpha particles at beryllium.
- 1938: Discovery of nuclear fission
- 1956: Neutrinos were detected

1.1 Proton Discovery: The Rutherford Scattering Experiment

Thomson's model of the atom was a positive sphere with electrons embedded in it. Rutherford wanted to test this model by shooting alpha particles at a thin gold foil surrounded by a detector foil. The alpha particles were shot from a radioactive source and when the alpha particles exited, they hit the foil and emitted light.

1.1.1 Conclusion

- Most alpha particles went straight through the foil. This implies the atom is mostly empty space.
- Some alpha particles were deflected by a small angle. This implies the positive charge is concentrated in a small volume.
- Sometimes the particles travel backwards. This implies the positive center has most of the mass of the atom.

1.2 Discovery of the Neutron

- Shooting alpha particles on beryllium which is much lighter than gold. This

2 Nucleus

- Very dense. Carries all the mass. $2.7 \cdot 10^{14}$ times denser than water.
- The atom is mostly empty space. If the nucleus was the size of a coin, the atom would be 2-3 km in radius.

2.1 Notation

- **Notation**: ${}^{A}_{Z}X_{N}$
- Isotope: Same **proton** number Z
- Isotone: Same **neutron** number N
- Isobar: Same **atomic** mass number $A = Z + N$

2.2 Nuclides

- 92 stable elements
- 280 stable isotopes
- 3000 unstable isotopes
- 6000 more predicted to exist

2.2.1 Stable Numbers

$$N = 2, 8, 20, 28, 50, 82, 126 \tag{1}$$

$$Z = 2, 8, 20, 28, 50, 82, \ldots \tag{2}$$

3 Units and Dimensions in Nuclear Physics

3.1 Length

The order of 10^{-15}m = 1fm (fermi/femtometer) meter. This is the distance between nucleons.

3.2 Time Scale

- 10^{-20}s: Unbound, in the case of nuclear reactions and decays.
- $10^{-9}/10^{-12}$s: lifetimes of excited nuclear states through gamma decays.
- Minutes/hours/millions of years: Alpha and beta decays.

3.3 Energy

MeV in nuclear physics.

$$1 \text{ MeV} = 1.6 \times 10^{-13}\text{J}. \tag{3}$$

$$1\text{eV} = 1.6 \times 10^{-19}\text{J} \tag{4}$$

Particle	Mass (kg)	Mass (u)	Mass (Mev/c^2)
1 atomic mass unit	1.660540×10^{-27} kg	1.000 u	931.5 MeV/c^2
neutron	1.674929×10^{-27} kg	1.008664 u	939.57 MeV/c^2
proton	1.672623×10^{-27} kg	1.007276 u	938.28 MeV/c^2
electron	9.109390×10^{-31} kg	0.00054858 u	0.511 MeV/c^2

Figure 1: Table of the masses of the nucleons. $c^2 = 931.5\text{MeV/u}$. In reality, the mass of the proton is slightly less than the mass of the neutron. The proton is 2000 times more massive than the electron.

3.4 Mass

u = unified atomic mass unit. 1 u is defined as 1/12 of the mass of an unbound ^{12}C atom. Mass is equivalent with energy. Therefore:

$$u = 931.5 \text{ MeV}/c^2 = 1.66 \times 10^{-27}\text{kg} \tag{5}$$

4 Nuclear Properties

The parameters which describe the nucleus are. There are two types of nuclear properties: static and dynamic.

- Static: Charge, Radius, mass, Binding energy, Angular momentum, Parity, Magnetic dipole moment, Electric quadrupole moments, Exited states and their energies.
- Dynamic: Shape, Decay

4.1 Connected Terms

- Charge/Charge Distribution: Protons. Found via electron scattering Section 4.3 by the Coulomb interaction.
- Matter/Mass Distribution: Nucleons. Found via hadron scattering Section 4.5, alpha particles (Rutherford), protons and neutrons by using the strong force.
- Radius: Size of the nucleus (nucleons)

4.2 Charge Distribution

To probe the charge distribution of the nucleus, we use charged particles. We also need the following:

- A beam of charged particles (often protons)
- Wavelength should be similar or smaller than the nucleus (about 10fm in diameter).
- Electrons were popular in the 50's.
- An energy of 100 Mev to 1 GeV is needed.
- Calculating the energy needed is done by using the de Broglie wavelength where $\lambda = h/p$ with $\lambda \leq 10$fm.

4.3 Nuclear Charge Distribution from Electron Scattering

- Radius increases with mass number A
- The central nuclear charge density is nearly the same for all nuclei. Nucleons do not seem to concentrate near the center of the nucleus, but instead have a constant distribution along the surface.
- The number of nucleons per unit volume is roughly constant:

$$\frac{A}{\frac{4}{3}\pi R^3} \approx \text{const} \tag{6}$$

- The radius of the nucleus is proportional to $A^{1/3}$.

$$R = R_0 A^{1/3} \quad , \quad R_0 \approx 1.2 \text{ fm} \tag{7}$$

4.4 Nuclear Size

We can find the radius of a nucleus by using the scattering angle of the local minimum of the Rutherford cross-section, see Fig. 2. The diffraction pattern is not exactly that of a circular disk, as the nucleus does not have a well-defined surface.

$$\sin\theta = \frac{1.22\lambda}{d} \Rightarrow R = \frac{d}{2} = \frac{1.22\lambda}{2\sin\theta} \tag{8}$$

This is only a rough estimate as the angle is calculated in two dimensions, instead of three.

4.5 Nuclear Mass Distribution from Hadron Scattering

- Electrons only mostly interact with protons. We therefore use hadrons to study the mass distribution of the nucleus.
- The radius is proportional to the nuclear rather than the Coulomb force.
- The Rutherford experiment showed that the nucleus is a point-like object.

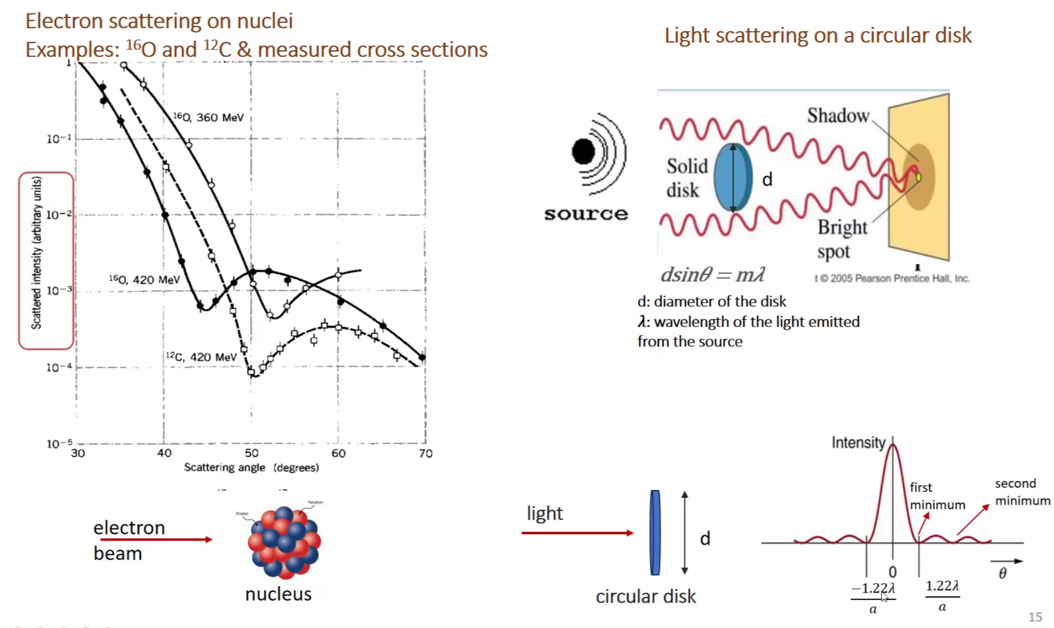


Figure 2: Example of the local minimum of the Rutherford cross-section. The angle is used to calculate the radius of the nucleus.

4.5.1 Fixed Angle of Observation with Changing Energy

- At low energies the alpha particles and the ^{208}Pb nucleus interact with the Coulomb interacting as with Rutherford scattering.
- With increasing energy, the repulsion from the Coulomb force is overcome, and the strong force becomes the dominant force. The Rutherford formula no longer holds.
- The alpha particles became absorbed by the nucleus and only a small fraction is scattered.
- When energy is high enough, we get the diffraction pattern.

4.6 Conclusion from Charge Radius Experiments

- The charge and mass radii of nuclei is nearly equal to within about 0.1fm.
- Both show the same $A^{1/3}$ dependence with $R_0 = 1.2$fm.
- As heavy nuclei have about 50 % more neutrons than protons, we might expect the neutron distribution to be more extended than the proton distribution. This is not the case as the neutrons pulls inwards, and the protons push outwards, until they are mixed such that the radius is the same.

4.7 Nuclear Mass

4.8 Deflection Spectrometer

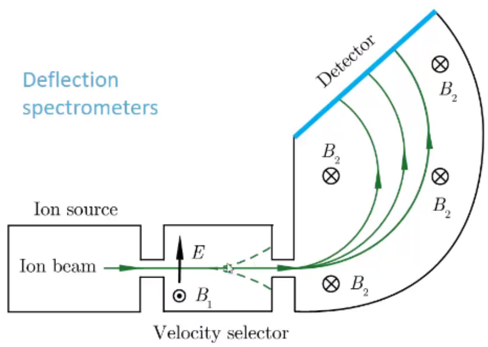


Figure 3: Experimental setup for measuring the mass of a particle.

- Shooting a ray of charge particles affected by a magnetic field and measuring the deflection we can calculate its mass.
- To measure an entire particle they must be ionized. The electrons carry so little mass that they are neglected.
- After ionization, the particles travel through an electric and magnetic field.
- Only the particles with the right velocity will pass through the fields and be subjected to the new magnetic field.
- The new field will deflect the particles according to their m/q value.

4.8.1 Calculating the Mass

$$F_B = q\vec{v} \times \vec{B} \tag{9}$$

The field and velocity are perpendicular.

$$F_B = qvB \tag{10}$$

$$F_E = F_B \Rightarrow qE = qvB \Rightarrow v = \frac{E}{B_1} \tag{11}$$

B_1 is the first magnetic field as seen in Fig. 3. The force from the magnetic field centripetal force.

$$F_B = \frac{mv^2}{r} = qvB_2 \tag{12}$$

$$\frac{mv}{r} = qB_2 \tag{13}$$

$$\frac{m}{q} = \frac{B_2\rho}{v} \tag{14}$$

The radius of the circle is given by $r = \rho$. Setting $B_1 = B_2$ gives the following for the mass.

$$m = \frac{B_1 B_2 \rho}{E} = \frac{B^2 \rho q}{E} \tag{15}$$

where q is the charge of the particle.

Accuracy

- These measurements are very important for mass models used in other parts of physics.
- The accuracy is about $\Delta m/m = 10^{6-}$, but that is not enough.
- The mass doublet technique gives a precision of 10^{-8} / 10^{-9}

5 Binding Energy

5.1 Formulas and Definitions

- Binding Energy: The energy required to keep the nucleus together. The mass of the nucleus is not equal to the sum of its parts. The mass of the individual nucleons is higher than the mass of the nucleus. The difference is the binding energy.

$$Zm_p + Nm_n - M_{\text{Nucleus}} = \text{Binding Energy} \quad \Rightarrow \quad Zm_p + Nm_n > M_{\text{Nucleus}} \tag{16}$$

5.2 Mass of the Nucleus

The total mass of the atom is the mass of the nucleus and electrons, minus the binding energy of the electrons.

$$M_{\text{Atom}} = M_{\text{Nucleus}} + Zm_e - \underbrace{\sum_{i=1}^{Z} B_i/c^2}_{\text{Often negligible}} \tag{17}$$

$$M_{\text{Atom}} = M_{\text{Nucleus}} + Zm_e \tag{18}$$

M usually refers to the mass of the entire atom, and so the subscript "Atom" is often omitted. We usually write the atom using the following notation:

$$M\left({}^A_Z X_N\right) = M_{\text{Nucleus}}\left({}^A_Z X_N\right) + Zm_e \tag{19}$$

Multiplying by c^2 we get the mass in energy units ($E = mc^2$):

$$M_{\text{Nucleus}}\left({}^A_Z X_N\right) = M\left({}^A_Z X_N\right) - Zm_e c^2 \tag{20}$$

$$\underline{\underline{M_{\text{Nucleus}}\left({}^A_Z X_N\right) c^2 = M\left({}^A_Z X_N\right) c^2 - Zm_e c^2}} \tag{21}$$

5.3 Nuclear Binding Energy (B.E.)

This energy is very small compared to the mass energy of the nucleus. We can derive this from the mass of the nucleus.

$$
\begin{aligned}
B.E. &= \left(Zm_p + Nm_n - M_N\left({}^A_Z X_N\right)\right)c^2 && (22)\\
&= \left(Zm_p + Nm_n - \left(M\left({}^A_Z X_N\right) - Zm_e\right)\right)c^2 && (23)\\
&= \left(Z\underbrace{(m_p + m_e)}_{\text{Hydrogen}} + Nm_n - M\left({}^A_Z X_N\right)\right)c^2 && (24)\\
& && (25)\\
B.E. &= \underline{\underline{\left(Zm\left({}^1H\right) + Nm_n - M\left({}^A_Z X_N\right)\right)c^2}} && (26)
\end{aligned}
$$

As the units so far has been energy (mc^2) we can switch to MeV.

$$B.E. = \left[mc^2\right] = \left[uc^2\right] = u931.5\text{MeV }/u \quad \Rightarrow \quad c^2 = 931.5\text{MeV/u} \tag{27}$$

$$B.E. = \underline{\underline{\left(Zm\left({}^1H\right) + Nm_n - M\left({}^A_Z X_N\right)\right)931.5\text{MeV/u}}} \tag{28}$$

5.3.1

Example: Helium ${}^4_2 H_2$ We use the formula for binding energy from Eq. (28) to calculate the binding energy of the hydrogen atom ${}^4_2 He_2$.

$$
\begin{aligned}
B.E. &= \left(2m_p + 2m_n - M\left({}^4_2 He_2\right)\right)931.5\text{MeV/u} && (29)\\
&= (2 \cdot 1.007825u + 2 \cdot 1.008664u - 4.002603u) \cdot 931.5\text{MeV/u} && (30)\\
&= \underline{\underline{0.0304 \cdot 931.5 \text{ MeV} = 28.3 \text{ MeV}}} && (31)
\end{aligned}
$$

The ratio between the binding energy and the rest mass of the nucleus is very small. Using the binding energy from Eq. (31) and the mass of the helium nucleus, we can calculate the ratio:

$$\frac{28.3}{3728} = 0.75\% \tag{32}$$

5.4 Nuclear Separation Energy

The energy required to separate a proton S_p or a neutron S_n from the nucleus.

