Green's Theorem in Reverse

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Green's theorem is often used to convert a line integral problem into an equivalent double integral problem. Green's theorem relates a line integral around a simple closed curve C to a double integral over the planar region R bounded by curve C. In some cases though, the double integral is a difficult problem, where an equivalent line integral may prove simple. We will go through the steps to utilize *Green's theorem in reverse* in this short document.

1 Vector Field Parametrization

We first define a vector field $\vec{F}$ which returns a force acting upon any given Cartesian coordinate (x, y). $\vec{F}$ returns the force vector with component magnitudes relative to the basis vectors $\hat{\imath}$ and $\hat{\jmath}$. We will parameterize the actions on each basis vector as separate functions M and N, given the x and y coordinates.

$$\vec{F}(x,y) = M(x,y)\hat{\imath} + N(x,y)\hat{\jmath} \tag{1}$$

$M(x, y)$ represents the influence in the $\hat{\imath}$ direction, while $N(x, y)$ represents the influence in the $\hat{\jmath}$ direction.

2 Green's Theorem

Green's theorem is defined as such.

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \tag{2}$$

The left integral can be interpreted as the work or "effort" done *by* the vector field $\vec{F}$, onto any position (x, y), while taking a infinitesimally small step in direction $d\vec{r}$, along the path C. The *dot product* tracks both the force strength, and how *aligned* the direction $d\vec{r}$ is with the flow of the vector field $\vec{F}$ at position (x, y). This dot product is expanded to the sum of vector components, as shown in the middle integral. The right side reframes the line integral form as the sum of *circulation densities* or the *k-th component of curl* for region R. Curl won't be the guiding topic here, but it's defined as the cross product or *outer product* of a vector fields gradient ∇ in basis $\hat{\imath}, \hat{\jmath}, \hat{k}$.

$$\begin{aligned} \textbf{curl}\,\vec{F} &= \nabla \times \vec{F} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\hat{\imath} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k} \\ (\textbf{curl}\,\vec{F}) \cdot \hat{k} &= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \end{aligned} \tag{3}$$

The definitions of functions M and N may omit their full signatures for clarity when denoted.

3 An Example

3.1 Double Integral Method

Let's execute on an integral that's easy to evaluate for both methods, starting with the traditional double integral method. You may soon notice that this particular example is *much easier* to evaluate through double integration, but *we have different tools for different reasons after all.*

$$\int_1^2 \int_1^2 x^2 + y^2 \, dy \, dx \tag{4}$$

Since the rules of *Fubini's theorem* apply here, we may solve by *iterated integration* like so.

$$
\begin{aligned}
I &= \int_1^2 \left[\int_1^2 x^2 + y^2 \, dy\right] dx \\
&= \int_1^2 \left[\int_1^2 {\color{blue}y^0} x^2 + y^2 \, dy\right] dx && \text{Introduce hidden constant} \\
&= \int_1^2 \left[x^2 \int_1^2 y^0 \, dy + \int_1^2 y^2 \, dy\right] dx && \text{Separate terms, constant factor} \\
&= \int_1^2 \left[x^2 \left[\frac{y^1}{1}\right]_1^2 + \left[\frac{y^3}{3}\right]_1^2\right] dx && \text{Power rule for integration} \\
&= \int_1^2 \left[x^2 \left[\frac{2}{1} - \frac{1}{1}\right] + \left[\frac{2^3}{3} - \frac{1^3}{3}\right]\right] dx && \text{F.T.C (Part 2)} \\
&= \int_1^2 \left[x^2 + \frac{7}{3}\right] dx && \text{Evaluate} \\
&= \int_1^2 \left[x^2 + \frac{7}{3}{\color{blue}x^0}\right] dx && \text{Introduce hidden constant} \\
&= \int_1^2 x^2 \, dx + \frac{7}{3}\int_1^2 \, dx && \text{Separate terms, constant factor} \\
&= \left[\frac{x^3}{3}\right]_1^2 + \frac{7}{3}\left[\frac{x}{1}\right]_1^2 && \text{Power rule for integration} \\
&= \left[\frac{2^3}{3} - \frac{1^3}{3}\right] + \frac{7}{3}[2-1] = \frac{7}{3} + \frac{7}{3} = \frac{14}{3} + C && \text{Evaluate}
\end{aligned}
\tag{5}
$$

$$
\int_1^2 \int_1^2 \left[x^2 + y^2\right] \, dy \, dx = \frac{14}{3} + C \tag{6}
$$

3.2 Line Integral Method

Recall the of the circulation density integral from Green's theorem (2)(3). We can restructure this integral to match our original double integral (4). First, let's expand the notation to realize the double integrals are the same but abbreviated.

$$
\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \to \int_a^b \int_c^d \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy \, dx \quad R \in [a,b] \times [c,d] \tag{7}
$$

To use the relationships proposed in Green's theorem, we must convert our vector field function $\vec{F}$ to an equivalent curl formula (3).

