Nordic probabilistic AI school

Variational Inference and Optimization

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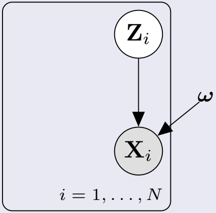
Deep Bayesian Learning – The VAE

Model of interest



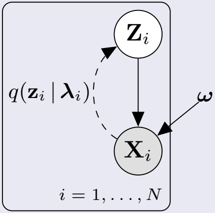
- $p(\mathbf{z}_i)$ is (usually) an isotropic Gaussian distribution.
- $p_{\boldsymbol{\omega}}(\mathbf{x}_i \mid g_{\boldsymbol{\omega}}(\mathbf{z}_i))$, where g is a deep neural network.

$$p_{\boldsymbol{\omega}}(\mathbf{x}_i|\mathbf{z}_i) \sim \text{Bernoulli}(\text{logits} = g_{\boldsymbol{\omega}}(\mathbf{z}_i))$$

- $g_{\boldsymbol{\omega}}(\mathbf{z}_i)$ plays the role of a **DECODER NETWORK**.
- **Goal:** Learn $\boldsymbol{\omega}$ to maximize the model's fit to $\mathcal{D}$.
    - We will cheat and find a **point estimate** for $\boldsymbol{\omega}$.

The Variational Auto Encoder (VAE)

Model of interest



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Variational Inference

- We will need $p_{\boldsymbol{\omega}}(\mathbf{z}_i \mid \mathbf{x}_i)$ for each data-point $\mathbf{x}_i$:

$$p_{\boldsymbol{\omega}}(\mathbf{z}_i \mid \mathbf{x}_i) = \frac{p_{\boldsymbol{\omega}}(\mathbf{z}_i) \cdot p_{\boldsymbol{\omega}}(\mathbf{x}_i \mid g_{\boldsymbol{\omega}}(\mathbf{z}_i))}{\int_{\mathbf{z}_i} p_{\boldsymbol{\omega}}(\mathbf{z}_i) \cdot p_{\boldsymbol{\omega}}(\mathbf{x}_i \mid g_{\boldsymbol{\omega}}(\mathbf{z}_i))\, \mathrm{d}\mathbf{z}_\mathrm{i}}.$$

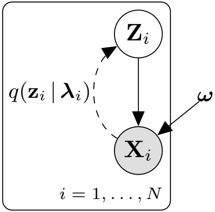
- **Initial plan:** Fit $q(\mathbf{z}_i \mid \boldsymbol{\lambda}_i)$ to $p_{\boldsymbol{\omega}}(\mathbf{z}_i \mid \mathbf{x}_i)$ using variational inference.

Initial plan:

- Optimize the ELBO

$$\mathcal{L}(\boldsymbol{\omega}, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_N) = -\mathbb{E}_q\left[\log \frac{\prod_{i=1}^N q(\mathbf{z}_i \mid \boldsymbol{\lambda}_i)}{\prod_{i=1}^N p_{\boldsymbol{\omega}}(\mathbf{z}_i, \mathbf{x}_i)}\right].$$

- A natural model for $q(\mathbf{z}_i \mid \boldsymbol{\lambda}_i)$ is a Gaussian with parameters $\boldsymbol{\lambda}_i = \{\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i\}$.
- If $\mathbf{Z}_i$ is d-dim and we for simplicity assume diagonal $\boldsymbol{\Sigma}_i$, this still gives $2Nd$ variational parameters to learn.



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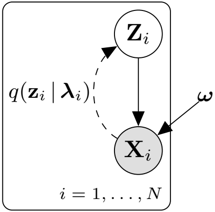

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A better plan

- Assume $g_{\boldsymbol{\omega}}(\mathbf{z})$ is "smooth": if $\mathbf{z}_i$ and $\mathbf{z}_j$ are "close", then so are $\mathbf{x}_i$ and $\mathbf{x}_j$.