5.4.1 Neutron Separation Energy

It requires lower energy to remove a neutron from a nucleus with an odd number of neutrons. This is because one is unpaired.

$$S_n = \left(M\left({}^{A-1}_Z X_{N-1}\right) - M({}^A_Z X_N + m_n)\right)c^2 \tag{33}$$

This can also be expressed using binding energies as mass and energy are equivalent through $E = mc^2$:

$$S_n = B\left({}^A_Z X_N\right) - B\left({}^{A-1}_Z X_{N-1}\right) \tag{34}$$

5.4.2 Proton Separation Energy

Using the same logic as for the neutron separation energy Section 5.4.1, we can express the proton separation energy through the binding energies. It's important to keep in mind that after loosing a proton, the element changes.

$$S_p = \left(M\left({}^{A-1}_{Z-1}Y_N\right) - M({}^{A}_{Z}X_N + \underbrace{m_p + m_n}_{{}^1_1H}) \right) c^2 \tag{35}$$

$$S_p = B\left({}^{A}_{Z}X_N\right) - B\left({}^{A-1}_{Z-1}Y_N\right) \tag{36}$$

- Except for very light nuclei, the binding energy per nucleon is linear. It's almost constant at around 8 MeV/nucleon.
- The highest binding energy per nucleon is around $A = 60$ with the highest binding energy per nucleon at ^{56}Fe.
- When going from heavier elements towards iron we get nuclear fission
- When going from lighter elements towards iron we get nuclear fusion

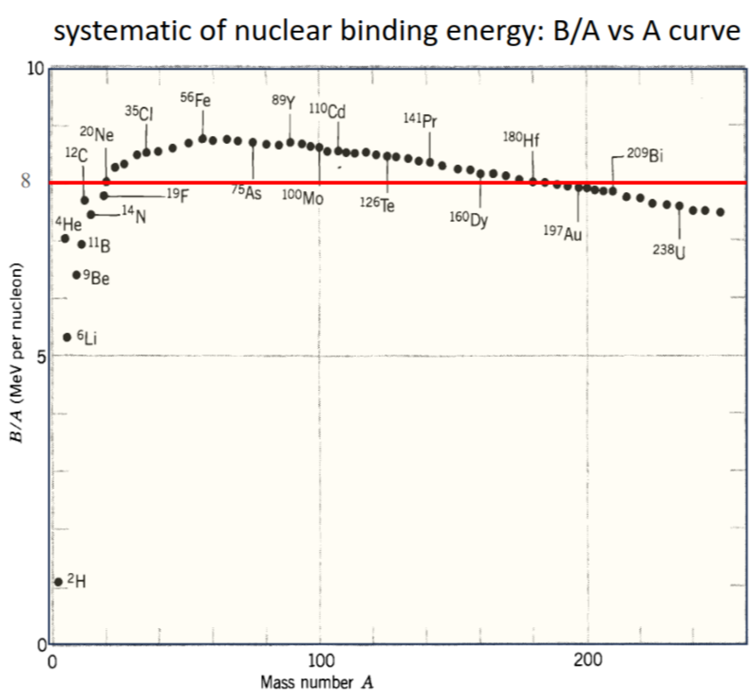


Figure 4

5.5 Semi-Empirical Mass Formula

- Sets out to explain the binding energies of nuclei.
- It is semi-empirical as the five of its constant are found by experiment.
- Tries to recreate the binding energy per nucleus graph in Fig. 4 by using the *liquid drop model*.

$$B = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_{\text{asym}} \frac{(A-2Z)^2}{A} + \delta \tag{37}$$

5.5.1 Explanation of the Terms in the Semi-Empirical Mass Formula

- $\mathbf{a_v A}$: **Volume term**. The binding energy is proportional to the volume of the nucleus approximated to a sphere $\big(V = 4\pi R^3/3\big)$. This dominates the binding energy for large nuclei.
$$a_v \approx 15.8 \text{ MeV} \tag{38}$$
The linear dependence of the binding energy on the number of nucleons tells us that the strong force is short range as each nucleon only interacts with its nearest neighbors.
- $\mathbf{a_s A^{2/3}}$: **Surface term**. The volume term is not quite accurate as the nucleons on the surface have fewer neighbors. This term corrects for that. The binding energy is proportional to πR^2
$$a_s \approx 16.8 \text{ MeV} \tag{39}$$
- $\mathbf{a_c Z(Z-1) A^{-1/3}}$: **Coulomb term**. The binding energy is reduced by the repulsion between the protons. It is therefore detracted. The Coulomb force is long range and is therefore proportional to $Z(Z-1)$ as all protons interact.
$$a_c \approx 0.72 \text{ MeV} \tag{40}$$
- $\mathbf{a_{\text{asym}} (A-2Z)^2 A^{-1}}$: **Asymmetry term**. Stable nuclei have a balance between protons and neutrons. As the ratio of protons to neutrons deviate from 1, the nuclei becomes less stable (lower binding energy). This inhibits Hydrogen or Helium atoms with many neutrons. It is caused by the Pauli exclusion principle as nucleons are fermions and therefore can not occupy the same state at once.
$$a_{\text{asym}} \approx 23 \text{ MeV} \tag{41}$$
Heavier nuclei must have more neutrons to fight the Coulomb repulsion. The term gets relatively small as the number of nucleons increases.
- δ: **Pairing term**. This term is not included in the original formula, but is added to account for the fact that nuclei with an even number of protons and neutrons are more stable. This is because the nucleons in the same space-state can be coupled to have a total spin of 0. They are therefore closer together and therefore more tightly bound with a higher binding energy. This is called even-even nuclei.
$$\delta = \begin{cases} +a_p S^{-3/4}, & \text{if even(N)-even(Z)} \\ 0, & \text{if odd(A)} \\ -a_p S^{-3/4}, & \text{if odd(N)-odd(Z)} \end{cases} \tag{42}$$
$$a_p \approx 34 \text{ MeV} \tag{43}$$

5.5.2 SEMF Conclusion

- The semi-empirical mass formula was a first attempt at understanding how binding energy works.
- It is semi-empirical as the constants are found by experiment.
- A negative binding energy means the nucleus is not bound and is therefore not stable.

$$B = \underbrace{a_v A - a_s A^{2/3} - a_c Z(Z-1)A^{-1/3}}_{\text{Liquid-drop model for energy calculations}} - \underbrace{a_{\text{asym}}(A-2Z)^2 A^{-1} + \delta}_{\text{Interactions between nucleons}} \tag{44}$$

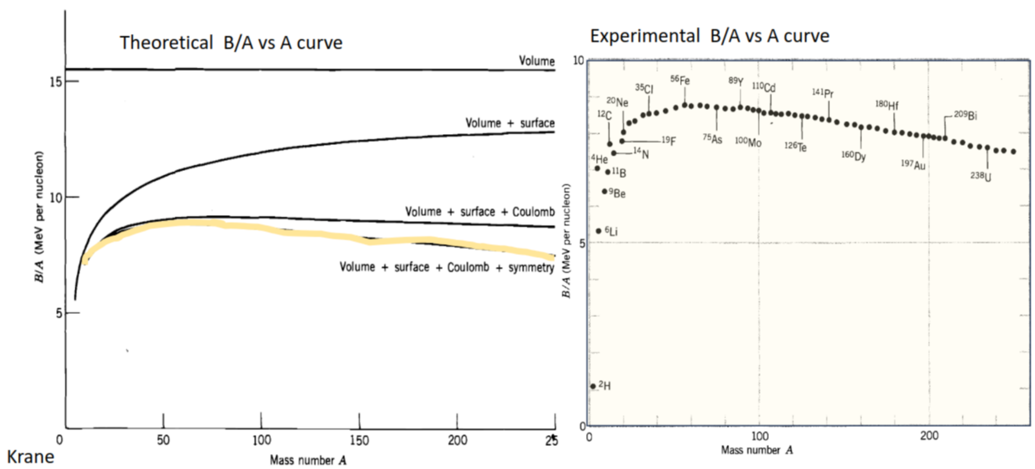


Figure 5: Plot of how the different terms in the semi-empirical mass formula Eq. (37) gets us closer to the experimental values

5.6 Mass Parabolas of Isobars

Isobars have the same number of nucleons (A).

$$M(A,Z) = Z\overbrace{(m_p + m_e)}^{M(^1H)} + (\underbrace{A-Z}_{\text{neut. num.}})m_n - B(A,Z)/c^2 \tag{45}$$

$$B(A,Z) = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_a \frac{(A-2Z)^2}{A} + \delta(A,Z) \tag{46}$$

5.6.1 Finding the Minimum of the Mass Parabola

As the parabola is mass M as a function of Z, we can find the minimum by taking the derivative with respect to Z and setting it equal to zero.

$$\frac{\partial M}{\partial Z} = 0 \tag{47}$$

$$Z_{\text{min}} = \frac{(m_n - m_p - m_e) + a_c A^{-1/3} + 4a_{\text{sym}}}{2a_c A^{-1/3} + 8a_{\text{sym}} A^{-1}} \tag{48}$$

We can approximate this as the following:

$$Z_{\text{min}} \approx \frac{A}{2} \frac{1}{1 + \frac{1}{4} A^{2/3} a_c / a_{\text{sym}}} \quad , \quad a_{\text{sym}} \approx 23 \text{ MeV} \ , \ a_c \approx 0.72 \text{ MeV} \tag{49}$$

Example: A = 10 This is stable for smaller nuclei.

$$Z_{\text{min}} \approx 5 \quad \text{and} \quad \frac{Z_{\text{min}}}{A} \approx 0.5 \tag{50}$$

Example: A = 200 A lower ratio is stable for larger nuclei.

$$Z_{\text{min}} \approx 79 \quad \text{and} \quad \frac{Z_{\text{min}}}{A} \approx 0.4 \tag{51}$$

5.6.2 Valley of (beta) stability

- As can be seen in Fig. 6, we have two parabolas for $A = 128$ as it can be odd-odd or even-even. Higher binding energy is more stable.
- The even-even isobar is more stable as explained in Section 5.5.1, because the nucleons can pair up in the same space-state with opposite spins and therefore be closer to each other and thus more stable.
- Only the atom in the bottom of the valley is stable. The others are prompt to beta decay downwards.
- Double beta decay can happen with even numbers of nucleons as can be seen for $A = 128$ with $Z = 52$, as $Z = 53$ has higher energy, and it is therefore forced to decay all the way up to $Z = 54$.

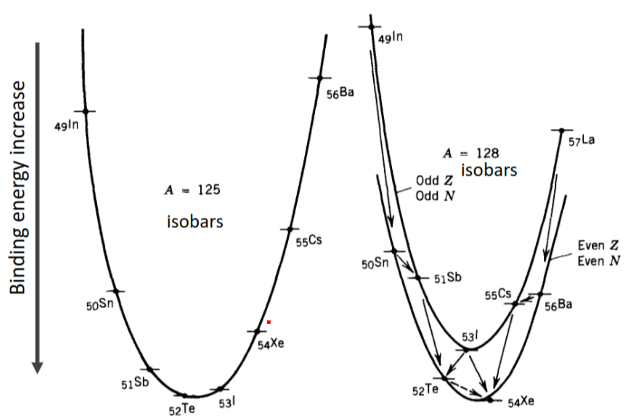


Figure 6: Valley of (beta) stability for different isobars with $A = 125$ and $A = 128$. The higher the binding energy, the more stable the isobar.

Beta Decay

- $\beta+$: Proton rich nuclei decay by converting a proton into a neutron, a positron and a neutrino.
- $\beta-$: Neutron rich nuclei decay by converting a neutron into a proton, an electron and an antineutrino.

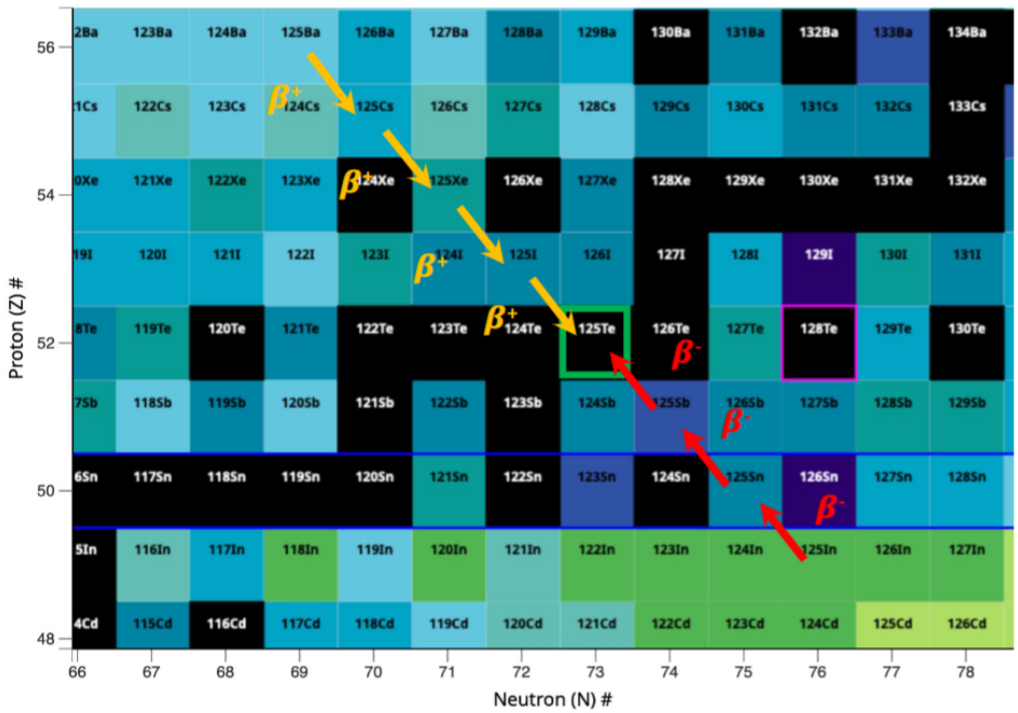


Figure 7: Chart showing different elements and their decays.