$$
(\textbf{curl}\, \vec{F}) \cdot \hat{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = x^2 + y^2
$$

$$
\begin{aligned}
-\frac{\partial M}{\partial y} &= y^2 \\
-\int \frac{\partial M}{\partial y} \, dy &= \int y^2 \, dy \\
-M &= \frac{y^3}{3} \\
M(x,y) &= -\frac{y^3}{3}
\end{aligned}
\qquad\qquad
\begin{aligned}
\frac{\partial N}{\partial x} &= x^2 \\
\int \frac{\partial N}{\partial x} \, dx &= \int x^2 \, dx \\
N(x,y) &= \frac{x^3}{3}
\end{aligned}
$$

We can verify these function choices will work by deriving to undo the work we did. I'd like to note the fact that there is more than one solution for N and M, I've just chosen to cancel the partial derivatives through integration.

$$
M(x,y) = -x^2 y \quad N(x,y) = y^2 x
$$

These choices are equally valid, but we will continue nonetheless. We can verify the the new form is correct by computing the derivatives as laid out in their definition.

$$
\begin{aligned}
\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} &= \frac{\partial}{\partial x}\left[\frac{x^3}{3}\right] - \frac{\partial}{\partial y}\left[-\frac{y^3}{3}\right] \\
&= \frac{1}{\cancel{3}}\left(\cancel{3}x^2\right) - \frac{1}{\cancel{3}}\left(-\cancel{3}y^2\right) \\
&= x^2 - \left(-y^2\right) \\
&= x^2 + y^2 \\
\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} &= \vec{F}(x,y)
\end{aligned}
$$

Now we can comfortably define a line integral with these terms.

$$
\oint_C \vec{F} \cdot d\vec{r} = \oint_C (M \, dx + N \, dy) = \oint_C \left(\left[-\frac{y^3}{3}\right] dx + \left[\frac{x^3}{3}\right] dy\right) \tag{8}
$$

3.2.1 Piecewise Linear Sum

The diagram in Figure 1 visualizes the bounding segments around region R. Technically, the region is defined entirely by curve C, but because this region is simple and closed, we can represent the entire curve as a sum of segments. It is *imperative* to evaluate the curve *counter-clockwise*, since the curl vector will face backwards if the direction is reversed.

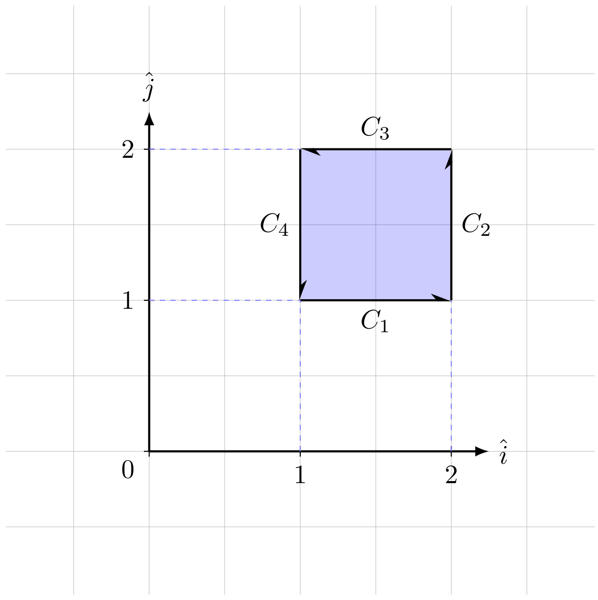

Figure 1

Because our region is square in shape, we will have four segments in total. As mentioned beforehand, the sum of those segments will represent the entire curve *en masse*.

$$\oint_C \vec{F} \cdot d\vec{r} = \sum_{i=1}^{4} \oint_{C_i} \vec{F} \cdot d\vec{r} \tag{9}$$

To convert our segments to integrable lines, we need to parameterize the domain of each segment C_n as a parametric vector $\vec{r}_n$, dependent on t. In essence, we are doing a *change of variables* to make components x and y linearly dependent on t. We can also bring the derivatives dx and dy to the dt world.

We define these vectors based on the endpoints we want for given values t_0 and t_1 through *linear interpolation*. For example, C_1 is a vector constrained by the following.

$$\begin{aligned}
\vec{r}_1(t_0) &= (1, 1) & \vec{r}_1(t_1) &= (2, 1) \\
\vec{r}_1(t = 1) &= 1\,\hat{\imath} + 1\,\hat{\jmath} & \vec{r}_1(t = 2) &= 2\,\hat{\imath} + 1\,\hat{\jmath} \\
f_x(t) &= \begin{cases} 1 & t = 1 \\ 2 & t = 2 \end{cases} & f_y(t) &= \begin{cases} 1 & t = 1 \\ 1 & t = 2 \end{cases} \\
f_x(t) &= t & f_y(t) &= 1 \\
\vec{r}_1(t) &= t\,\hat{\imath} + 1\,\hat{\jmath}
\end{aligned}$$

