4.22 $S(w) = \int_{-\infty}^{+\infty} R(\tau) e^{-jw\tau} d\tau$

$$= \frac{a^2}{2}\int_{-\infty}^{+\infty} \cos w_0\tau \, e^{-jw\tau} d\tau + b^2 \int_{-\infty}^{+\infty} e^{-a|\tau|} e^{-jw\tau} d\tau$$

$$= \frac{a^2}{4}\int_{-\infty}^{+\infty} (e^{jw_0\tau} + e^{-jw_0\tau}) e^{-jw\tau} d\tau + 2b^2 \int_{0}^{+\infty} \cos w\tau \, e^{-a|\tau|} d\tau$$

$$\cancel{\frac{a^2}{4}\int_{-\infty}^{+\infty} 1 \, d\tau + \frac{a^2}{4}\int_{-\infty}^{+\infty} e^{-j2w\tau} d\tau + \cdots}$$

$$= \frac{\pi a^2}{2}\left(\delta(w-w_0) + \delta(w+w_0)\right) + \frac{2ab^2}{a^2+w^2}$$

4.23. $EX = 0$, $Var X = R_X(0) = A$ $\quad \therefore EX^2 = EY = A$

$$\therefore R_Y(\tau) = E(Y(t)-A)(Y(t+\tau)-A) = EY(t)Y(t+\tau) - A^2$$

$$= EX^2(t)X^2(t+\tau) - A^2$$

$$= EX^2(t)\,EX^2(t+\tau) + E(X(t)X(t+\tau))E(X(t)X(t+\tau)) + E(X(t)X(t+\tau))\,E(X(t)X(t+\tau)) - A^2$$

$$= 2R_X^2(\tau)$$

$$\therefore R_Y(\tau) = 2A^2 e^{-2a|\tau|}\cos^2\beta\tau = A^2 e^{-2a|\tau|}(1+\cos 2\beta\tau)$$

$$S_Y(w) = \int_{-\infty}^{+\infty} A^2 e^{-2a|\tau|}(1+\cos 2\beta\tau) e^{-jw\tau} d\tau$$

$$= A^2\int_{-\infty}^{+\infty} e^{-2a|\tau|} e^{-jw\tau} d\tau + \frac{A^2}{2}\int_{-\infty}^{+\infty} (e^{j2\beta\tau} + e^{-j2\beta\tau}) e^{-2a|\tau|} e^{-jw\tau} d\tau$$

$$= \frac{4aA^2}{4a^2+w^2} + \frac{2aA^2}{4a^2+(w-2\beta)^2} + \frac{2aA^2}{4a^2+(w+2\beta)^2}$$

$\int f \cos 2\beta\tau = 2\sin^2\beta\tau - 1$ $\quad$ 的 $\frac{4a}{4a^2+w^2}$

4.24 解: $R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{1+w^2} e^{jw\tau}\, d\tau = \frac{e^{-|\tau|}}{2}$

$\therefore \operatorname{Var} X(t) = R_X(0) = \frac{1}{2}$

$\therefore X(t) \sim N(0, \frac{1}{2})$

$\therefore P(0.5 \le X(t) \le 1) = P\left(\frac{0.5-0}{\sqrt{\frac{1}{2}}} \le \frac{X(t)-0}{\sqrt{\frac{1}{2}}} \le \frac{1-0}{\sqrt{\frac{1}{2}}}\right)$

$= \Phi(\sqrt{2}) - \Phi(\frac{\sqrt{2}}{2}) = 0.1613$

4.28 解:

(1) $S(w) = \frac{w^2+64}{w^4+29w^2+100} = \frac{w^2+64}{(w^2+25)(w^2+4)}$

$= \frac{-13/7}{w^2+25} + \frac{20/7}{w^2+4}$

$\therefore R_X(\tau) = -\frac{13}{70} e^{-5|\tau|} + \frac{5}{7} e^{-2|\tau|}$

(2) $S(w) = \frac{1}{(1+w^2)^2} = \frac{1}{w^2+1} \cdot \frac{1}{w^2+1}$

$\therefore R_X(\tau) = \frac{e^{-|\tau|}}{2} * \frac{e^{-|\tau|}}{2}$

$= \frac{1}{4}\int_{-\infty}^{+\infty} e^{-|t|} e^{-|\tau-t|}\, dt$

$= \begin{cases} \frac{1}{4}(1-\tau)e^{\tau}, & \tau \le 0 \\ \frac{1}{4}(1+\tau)e^{-\tau}, & \tau > 0 \end{cases}$

$= \frac{1}{4}(1+|\tau|)e^{-|\tau|}$

4.31 解:

(1) 预报误差为 $E_r(a) = E\left(X_{n+1} - \hat{X}_{n+1}^{(1)}\right)^2 = E\left(X_{n+1} - aX_n\right)^2$