6 Angular Momentum & Parity

6.1 Angular Momentum of the Nucleus

- Total angular momentum $j = l + s$ is the sum of the orbital angular momentum l and the spin s.
- Both l and s are quantized, and the total angular momentum j is also quantized.
- Nucleons are fermions and therefore spin half particles.
- Fermions can't rotate, but still have spin s. There is no classical analogy for this. l is the orbital angular momentum and is just like the classical angular momentum.

6.1.1 Orbital Angular Momentum

Angular momentum is a vector and thus has both magnitude and direction. As the values are quantized we use the quantum numbers l, s and j to describe the magnitude and direction.

- l is the orbital angular momentum and can take the values $0, 1, 2, 3, \ldots$.
- Magnitude:

$$l = \sqrt{l(l+1)}\hbar \tag{52}$$
$$l_z = m_l\hbar \quad , \quad m_l \in \{-l, -l+1, \ldots, l-1, l\} \tag{53}$$

- Direction Fig. 8:

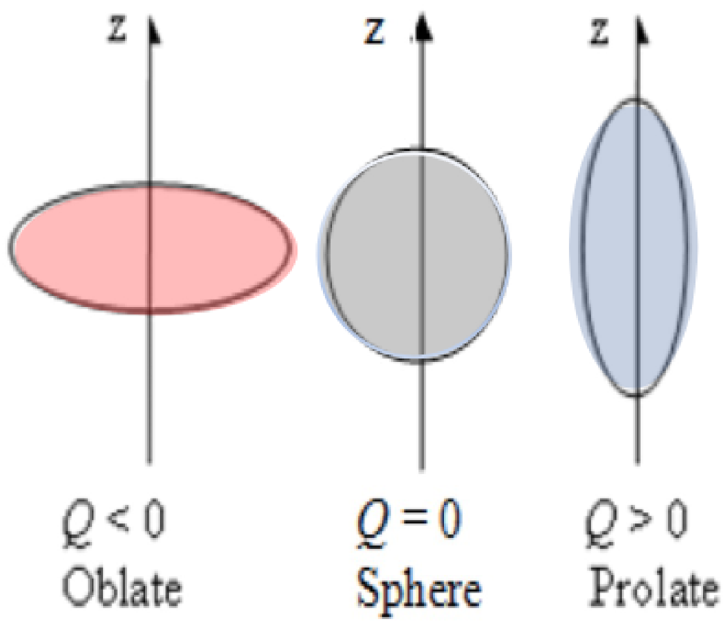


Figure 8: Orbital angular momentum vector visualized on a sphere.

6.1.2 Spin

- Spin is a property of particles and is not related to the motion of the particle.
- Spin is quantized and can take the values $s = \frac{1}{2}$ or $s = -\frac{1}{2}$.
- Magnitude:

$$s = \sqrt{s(s+1)}\hbar \tag{54}$$
$$s_z = m_s\hbar \quad , \quad m_s \in \{-s, -s+1, \ldots, s-1, s\} \tag{55}$$

- As the spin s can only be 1/2, the magnetic quantum number m_s can only be $\pm 1/2$

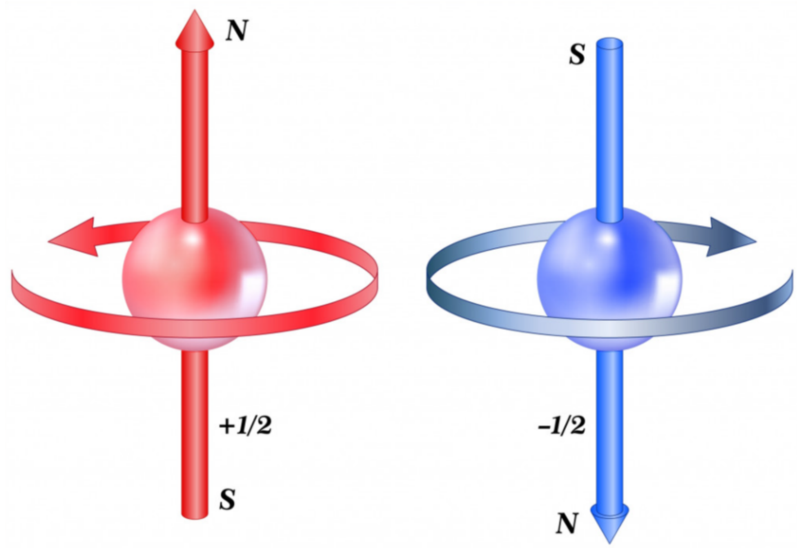


Figure 9: Visual representation of the spin of a nucleon in a magnetic field.

- Direction Fig. 9: There is no classical analogy for direction, but if in a magnetic field, the spin will align with or against the field

6.1.3 Total Angular Momentum

- The total angular momentum j is the sum of the orbital angular momentum l and the spin s.
- Magnitude:

$$j = \sqrt{j(j+1)}\hbar \tag{56}$$

$$j_z = m_j\hbar \quad , \quad m_j \in \{-j, -j+1, \dots, j-1, j\} \tag{57}$$

$$m_j = m_l + m_s = m_l \pm \frac{1}{2} \tag{58}$$

- Direction Fig. 10:

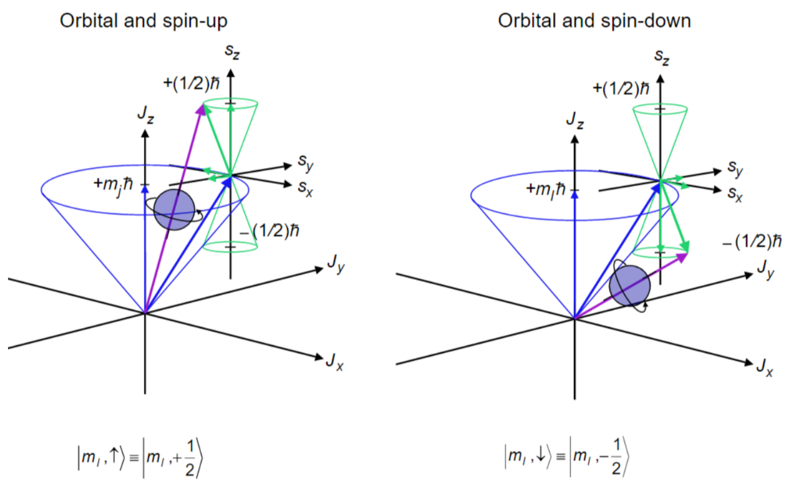


Figure 10: Total angular momentum visualized in 3D.

6.1.4 Total Angular Momentum of the Nucleus

- The sum of the angular momentum of all the nucleons in the nucleus.

$$\vec{I} = \sum_{i=1}^{A} \vec{j}_i \quad , \quad \vec{j}_i = \vec{l}_i + \vec{s}_i \tag{59}$$

$$I = \sqrt{I(I+1)}\hbar \tag{60}$$

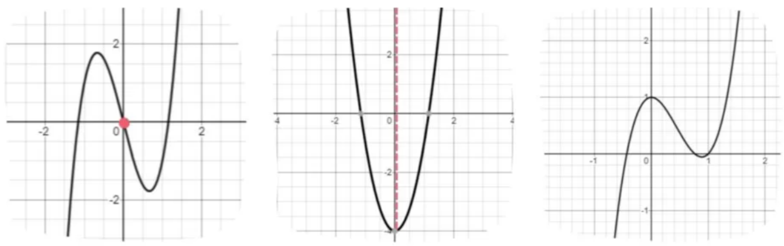
$$I_z = m\hbar \quad , \quad m \in \{-I, -I+1, \dots, I-1, I\} \tag{61}$$

- As each nucleus has half-integer total angular momentum, odd number of nucleons A will have half-integer total angular momentum, and even number of nucleons will have integer total angular momentum.
- All the known even-even nuclei have spin-0 ground states.
- As a result, the ground state of an odd A nucleus must be the j-value of the odd proton or neutron.

6.2 Parity

Parity is the behavior of a system under the inversion of all spatial coordinates $\vec{r} \to -\vec{r}$

- Cartesian coordinates: $r \to (-x, -y, -z)$.
- Spherical coordinates: $r \to (-r, \pi - \theta, \varphi + \pi)$.
- The parity operator is $\hat{P}$ and has two effects on the wave function:
  - Even parity (+): $\hat{P}\psi(\vec{r}) = \psi(\vec{r})$.
  - Odd parity (-): $\hat{P}\psi(\vec{r}) = -\psi(\vec{r})$.
  - An even function is symmetric around the origin and an odd function is antisymmetric around the origin. This means $\psi(-r) = \psi(r)$ or $\psi(-r) = -\psi(r)$.
  - Visual representation Fig. 11:



odd even neither

Figure 11: Visual representation of even and odd functions.

6.2.1 Splitting the Wave Function

- The wave function can be split into its radial and angular parts.

$$\Psi(\vec{r}) = R(r)Y(\theta, \varphi) \tag{62}$$
$$\hat{P}R(r) = R(r) \tag{63}$$
$$\hat{P}Y(\theta, \varphi) = (-1)^l Y_l^m(\theta, \phi) \tag{64}$$

- Parity of state with orbital angular momentum l

$$\pi(-1)^l \tag{65}$$

- By convention, the intrinsic parity of the nucleon is $\pi = +1$, because they are fermions. Anti-fermions (like positron) have $\pi = -1$.
- For a composite system, the parity is the product of the intrinsic parities of the constituents.

$$\pi_{\text{total}} = (-1)^L \pi_1 \pi_2 \pi_3 \ldots \quad , \quad L = l_1 + l_2 + l_3 + \ldots \tag{66}$$

7 Electric and Magnetic Moments

- The protons create a magnetic and electric fields.
- A distribution of charge is assigned an electric dipole moment of either monopole, dipole, quadrupole, octopole, etc.
- A spherical charge distribution gives only a monopole.
- A circular current only gives a magnetic dipole.
- Nuclei tend to have as simple of dipole moments as possible.
  - L = 0: Monopole
  - L = 1: Dipole
  - L = 2: Quadrupole
  - L = 3: Octopole

7.1 Parity Selection Rules

7.1.1

Electric Dipole Moments E_0

$$L = 0, 2 \tag{67}$$

Allowed values are $L \in 0, 2$ with a parity of $(-1)^L$. A dipole is a measure of the separation of positive and negative charge. In the nucleus there is no separation.
The electric monopole moment is just the charge of the nucleus $Z \cdot e$.

7.1.2

Magnetic Dipole Moments M_1

$$L = 1 \tag{68}$$

Allowed values are $L = 1$ with a parity of $(-1)^{L+1} = 1$. The magnetic monopole has not been observed.
As the charged particles are moving, they create a magnetic field. For an electron orbiting a nucleus, we get the following:

$$|\vec{\mu}| = \frac{e}{2\pi r/v}\pi r^2 = \frac{e}{2m}\left|\vec{l}\right| \tag{69}$$

This connects the magnetic moment to the mass of the particle. The same goes for the protons in the nucleus. We know the z-component of the orbital angular momentum and can be inserted to the equation:

$$\mu = \frac{e\hbar}{2m} l \tag{70}$$

7.2 Bohr Magneton & Nuclear Magneton

For atomic motion, the electron mass is used.

$$\mu_B = \frac{e\hbar}{2m_e} = 5.788 \cdot 10^{-5} \text{ eV/T} \tag{71}$$

For nuclear motion, the proton mass is used.

$$\mu_N = \frac{e\hbar}{2m_p} = 3.152 \cdot 10^{-8} \text{ eV/T} \tag{72}$$

As $\mu_B \gg \mu_N$, the nuclear magnetic moment plays much smaller role in atomic physics.

7.3 Magnetic Moments of Nuclei

- Magnetic Dipole Moment:
  - The magnetic dipole moment of the nucleons is caused by their orbital motion. $\mu = g_l \mu_N l$.
  - The g-factor g_l is a dimensionless quantity characterizing the magnetic moment of the atom, nucleus or other particle in question.
  - Protons have $g_l = 1$
  - Neutrons have $g_l = -0.5$. It was believed to be zero, but it proves it's not a point particle.
- Spin Magnetic Dipole Moment:
  - The magnetic dipole moment of the nucleons is caused by their spin. $\mu = g_s \mu_N s$.
  - The g-factor g_s is a dimensionless quantity characterizing the magnetic moment of the atom, nucleus or other particle in question.
  - Protons have $g_s = 5.59 \pm 0.0000022$
  - Neutrons have $g_s = -3.82 \pm 0.0000022$. This is unexpected as the neutron is a neutral particle. This shows there charge inside the neutron, and it is not a point particle.
  - Electrons have $g_s = 2$

7.3.1 Nuclear Structure from Magnetic Moments

- The pairing force favors the coupling of the nucleons such that the sum of their total angular momentum is zero.
- As a result, the magnetic moment of the nucleus is determined by the unpaired nucleons.
- Example Fig. 12:

Nuclide	$\mu(\mu_N)$
n	-1.9130418
p	$+2.7928456$
^{2}H (D)	$+0.8574376$
^{17}O	-1.89379
^{57}Fe	$+0.09062293$
^{57}Co	$+4.733$
^{93}Nb	$+6.1705$

Figure 12: Table showing the magnetic dipole moments of different nuclei. The box in red shows how larger atoms have a larger magnetic dipole moment, caused by more unpaired nucleons.