  $\rightsquigarrow$ $\boldsymbol{\lambda}_i$ and $\boldsymbol{\lambda}_j$ should be "close" if $\mathbf{x}_i$ and $\mathbf{x}_j$ are "close".

- **Therefore:** Let's assume there exists a (smooth) function $h(\mathbf{x})$ so that $h(\mathbf{x}_i) = \boldsymbol{\lambda}_i$.
- $h(\cdot)$ is unavailable, so represent it using a deep neural net and learn the weights.
- $h(\mathbf{x}_i)$ plays the role of an **ENCODER NETWORK**.

Amortized inference:
To learn a model $h(\cdot)$, typically a deep neural network, so that $h(\mathbf{x}_i) = \boldsymbol{\lambda}_i$.
$h(\cdot)$ is parameterized with weights, often (abusing notation) denoted by $\boldsymbol{\lambda}$.

Note! Amortized inference is useful also outside VAEs!

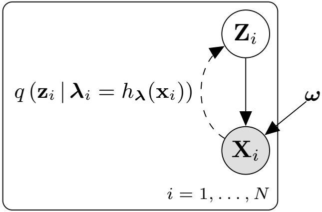
Benefits:

- The $2Nd$ parameters $\{\boldsymbol{\lambda}_i\}_{i=1}^N$ are replaced by the fixed-sized vector $\boldsymbol{\lambda}$.
  - If N is large we may get a simpler learning problem.
- Smoothness of $h(\cdot)$ implies regularization.
- We only change the **parameterization**, not the model itself!

VAE: Full setup

The full VAE approach:

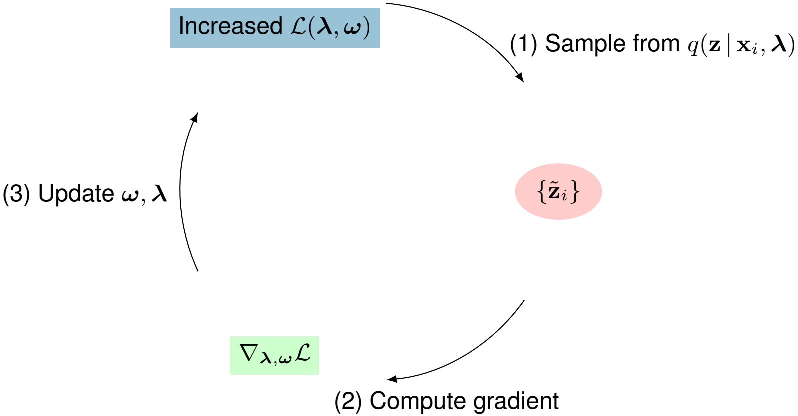
- $p(\mathbf{z}_i)$ is an isotropic Gaussian distribution.
- $p_{\boldsymbol{\omega}}(\mathbf{x}_i|\mathbf{z}_i) \sim$ Bernoulli(logits $= g_{\boldsymbol{\omega}}(\mathbf{z}_i)$), where $g_{\boldsymbol{\omega}}$ is a DNN with weights $\boldsymbol{\omega}$.
- $q(\mathbf{z}_i \,|\, \mathbf{x}_i, \boldsymbol{\lambda}) \sim \mathcal{N}(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$, where $\{\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i\}$ is given by $h_{\boldsymbol{\lambda}}(\mathbf{x}_i)$. $h_{\boldsymbol{\lambda}}$ is a DNN with weights $\boldsymbol{\lambda}$.



