Automatic Geometric Decomposition for Analytical Inverse Kinematics: Derivations on 1R, 2R, 3R, 4R, and 5R Manipulators

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TABLE I

SUBSET OF THE SUBPROBLEMS PRESENTED BY ELIAS ET AL. [1].

Subproblem	Formulation
SP1	$\min \left\| \boldsymbol{R}(\boldsymbol{h}_i, \theta_i)\boldsymbol{x}_1 - \boldsymbol{x}_2 \right\|_2$
SP2	$\min \left\| \boldsymbol{R}(\boldsymbol{h}_i, \theta_i)\boldsymbol{x}_1 - \boldsymbol{R}(\boldsymbol{h}_j, \theta_j)\boldsymbol{x}_2 \right\|_2$
SP3	$\min \left| \left\| \boldsymbol{R}(\boldsymbol{h}_i, \theta_i)\boldsymbol{x}_1 - \boldsymbol{x}_2 \right\|_2 - d \right|$
SP4	$\min \left| \boldsymbol{h}_i^T \boldsymbol{R}(\boldsymbol{h}_j, \theta_j)\boldsymbol{x}_1 - d \right|$
SP5	$\boldsymbol{x}_0 + \boldsymbol{R}(\boldsymbol{h}_i, \theta_i)\boldsymbol{x}_1 = \boldsymbol{R}(\boldsymbol{h}_j, \theta_j)\left(\boldsymbol{x}_2 + \boldsymbol{R}(\boldsymbol{h}_k, \theta_k)\boldsymbol{x}_3\right)$
SP6	$\boldsymbol{h}_i^T \boldsymbol{R}(\boldsymbol{h}_j, \theta_j)\boldsymbol{x}_1 + \boldsymbol{h}_l^T \boldsymbol{R}(\boldsymbol{h}_k, \theta_k)\boldsymbol{x}_2 = d_1$ $\boldsymbol{h}_m^T \boldsymbol{R}(\boldsymbol{h}_j, \theta_j)\boldsymbol{x}_3 + \boldsymbol{h}_n^T \boldsymbol{R}(\boldsymbol{h}_k, \theta_k)\boldsymbol{x}_4 = d_2$

We denote a vector displacement as $\boldsymbol{x}_n \in \mathbb{R}^3$, scalars as $d_n \in \mathbb{R}$, unit vectors of rotation as $\boldsymbol{h}_n \in \mathbb{R}^3$ and corresponding angles as θ_n. The given formulations are independent of any particular manipulator kinematics.

I. EXTENDED APPENDIX: AN OVERVIEW

Manipulators with fewer than four joint axes can be parametrized using only position or orientation constraints. For manipulators with four or more joints, we use a full 6DOF pose for our input constraints. As this parametrization can impose more constraints than needed for 4R and 5R manipulators, it is up to the user to ensure the given rotation and position are compatible for the respective manipulator. Unreachable poses result in least-squares solution sets. As we perform the inverse kinematic calculations for some angles based on orientation constraints and some based on position constraints, some solutions will be viable to achieve *either* the desired position or orientation, but not both. Hence, we compute the forward kinematics based on the obtained solution set and rule out (denote as approximate) solutions that do not result in the desired pose.

In the following derivations, we only consider cases of intersecting/parallel axes that resemble an analytically solvable manipulator, i.e., one that is not redundant in any pose with respect to the chosen parametrization (position/orientation IK). All proposed cases are hereby strictly necessary to check for, as additional intersecting/parallel axes can lead to solution continuities within the suproblems if not considered.

All analytically solvable manipulators with four (or fewer) rotational axes are solvable by the proposed subproblem decompositions (sometimes only after kinematic inversion). Contrarily, 5R manipulators must meet at least one of the following sufficient criteria to be analytically solvable:

- The last or first two axes 1,2 (5,6) intersect.
- The intermediate axes 2,3 (3,4) intersect while the axes 3,4 (2,3) are parallel.
- Any three consecutive axes are parallel.

Additional intersecting/parallel axes – as long as they do not lead to implicit redundancies – lead to further simplifications, which we also account for.

II. 1R MANIPULATORS

The forward kinematics for a 1R manipulator are given via:

$$ {}^0\boldsymbol{p}_{EE} = {}^0\boldsymbol{p}_1 + {}^0\boldsymbol{R}_1\, {}^1\boldsymbol{p}_{EE} \tag{1} $$
$$ {}^0\boldsymbol{R}^*_{EE} = {}^0\boldsymbol{R}_1\, {}^1\boldsymbol{R}^*_{EE}\,. \tag{2} $$

The IK can be solved either given a desired position by applying SP1 to (3) or via a desired orientation by applying SP1 to (4).

$$ \left\| {}^0\boldsymbol{R}_1\, {}^1\boldsymbol{p}_{EE} - \left({}^0\boldsymbol{p}_{EE} - {}^0\boldsymbol{p}_1\right) \right\| = 0 \tag{3} $$
$$ \left\| {}^0\boldsymbol{R}_1 \boldsymbol{h}_n - {}^0\boldsymbol{R}^*_{EE}\, {}^1\boldsymbol{R}^{*T}_{EE} \boldsymbol{h}_n \right\| = 0 \tag{4} $$

III. 2R MANIPULATORS

The forward kinematics for a 2R manipulator are given via:

$$ {}^0\boldsymbol{p}_{EE} = {}^0\boldsymbol{p}_1 + {}^0\boldsymbol{R}_1\left({}^1\boldsymbol{p}_2 + {}^1\boldsymbol{R}_2\, {}^1\boldsymbol{p}_2\right) \tag{5} $$
$$ {}^0\boldsymbol{R}^*_2 = {}^0\boldsymbol{R}^*_{EE}\, {}^2\boldsymbol{R}^{*T}_{EE} = {}^0\boldsymbol{R}_1\, {}^1\boldsymbol{R}_2\,. \tag{6} $$

a) Position IK: Two cases must be distinguished when solving the IK for a given position: the axes intersect, or they do not. If the two axes of the manipulator intersect, we can choose ${}^1\boldsymbol{p}_2 = 0$ to obtain (7) from (5). We subsequently use SP2 on (7) to obtain θ_1, θ_2.

$$ {}^0\boldsymbol{R}_1^T\left({}^0\boldsymbol{p}_{EE} - {}^0\boldsymbol{p}_1\right) - {}^1\boldsymbol{R}_2\, {}^2\boldsymbol{p}_{EE} = 0 \tag{7} $$

If, on the other hand, the two axes do not intersect, we leverage norm-preservation on (5) to obtain (8). We apply SP3 to (8) to obtain θ_2.

$$ \left\| {}^1\boldsymbol{R}_2\, {}^2\boldsymbol{p}_{EE} + {}^1\boldsymbol{p}_2 \right\| - \left\| {}^1\boldsymbol{p}_{EE} \right\| = 0 \tag{8} $$

By reformulating (5) and inserting the calculated θ_2, we obtain (9). We obtain θ_1 by applying SP1 to (9).

$$ {}^0\boldsymbol{R}_1\left({}^1\boldsymbol{p}_2 + {}^1\boldsymbol{R}^*_2\, {}^2\boldsymbol{p}_{EE}\right) - \left({}^0\boldsymbol{p}_{EE} - {}^0\boldsymbol{p}_1\right) = 0 \tag{9} $$