7.4

Electric Quadrupole Moments E_2 & Shape of the Nucleus Visual representation of the electric quadrupole moments effect on the shape of the nucleus in Fig. 13.

$$eQ = e\int \psi^*(3z^2 - r^2)\psi \, \mathrm{d}v \tag{73}$$

Experiment shows that large nuclei like Barium, has a pear-like shape.

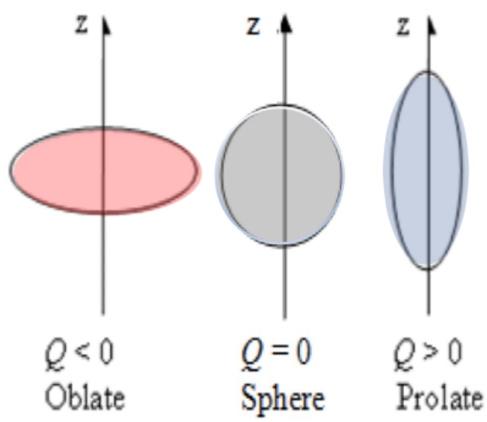


Figure 13: Shape of the nucleus as a function of the electric quadrupole moment.

7.5 Example: Calculating Parity of State

Case 1: Calculate the parity of two nucleons in the $p_{3/2}$ orbital In the $p_{3/2}$ orbital, we know $l = 1$. As all nucleons are fermions, the parity π of the orbital is $(-1)^l = -1$.

Case 2: Calculate the parity of two nucleons in the $g_{9/2}$ orbital In the $g_{9/2}$ orbital, we know $l = 4$. As all nucleons are fermions, the parity π of the orbital is $(-1)^l = 1$.

7.6 Level Schemes & Excited States

- Some nuclei have more excited states than others. This is regularly associated with even-Z and even-N nuclei in the interval $150 \leq A \leq 190$.
- Comparing the level schemes of different nuclei :

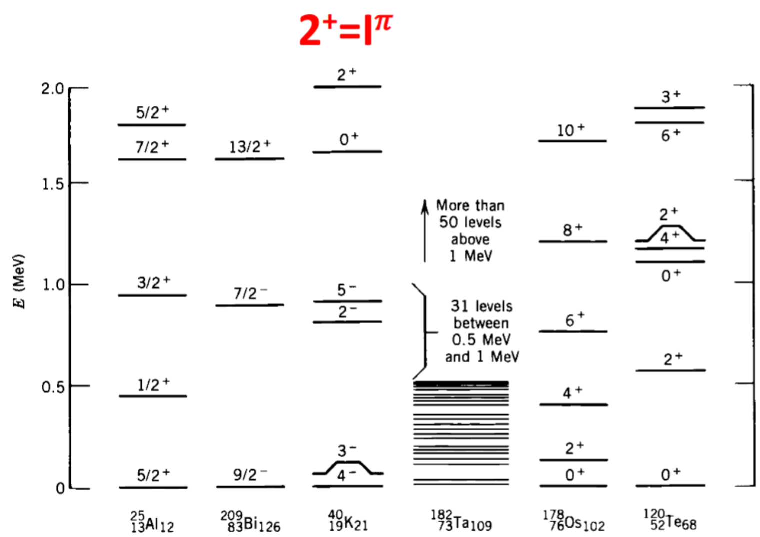


Figure 14: Some nuclei have more complex level schemes than others

8 Nuclear Force

- The strong force is very attractive at short distances. Even stronger than the Coulomb force.
- Negligible at greater distances than 1-2 fm.
- Some particles are immune, such as electrons. Electrons are 100,000 fm away from the nucleus.
- The strong force becomes very repelling at distances smaller than 1 fm.
- Nuclear force is nearly charge independent. We know this from experiments on excited states of *mirror nuclei* (same A, but opposite N and Z) as seen in Fig. 15.

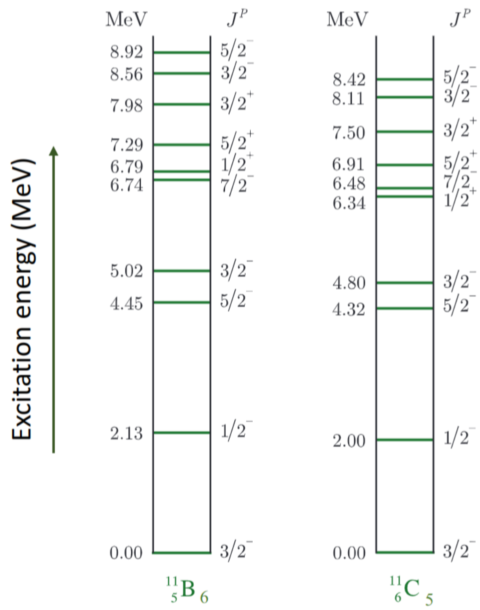


Figure 15: Comparison of the excitation levels of mirror nuclei. In this case we have $^{11}_{5}\mathrm{B}_6$ and $^{11}_{6}\mathrm{C}_5$

8.1 Effects of the Short Range of the Strong Force

- When shooting alpha particles at a nucleus, the alpha particles are repelled by the Coulomb force if they do not have enough energy to get close enough to the nucleus. Then the strong force takes over and the alpha particles are attracted to the nucleus. This is why the Rutherford model does not work at lower energies.
- The linear dependence on the binding energy per nucleon shows that the strong force is short range. If it were long range, each nucleon would attract all the others. Then the term in the binding energy as seen in the first therm of Eq. (37), would be quadratic and not linear ($\alpha_v A$)

8.2 Deuteron

- Consist of a proton and a neutron (nucleus of deuterium).
- To understand the structure of the atoms we would need to study its excited states. The problem is that deuteron is weakly bound and has no excited states.

8.2.1 Deuteron Binding Energy

There are multiple ways of calculating the binding energy of the deuteron.

1. **Mass spectroscopy**: Find the difference in mass between the deuteron and the proton and neutron.

$$B = \left(M\left({}^{1}\mathrm{H}\right) + m_n - m\left({}^{2}\mathrm{H}\right)\right)c^2 = 2.225\ \mathrm{MeV} \tag{74}$$

2. **Nuclear reaction**: The gamma ray emitted when a neutron is captured by a proton is almost the binding energy. It has only energy, but can be converted to mass through $E = mc^2$.

$$^{1}\mathrm{H} + n \rightarrow {}^{2}\mathrm{H} + \gamma \tag{75}$$

$$E_\gamma \approx B = M_{\text{initial}} - M_{\text{final}} \tag{76}$$

$$B = \left(M\left({}^{1}\mathrm{H}\right) + m_n - M\left({}^{2}\mathrm{H}\right)\right)c^2 = 2.224\ \mathrm{MeV} \tag{77}$$

8.2.2 Nucleon-Nucleon Potential

- We assume that the potential between the nucleons is a finite square well with a potential depth of $-V_0$
- Solving for the Schrödinger equation for specific energy values and applying the boundary conditions we get the following results:

$$k_1 \cot\left(k_1 \vec{R}\right) = -k_2 \tag{78}$$

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}} \quad k_2 = \sqrt{\frac{2m\left(V_0 + E\right)}{\hbar^2}} \tag{79}$$

- The radius R is now connected to the energy, and we know from scattering experiments that the radius is around 2.1 fm.
- The solution gives a potential of $V_0 = 35$ MeV.
- The binding energy of the deuteron is just below the potential depth as seen in Fig. 16.

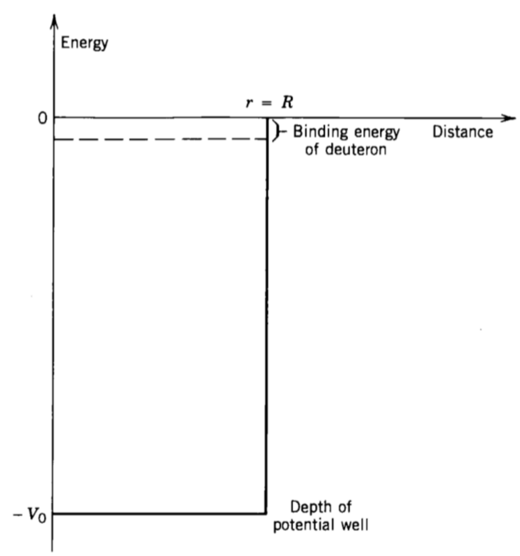


Figure 16: Square well potential for the deuteron.

8.2.3 Deuteron Spin and Parity

Total Spin:

$$\vec{I} = \vec{S}_p + \vec{S}_n + \vec{l} \tag{80}$$

Spin Configuration with $I = 1$

1. Aligning the spins gives $I = 1$ and $S = 1$ with $l = 0$. This is a positive parity state with $\pi = (-1)^l = 1$. We then get I^π
2. Aligning the spins with gives $I = 3$

Electric Quadrupole Moment The deuteron has a small non-zero electric quadrupole moment. This makes it so about 4% of the time the deuteron is in an excited state with $l = 2$.

9 The Standard Model

- The periodic table of particle physics.
- Categorized as seen in Fig. 17.

9.1 The Particles

- **Quarks**:
  - Each quark pair from each generation has one up-type and one down-type quark. The up-type quarks have a charge of $+\frac{2}{3}e$, and the down-type quarks have a charge of $-\frac{1}{3}e$.
  - The quarks are bound together by the strong force, mediated by the gluons.
  - They are all spin-half particles.
  - They can interact through the weak force, electromagnetic force, and the strong force.
  - Each quark is considered a quark flavor.
- **Leptons**:
  - The leptons are divided into three generations, each with a charged lepton and a neutrino.
  - The charged leptons all have a charge of $-e$, and the neutrinos are neutral.
  - They all have half-integer spin.
  - The charged leptons are much lighter than the neutrinos.
  - The charged leptons interact with the weak force and the electromagnetic force, while the neutrinos only interact with the weak force.
  - Each lepton is considered a lepton flavor.
- **Bosons**:

- **Fermions**: All quarks and leptons are fermions, which means they have half-integer spin.
- **Higgs Boson**: Gives mass to all the particles. Can interact with itself, as it has mass.

9.2 The Forces

- **Electromagnetic Force**:
  - Mediated by the photon.
  - Acts between particles with electric charge.
- **Weak Force**:
  - Mediated by the $W^{\pm}$ and Z^0 bosons.
  - The $W^{\pm}$ has electric charge and can interact with itself and other particles with electric charge.
  -
- **Strong Force**:
  - Mediated by gluons with strong force charge.
  - The gluon has a strong force charge, making it able to interact with itself.

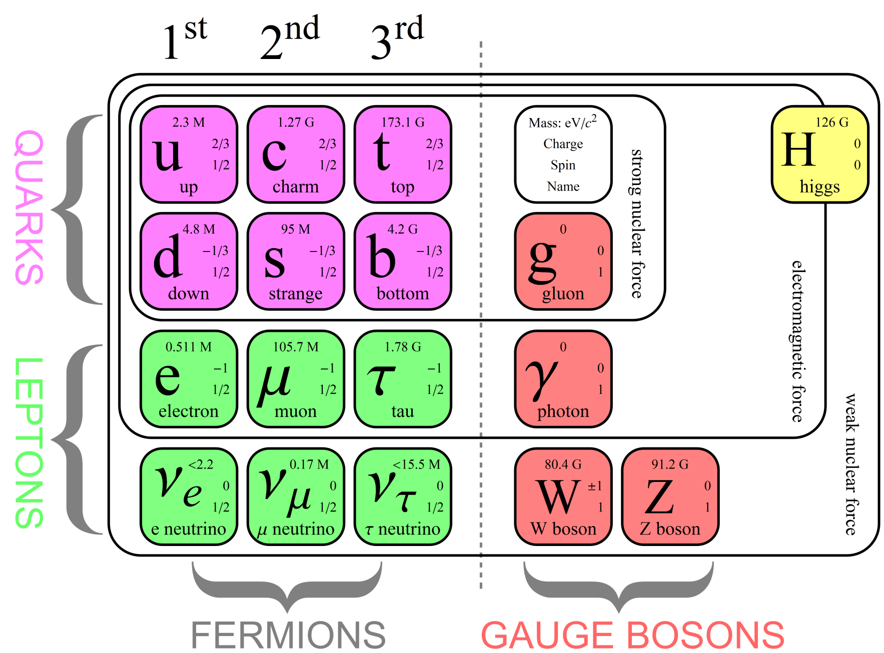


Figure 17: Figure showing the particles of the Standard Model, and their respective forces. Each column represents a generation of particles.

10 Feynman Diagrams

- A graphical representation of particle interactions.
- Time goes from left to right.
- The particles are represented by lines, and the interactions are represented by vertices.
- **Fermions**:
  - Represented by straight lines with arrows
  - Anti-particles are represented by straight lines with arrows pointing in the opposite direction.
- **Electromagnetic/Weak Interactions**:
  - Represented by wavy lines
  - Virtual photons (often noted by an asterisk) may have mass and may not move at the speed of light
- **Strong Interactions**: Represented by curly lines.
- **Higgs Boson**: Represented by a dashed line.

10.1 Charges

- The charges are $\alpha_{\mathrm{W}}, \alpha_{\mathrm{W}}, \alpha_{\mathrm{S}}$, for the weak, electromagnetic, and strong force, respectively.
- The product of the charges show how likely an interaction is to happen.