Goal:

Learn **both** $\boldsymbol{\omega}$ and $\boldsymbol{\lambda}$ by maximizing the ELBO:

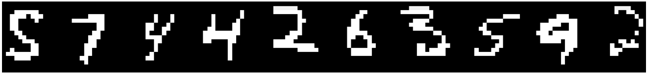
$$\mathcal{L}(\boldsymbol{\lambda}, \boldsymbol{\omega}) = -\mathbb{E}_q \left[\log \frac{q(\mathbf{z} \,|\, \mathbf{x}, \boldsymbol{\lambda})}{p_{\boldsymbol{\omega}}(\mathbf{z}, \mathbf{x} \,|\, \boldsymbol{\omega})} \right].$$



1. For each $\mathbf{x}_i$, sample M (typically 1) $\mathbf{z}$-values to approximate expectation in $\nabla\mathcal{L}$.
2. Calculate $\nabla_{\boldsymbol{\lambda},\boldsymbol{\omega}}\, \mathcal{L}(\boldsymbol{\lambda}, \boldsymbol{\omega})$ using the reparameterization-trick.
3. Update parameters using a standard DL optimizer (like Adam).

Fun with MNIST – The model

- The model is learned from $N = 55.000$ training examples.
- Each $\mathbf{x}_i$ is a binary vector of 784 pixel values.
- When seen as a 28×28 array, each $\mathbf{x}_i$ is a picture of a handwritten digit ("0" – "9").



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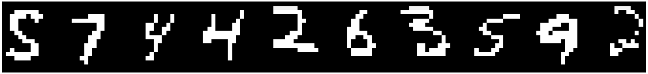

- Encoding is done in **two** dimensions. $p(\mathbf{z}_i) = \mathcal{N}\left(\mathbf{0}_2, \mathbf{I}_2\right)$.

- The **encoder network** $\mathbf{X} \rightsquigarrow \mathbf{Z}$.

$$\begin{tikzcd}
 & & & \log \boldsymbol{\Sigma}_{\boldsymbol{\lambda}}(\mathbf{x}_i),\ \text{2-d} \arrow[dr] & \\
\mathbf{x}_i : 784 \text{ dim} \arrow[r, "\text{ReLU}"] & \text{Hidden, 256-d} \arrow[r, "\text{ReLU}"] & \text{Hidden, 64-d} \arrow[ur, "\text{Linear}"] \arrow[dr, "\text{Linear}"'] & & q_{\boldsymbol{\lambda}}(\mathbf{z}_i|\mathbf{x}_i, \boldsymbol{\lambda}) = \mathcal{N}\left(\boldsymbol{\mu}_{\boldsymbol{\lambda}}(\mathbf{x}_i), \boldsymbol{\Sigma}_{\boldsymbol{\lambda}}(\mathbf{x}_i)\right),\ \text{2-d} \\
 & & & \boldsymbol{\mu}_{\boldsymbol{\lambda}}(\mathbf{x}_i),\ \text{2-d} \arrow[ur] &
\end{tikzcd}$$

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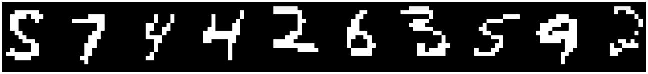
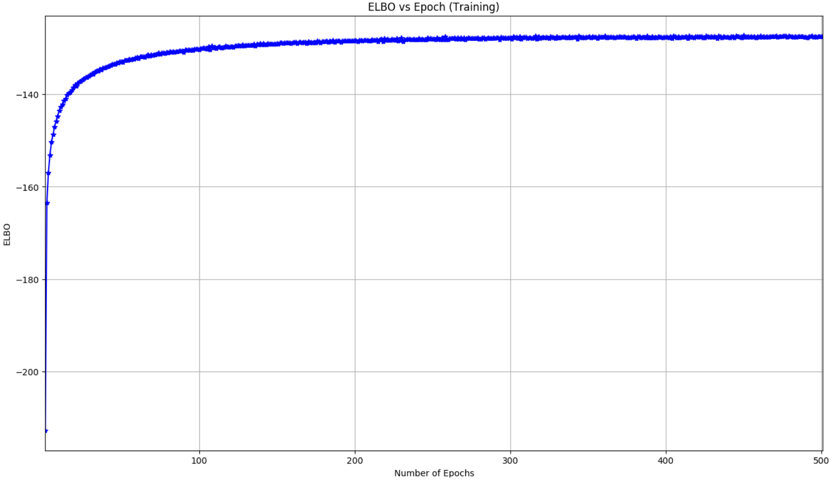