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b) Orientation IK: The orientation kinematics can only be analytically solved if the two axes are not parallel. Otherwise, the manipulator is redundant in every configuration for its orientation IK. We obtain a vector $\boldsymbol{h}_n$ that is normal to the rotated axis $\boldsymbol{h}_1$ and the axis $\boldsymbol{h}_2$ via (10). After reformulating (6) and right-multiply with $\boldsymbol{h}_n$, we obtain (11). Applying SP2 to (11) subsequently yields θ_1, θ_2.

$$\boldsymbol{h}_n = {}^0\boldsymbol{R}_{EE}^{*T}\boldsymbol{h}_1 \times \boldsymbol{h}_2 \tag{10}$$

$$ {}^0\boldsymbol{R}_1^T\,{}^0\boldsymbol{R}_2^*\boldsymbol{h}_n - {}^1\boldsymbol{R}_2\boldsymbol{h}_n = 0 \tag{11}$$

IV. 3R MANIPULATORS

The forward kinematics for a 3R manipulator are given via (12). In the following, we will use ${}^1\boldsymbol{p}_{EE}$ for conciseness as denoted in (13).

$$ {}^0\boldsymbol{p}_{EE} = {}^0\boldsymbol{p}_1 + {}^0\boldsymbol{R}_1\,{}^1\boldsymbol{p}_2 + {}^0\boldsymbol{R}_2\,{}^2\boldsymbol{p}_3 + {}^0\boldsymbol{R}_3\,{}^3\boldsymbol{p}_{EE} \tag{12}$$

$$ {}^1\boldsymbol{p}_{EE} = {}^0\boldsymbol{p}_{EE} - {}^0\boldsymbol{p}_1 \tag{13}$$

$$ {}^0\boldsymbol{R}_3 = {}^0\boldsymbol{R}_{EE}\,{}^3\boldsymbol{R}_{EE}^{*T} = {}^0\boldsymbol{R}_1\,{}^1\boldsymbol{R}_2\,{}^2\boldsymbol{R}_3 \tag{14}$$

a) Position IK: **If no two consecutive axes are parallel and no axes intersect**, we can directly apply SP5 to (12) and obtain $\theta_1, \theta_2, \theta_3$.

If the first two axes intersect we can choose ${}^1\boldsymbol{p}_2 = 0$, rephrase (12) according to (15), and then leverage norm-preservation to obtain (16). Applying SP3 to (16) yields θ_3.

$$ {}^1\boldsymbol{R}_2^T\,{}^0\boldsymbol{R}_1^T\,{}^1\boldsymbol{p}_{EE} = {}^2\boldsymbol{R}_3\,{}^3\boldsymbol{p}_{EE} + {}^2\boldsymbol{p}_3 \tag{15}$$

$$ \left\|{}^1\boldsymbol{p}_{EE}\right\| = \left\|{}^2\boldsymbol{R}_3\,{}^3\boldsymbol{p}_{EE} + {}^2\boldsymbol{p}_3\right\| \tag{16}$$

Inserting the calculated value for θ_3 into a rephrased version of (12) yields (17). Applying SP2 to (17) yields θ_1, θ_2.

$$ {}^0\boldsymbol{R}_1^T\,{}^1\boldsymbol{p}_{EE} = {}^1\boldsymbol{R}_2\left({}^2\boldsymbol{p}_3 + {}^2\boldsymbol{R}_3^*\,{}^3\boldsymbol{p}_{EE}\right) \tag{17}$$

If the first two axes are parallel we can choose $\boldsymbol{h}_1 = \boldsymbol{h}_2$. We left-multiply (12) with $\boldsymbol{h}_1$ to obtain (18). Applying SP4 to (18) yields θ_3. By inserting θ_3 into a rephrased version of (12), we get (19), which simplifies to (20) after leveraging the norm-preserving property.

$$ \boldsymbol{h}_1^T\left({}^1\boldsymbol{p}_{EE} - {}^1\boldsymbol{p}_2 - {}^2\boldsymbol{p}_3\right) = \boldsymbol{h}_1^T\,{}^2\boldsymbol{R}_3\,{}^3\boldsymbol{p}_4 \tag{18}$$

$$ {}^0\boldsymbol{R}_1^T\,{}^1\boldsymbol{p}_{EE} = {}^1\boldsymbol{R}_2\left({}^2\boldsymbol{p}_3 + {}^2\boldsymbol{R}_3^*\,{}^3\boldsymbol{p}_{EE}\right) + {}^1\boldsymbol{p}_2 \tag{19}$$

$$ \left\|{}^1\boldsymbol{p}_{EE}\right\| = \left\|{}^1\boldsymbol{R}_2\left({}^2\boldsymbol{p}_3 + {}^2\boldsymbol{R}_3^*\,{}^3\boldsymbol{p}_{EE}\right) + {}^1\boldsymbol{p}_2\right\| \tag{20}$$

We apply SP3 to (20) to obtain θ_2. Plugging θ_2, θ_3 into (12) yields (21), from which we obtain θ_1 via SP1.

$$ {}^0\boldsymbol{R}_1^T\,{}^1\boldsymbol{p}_{EE} - {}^1\boldsymbol{R}_2^*\left({}^2\boldsymbol{p}_3 + {}^2\boldsymbol{R}_3^*\,{}^3\boldsymbol{p}_{EE}\right) = 0 \tag{21}$$

b) Orientation IK: **If no consecutive two axes are parallel** we rephrase (14), left-multiply with $\boldsymbol{h}_3^T$ and right-multiply with $\boldsymbol{h}_1$ to obtain (22), which simplifies to (23). Applying SP4 to (23) yields θ_2.

$$ \boldsymbol{h}_3^T\,{}^0\boldsymbol{R}_3^{*T}\,{}^0\boldsymbol{R}_1\boldsymbol{h}_1 = \boldsymbol{h}_3^T\,{}^2\boldsymbol{R}_3^T\,{}^1\boldsymbol{R}_2^T\boldsymbol{h}_1 \tag{22}$$

$$ \boldsymbol{h}_3^T\,{}^0\boldsymbol{R}_3^{*T}\boldsymbol{h}_1 = \boldsymbol{h}_3^T\,{}^1\boldsymbol{R}_2^T\boldsymbol{h}_1 \tag{23}$$

We insert θ_2 into a rephrased version of (14) and right-multiply with $\boldsymbol{h}_3$. This yields (24), which simplifies to (25). We obtain θ_1 from (25) by applying SP1.

$$ {}^0\boldsymbol{R}_1^T\,{}^0\boldsymbol{R}_3^*\,{}^2\boldsymbol{R}_3^T\boldsymbol{h}_3 = {}^1\boldsymbol{R}_2^*\boldsymbol{h}_3 \tag{24}$$

$$ {}^0\boldsymbol{R}_1^T\,{}^0\boldsymbol{R}_3^*\boldsymbol{h}_3 = {}^1\boldsymbol{R}_2^*\boldsymbol{h}_3 \tag{25}$$

Let $\boldsymbol{h}_n$ denote a vector normal to $\boldsymbol{h}_3$. We again rephrase (14), insert θ_1, θ_2, and right-multiply with $\boldsymbol{h}_n$ to obtain (26). Applying SP1 to (26) finally yields θ_3.