10.1.1 Examples

At the vertices in Eq. (81), we place the charge, α_{EM}, being proportional to the electromagnetic charge squared.

$$e^-\gamma \overset{\alpha_{\mathrm{EM}}}{\to} e^-\gamma \tag{81}$$

We do the same for the strong force in Eq. (82).

$$g \overset{\alpha_{\mathrm{S}}}{\to} gg \tag{82}$$

10.2 Allowed Vertices (Fig. 18)

- A vertices is where the particle lines meet.
- Processes can in theory be written in reverse.
- Not all vertices are allowed. The Q-value must be negative, meaning the process must be energetically allowed.
- Quantities like charge, baryon number, lepton number,energy and strangeness must be conserved.

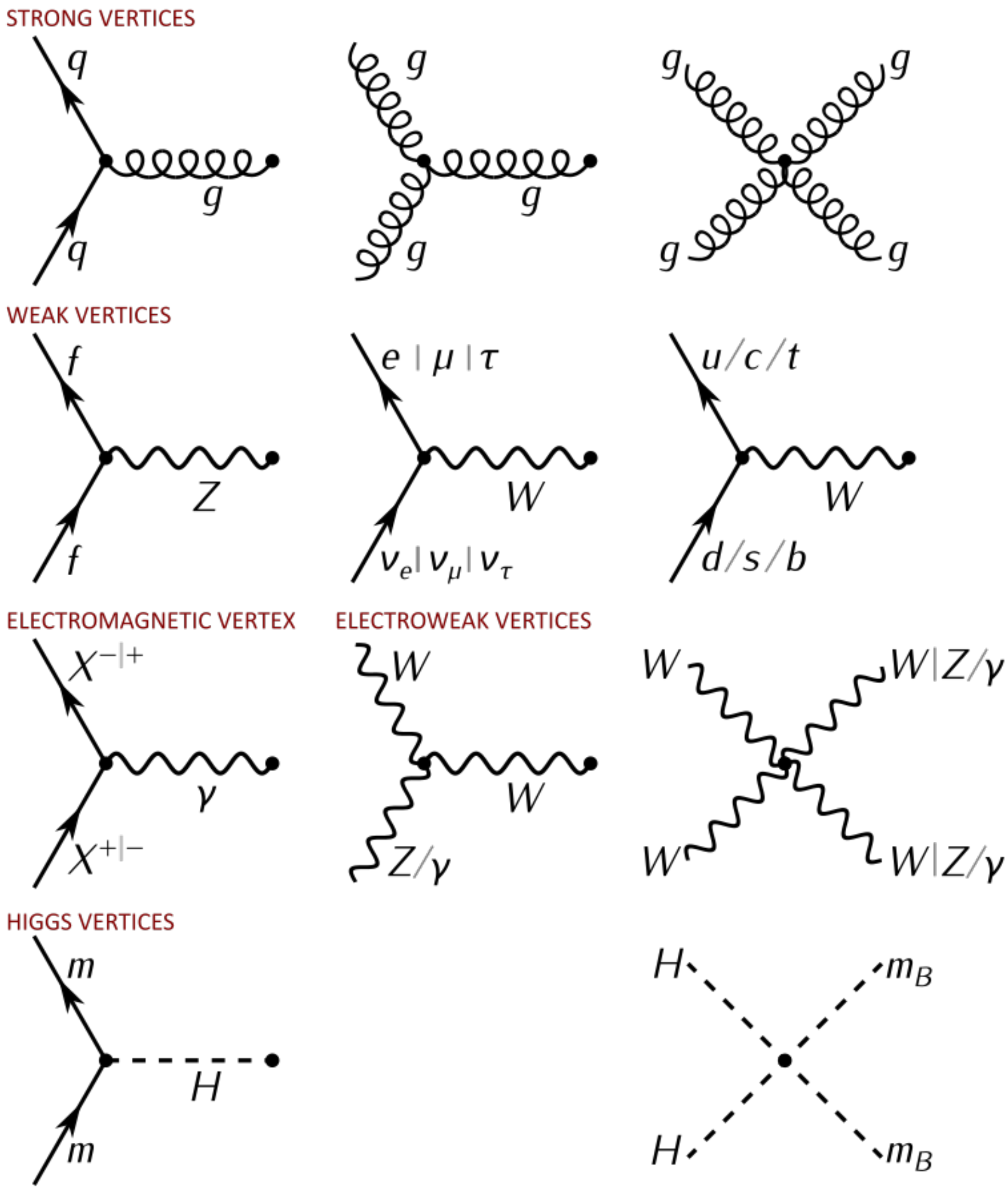


Figure 18: Figure of all allowed vertices in the Standard Model.

- $e^+ \to e^+\gamma$ would not be allowed as we can pick a frame of reference where the electron is at rest, and the photon would have energy less than or equal to 0.

We can change the time ordering in Eq. (81), to the following in Eq. (83). Here the photons comes in and interacts with the electron. This is a non-physical process, as the mass/energy of the system in a rest frame is not conserved.

$$\gamma e^- \overset{\alpha_{\rm EM}}{\rightarrow} e^- \gamma \tag{83}$$

10.2.1 Allowed EM Vertices

Eq. (84) is allowed as a part of a larger process, but not on its own. This is because the mass of the tauon can't increase or decrease, meaning the photon must have a mass (energy) of 0.

$$\tau^+ \rightarrow \tau^+ \gamma \tag{84}$$

Eq. (85) is allowed as a part of a larger process, but not on its own. Same as Eq. (84), where the mass of the anti charm quark can't increase or decrease, meaning the photon must have a mass (energy) of 0.

$$y\bar{c} \rightarrow \bar{c} \tag{85}$$

Eq. (86) is allowed as a part of a larger process, but not on its own. A real photon does not have mass, meaning it cannot create a top quark and an anti-top quark. We could flip one of the vertices to be $\gamma\bar{t} \rightarrow \bar{t}$ and we see this to be impossible.

$$\gamma \rightarrow t\bar{t} \tag{86}$$

10.2.2 Allowed Strong Vertices

Eq. (87) Could be a part of a larger diagram, but can't represent a physical process on its own due to conservation of energy.

$$g\bar{c} \rightarrow \bar{c} \tag{87}$$

Eq. (88) This is a possible process, as the gluon can have a high enough mass for this to be possible.

$$g \rightarrow t\bar{t} \tag{88}$$

Eq. (89) This is a possible process, as the gluon can interact with it self

$$g \rightarrow gg \tag{89}$$

10.2.3 Neutral Weak Vertices

All vertices with the photon can be interchanged with the Z^0-boson but NOT the other way around. See Eq. (90) for an example.

$$\gamma/Z^0 \rightarrow e^+e^- \tag{90}$$

The neutrino is electrically neutral, and does not interact with the photon, meaning the process in Eq. (91) only works with a Z^0-boson.

$$\nu_\mu \rightarrow \nu_\mu Z^0 \tag{91}$$

As the process is electrically neutral, the Z^0-boson can be created from the annihilation of a down-type quark and an anti-down-type quark, as seen in Eq. (92).

$$d\bar{d} \rightarrow Z^+ \tag{92}$$

10.2.4 Charged Weak Vertices

The $W^{\pm}$-boson can interact with all fermions, but as it has a charge, it changes the charge of the particles it interacts with. Net charge must be conserved, meaning the process in Eq. (93) is allowed.

$$W^- \to \bar{u}d \tag{93}$$

To get zero charge, but keep the number of anti-fermions we can have the process in Eq. (94).

$$e^+W^- \to \bar{\nu}_e \tag{94}$$

The electric charge of the $W^{\pm}$-boson can interact with the photon. This is seen in Eq. (95).

$$W^+ \to W^+\gamma \tag{95}$$

Net charge is conserved, which allows the process in Eq. (96).

$$Z^0 \to W^+W^- \tag{96}$$

10.2.5 Higgs Boson Vertices

The Higgs boson interacts with the mass of fermions, but maybe not the mass of neutrinos. Eq. (97) is a possible process, as the Higgs boson can interact with the mass of the top quark.

$$t\bar{t} \to H \tag{97}$$

Eq. (98) is a possible process, as the Higgs boson can create the muons, as charge is conserved.

$$H \to \mu^+\mu^- \tag{98}$$

Eq. (99) is possible, and dependant on the mass of the Z^0-boson.

$$Z_0 \to Z^0H \tag{99}$$

The Higgs boson can interact with itself, as seen in Eq. (100).

$$H \to HH \tag{100}$$

10.2.6 4-Line Vertices

- There is no complete list of all possible 4-line vertices.
- As seen in Fig. 18, we know that both gluons, Higgs bosons and $W/Z/\gamma$ bosons can have 4-line vertices. This means two particles can scatter, creating two new particles, without an intermediate particle.
- This is caused by particles being able to interact with themselves.
- These are rarely used

10.3 Creating Feynman Diagrams

Often there are multiple ways to describe the same process. There can be multiple different time orders. Example can be seen in Eq. (101), visualized in two different ways in Fig. 19. This is not always done as it is implied that both are possible, and you need to calculate both. A time independent version can be seen in Fig. 20.

$$e^-e^- \to e^-e^- \tag{101}$$

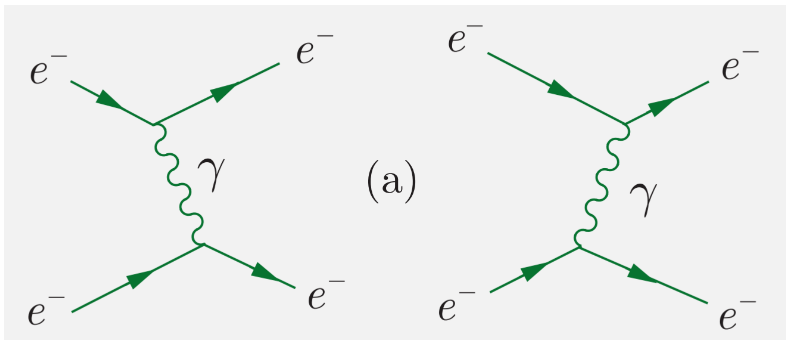


Figure 19: Two different ways to visualize Møller scattering as seen in Eq. (101).

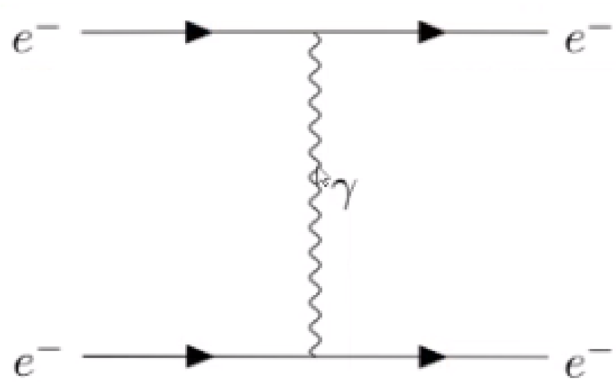


Figure 20: A time independent version of Møller scattering as seen in Eq. (101).

10.3.1 Tip for Choosing Time-Ordering

Just pick one and see if the charge and all other properties are conserved. As seen in Fig. 21, we can choose the time ordering based on the charge conservation. Only the electron can emit the negatively charged W^--boson. Therefore, the emission on the bottom happens before absorption on the top. The unambiguous time ordering can be seen in Fig. 22. We could of course have flipped the figure horizontally.

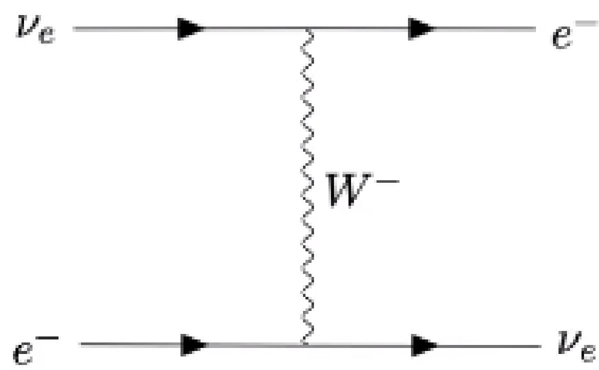


Figure 21: Example showing a case were logic is used to choose the time ordering.

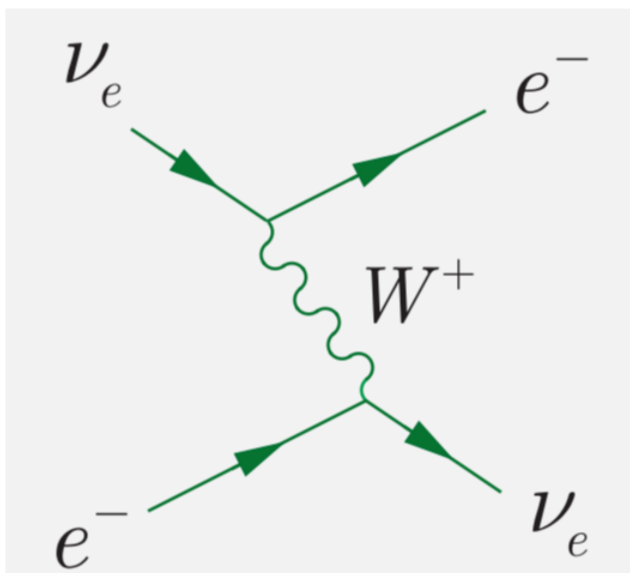


Figure 22: Solved case of ambiguity in time ordering from Fig. 21.

10.3.2 Current interactions

- Fermion lines must be continuous.
- Anti-fermion lines must be continuous.

11 Relativity and Anti-Particles

11.1 Dirac Equation

- A relativistic equation for the electron.
- Makes a prediction of the positron (anti-electron).

- All charged particles have an anti-particle $(p, \bar{p})$, (μ^-, μ^+)
- For neutral particles, their respective anti-particles have the same charge, but their quarks are opposites (neutrons), or they are their own anti-particle (photons).
- Predicts a relation between spin $\vec{S}$ and magnet moment $\vec{\mu} = q\vec{S}/m$. This has been confirmed for the electron and muon, but not for protons and neutrons. This hints at the existence of substructure in protons and neutrons.