- Encoding is done in **two** dimensions. $p(\mathbf{z}_i) = \mathcal{N}\left(\mathbf{0}_2, \mathbf{I}_2\right)$.
- The **encoder network** $\mathbf{X} \rightsquigarrow \mathbf{Z}$.
- The **decoder network** $\mathbf{Z} \rightsquigarrow \mathbf{X}$ is a $64 + 256$ neural net with ReLU units.

$$\mathbf{z}_i : 2 \text{ dim} \xrightarrow{\text{ReLU}} \text{Hidden, 64-d} \xrightarrow{\text{ReLU}} \text{Hidden, 256-d} \xrightarrow{\text{Linear}} \text{logit}(\mathbf{p}_i)\text{, 784-d} \longrightarrow p_{\boldsymbol{\omega}}(\mathbf{x}_i \,|\, \mathbf{z}_i, \boldsymbol{\omega}) = \text{Bernoulli}\left(\mathbf{p}_i\right)\text{, 784-d}$$

Learning progress; learning rate $\eta = 10^{-4}$, $M = 1$



Note! SGD algorithm uses the negative ELBO as loss.

Trying to reconstruct $\mathbf{x}_i$ by $\mathbb{E}_{p_\omega}\left[\mathbf{X} \mid \mathbf{Z} = \mathbb{E}_{q_\lambda}\left[\mathbf{Z} \mid \mathbf{x}_i\right]\right]$