$$ {}^2\boldsymbol{R}_3\boldsymbol{h}_n = {}^1\boldsymbol{R}_2^{*T}\,{}^0\boldsymbol{R}_1^{*T}\,{}^0\boldsymbol{R}_3^*\boldsymbol{h}_n \tag{26}$$

If the first two axes are parallel We rephrase according to (22), but without left-multiplication with $\boldsymbol{h}_3^T$, which, after simplification, yields (27). Applying SP1 to (27) yields θ_3.

$$ {}^0\boldsymbol{R}_3^{*T}\boldsymbol{h}_1 = {}^2\boldsymbol{R}_3^T\boldsymbol{h}_1 \tag{27}$$

Inserting θ_3 into a rephrased version of (14) yields (28), which simplifies to (29). We obtain θ_1 by applying SP1 to (29).

$$ {}^0\boldsymbol{R}_1^T\,{}^0\boldsymbol{R}_3^*\,{}^2\boldsymbol{R}_3^{*T}\boldsymbol{h}_2 = {}^1\boldsymbol{R}_2\boldsymbol{h}_2 \tag{28}$$

$$ {}^0\boldsymbol{R}_1^T\,{}^0\boldsymbol{R}_3^*\,{}^2\boldsymbol{R}_3^{*T}\boldsymbol{h}_2 = \boldsymbol{h}_2 \tag{29}$$

$$ {}^1\boldsymbol{R}_2\boldsymbol{h}_3 = {}^0\boldsymbol{R}_1^{*T}\,{}^0\boldsymbol{R}_3^*\,{}^2\boldsymbol{R}_3^{*T}\boldsymbol{h}_3 \tag{30}$$

Inserting θ_1, θ_3 into a rephrased version of (14) yields (30). We apply SP1 to (30) for θ_2.

If the manipulator does not match any of the above cases, we use kinematic inversion and solve the inverted kinematic chain with the previous derivations.

V. 4R MANIPULATORS

In contrast to the prior manipulators with less than four degrees of freedom, we now consider both the end effector position and orientation as given.

The forward position kinematics for a 4R manipulator are given by (31). As the rotation ${}^0\boldsymbol{R}_4^*$ is known as per the orientation kinematics in (33), we can break down (31) to the translation ${}^1\boldsymbol{p}_4$ according to (32).

$$\begin{aligned} {}^0\boldsymbol{p}_{EE} &= {}^0\boldsymbol{p}_1 + {}^0\boldsymbol{R}_1\,{}^1\boldsymbol{p}_2 + {}^0\boldsymbol{R}_2\,{}^2\boldsymbol{p}_3 \\ &\quad + {}^0\boldsymbol{R}_3\,{}^3\boldsymbol{p}_4 + {}^0\boldsymbol{R}_4\,{}^4\boldsymbol{p}_{EE} \end{aligned} \tag{31}$$

$$\begin{aligned} {}^1\boldsymbol{p}_4 &= {}^0\boldsymbol{p}_{EE} - {}^0\boldsymbol{p}_1 - {}^0\boldsymbol{R}_4^*\,{}^4\boldsymbol{p}_{EE} \\ &= {}^0\boldsymbol{R}_1\,{}^1\boldsymbol{p}_2 + {}^0\boldsymbol{R}_2\,{}^2\boldsymbol{p}_3 + {}^0\boldsymbol{R}_3\,{}^3\boldsymbol{p}_4 \end{aligned} \tag{32}$$

$$ {}^0\boldsymbol{R}_4^* = {}^0\boldsymbol{R}_{EE}^*\,{}^4\boldsymbol{R}_{EE}^{*T} = {}^0\boldsymbol{R}_1\,{}^1\boldsymbol{R}_2\,{}^2\boldsymbol{R}_3\,{}^3\boldsymbol{R}_4 \tag{33}$$

If no consecutive pair of axes is parallel or intersects, we can rewrite (32) according to (34), to which we directly apply SP5 to obtain $\theta_1, \theta_2, \theta_3$.

$$ {}^0\boldsymbol{R}_1^T\,{}^1\boldsymbol{p}_4 - {}^1\boldsymbol{p}_2 = {}^1\boldsymbol{R}_2\left({}^2\boldsymbol{p}_3 + {}^2\boldsymbol{R}_3\,{}^3\boldsymbol{p}_4\right) \tag{34}$$

$$ {}^3\boldsymbol{R}_4\boldsymbol{h}_n - {}^2\boldsymbol{R}_3^{*T}\,{}^1\boldsymbol{R}_2^{*T}\,{}^0\boldsymbol{R}_1^{*T}\,{}^0\boldsymbol{R}_4^*\boldsymbol{h}_n = 0 \tag{35}$$

Further, we insert the obtained joint values into (33) and, after rephrasing, obtain (35). The vector $\boldsymbol{h}_n$ is hereby chosen normal to $\boldsymbol{h}_4$. We obtain θ_4 by applying SP1 to (35).

If the first two axes are parallel, we can choose $\boldsymbol{h}_1 = \boldsymbol{h}_2$. We left-multiply (32) with $\boldsymbol{h}_1^T$ to obtain (36), from which we obtain θ_3 via SP4.

$$\boldsymbol{h}_1^T \, {}^2\boldsymbol{R}_3 \, {}^3\boldsymbol{p}_4 = \boldsymbol{h}_1^T \left({}^1\boldsymbol{p}_4 - {}^1\boldsymbol{p}_2 - {}^2\boldsymbol{p}_3 \right) \tag{36}$$

Rephrasing (32) and inserting θ_3 yields (37), which, via norm-preservation, results in (38). Applying SP3 to (38) yields θ_1.

$${}^0\boldsymbol{R}_1^T \, {}^1\boldsymbol{p}_4 - {}^1\boldsymbol{p}_2 = {}^1\boldsymbol{R}_2 \left({}^2\boldsymbol{p}_3 + {}^2\boldsymbol{R}_3^* \, {}^3\boldsymbol{p}_4 \right) \tag{37}$$
$$\left\| {}^0\boldsymbol{R}_1^T \, {}^1\boldsymbol{p}_4 - {}^1\boldsymbol{p}_2 \right\| = \left\| {}^2\boldsymbol{p}_3 + {}^2\boldsymbol{R}_3^* \, {}^3\boldsymbol{p}_4 \right\| \tag{38}$$

Inserting θ_1 back into (37) yields (39), from which we obtain θ_2 via SP1. Finally, we insert $\theta_1, \theta_2, \theta_3$ into (33) and right-multiply by the vector $\boldsymbol{h}_n$ – which we choose such that it is normal to $\boldsymbol{h}_4$ – to obtain (40). We retrieve θ_4 by applying SP1 to (40).

$${}^1\boldsymbol{R}_2 \left({}^2\boldsymbol{p}_3 + {}^2\boldsymbol{R}_3^* \, {}^3\boldsymbol{p}_4 \right) - \left({}^0\boldsymbol{R}_1^{*T} \, {}^1\boldsymbol{p}_4 - {}^1\boldsymbol{p}_2 \right) = 0 \tag{39}$$
$${}^3\boldsymbol{R}_4^T \boldsymbol{h}_n - {}^0\boldsymbol{R}_4^{*T} \, {}^0\boldsymbol{R}_1^* \, {}^1\boldsymbol{R}_2^* \, {}^2\boldsymbol{R}_3^* \boldsymbol{h}_n = 0 \tag{40}$$