11.2 Symmetries and Conservation Laws

11.2.1 Symmetries and Conserved Quantities

Noethers theorem states that for every continuous symmetry, there is a conserved quantity. A list of symmetries and their conserved quantities can be seen in Section 11.2.1. They all have their quantum operators. If the physics remains unchanged under a transformation, the system is said to be symmetric under that transformation.

Symmetry	**Conserved Quantity**	**Interactions**
Space translation	Linear momentum	All
Space rotation	Angular momentum	All
Time displacement	Energy	all
Space inversion	Parity	Not weak
Charge inversion	C-parity	Not weak
u ¡-¿ d	Isospin	String

11.2.2 Intrinsic Parity

- All particles at rest have intrinsic parity. This comes from a the only eigenvalue of the parity operator $\hat{P}$ acting on a plane wave function $\Psi(\vec{r}, t) = \exp(i(\vec{r} \cdot \vec{p} - Et)/h)$, can only return the same wavefunction if $\vec{p} = 0$.
- By convention, we say fundamental fermions have $P = +1$, and their antiparticles have $P = -1$. This is arbitrary.

11.2.3 Parity of Bound States

In quantum mechanics we can split the spatial part of the wave function into a radial part $R(r)$ and a spherical harmonic $Y_l^m(\theta, \phi)$.

$$\Psi = R_{nl}(r) Y_l^m(\theta, \phi) \tag{102}$$

$$\vec{r} \to -\vec{r} \quad \Rightarrow \quad r \to r, \theta \to \pi - \theta, \phi \to \phi + \pi \tag{103}$$

The parity of the spherical harmonic is $(-1)^l$.

11.2.4 Charge Conjugation

- The charge conjugation operator $\hat{C}$ changes all particles to their antiparticles.
- Eigenstates of $\hat{C}$ are particles that are their own antiparticles, like photons.
- Eigenvalues are ± 1.

11.2.5 Natural Units

- To make life simple we let $\hbar = c = 1$, and only add them back at the end when knowing the units.
- All quantities are measured in energy $(eV)^n$.
- Energy: eV
- Momentum: eV/c
- Mass: eV/c^2
- Time and length: $1/eV$
- It is useful to memorize $\hbar c \approx 197MeVF$

11.3 Scattering

- Scattering experiments is the core of nuclear and particle physics.
- Elastic Scattering: Same particles in the initial and final state. An example can be seen in Eq. (104), where a proton and electron collide. The electron might lose some energy in the form of momentum to the proton, but the particles are the same and their center of mass is unchanged.

$$e^- p \to e^- p \tag{104}$$

- Inelastic Scattering: Different particles in the initial and final state. An example can be seen in Eq. (105), where a particle can collide with an atom and excite it to a higher energy state.

$$aA \to a + A^* \tag{105}$$

11.3.1 Decay

- Unstable parent particles decay into stable and lighter daughter particles.
- Atom decay: $A^* \to A + \gamma$
- Neutron decay: $n \to p + e^- + \bar{\nu}_e$
- Z^0 boson decay: $Z^0 \to e^+ + e^-$.
- For free particles, we must have a positive Q-value defined in Eq. (106).

$$Q = \left(M_{\mathrm{P}} - \sum_i M_{\mathrm{D}_i} \right) c^2 \tag{106}$$

- For unstable nuclei and particles we have a decay rate as seen in Eq. (107).

$$\frac{dN(t)}{dt} = -\lambda N(t) \tag{107}$$

  The decay constant λ is related to the half-life $T_{1/2}$ by $\lambda = \ln(2)/T_{1/2}$, with units of s^{-1}.

- The mean lifetime is given by $\tau = 1/\lambda$. This represent the average time a particle will live before decaying. The mean lifetime refer to the rest frame of the particle.
- Decay Width Γ: The decay rate in natural units. It is related to the decay constant by $\Gamma = \hbar\lambda$.
- Decay Channels: The different particles a particle can decay into. Each has its own decay width, where the total decay width is the sum of all the decay widths $\Gamma = \sum_i \Gamma_i$. Each channel has the same mean lifetime. They live for the same amount of time, but the probability of decaying into a specific channel is different.
- Branching Fraction: The probability of a particle decaying into a specific channel, given by $B = \Gamma_i/\Gamma_{\text{tot}}$.

Tau Particle

- A heavy particle which always decays ($B = 100\%$).
- A list of some decay channels, responsible for approximately 60% of all decays:
  - $B(\tau^- \to e^- \bar{\nu}_e \nu_\tau) = 17.8\%$
  - $B(\tau^- \to \mu^- \bar{\nu}_\mu \nu_\tau) = 17.4\%$
  - $B(\tau^- \to \pi^- \pi^0 \nu^\tau)?25.5\%$

11.4 Cross-Section

- We measure the rate W of a process happening.
- The number of particles hitting a target per unit area is called flux $J = n_b v_i$, where n_b is the number of particles per unit volume and v_i is the velocity of the beam.
- The area per target particle is called the cross section σ. This represents the area in which the particles interact in the relevant process.
- The rate of particles hitting the target is $W = JN\sigma$. where N is the number of particles.
- The rate W is proportional to the luminosity L by $W = L(t)\sigma$.
- The cross section σ is invariant under Lorentz transformations in the direction of the beam, meaning it is independent of the reference frame.
- Colliding two beams beams of N_1 number of particles and N_2 number of particles, we get a luminosity $L(t) = fN_1N_2/A$, where f is the frequency of the beam (how often they collide) and A is the area of the beam.

11.4.1 Differential Cross-Section

Measured time-integrated rate as a function of some observable. This could be momentum:

$$\sigma = \int_{p_{\min}}^{p_{\max}} \frac{\mathrm{d}\sigma(p)}{\mathrm{d}p}\ \mathrm{d}p \tag{108}$$

11.4.2 Measured Cross-Section

1. Define the process of interest.
2. Count the number of times the process happens.
3. Remove the background noise as measurement is imperfect.
4. Find the time-integrated luminosity $\mathcal{L} \equiv \int L(t)\mathrm{d}t$.
5. For the total cross-section, we have $N = \sigma\mathcal{L}\epsilon + b$ or $\sigma = \dfrac{N-b}{\mathcal{L}\epsilon}$, where ϵ is the efficiency of the detector and b is the background noise.

11.4.3 Integrated Luminosity

- One can either precisely control the details of the beam and target to determine $\mathcal{L}$.
- One can also used well-known processes to determine $\mathcal{L} = N_1/\sigma_1$, if this event has a known cross-section. After that, one can correct for noise and efficiency, to then find the cross-section of the process of interest by $\sigma_2 = \frac{N_2}{\mathcal{L}} = \frac{N_2}{N_1}\sigma_1$
- Barn: Unit of cross-section, $1\mathrm{b} = 10^{-24}\mathrm{cm}^2$ or $1\mathrm{b} = 10^{-28}\mathrm{m}^2$. Integrated luminosity is often given in fb^{-1} or pb^{-1}, where f (10^{-15}) and p (10^{-12}) stands for femto and pico, respectively. $1\mathrm{pb} \approx 10^{-40}\mathrm{m}^2 = (10^{-5}\mathrm{fm})^2$. In comparison the proton has a radius of about 0.84fm.

11.4.4 Single Beam Particle and Target

- Again we use that the rate of observing some scattering process r is $W_r = JN\sigma r$.
- In the case of a single target particle we have $N = 1$.
- The flux $J = n_b v_i$ where n_b is the number of beam particles per unit volume V and v_i is the velocity of the beam.
- The differential rate of observing the process r in a solid angle at direction (θ, ϕ), is therefore $\mathrm{d}W_r = v_{\frac{i}{V}} \dfrac{\mathrm{d}\sigma_r(\theta,\phi)}{\mathrm{d}\Omega} d\Omega$

11.4.5 Born Approximation

- Assuming the particle has a wavefunction traveling a large distance around the accelerator, we can describe it as a plane wave ψ_i.
- When disturbed by a potential $V(r)$, we can calculate the rate of scattering by the Born approximation. This is also known as "Fermi's Golden Rule". If we define the probability amplitude $\mathcal{M}(\vec{q})$, we get from Eq. (109) that the rate of scattering is given by Eq. (110). Here $\vec{q}$ is the momentum transferred.

$$\mathrm{d}W_r = \frac{2\pi}{\hbar}\left|\int \psi_f^* V(\vec{r})\psi_i \mathrm{d}^3\vec{r}\right| \rho(E_f) \tag{109}$$

$$\mathrm{d}W_r = \frac{2\pi}{\hbar}\left|M(\vec{q})\right|^2 \rho(E_f), \quad M \equiv \int V(\vec{r}) \exp\left(i\vec{q}\cdot\vec{r}/\hbar\right) d^3\vec{r} \tag{110}$$

- Multiplying by $\hbar^2 c^2$ which we know to be about $(197 \text{ MeV F})^2$ we cancel out the energy and get σ in units of F^2 aka barn.

11.4.6 Natural Units and Cross Section

- Cross-sections σ has unit of area, barn (b) of 10^{-28} m^2.
- In natural units this is proportional to $1/E^2$ with units $1/\text{MeV}^2$

11.5 Density of States (Energy States)

The number of states in a range of momenta $d^3\vec{p}$. This is derived from a wavefunction in a box of volume $V = L^3$ with walls of infinite potential. The coordinate system is places in one of the corners such that the walls are at $x = L$, $y = L$ and $z = L$. From this we get the density of states in momentum space $d^3\vec{p}$ as seen in Eq. (111).

$$\rho(E)\mathrm{d}q = \frac{L^3}{(2\pi\hbar)^3} d^3\vec{p} \quad , \quad \rightarrow \mathrm{d}^3\vec{p} = q^2 \ \mathrm{d}q \ \mathrm{d}\phi \ \mathrm{d}(\cos\phi) \equiv q^2 \mathrm{d}q\mathrm{d}\Omega \tag{111}$$

Eq. (111) can be used further where Eq. (112) is defined. Now we need to find $\frac{\mathrm{d}q}{\mathrm{d}E}$

$$\rho(q)\mathrm{d}q \equiv \rho(E)\mathrm{d}E \rightarrow \rho(E) = \rho(q)\frac{\mathrm{d}q}{\mathrm{d}E} = \frac{L^3}{(2\pi\hbar)^3} q^2 \frac{\mathrm{d}q}{\mathrm{d}E}\mathrm{d}\Omega \tag{112}$$

Non-Relativistic Case:

- $E = q^2/2m$
- $\mathrm{d}E = 2q\mathrm{d}q/2m = q\mathrm{d}q/m$
- **Conclusions:** $\frac{\mathrm{d}q}{\mathrm{d}E} = m/q = 1/v$

Relativistic Case:

- $E^2 = m^2c^4 + q^2c^2$
- $2E\mathrm{d}E = 2c^2 q\mathrm{d}q$
- **Conclusions:** $\frac{\mathrm{d}q}{\mathrm{d}E} = E/qc^2 = m\gamma c^2/m\gamma\beta c^3 = \frac{1}{\beta c} = 1/v$ The result is the same!

11.5.1 Conclusion

$$\frac{\mathrm{d}\sigma_r(\theta,\phi)}{\mathrm{d}\Omega} = \frac{1}{(2\pi^2)} \frac{1}{\hbar^4} \frac{q_f^2}{v_f v_i} \left|M(\vec{q})\right|^2 \tag{113}$$

11.6 Theoretical Cross-Section for $ab \to cd$

- The differential cross-section for a process $ab \to cd$ is given by Eq. (114), where g_i and g_f are number of spin-states for the initial and final states particles, respectively. $g_f = (2S_c + 1)(2S_d + 1)$ and $g_i = (2S_a + 1)(2S_b + 1)$. k represent all possible spin states.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{g_f}{4\pi^2\hbar^4}\frac{q_f^2}{v_i v_f}\overline{|M(\vec{q})|^2} \quad , \quad \overline{|M(\vec{q})|^2} = \frac{1}{g_i g_f}\sum_k |M_k|^2 \tag{114}$$

- In general, the particle beams are unpolarized and we therefore do not measure the spin of the final state particles.
- We average over the initial spin states, and sum over the final ones.

12 Particle Phenomenology

12.1 Leptons

- 3 generations increasing in mass.
- Neutrinos has only weak interactions due to lack of charge. This makes it has to detect and is a hint of a weak interaction.
- Except a few cases, the generations of leptons are conserved in interactions.
- The weakness of the weak force comes from the massive mediating particles $W^{\pm}$ and Z^0.
- The flavour of the lepton does not affect the strength of the interaction.