The left section of the following holds each segment C_n as the parameterized vector $\vec{r}_n$, used for the corresponding line integrals on the right.

$$\begin{aligned}
\vec{r}_1(t) &= t\,\hat{\imath} + 1\,\hat{\jmath} \quad & t &\in [1, 2] \\
x &= t & y &= 1 \\
dx &= 1\,dt & dy &= 0\,dt \\
M(x, y) &= \frac{-y^3}{3} \to \frac{-1^3}{3} \\
N(x, y) &= \frac{x^3}{3} \to \frac{t^3}{3}
\end{aligned}$$

$$\begin{aligned}
C_1 &= \int_1^2 (M\,dx + N\,dy) \\
&= \int_1^2 \left(\left[\frac{-1^3}{3}\right] (1\,dt) + \cancel{\left[\frac{t^3}{3}\right] (0\,dt)} \right) \\
&= \int_1^2 \left(\frac{-1}{3}\right) dt \\
&= \frac{-1}{3} \int_1^2 t^0\,dt = \frac{-1}{3} \left[\frac{t}{1}\right]_1^2 = \frac{-1}{3} [2 - 1] \\
&= \frac{-1}{3} + C
\end{aligned}$$

$$
\begin{aligned}
\vec{r}_2(t) &= 2\,\hat{i} + t\,\hat{j} \qquad t \in [1, 2] \\
x &= 2 \qquad y = t \\
dx &= 0\,dt \qquad dy = 1\,dt \\
M(x, y) &= \frac{-y^3}{3} \to \frac{-t^3}{3} \\
N(x, y) &= \frac{x^3}{3} \to \frac{2^3}{3}
\end{aligned}
$$

$$
\begin{aligned}
C_2 &= \int_1^2 (M\,dx + N\,dy) \\
&= \int_1^2 \left(\cancel{\left[\frac{-t^3}{3}\right] (0\,dt)} + \left[\frac{2^3}{3}\right] (1\,dt) \right) \\
&= \int_1^2 \left(\frac{2^3}{3} \right) dt \\
&= \frac{8}{3} \int_1^2 t^0\,dt = \frac{8}{3} \left[\frac{t}{1}\right]_1^2 \\
&= \frac{8}{3} [2 - 1] = \frac{8}{3} + C
\end{aligned}
$$

$$
\begin{aligned}
\vec{r}_3(t) &= -t\,\hat{i} + 2\,\hat{j} \qquad t \in [1, 2] \\
x &= -t \qquad y = 2 \\
dx &= -1\,dt \qquad dy = 0\,dt \\
M(x, y) &= \frac{-y^3}{3} \to \frac{-2^3}{3} \\
N(x, y) &= \frac{x^3}{3} \to \frac{-t^3}{3}
\end{aligned}
$$

$$
\begin{aligned}
C_3 &= \int_1^2 (M\,dx + N\,dy) \\
&= \int_1^2 \left(\left[\frac{-2^3}{3}\right] (-1\,dt) + \cancel{\left[\frac{-t^3}{3}\right] (0\,dt)} \right) \\
&= \frac{8}{3} \int_1^2 t^0\,dt = \frac{8}{3} \left[\frac{t}{1}\right]_1^2 = \frac{8}{3} [2 - 1] \\
&= \frac{8}{3} + C
\end{aligned}
$$

$$
\begin{aligned}
\vec{r}_4(t) &= 1\,\hat{i} - t\,\hat{j} \qquad t \in [1, 2] \\
x &= 1 \qquad y = -t \\
dx &= 0\,dt \qquad dy = -1\,dt \\
M(x, y) &= \frac{-y^3}{3} \to \frac{t^3}{3} \\
N(x, y) &= \frac{x^3}{3} \to \frac{1^3}{3}
\end{aligned}
$$

$$
\begin{aligned}
C_4 &= \int_1^2 (M\,dx + N\,dy) \\
&= \int_1^2 \left(\cancel{\left[\frac{t^3}{3}\right] (0\,dt)} + \left[\frac{1^3}{3}\right] (-1\,dt) \right) \\
&= \int_1^2 \left(\frac{-1^3}{3} \right) dt \\
&= \frac{-1}{3} \int_1^2 t^0\,dt = \frac{-1}{3} \left[\frac{t}{1}\right]_1^2 = \frac{-1}{3} [2 - 1] \\
&= \frac{-1}{3} + C
\end{aligned}
$$

Finally, the sum of all integrated segments work out to be equivalent to the double integral result (6).

$$
\begin{aligned}
\sum_{i=1}^{4} \oint_{C_i} \vec{F} \cdot d\vec{r} &= C_1 + C_2 + C_3 + C_4 \\
&= \frac{-1}{3} + \frac{8}{3} + \frac{8}{3} + \frac{-1}{3} \\
&= \frac{14}{3} + C
\end{aligned} \tag{10}
$$