If the second and third axis are parallel, we can choose $\boldsymbol{h}_2 = \boldsymbol{h}_3$. We start off by rephrasing (32) according to (41). When left-multiplying (41) with $\boldsymbol{h}_1$, it simplifies to (42), to which we apply SP4 to obtain θ_1.

$${}^2\boldsymbol{R}_3^T \, {}^1\boldsymbol{R}_2^T \, {}^0\boldsymbol{R}_1^T \, {}^1\boldsymbol{p}_4 = {}^2\boldsymbol{R}_3^T \left({}^1\boldsymbol{R}_2^T \, {}^1\boldsymbol{p}_2 + {}^2\boldsymbol{p}_3 \right) + {}^3\boldsymbol{p}_4 \tag{41}$$
$$\boldsymbol{h}_2^T \, {}^0\boldsymbol{R}_1^T \, {}^1\boldsymbol{p}_4 = \boldsymbol{h}_2^T \left({}^1\boldsymbol{p}_2 + {}^2\boldsymbol{p}_3 + {}^3\boldsymbol{p}_4 \right) \tag{42}$$

We rephrase (32) according to (34), insert θ_1, and leverage norm-preservation to obtain (43). Applying SP3 to (43) yields θ_3. Inserting θ_1, θ_3 back into (34) yields (44), which we can solve via SP1 for θ_2.

$$\left\| {}^0\boldsymbol{R}_1^{*T} \, {}^1\boldsymbol{p}_4 - {}^1\boldsymbol{p}_2 \right\| = \left\| {}^2\boldsymbol{R}_3 \, {}^3\boldsymbol{p}_4 + {}^2\boldsymbol{p}_3 \right\| \tag{43}$$
$${}^1\boldsymbol{R}_2 \left({}^2\boldsymbol{p}_3 + {}^2\boldsymbol{R}_3^* \, {}^3\boldsymbol{p}_4 \right) - \left({}^0\boldsymbol{R}_1^{*T} \, {}^1\boldsymbol{p}_4 - {}^1\boldsymbol{p}_2 \right) = 0 \tag{44}$$

Just like before, we insert $\theta_1, \theta_2, \theta_3$ into (33) and right-multiply by the vector $\boldsymbol{h}_n$ (normal to $\boldsymbol{h}_4$) to obtain (40). We retrieve θ_4 by applying SP1 to (40).

If the second and third axis intersect, we can choose ${}^2\boldsymbol{p}_3 = 0$, which simplifies (32) to (45). Employing norm-preservation on (45) results in (46), from which we obtain θ_1 through SP3.

$${}^1\boldsymbol{p}_4 - {}^0\boldsymbol{R}_1 \, {}^1\boldsymbol{p}_2 = {}^0\boldsymbol{R}_1 \, {}^1\boldsymbol{R}_2 \, {}^2\boldsymbol{R}_3 \, {}^3\boldsymbol{p}_4 \tag{45}$$
$$\left\| {}^1\boldsymbol{p}_4 - {}^0\boldsymbol{R}_1 \, {}^1\boldsymbol{p}_2 \right\| = \left\| {}^3\boldsymbol{p}_4 \right\| \tag{46}$$

Rephrasing (45) and inserting θ_1 yields (47), to which we apply SP2 to obtain θ_2, θ_3.

$${}^1\boldsymbol{R}_2^T \left({}^0\boldsymbol{R}_1^{*T} \, {}^1\boldsymbol{p}_4 - {}^1\boldsymbol{p}_2 \right) = {}^2\boldsymbol{R}_3 \, {}^3\boldsymbol{p}_4 \tag{47}$$

Just as in the two prior cases, we insert $\theta_1, \theta_2, \theta_3$ into (33) and right-multiply by the vector $\boldsymbol{h}_n$ (normal to $\boldsymbol{h}_4$) to obtain (40). We then retrieve θ_4 by applying SP1 to (40).

If the third and fourth axis intersect, we can choose ${}^3\boldsymbol{p}_4 = 0$ and hence simplify (32) to (48), which can be rephrased to (49). Imposing the L2 norm on both sides of (49) results in (50), from which we obtain θ_1 via SP3.

$${}^1\boldsymbol{p}_4 = {}^0\boldsymbol{R}_1 \, {}^1\boldsymbol{p}_2 + {}^0\boldsymbol{R}_2 \, {}^2\boldsymbol{p}_3 \tag{48}$$
$${}^0\boldsymbol{R}_1^T \, {}^1\boldsymbol{p}_4 - {}^1\boldsymbol{p}_2 = {}^1\boldsymbol{R}_2 \, {}^2\boldsymbol{p}_3 \tag{49}$$
$$\left\| {}^0\boldsymbol{R}_1^T \, {}^1\boldsymbol{p}_4 - {}^1\boldsymbol{p}_2 \right\| = \left\| {}^2\boldsymbol{p}_3 \right\| \tag{50}$$

After inserting θ_1 back into (49), we can directly apply SP1 to retrieve θ_2. Finally, we rephrase (33), insert θ_1, θ_2, and right-multiply with a vector $\boldsymbol{h}_n$ that is normal to $\boldsymbol{h}_4$ to obtain (51). Applying SP2 to (51) then yields θ_3, θ_4.

$${}^2\boldsymbol{R}_3^T \, {}^1\boldsymbol{R}_2^{*T} \, {}^0\boldsymbol{R}_1^{*T} \, {}^0\boldsymbol{R}_4^* \boldsymbol{h}_n = {}^3\boldsymbol{R}_4 \boldsymbol{h}_n \tag{51}$$

If the first and last two axes intersect in a separate point, we can choose ${}^1\boldsymbol{p}_2 = 0$ and ${}^3\boldsymbol{p}_4 = 0$ to simplify (32) to (52). We rephrasing (52) according to (53), to which we apply SP2 to obtain θ_1, θ_2.

$${}^1\boldsymbol{p}_4 = {}^0\boldsymbol{R}_2 \, {}^2\boldsymbol{p}_3 \tag{52}$$
$${}^0\boldsymbol{R}_1^T \, {}^1\boldsymbol{p}_4 = {}^1\boldsymbol{R}_2 \, {}^2\boldsymbol{p}_3 \tag{53}$$

We rephrase (33), insert θ_1, θ_2, and right-multiply by a vector $\boldsymbol{h}_n$ that is normal to $\boldsymbol{h}_4$ to obtain (54). Applying SP2 to (54) subsequently yields θ_3, θ_4.

$${}^2\boldsymbol{R}_3^T \, {}^1\boldsymbol{R}_2^{*T} \, {}^0\boldsymbol{R}_1^{*T} \, {}^0\boldsymbol{R}_4^* \boldsymbol{h}_n = {}^3\boldsymbol{R}_4 \boldsymbol{h}_n \tag{54}$$

If the last three axes intersect in a common point (i.e., form a spherical wrist), we can choose ${}^2\boldsymbol{p}_3 = 0$ and ${}^3\boldsymbol{p}_4 = 0$ to simplify (32) to (55). We directly obtain θ_1 from (55) by applying SP1.

$${}^1\boldsymbol{p}_4 = {}^0\boldsymbol{R}_1 \, {}^1\boldsymbol{p}_2 \tag{55}$$

Rephrasing (33) and inserting θ_1 yields (56). Right-multiplying (56) with $\boldsymbol{h}_4$ simplifies it to (56). Applying SP2 to (56) yields θ_2, θ_3.

$${}^1\boldsymbol{R}_2^T \, {}^0\boldsymbol{R}_1^{*T} \, {}^0\boldsymbol{R}_4^* = {}^2\boldsymbol{R}_3 \, {}^3\boldsymbol{R}_4 \tag{56}$$
$${}^1\boldsymbol{R}_2^T \, {}^0\boldsymbol{R}_1^{*T} \, {}^0\boldsymbol{R}_4^* \boldsymbol{h}_4 = {}^2\boldsymbol{R}_3 \boldsymbol{h}_4 \tag{57}$$

Again, we insert $\theta_1, \theta_2, \theta_3$ into (33) and right-multiply by the vector $\boldsymbol{h}_n$ (normal to $\boldsymbol{h}_4$) to obtain (40). We then retrieve θ_4 by applying SP1 to (40).