13 Range of Forces and Amplitudes

13.1 Yukawa Potential

Looking at the Schrödinger equation:

$$H\Psi = E\Psi \tag{115}$$

where the energy solution must satisfy $E^2 = m^2c^4 + p^2c^2$. The energy and momentum operators are given by:

$$E = i\hbar\frac{\partial}{\partial t} \quad , \quad \vec{p} = -i\hbar\vec{\nabla} \tag{116}$$

The solution should ϕ be a static one, representing a radial potential:

$$\nabla^2\phi(r) = \frac{m^2c^2}{\hbar^2}\phi(r) \tag{117}$$

As the factors have units of length squared, we can create a new variable $R = \hbar^2/m^2c^2$. Guessing a solution on the form $\phi(r) = u(r)/r$, we get the following solution not diverging for large r:

$$\frac{\partial^2 u(r)}{\partial r^2} = \frac{u(r)}{R^2} \to u(r) = ae^{-r/R} \to \phi(r) = a\frac{e^{-r/R}}{r} \tag{118}$$

This gives a potential V as seen in Eq. (119), with a potential vanishing for large distances. For complex structures we need to take into the internal structure as well. This assumes an interaction between a particle with charge $-g$, and another much heavier particle with charge $+g$.

$$V(r) = -\frac{g^2}{4\pi}\frac{e^{-r/R}}{r} \tag{119}$$

13.2 Yukawa Amplitude

Amplitude from the Bourne approximation:

$$\mathcal{M} = \int V(\vec{r})e^{i\vec{q}\cdot\vec{r}/\hbar}d^3r \quad , \quad \vec{q} = \vec{p}_f - \vec{p}_i \tag{120}$$

Solving this with the Yukawa potential, we get the following result:

$$\vec{q}\cdot\vec{r} = qr\cos\theta \quad , \quad \mathrm{d}^3r = r^2\mathrm{d}r = \sin\mathrm{d}\theta\mathrm{d}r\mathrm{d}\phi \tag{121}$$

This gives the following result in Eq. (122), only valid when $|\vec{q}_i| = |\vec{q}_f|$ and energy is constant (elastic scattering):

$$\mathcal{M}(q^2) = -\frac{g^2\hbar^2}{q^2 + m^2c^2} \tag{122}$$

To make this Lorentz invariant (as momentum is dependant on the frame of reference), we get the following result in Eq. (124), when $Q^2 \ll m^2$

$$Q^2(\vec{q}_i - \vec{q}_f)^-(E_f - E_i)^2/c^2 \tag{123}$$

$$\mathcal{M}(Q^2) = -\frac{g^2\hbar^2}{Q^2 + m^2c^2} \tag{124}$$

If the range of the exchange is very small, then Q^2 is also very small, giving us the following approximation:

$$\mathcal{M}(Q^2) = -\frac{g^2\hbar^2}{m^2c^2} \equiv -G \tag{125}$$

We often visualize this in Feynman diagrams as shown in Figs. 23 and 24 . Both Q^2 and G are Lorentz invariant.

13.3 Amplitude of W-boson Exchange

As the result of the short-range approximation shows in Eq. (124), we see it is dependant on mass, an therefore we can use this to calculate the mass of the W-boson. The amplitude of the W-boson exchange is given by:

$$\alpha_W \equiv \frac{g_W^2}{4\pi\hbar c} \quad , \quad \alpha_{EM} = \frac{e^2}{4\pi\epsilon_0\hbar c} \tag{126}$$

Taking into account the spins of the particles gives a factor of $\sqrt{2}$.

$$G_F ? \frac{g_W^2\hbar^2}{m_W^2c^2}\sqrt{2} \rightarrow \frac{G_F}{(\hbar c)^3} = \frac{4\pi\alpha_W}{(m_{Wc^2})^2}\sqrt{2} \tag{127}$$

By measurement of $G_F/(\hbar c)^3 = 1.166 \cdot 10^{-5}$ GeV^{-2}, we can calculate the mass of the W-boson to be $m_W \approx 105$ GeV/c^2, which is close to the actual value of $m_W = 80.4$ GeV/c^2. This assumes $\alpha_W \approx \alpha_{EM}$.

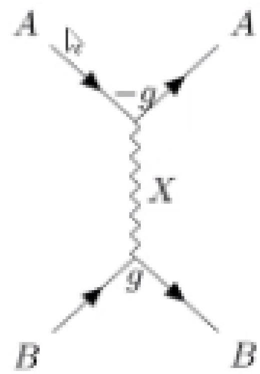


Figure 23: Feynman diagram of a non-approximated exchange of particle X

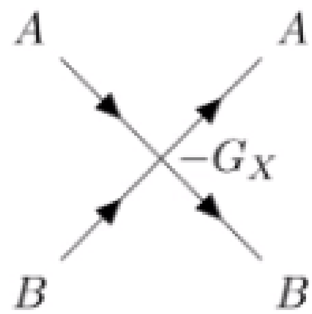


Figure 24: Feynman diagram of an short-range approximated exchange of particle X

13.4 Amplitude of Other exchanges

Each vertex has a factor charge and a cross section decay rate which is proportional to $|\mathcal{M}|^2$. We can count the number of exchanges of particle X as orders of α_X ($\sqrt{\alpha_X}$, per vertex before squaring). This typically represent small corrections as $\alpha_X \ll 1$.

13.5 Muon vs Tau Decays

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu \tag{128}$$

$$\tau^- \to e^- + \bar{\nu}_e + \nu_\tau \tag{129}$$

The width of the decay rate is proportional to the matrix elements square: $\Gamma \propto |\mathcal{M}|^2$ and $\mathcal{M} \propto G_F, G_f = 1.166 \cdot 10^{-5} (\hbar c)^3/\text{GeV}^2$. To get units of energy we must have

$$\Gamma = K G_F^2 (m_l c^2)^5 / (\hbar c)^6 \tag{130}$$

where m_l is the mass of the initial lepton. We do not need to know K as we only want to compare the two processes. In natural units we get:

$$K G_F^2 m_l^5 \tag{131}$$

The ratio then becomes as shown in Eq. (132). As the muon only have one decay channel, it is easy to find the decay width. This does not hold for the tau, as it only becomes an electron about 18% of the time.

$$\frac{\Gamma(\tau^- \to e^- \bar{\nu}_e \nu_\tau)}{\Gamma(\mu^- \to e^- \bar{\nu}_e \nu_\mu)} \approx \left(\frac{m_\tau}{m_\mu}\right)^5 \approx 1.354 \cdot 10^6 \tag{132}$$

We must correct for the fact that the tau decays into a muon about 17.8% of the time, by using their respective lifetime $\tau \equiv \frac{1}{\Gamma}$. Fort he muon we have $\frac{1}{\tau_\mu} = 1/2.2\mu$ seconds, and fore the tau we have $\frac{1}{\tau_\tau} = 1/0.29p$ seconds. Using experiment telling us the branching fraction of the tau, we find we a good estimate of the ratio in Eq. (132).

$$\frac{\Gamma(\tau^- \to e^- \bar{\nu}_e \nu_\tau)}{\Gamma(\mu^- \to e^- \bar{\nu}_e \nu_\mu)} = \frac{\tau_\mu}{\tau_\tau} \underbrace{B(\tau^- \to e^- \bar{\nu}_e \nu_\tau)}_{17.8\%} = 1.35 \cdot 10^6 \tag{133}$$

which is very good estimate within 2% of the predicted value.

14 Resonance Formation and Decay

- Cross-section is proportional to the following:

$$\sigma \propto \chi^*(E)\chi(E) \quad , \quad \chi(E) \int_0^\infty \Psi(t) e^{i\omega t} \, \mathrm{d}t \omega = E/\hbar \tag{134}$$

- The center of mass of the particle has 0 momentum. This gives:

$$\Psi(t) = \Psi(0) e^{-i\omega_R t} e^{-\Gamma t/2} \quad , \quad \omega_R = E_r/\hbar = m_R c^2/\hbar \tag{135}$$

- The result is

$$\chi^*(E)\chi(E) \propto \frac{1}{(E - m_R c^2)^2 + \hbar^2 \Gamma^2/4} \tag{136}$$

- Visualized it looks like a Gaussian curve, but with a much higher peak. This is called a Breit-Wigner formula as shown in Fig. 25. It tells us the width of the decay rate Γ in energy units, and lets us look at decay rates of particles with very short lifetimes.

14.0.1 Width and Lifetime

- In practical units: $\lambda \equiv \Gamma = 1/\tau$.
- In natural units (energy in this case) : $\Gamma = \hbar/\tau = \hbar c/\tau c$
- The wider the width, the shorter the lifetime.

15 Neutrino Mixing and Oscillations

- Slowly, the neutrinos changes into each other.
- The flavour eigenstates are not the same as their mass eigenstates.
- The neutrinos propagate as mass eigenstates, but are produced and detected as flavour eigenstates of the weak interaction.

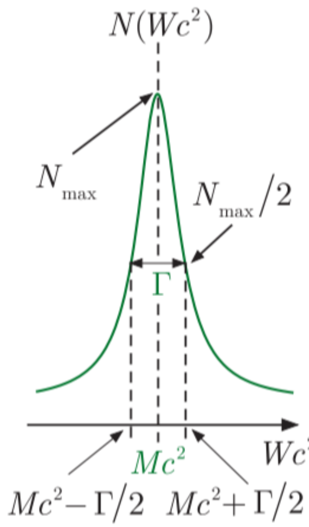


Figure 25: The Breit-Wigner formula for resonance formation and decay

15.1 Neutrino Mixing

- This has an extremely low probability of happening, and has not been observed in the lab.

15.1.1 3-generation mixing

- Easier to understand with 2 generations.
- If we produce a beam of ν_e with momentum p, we can observe it later at some distance x.
- As the neutrinos are produced as a mix of mass eigenstates, they will propagate slightly differently because of their mass difference. Some will turn into ν_μ, written as: $\nu_e \to \otimes \to \nu_\mu$

15.2 Probability of Mixing

Starting with a beam of only ν_e at $t = 0$. We can calculate the time dependent state.

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix} \tag{137}$$

$$|\nu(t)\rangle = a(t)\cos\theta\,|\nu_1\rangle + b(t)\sin\theta\,|\nu_2\rangle \quad , \quad a(t) = \exp(-iE_1 t) \ , \ b(t) = \exp(-iE_2 t) \tag{138}$$

As we can both detect ν_e and ν_μ, we can calculate each probability, with a superposition of their wavefunctions instead of ν_1 and ν_2.

$$|\nu(t)\rangle = a(t)\cos\theta\,(\cos\theta\,|\nu_e\rangle - \sin\theta\,|\nu_\mu\rangle) + b(t)\sin\theta\,(\sin\theta\,|\nu_e\rangle + \cos\theta\,|\nu_\mu\rangle) \tag{139}$$

$$|\nu(t)\rangle = \Big(a(t)\cos^2\theta + b(t)\sin^2\theta\Big)\,|\nu_e\rangle + \Big(b(t)\sin\theta\cos\theta - a(t)\sin\theta\cos\theta\Big)\,|\nu_\mu\rangle \tag{140}$$

Only if $a = b$ for all t, there will be no mixing. The normalization is defined as:

$$|\langle \nu_e | \nu_e \rangle|^2 \equiv |\langle \nu_\mu | \nu_\mu \rangle|^2 \equiv 1 \tag{141}$$

The probability of finding the muon neutrino is given by:

$$P(\nu_\mu) = |\langle \nu_\mu | \nu(t) \rangle|^2 = \left| \frac{b(t) - a(t)}{2} \right|^2 \sin^2 2\theta \quad , \quad \text{using } \sin 2\theta = 2 \sin\theta \cos\theta \tag{142}$$

$$|b(t) - a(t)|^2 = b^* b + a^* a - (a^* b + b^* a) = 2 - \exp\left(i(E_2 - E_1)t\right) - \exp\left(-i(E_2 - E_1)t\right) \tag{143}$$

Using the fact that $\exp ix + \exp -ix = 2\cos x$ and $\sin^2 x/2 = (1 - \cos x)/2$ we get:

$$2 - 2\cos\left(E_2 - E_1\right)t = 4\sin^2\left(\frac{E_2 - E_1}{2} t\right) \tag{144}$$

We finally get the probability:

$$\underline{\underline{P(\nu_\mu) = |\langle \nu_\mu | \nu(t) \rangle|^2 = \sin^2\theta \sin^2\left(\frac{(E_2 - E_1)t}{2}\right)}} \tag{145}$$

As long as θ is nonzero, or the energies are equal, we get a probability oscillating between 0 and 1.

15.2.1 Conclusion

- As the masses of the neutrinos are very small in comparison to their momentum, we say the energy is the momentum. This gives a new difference in energy:

$$E_2 - E_1 = \frac{m_2^2 - m_1^2}{2E} \tag{146}$$

- for small masses, $v \approx c$, so the position $L(t) \approx ct$.
- We define a new constant in natural units $L_0 \equiv 4E/(m_2^2 - m_1^2)$
- We have a final result of the probability of oscillation as:

$$P(\nu_\mu) = \sin^2 2\theta \sin^2\left(\frac{L}{L_0}\right) \tag{147}$$

- The scale of these oscillations are very large, and therefore very hard to detect.

16 Nuclear Models

- Two main categories, single particle, and collective.
- Collective models looks at the nucleas as a whole. Single particle models looks at the nucleas as a collection of individual particles.

16.1 Liquid-Drop Model

- A collective model.
- Assumes the nucleus behaves like molecules in an oscillating drop of liquid.
- All molecules attract eachother and are held together by surface tension.
- The droplet is charged which destabilizes the oscillations.
- Heavy droplets are shaped like dumbbells, as they are almost split in two.
- The models explains the binding energy and mass of the nuclei. It also explains the fission of heavy nuclei.
- The model does not explain the shell structure or magic numbers

16.2 Fermi-Gas model Section 16.2

- A single particle model.
- Consider a system of completely non-interacting nucleons in a three dimensional box potential.
- The potential is a well-potential, which treats protons and neutrons differently. The protons has a lower potential than the neutrons.
- The nucleons are arranged in pairs because of the Pauli exclusion principle.
- The Fermi energy E_F is the energy of the highest occupied state.
- The Fermi momentum p_F is the momentum of the highest occupied state.
- To calculate E_F, we must first calculate the number of states in the box of volume V.

$$\mathrm{d}A = 4\frac{V}{\Delta x^3} \tag{148}$$

- The model can not explain the shell structure or magic numbers.