$\therefore E_r'(a) = -2E\left(X_{n+1} - aX_n\right)X_n$

令 $E_r'(a) = 0 \quad \therefore E\left(X_{n+1} - aX_n\right)X_n = 0$

$\therefore EX_{n+1}X_n - aEX_n^2 = 0$

$\therefore a = \dfrac{R(1)}{R(0)}$

(2) 预报误差为 $E_r(a,b) = E\left(X_{n+1} - \hat{X}_{n+1}^{(2)}\right)^2 = E\left(X_{n+1} - aX_n - bX_{n-1}\right)^2$

$$\therefore \begin{cases} \dfrac{\partial E_r(a,b)}{\partial a} = -2E\left(X_{n+1} - aX_n - bX_{n-1}\right)X_n = 0 \\ \dfrac{\partial E_r(a,b)}{\partial b} = -2E\left(X_{n+1} - aX_n - bX_{n-1}\right)X_{n-1} = 0 \end{cases}$$

$$\therefore \begin{cases} R(1) - aR(0) - bR(1) = 0 \\ R(2) - aR(1) - bR(0) = 0 \end{cases}$$

$$\therefore \begin{cases} a = \dfrac{R(1)\left(R(0) - R(2)\right)}{R^2(0) - R^2(1)} \\ b = \dfrac{R(0)R(2) - R^2(1)}{R^2(0) - R^2(1)} \end{cases}$$

4.34. 证明 $R_X(s) = \begin{cases} \sigma^2 2 & s = 0 \\ 0 & s \geq 1 \end{cases}$

$R_Y(s) = \begin{cases} \dfrac{\sigma^2}{2} & s = 0 \\ 0 & s \geq 1 \end{cases}$

$\therefore \rho_x(v) = \dfrac{R_x(s)}{R_x(0)} = \begin{cases} 1 & s=0 \\ 0 & s\geq 1 \end{cases}$

$\rho_Y(v) = \dfrac{R_Y(s)}{R_Y(0)} = \begin{cases} 1 & s=0 \\ 0 & s>0 \end{cases}$

4.35.

$E(X_n X_{n-h}) = \alpha_1 E(X_{n-1}X_{n-h}) + \dots + \alpha_p E(X_{n-p}X_{n-h}) + E(\varepsilon_n X_{n-h})$

$\therefore R(h) = \alpha_1 R(h-1) + \dots + \alpha_p R(h-p) + 0$

$= \alpha_1 R(h-1) + \dots + \alpha_p R(h-p),\ h>0$

4.37.(1) $E(X_n X_{n-1}) = 0.5 E X_{n-1}^2 + 0.3 E X_{n-2}X_{n-1} + E(\varepsilon_n X_{n-1})$

$\therefore R(1) = 0.5R(0) + 0.3R(1)$ $\quad \therefore R(1) = \frac{5}{7}R(0)$

同理 $R(2) = 0.5R(1) + 0.3R(0) = \frac{23}{35}R(0)$

$R(3) = 0.5R(2) + 0.3R(1) = \frac{19}{35}R(0)$

$\therefore E X_n X_{n-\tau} = 0.5 E X_{n-1}X_{n-\tau} + 0.3 E X_{n-2}X_{n-\tau} + E\varepsilon_n X_{n-\tau}$

$\therefore R(\tau) = 0.5R(\tau-1) + 0.3R(\tau-2)$

4.41. $\hat{X}_{n+1|n} = E(X_{n+1} | X_n, X_{n-1}, \dots, X_1) = 1.8X_n + 0.8X_{n-1}$

$\hat{X}_{n+2|n} = E(X_{n+2} | X_n, X_{n-1}, \dots, X_1) = 1.8\hat{X}_{n+1|n} + 0.8X_n$

$= 4.04X_n + 1.44X_{n-1}$

$$\hat{X}_{n+l|n} = 1.8\hat{X}_{n+l-1|n} + 0.8\hat{X}_{n+l-2|n} \quad . \quad l \geq 3$$

10.

5.2 $W(0)=0$ $EW(t)=0$

(1) $ET(t) = E\,tW(1/t) = t\,EW(1/t) = 0$

$ET(t)T(s) = st\,E\,W(1/t)W(1/s) = st \cdot \frac{1}{t} = s$. $t \geq s \geq 0$

$\therefore T(t) = tW(1/t)$ 是 $[0.\infty)$ 上的 Brown 运动

(2) $EW(t) = \frac{1}{a} EW(a^2t) = 0$

$EW(t)W(s) = \frac{1}{a^2} EW(a^2t)W(a^2s) = \frac{1}{a^2}a^2s = s$. $t \geq s \geq 0$

$\therefore EW(t) = W(a^2t)/a$. $a>0$ 是 $[0.\infty)$ 上的 Brown 运动

5.5. $Ez(t) = 0$

$\therefore EW(t) = (t+1)\,E\,z\left(\frac{t}{t+1}\right) = 0$

$EW(t)W(s) = (t+1)(s+1)\,E\,z\left(\frac{t}{t+1}\right)z\left(\frac{s}{s+1}\right)$

$= (t+1)(s+1)\frac{s}{s+1}\left(1-\frac{t}{t+1}\right)$

$= s$ $t \geq s \geq 0$

$\therefore W(t)$ 是 ~~[0.1] 上的 Brown 运动~~

Brown 运动

10.