In any case, we use kinematic inversion and check if the inverted kinematic chain matches a case that is more specialized (i.e., represents a case that employs *simpler* subproblems) than that of the non-inverted one. We hereby first check for a spherical wrist (i.e., three axes at either end of the manipulator intersecting in a common point) and then all the other cases. The most general case (no consecutive intersecting or parallel axes) is only chosen if, and only if, no other case matches.

VI. 5R MANIPULATORS

As mentioned Section I, only certain 5R manipulators are currently known to us to be analytically solvable per our method. Hence, the following list of cases is not exhaustive. Non-redundant (analytically solvable) 5R manipulators might exist that fall into neither of the following categories.

Like for 4R manipulators, we consider both the desired end effector position and orientation as given. When leveraging the same simplifications as in (32), we obtain a concise formulation for the forward kinematics of a 5R manipulator via (58) and (59).

$$\begin{aligned} {}^1\boldsymbol{p}_5 = {}^0\boldsymbol{R}_1\,{}^1\boldsymbol{p}_2 + {}^0\boldsymbol{R}_2\,{}^2\boldsymbol{p}_3 \\ + {}^0\boldsymbol{R}_3\,{}^3\boldsymbol{p}_4 + {}^0\boldsymbol{R}_4\,{}^4\boldsymbol{p}_5 \end{aligned} \tag{58}$$

$${}^0\boldsymbol{R}_5^* = {}^0\boldsymbol{R}_{EE}^*\,{}^5\boldsymbol{R}_{EE}^{*T} = {}^0\boldsymbol{R}_1\,{}^1\boldsymbol{R}_2\,{}^2\boldsymbol{R}_3\,{}^3\boldsymbol{R}_4\,{}^4\boldsymbol{R}_5 \tag{59}$$

If the last two axes intersect, we can choose ${}^4\boldsymbol{p}_5 = 0$ such that (58) simplifies to (60). We obtain $\theta_1, \theta_2, \theta_3$ by applying SP5 to (60). We rephrase (59), insert $\theta_1, \theta_2, \theta_3$, and right-multiply with a vector $\boldsymbol{h}_n$ that is normal to $\boldsymbol{h}_5$ to obtain (61). Applying SP2 to (61) yields θ_4, θ_5.

$${}^0\boldsymbol{R}_1^T\,{}^1\boldsymbol{p}_5 = {}^1\boldsymbol{p}_2 + {}^1\boldsymbol{R}_2\,{}^2\boldsymbol{p}_3 + {}^1\boldsymbol{R}_2\,{}^2\boldsymbol{R}_3\,{}^3\boldsymbol{p}_4 \tag{60}$$

$${}^3\boldsymbol{R}_4^T\,{}^2\boldsymbol{R}_3^{*T}\,{}^1\boldsymbol{R}_2^{*T}\,{}^0\boldsymbol{R}_1^{*T}\,{}^0\boldsymbol{R}_5^*\boldsymbol{h}_n = {}^4\boldsymbol{R}_5\boldsymbol{h}_n \tag{61}$$

If the last two axes intersect while also the second and third axis intersect, we can choose ${}^4\boldsymbol{p}_5 = 0$ and ${}^2\boldsymbol{p}_3 = 0$ such that (58) simplifies to (62). Using the norm-preservation property, we further simplify (62) to (63). We apply SP3 to (63) to obtain θ_1.

$${}^0\boldsymbol{R}_1^T\,{}^1\boldsymbol{p}_5 - {}^1\boldsymbol{p}_2 = {}^1\boldsymbol{R}_2\,{}^2\boldsymbol{R}_3\,{}^3\boldsymbol{p}_4 \tag{62}$$

$$\left\|{}^0\boldsymbol{R}_1^T\,{}^1\boldsymbol{p}_5 - {}^1\boldsymbol{p}_2\right\| = \left\|{}^3\boldsymbol{p}_4\right\| \tag{63}$$

Inserting θ_1 into a rephrased version of (62) yields (64), to which we apply SP2 to retrieve θ_2, θ_3.

$${}^1\boldsymbol{R}_2^T\left({}^0\boldsymbol{R}_1^{*T}\,{}^1\boldsymbol{p}_5 - {}^1\boldsymbol{p}_2\right) = {}^2\boldsymbol{R}_3\,{}^3\boldsymbol{p}_4 \tag{64}$$

Finally, we insert $\theta_1, \theta_2, \theta_3$ into (59), rephrase it, and right-multiply with a vector $\boldsymbol{h}_n$ that is normal to $\boldsymbol{h}_5$ to obtain (65). We then apply SP2 to (65) to obtain θ_4, θ_5

$${}^4\boldsymbol{R}_5\boldsymbol{h}_n = {}^3\boldsymbol{R}_4^T\,{}^2\boldsymbol{R}_3^{*T}\,{}^1\boldsymbol{R}_2^{*T}\,{}^0\boldsymbol{R}_1^{*T}\,{}^0\boldsymbol{R}_5^*\boldsymbol{h}_n \tag{65}$$

If the last two axes intersect while also the first two axes intersect, we can choose ${}^4\boldsymbol{p}_5 = 0$ and ${}^1\boldsymbol{p}_2 = 0$ such that (58) simplifies to (66). Using norm-preservation on (66) yields (67), to which we apply SP3 for θ_3.

$${}^0\boldsymbol{R}_1^T\,{}^1\boldsymbol{p}_5 = {}^1\boldsymbol{R}_2\left({}^2\boldsymbol{p}_3 + {}^2\boldsymbol{R}_3\,{}^3\boldsymbol{p}_4\right) \tag{66}$$

$$\left\|{}^1\boldsymbol{p}_5\right\| = \left\|{}^2\boldsymbol{p}_3 + {}^2\boldsymbol{R}_3\,{}^3\boldsymbol{p}_4\right\| \tag{67}$$

After inserting θ_3 back into (66), we can directly apply SP2 to it and subsequently obtain θ_1, θ_2. For θ_4, θ_5, we follow the same procedure as in the previous case to end up at (65), to which we then obtain SP2 to yield θ_4, θ_5.