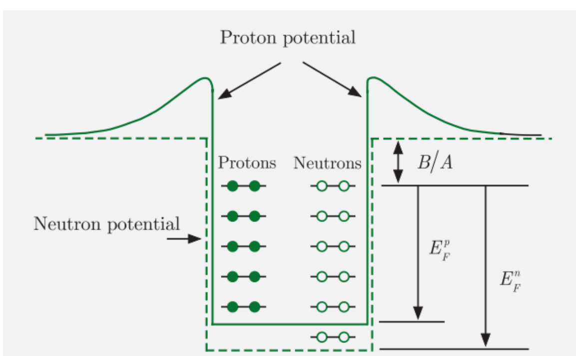


Figure 26: Visual representation of the Fermi-Gas model

16.3 Shell Model Fig. 27

- A single particle model. Takes inspiration from the shell structure of the atom.
- Explains the shell structure and properties of nuclei. Explains the magic numbers
- The model assumes the particles do not interact. They are affected by a radial central spherical potential created by all the nucleons. It is therefore easier to estimate the movement of any single nucleon.
- Nuclei with magic numbers are more stable. The magic numbers for protons are 2, 8, 20, 28, 50, 82. The magic numbers for neutrons are 2, 8, 20, 28, 50, 82, 126.
- Double magic nuclei are nuclei with magic numbers for both protons and neutrons.
- Instead of just having the nucleons pair up as in the Fermi-Gas model, the nucleons are arranged in shells.
- Each shell has a maximum number of nucleons.
- When certain magic number are reached, the separation energy is higher. They are therefore more stable.
- The electric quadrupole moment is the lowest for nuclei with magic numbers. This hints at them having a spherical shape, which is the most stable shape.
- Odd-Odd (Z-P) nuclei have only 4 stable isotopes.
- Odd-Even nuclei have 50 stable isotopes.
- Even-Odd nuclei have 53 stable isotopes.
- Even-Even nuclei have 165 stable isotopes.

16.3.1 Simplifying the Complex System to a Simple Model Fig. 28

16.3.2 Finding the Hamiltonian

- The system gets complex fast, when we have multiple unpaired nucleons. The Hamiltonian is then given by:
$$H = H_0 + H_{\text{res}} \tag{149}$$
where H_0 is the Hamiltonian for the system with paired nucleons, and H_{res} is the residual Hamiltonian for the unpaired nucleons.
- To find the Hamiltonian we need the potential $U(r)$ and finally reproduce the magic numbers.

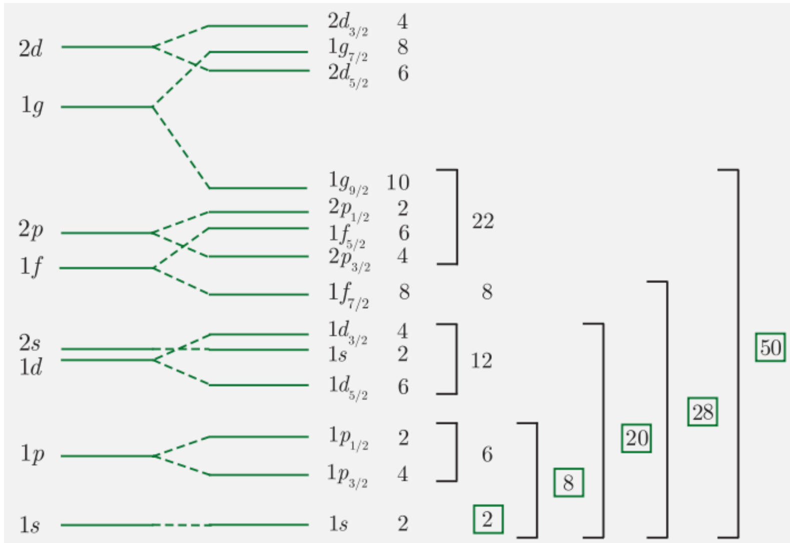


Figure 27: Lowest energy levels for nucleons with their spin-orbit term.

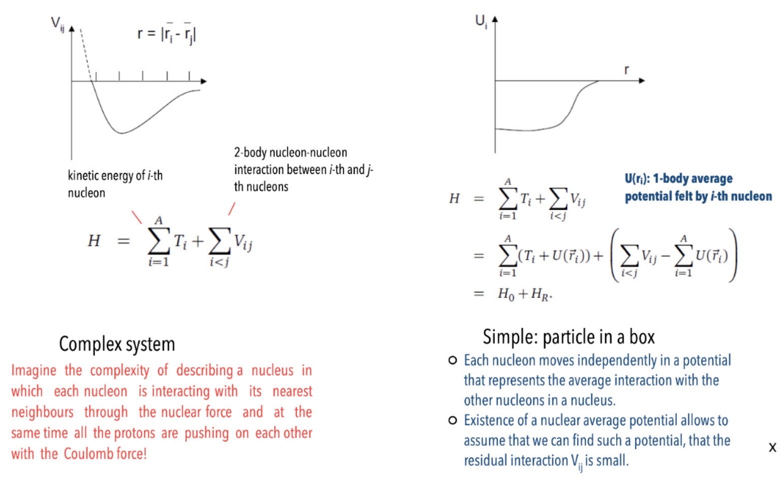


Figure 28: The transition from a complex system where all the particles interact, to a simple model where the nucleons are affected by a radial central spherical potential.

17 Relativistic Kinematics

The conversion from one reference frame to another is done by the linear Lorentz transformation. The Lorentz transformation is given by Eq. (150) where $\beta = v/c$ and $\gamma 1/\sqrt{1-\beta^2}$.

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \quad \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}' = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} \tag{150}$$

Example: For a particle at rest in coordinate system s, we have $\vec{p} = 0$ and $E = mc^2$. From here we can derive the energy and momentum in the s' system.

$$p'_z = -\frac{\gamma\beta E}{c} = -m\gamma\beta c = -mv\gamma \tag{151}$$

$$E'/c = \gamma E/c = ymc \to E' = \gamma mc^2 \tag{152}$$

In this case we have $p'_\perp = p_\perp$, as the transformation is in the z-direction.

17.1 Four-Vectors

$$\vec{A} = (A_0, a_x, a_y, a_z) \tag{153}$$

$$\vec{B} = (B_0, b_x, b_y, b_z) \tag{154}$$

17.1.1 Dot product

$$\vec{A} \cdot \vec{B} = A_0B_0 - a_xb_x - a_yb_y - a_zb_z)\vec{A}^T\eta\vec{B} \tag{155}$$

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{156}$$

17.1.2 Dot product in different reference frames

$$\vec{A}' \cdot \vec{B}' = \begin{pmatrix} A_0 \\ a_x \\ a_y \\ a_z \end{pmatrix}^T \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} B_0 \\ b_x \\ b_y \\ b_z \end{pmatrix} \tag{157}$$

Using the fact that $(ab)^T = b^Ta^T$ we get:

$$\vec{A}' \cdot \vec{B}' = \begin{pmatrix} A_0 \\ a_x \\ a_y \\ a_z \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} B_0 \\ b_x \\ b_y \\ b_z \end{pmatrix} = \vec{A} \cdot \vec{B} \tag{158}$$

17.1.3 Attributes Summary

Consider the following attributes of the dot product of two four-vectors $\vec{A} = (ct, \vec{x})$ and $\vec{B} = (E/c, \vec{p})$.

- The dot product is invariant under Lorentz transformations. This means that the dot product is the same in all reference frames.
- $\vec{A}^2 = \vec{A} \cdot \vec{A} = c^2t^2 - |\vec{x}|^2 =$ constant.
- $\vec{B}^2 = E^2/c^2 - |\vec{p}|^2 =$ constant. We define the invariant mass as mass in the rest frame.

  $$(Wc^2)^2 = E^2 - |\vec{p}|^2 c^2 \tag{159}$$

  If $\vec{p} = 0$ then $(Wc^2)^2 = E^2 = m^2c^4$, which makes $m = W$, the rest mass.

- The invariant mass for a collation of particles is defined as the sum of the invariant masses of the particles.

$$(Wc^2)^{\equiv}\left(\sum_i E_i\right)^2 - \left(\sum_i |\vec{p}_i|\right)^2 c^2 \tag{160}$$

This allows us to measure the mass in any frame we find convenient. This allowed us to detect the Higgs boson, by looking at the invariant mass of the decay products which were photons.

17.2 Four-Momentum Transfer

$$Q^2 \equiv -(E - E')^2 - (c\vec{p} - c\vec{p}')^2 = -c^2\left(\vec{P}_i - \vec{P}_f\right)^2 \tag{161}$$

Since Q^2 is dependant on the Lorenz invariant four-vectors, it is also Lorenz invariant. The probability amplitude for the Yukawa potential is therefore:

$$\mathcal{M}(Q^2) = \frac{-g^2\hbar^2}{Q^2 + m^2c^2} \tag{162}$$

which is also Lorenz invariant.

17.3 Invariant Mass of Virtual Photon

Consider the following reaction:

$$e^+e^- \to \gamma^*/Z^* \to f\bar{f} \tag{163}$$

We have a symmetric collider which $E_b \rightarrow\leftarrow E_b$. The momentum of the virtual photon is 0.

$$\vec{p}^*_\gamma = 0 \to Wc^2 = 2E_b \tag{164}$$

We can then find the center mass energy:

$$\sqrt{s} = Wc^2 = E_{\text{cm}} = 2E_b \tag{165}$$

17.3.1 Fixed-Target Experiment

Consider a particle collision between a beam b, and a target t at rest. How do we find the center mass energy?

$$E_b^2 = m_b^2c^4 + p_b^2c^2 \quad , \quad E_t = m_tc^2 \tag{166}$$

$$(Wc^2)^2 = \left(\sum_i E_i\right)^2 - \left(\sum_i |\vec{p}_i|\right)^2 c^2 = \left(E_b + m_tc^2\right)^2 - (\vec{p}_bc)^2 \tag{167}$$

If we let $c = 1$, we find W.

$$W = E_{\text{cm}} = \sqrt{m_b^2 + m_t^2 + 2E_bm_t} \tag{168}$$

If the momentum is very large, we can neglect the masses of the particles so $E_{\text{cm}} = \sqrt{2m_tE_b}$.

18 Particle Accelerators

- Particles are accelerated by a voltage. This has a 1-1 correspondence with the energy for the electrons. Using a voltage of 1 MeV, we can accelerate electrons to an energy of 1 MeV.

18.1 Linear Accelerators

- At high voltages, the field breaks down. Using an alternating current, we can avoid this.
- Radio frequency linear accelerators are used to accelerate particles to any energy, as the particle only feels the electric energy in the gaps. This is illustrated in Fig. 29.

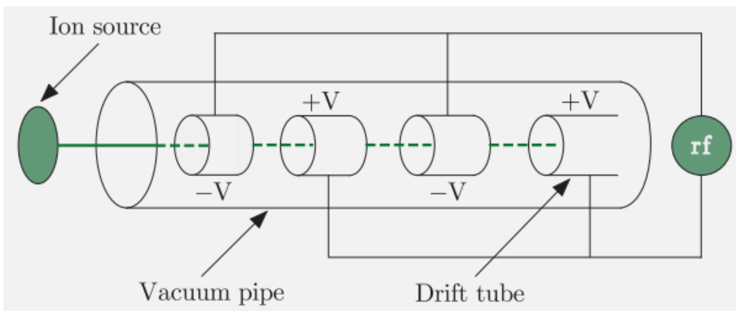


Figure 29: Illustration of a radio frequency linear accelerator. The voltages oscillate as the particles moves through the gaps. Only every second tube can be occupied.

18.2 Cyclotrons

- Using an oscillating current, the particle will start in the center, and spiral outwards. This is illustrated in Fig. 30.

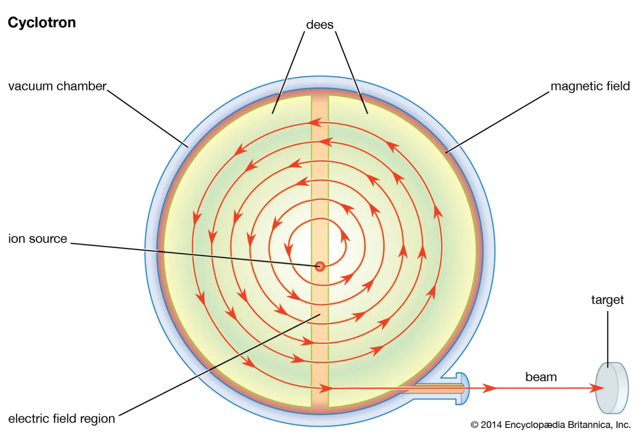


Figure 30: Illustration of a cyclotron.

18.3 Focusing of Particles

- Magnetic fields are used to focus the beam to a point.
- Quadrupole magnets (4 poles) are often used. They work analogously to a optical lense.
- The ability to focus the beam of some width $\sigma(s)$ is limited by the focusing properties $\beta(s)$ (the magnets) and how much each particle deviates from the beam ε.

$$\sigma(s) = \sqrt{\varepsilon\beta(s)} \tag{169}$$

18.4 Characteristics of Accelerators

- **Particle Type:** Protons, electrons, ions, etc.
- **Center of mass energy:** You need a certain amount of energy to create the different particles. You need $E_{\mathrm{cm}} \geq mc^2$. The wavelength of the particles is given by $\lambda = \frac{h}{p}$. You need a probe with an even smaller wavelength.
- **Luminosity:** $R = \mathcal{L}\sigma$ where $\mathcal{L} = fn_1n_2/4\pi\sigma_x\sigma_y$. This shows the production rate R needed. n_1 and n_2 are the number of particles, σ_x and σ_y are the transverse sizes of the beams. f is the rate of collisions.

18.5 Large Hadron Collider (LHC)

- Circumference of 27 km.
- Four interaction points where the beams collide.
- Eight straight sections of 530m, leading to the IPs (intersection points).
- 1200 superconducting dipole magnets are used to bend the beams.
- Can bunch $n = 1e11$ protons with a collision frequency $f = 40$MHz. These are focused to a width of $\sigma = 20\mu$m at the IP.

18.6 Particle Types

- **Proton-Proton Colliders:** Initial state is unknown. There is a lot of background noise, which must be filtered.
- **Electron-Positron Colliders:** Initial state is known. The background noise is lower.

18.7 Limitations

- **Circular proton-proton collider:** The strength of the magnetic fields limits the energy.
- **Circular electron-positron collider:** The radiation loss makes it hard to reach high energies.
- **Linear electron-positron collider:** The width of the beam is a lot larger, making the luminosity requirements harder to reach.