If the last two axes intersect while the first two axes are parallel, we can choose ${}^4\boldsymbol{p}_5 = 0$ and $\boldsymbol{h}_1 = \boldsymbol{h}_2$. Equation (58) then simplifies to (60). Left-multiplication of (60) by $\boldsymbol{h}_1^T$ further simplifies it to (68). We apply SP4 to (68) to obtain θ_3.

$$\boldsymbol{h}_1^T\,{}^2\boldsymbol{R}_3\,{}^3\boldsymbol{p}_4 = \boldsymbol{h}_1^T\left({}^1\boldsymbol{p}_5 - {}^1\boldsymbol{p}_2 - {}^2\boldsymbol{p}_3\right) \tag{68}$$

Inserting θ_3 into (60) yields (69), which, after using norm-preservation, simplifies to (70). Applying SP3 to (70) yields θ_1. We insert θ_1 back into (69), apply SP1 to it, and thus obtain θ_2.

$${}^0\boldsymbol{R}_1^T\,{}^1\boldsymbol{p}_5 - {}^1\boldsymbol{p}_2 = {}^1\boldsymbol{R}_2\left({}^2\boldsymbol{p}_3 + {}^2\boldsymbol{R}_3^*\,{}^3\boldsymbol{p}_4\right) \tag{69}$$

$$\left\|{}^0\boldsymbol{R}_1^T\,{}^1\boldsymbol{p}_5 - {}^1\boldsymbol{p}_2\right\| = \left\|{}^2\boldsymbol{p}_3 + {}^2\boldsymbol{R}_3\,{}^3\boldsymbol{p}_4\right\| \tag{70}$$

For θ_4, θ_5, we follow the same procedure as in the previous two case.

If the last two axes intersect while the second and third axis are parallel, we can choose ${}^4\boldsymbol{p}_5 = 0$ and $\boldsymbol{h}_2 = \boldsymbol{h}_3$. Equation (58) then simplifies to (60). Left-multiplication of (60) by $\boldsymbol{h}_2^T$ further simplifies it to (71). Applying SP4 to (71) yields θ_1.

$$\boldsymbol{h}_2^T\,{}^0\boldsymbol{R}_1^T\,{}^1\boldsymbol{p}_5 = \boldsymbol{h}_2^T\left({}^1\boldsymbol{p}_2 + {}^2\boldsymbol{p}_3 + {}^3\boldsymbol{p}_4\right) \tag{71}$$

We rephrase (60) and insert θ_1, such that we obtain (72). Leveraging norm-preservation, (72) further simplifies to (73). We apply SP3 to (73) to retrieve θ_3.

$${}^2\boldsymbol{R}_3\,{}^3\boldsymbol{p}_4 + {}^2\boldsymbol{p}_3 = {}^1\boldsymbol{R}_2^T\left({}^0\boldsymbol{R}_1^{*T}\,{}^1\boldsymbol{p}_5 - {}^1\boldsymbol{p}_2\right) \tag{72}$$

$$\left\|{}^2\boldsymbol{R}_3\,{}^3\boldsymbol{p}_4 + {}^2\boldsymbol{p}_3\right\| = \left\|{}^0\boldsymbol{R}_1^{*T}\,{}^1\boldsymbol{p}_5 - {}^1\boldsymbol{p}_2\right\| \tag{73}$$

We obtain $\theta_2, \theta_4, \theta_5$ by following the same procedure as in the previous case.

If the second and third axis are parallel while the third and fourth axis intersect, we can choose ${}^3\boldsymbol{p}_4 = 0$ and $\boldsymbol{h}_2 = \boldsymbol{h}_3$. This simplifies (58) to (74). Left-multiplication of (74) by $\boldsymbol{h}_2^T$ further simplifies it to (75).

$${}^0\boldsymbol{R}_1^T\,{}^1\boldsymbol{p}_5 = {}^1\boldsymbol{p}_2 + {}^1\boldsymbol{R}_2\left({}^2\boldsymbol{p}_3 + {}^2\boldsymbol{R}_3\,{}^3\boldsymbol{R}_4\,{}^4\boldsymbol{p}_5\right) \tag{74}$$

$$\boldsymbol{h}_2^T\,{}^0\boldsymbol{R}_1^T\,{}^1\boldsymbol{p}_5 - \boldsymbol{h}_2^T\,{}^3\boldsymbol{R}_4\,{}^4\boldsymbol{p}_5 = \boldsymbol{h}_2^T\left({}^1\boldsymbol{p}_2 + {}^2\boldsymbol{p}_3\right) \tag{75}$$

Further, we rephrase (59) as (76). Left-multiplication by $\boldsymbol{h}_2^T$ and right-multiply by $\boldsymbol{h}_5$ simplifies (76) to (77). The two equations (75) and (77) now resemble a system of equations that we can solve for θ_1, θ_4 through SP6.

$${}^1\boldsymbol{R}_2^T\,{}^0\boldsymbol{R}_1^T\,{}^0\boldsymbol{R}_5^*\,{}^4\boldsymbol{R}_5^T - {}^2\boldsymbol{R}_3\,{}^3\boldsymbol{R}_4 = 0 \tag{76}$$

$$\boldsymbol{h}_2^T\,{}^0\boldsymbol{R}_1^T\,{}^0\boldsymbol{R}_5^*\boldsymbol{h}_5 - \boldsymbol{h}_2^T\,{}^3\boldsymbol{R}_4\boldsymbol{h}_5 = 0 \tag{77}$$

We rephrase (58) as (78) and insert θ_1, θ_4. Leveraging norm-preservation (78) simplifies to (79). Applying SP3 to (79) yields θ_3.

$${}^1\boldsymbol{R}_2^T\left({}^0\boldsymbol{R}_1^{*T}\,{}^1\boldsymbol{p}_5 - {}^1\boldsymbol{p}_2\right) = {}^2\boldsymbol{R}_3\,{}^3\boldsymbol{R}_4^*\,{}^4\boldsymbol{p}_5 + {}^2\boldsymbol{p}_3 \tag{78}$$

$$\left\|{}^0\boldsymbol{R}_1^{*T}\,{}^1\boldsymbol{p}_5 - {}^1\boldsymbol{p}_2\right\| = \left\|{}^2\boldsymbol{R}_3\,{}^3\boldsymbol{R}_4^*\,{}^4\boldsymbol{p}_5 + {}^2\boldsymbol{p}_3\right\| \tag{79}$$

Inserting $\theta_1, \theta_3, \theta_4$ back into (74) results in (80), from which we obtain θ_2 via SP1.

$${}^0\boldsymbol{R}_1^{*T}\,{}^1\boldsymbol{p}_5 - {}^1\boldsymbol{p}_2 = {}^1\boldsymbol{R}_2\left({}^2\boldsymbol{p}_3 + {}^2\boldsymbol{R}_3^*\,{}^3\boldsymbol{R}_4^*\,{}^4\boldsymbol{p}_5\right) \tag{80}$$

Finally, we rephrase (59) and insert $\theta_1, \theta_2, \theta_3, \theta_4$ to obtain (81), where $\boldsymbol{h}_n$ represents a normal vector to $\boldsymbol{h}_5$. Applying SP1 to (81) then yields θ_5.

$$ {}^4\boldsymbol{R}_5 \boldsymbol{h}_n = {}^3\boldsymbol{R}_4^{*T}\, {}^2\boldsymbol{R}_3^{*T}\, {}^1\boldsymbol{R}_2^{*T}\, {}^0\boldsymbol{R}_1^{*T}\, {}^0\boldsymbol{R}_5^{*} \boldsymbol{h}_n \tag{81} $$

If the second and third axis are parallel while the third and fourth axis intersect, and the fourth and fifth axis are parallel, we can choose ${}^3\boldsymbol{p}_4 = 0$, $\boldsymbol{h}_2 = \boldsymbol{h}_3$, and $\boldsymbol{h}_4 = \boldsymbol{h}_5$. This simplifies (58) to (74). We rephrase (59) as (82). Left- and right-multiplication of (82) with $\boldsymbol{h}_2^T$ and $\boldsymbol{h}_5$ simplifies the equation to (83). Applying SP4 to (83) yields θ_1.

$$ {}^0\boldsymbol{R}_1^T\, {}^0\boldsymbol{R}_5^{*} = {}^1\boldsymbol{R}_2\, {}^2\boldsymbol{R}_3\, {}^3\boldsymbol{R}_4\, {}^4\boldsymbol{R}_5 \tag{82} $$

$$ \boldsymbol{h}_1^T\, {}^0\boldsymbol{R}_1^T\, {}^0\boldsymbol{R}_5^{*} \boldsymbol{h}_5 = \boldsymbol{h}_1^T \boldsymbol{h}_5 \tag{83} $$

We insert θ_1 into (74), and, after left-multiplication with $\boldsymbol{h}_2^T$, obtain (84). Applying SP4 to (84) yields $\theta 4$.

$$ \boldsymbol{h}_2^T\, {}^3\boldsymbol{R}_4\, {}^4\boldsymbol{p}_5 = \boldsymbol{h}_2^T \left({}^0\boldsymbol{R}_1^{*T}\, {}^1\boldsymbol{p}_5 - {}^1\boldsymbol{p}_2 - {}^2\boldsymbol{p}_3 \right) \tag{84} $$

We obtain $\theta_3, \theta_4, \theta_5$ by following the same procedure as in the previous case where the fourth and fifth axis were not parallel.

If the first three axes are parallel, we can choose $\boldsymbol{h}_1 = \boldsymbol{h}_2 = \boldsymbol{h}_3$. We rephrase (59) according to (85). Right-multiplication of (85) by $\boldsymbol{h}_1$ results in (86), to which we apply SP2 to obtain θ_4, θ_5.

$$ {}^4\boldsymbol{R}_5\, {}^0\boldsymbol{R}_5^{*T} = {}^3\boldsymbol{R}_4^T\, {}^2\boldsymbol{R}_3^T\, {}^1\boldsymbol{R}_2^T\, {}^0\boldsymbol{R}_1^T \tag{85} $$

$$ {}^4\boldsymbol{R}_5\, {}^0\boldsymbol{R}_5^{*T} \boldsymbol{h}_1 = {}^3\boldsymbol{R}_4^T \boldsymbol{h}_1 \tag{86} $$

As the first three axes are parallel, such that $\boldsymbol{h}_1 = \boldsymbol{h}_2 = \boldsymbol{h}_3$, their induced rotation can be described by a single rotation ${}^0\boldsymbol{R}_3$ about one of the equivalent axes. The rotation angle hereby consists of the sum of all three rotations, i.e., $\theta_{1,2,3} = \theta_1 + \theta_2 + \theta_3$. Using this definition, inserting θ_4, θ_5 into (59), and rephrasing it yields (87). We apply SP1 to (87) to obtain $\theta_{1,2,3}$.

$$ {}^0\boldsymbol{R}_3 \boldsymbol{h}_n = {}^0\boldsymbol{R}_5^{*}\, {}^4\boldsymbol{R}_5^{*T}\, {}^3\boldsymbol{R}_4^{*T} \boldsymbol{h}_n \tag{87} $$

Inserting the obtained values for θ_4, θ_5 and $\theta_{1,2,3}$ into (58) yields, after rephrasing, (88). Leveraging norm-preservation (88) simplifies to (89). We apply SP1 to (89) to obtain θ_2. After inserting θ_2 back into (88), we can directly appls SP1 to obtain θ_1.

$$ {}^1\boldsymbol{p}_5 - {}^0\boldsymbol{R}_3^{*} \left({}^3\boldsymbol{p}_4 + {}^3\boldsymbol{R}_4^{*}\, {}^4\boldsymbol{p}_5 \right) = {}^0\boldsymbol{R}_1 \left({}^1\boldsymbol{p}_2 + {}^1\boldsymbol{R}_2\, {}^2\boldsymbol{p}_3 \right) \tag{88} $$

$$ \left\| {}^1\boldsymbol{p}_5 - {}^0\boldsymbol{R}_3^{*} \left({}^3\boldsymbol{p}_4 + {}^3\boldsymbol{R}_4^{*}\, {}^4\boldsymbol{p}_5 \right) \right\| = \left\| {}^1\boldsymbol{p}_2 + {}^1\boldsymbol{R}_2\, {}^2\boldsymbol{p}_3 \right\| \tag{89} $$

Finally, we use the definition of ${}^0\boldsymbol{R}_3^{*} = {}^0\boldsymbol{R}_1\, {}^1\boldsymbol{R}_2\, {}^2\boldsymbol{R}_3$, insert θ_1, θ_2, and right-multiply by $\boldsymbol{h}_n$ – a normal vector to $\boldsymbol{h}_3$ – to obtain (90). We obtain θ_3 by applying SP1 to (90).

$$ {}^2\boldsymbol{R}_3 \boldsymbol{h}_n = {}^1\boldsymbol{R}_2^{*T}\, {}^0\boldsymbol{R}_1^{*T}\, {}^0\boldsymbol{R}_3^{*} \boldsymbol{h}_n \tag{90} $$

If the first three axes are parallel to each other, while the last two axes are also parallel, we can choose $\boldsymbol{h}_1 = \boldsymbol{h}_2 = \boldsymbol{h}_3$ and $\boldsymbol{h}_4 = \boldsymbol{h}_5$. First, we left-multiply (58) by $\boldsymbol{h}_1^T$ to obtain (91). We then apply SP4 to (91) to obtain θ_4.

$$ \boldsymbol{h}_1^T\, {}^3\boldsymbol{R}_4\, {}^4\boldsymbol{p}_5 = \boldsymbol{h}_1^T \left({}^1\boldsymbol{p}_5 - {}^1\boldsymbol{p}_2 - {}^2\boldsymbol{p}_3 - {}^3\boldsymbol{p}_4 \right) \tag{91} $$

We insert θ_4 into (59), rephrase according to (92), and right-multiply with $\boldsymbol{h}_1$ to obtain (93). Applying SP1 to (93) then yields θ_5.

$$ {}^4\boldsymbol{R}_5\, {}^0\boldsymbol{R}_5^{*T} = {}^3\boldsymbol{R}_4^{*T}\, {}^2\boldsymbol{R}_3^T\, {}^1\boldsymbol{R}_2^T\, {}^0\boldsymbol{R}_1^T \tag{92} $$

$$ {}^4\boldsymbol{R}_5\, {}^0\boldsymbol{R}_5^{*T} \boldsymbol{h}_1 = {}^3\boldsymbol{R}_4^{*T} \boldsymbol{h}_1 \tag{93} $$

Starting from (87), we then follow the same procedure as proposed in the last case – where $\boldsymbol{h}_4 \neq \boldsymbol{h}_5$ – to obtain $\theta_1, \theta_2, \theta_3$.

If the second, third, and fourth axis are parallel, we can choose $\boldsymbol{h}_2 = \boldsymbol{h}_3 = \boldsymbol{h}_4$. We rephrase (58) as done in (89) and left-multiply with $\boldsymbol{h}_2^T$ to obtain (94). Applying SP4 to (94) yields θ_1.

$$ \boldsymbol{h}_2^T\, {}^0\boldsymbol{R}_1^T\, {}^1\boldsymbol{p}_5 = \boldsymbol{h}_2^T \left({}^1\boldsymbol{p}_2 + {}^2\boldsymbol{p}_3 + {}^3\boldsymbol{p}_4 + {}^4\boldsymbol{p}_5 \right) \tag{94} $$

Rephrasing (59) and inserting θ_1 yields (95), which, after right-multiplication with $\boldsymbol{h}_2$ simplifies to (96). We apply SP1 to (96) to obtain θ_5.

$$ {}^4\boldsymbol{R}_5\, {}^0\boldsymbol{R}_5^{*T}\, {}^0\boldsymbol{R}_1^{*} = {}^3\boldsymbol{R}_4^T\, {}^2\boldsymbol{R}_3^T\, {}^1\boldsymbol{R}_2^T \tag{95} $$

$$ {}^4\boldsymbol{R}_5\, {}^0\boldsymbol{R}_5^{*T}\, {}^0\boldsymbol{R}_1^{*} \boldsymbol{h}_2 = \boldsymbol{h}_2 \tag{96} $$

Just like in (87), we can represent ${}^1\boldsymbol{R}_4 = {}^1\boldsymbol{R}_2\, {}^2\boldsymbol{R}_3\, {}^3\boldsymbol{R}_4$ via a single rotation about one of the three parallel axes (e.g., $\boldsymbol{h}_2$) by a single angle $\theta_{2,3,4} = \theta_2 + \theta_3 + \theta_4$. To obtain $\theta_{2,3,4}$, we insert θ_1, θ_5 into (59), which yields (97), where $\boldsymbol{h}_n$ is a vector normal to the parallel axes. Applying SP1 to (97) yields $\theta_{2,3,4}$.

$$ {}^1\boldsymbol{R}_4 \boldsymbol{h}_n = {}^0\boldsymbol{R}_1^{*T}\, {}^0\boldsymbol{R}_5^{*}\, {}^4\boldsymbol{R}_5^{*T} \boldsymbol{h}_n \tag{97} $$

We left-multiply (58) by ${}^0\boldsymbol{R}_4^T = {}^1\boldsymbol{R}_4^{*T}\, {}^1\boldsymbol{R}_0^{*T}$ and rephrase it such that we obtain (98). Using norm-preservation on (98) yields (99). Applying SP3 to (99) yields θ_3. We insert θ_3 back into (98) and then directly apply SP1 to obtain θ_4.

$$ {}^1\boldsymbol{R}_4^{*T} \left({}^0\boldsymbol{R}_1^{*T}\, {}^1\boldsymbol{p}_5 - {}^1\boldsymbol{p}_2 \right) - {}^4\boldsymbol{p}_5 = {}^3\boldsymbol{R}_4^T \left({}^2\boldsymbol{R}_3^T\, {}^2\boldsymbol{p}_3 + {}^3\boldsymbol{p}_4 \right) \tag{98} $$

$$ \left\| {}^1\boldsymbol{R}_4^{*T} \left({}^0\boldsymbol{R}_1^{*T}\, {}^1\boldsymbol{p}_5 - {}^1\boldsymbol{p}_2 \right) - {}^4\boldsymbol{p}_5 \right\| = \left\| {}^2\boldsymbol{R}_3^T\, {}^2\boldsymbol{p}_3 + {}^3\boldsymbol{p}_4 \right\| \tag{99} $$

Inserting θ_3, θ_4 into the definition of ${}^1\boldsymbol{R}_4$ yields (100), where $\boldsymbol{h}_n$ represent a normal vector to the three parallel axes. Applying SP1 to (100) finally yields θ_2.

$$ {}^1\boldsymbol{R}_2 \boldsymbol{h}_n = {}^1\boldsymbol{R}_4^{*}\, {}^3\boldsymbol{R}_4^{*T}\, {}^2\boldsymbol{R}_3^{*T} \boldsymbol{h}_n \tag{100} $$

If the last three axes intersect in a common point (i.e., form a spherical wrist), we can choose ${}^3\boldsymbol{p}_4 = {}^4\boldsymbol{p}_5 = 0$, which simplifies (58) to (101). Leveraging norm-preservation, we further obtain (102), to which we apply

SP3 to retrieve θ_1. We then insert θ_1 back into (101) and subsequently use SP1 to obtain θ_2.

$$
{}^0\boldsymbol{R}_1^T\,{}^1\boldsymbol{p}_5 - {}^1\boldsymbol{p}_2 = {}^1\boldsymbol{R}_2\,{}^2\boldsymbol{p}_3 \tag{101}
$$

$$
\left\|{}^0\boldsymbol{R}_1^T\,{}^1\boldsymbol{p}_5 - {}^1\boldsymbol{p}_2\right\| = \left\|{}^1\boldsymbol{R}_2\,{}^2\boldsymbol{p}_3\right\| \tag{102}
$$

We rephrase (59) as (103). Right-multiplication with $\boldsymbol{h}_5$ simplifies (103) to (104), to which we apply SP2 to obtain θ_3, θ_4

$$
{}^2\boldsymbol{R}_3^T\,{}^1\boldsymbol{R}_2^T\,{}^0\boldsymbol{R}_1^T\,{}^0\boldsymbol{R}_5^* = {}^3\boldsymbol{R}_4\,{}^4\boldsymbol{R}_5 \tag{103}
$$

$$
{}^2\boldsymbol{R}_3^T\,{}^1\boldsymbol{R}_2^T\,{}^0\boldsymbol{R}_1^T\,{}^0\boldsymbol{R}_5^*\boldsymbol{h}_5 = {}^3\boldsymbol{R}_4\boldsymbol{h}_5 \tag{104}
$$

We obtain θ_5 by following the same steps as for (81), where we insert $\theta_1, \theta_2, \theta_3, \theta_4$ into (59), right multiply with a normal vector to $\boldsymbol{h}_5$, and apply SP1.

If the last three axes intersect to form a spherical wrist, while the first two axes intersect in a point separate to that of the wrist, we can choose ${}^1\boldsymbol{p}_2 = 0$ in addition to ${}^3\boldsymbol{p}_4 = {}^4\boldsymbol{p}_5 = 0$, which simplifies (101) to (105). We apply SP2 to (105) to obtain θ_1, θ_2.

$$
{}^0\boldsymbol{R}_1^T\,{}^1\boldsymbol{p}_5 = {}^1\boldsymbol{R}_2\,{}^2\boldsymbol{p}_3 \tag{105}
$$

For $\theta_3, \theta_4, \theta_5$ we follow the same procedure as in the previous case where the first two axes are non-intersecting.

In any case, we use kinematic inversion and check if the inverted kinematic chain matches a case that is more specialized (i.e., represents a case that employs *simpler* subproblems) than that of the non-inverted one. We hereby first check for three parallel axes, then for two intersecting axes at either end of the manipulator and then all the others. E.g., if the first three axes of a manipulator are parallel and the last two axes intersect, we assign the manipulator to the kinematic family of three initial parallel axes. If, even after kinematic inversion, the manipulator does not match any of the proposed cases, it is to our current knowledge not solvable via our method.

REFERENCES

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