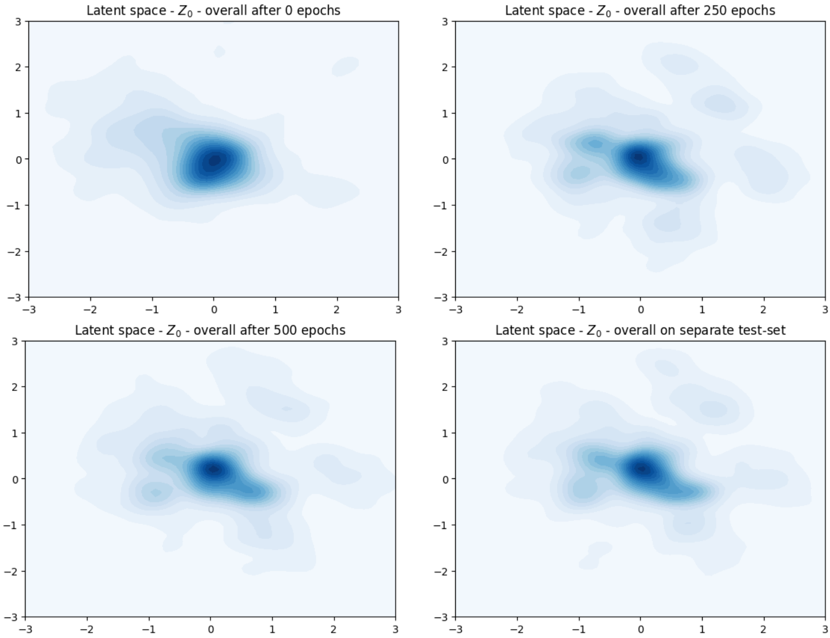
After 1 epoch

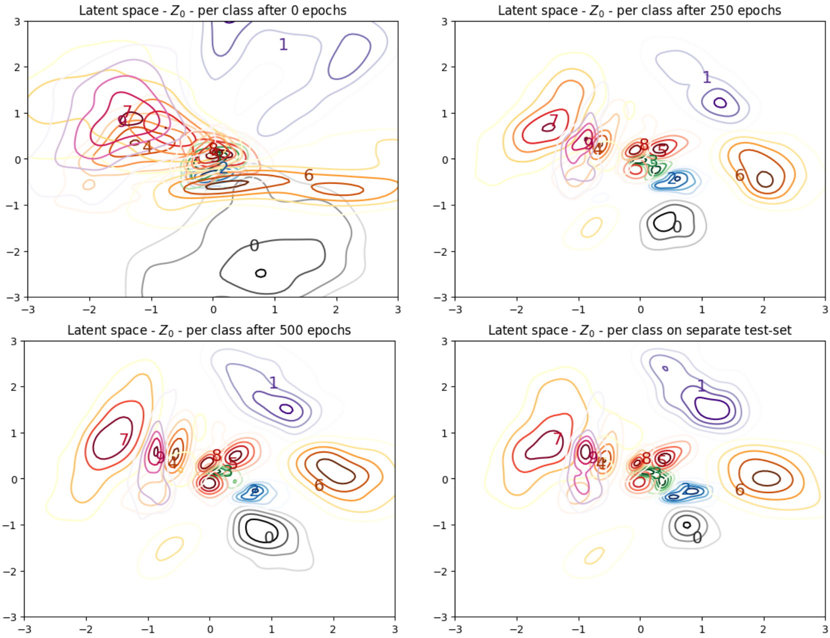
After 250 epochs

After 500 epoch

Using separate test-set

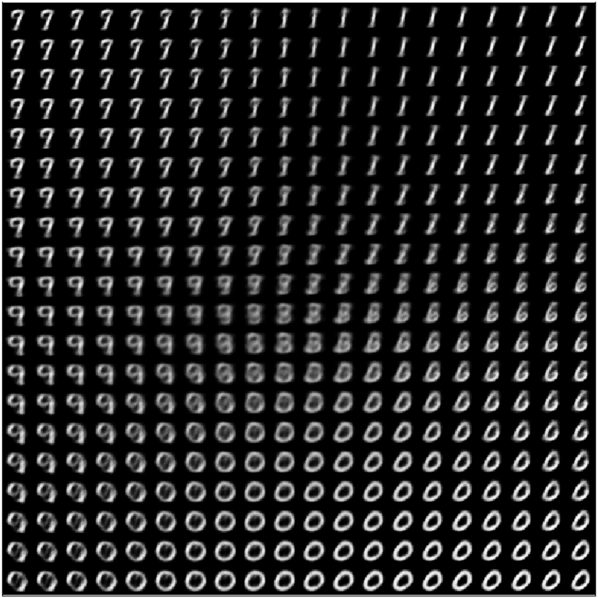
Averaged distribution over $\mathbf{Z}$



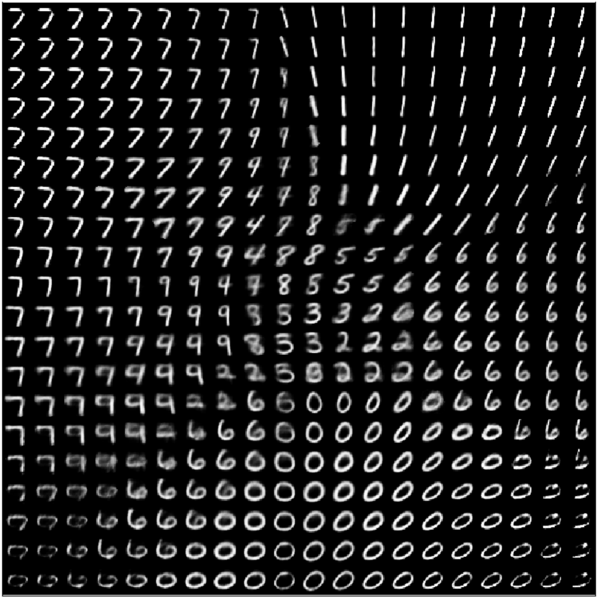


Latent space - Z_0 - per class after 0 epochs
Latent space - Z_0 - per class after 250 epochs
Latent space - Z_0 - per class after 500 epochs
Latent space - Z_0 - per class on separate test-set
0
1
2
3
4
5
6
7
8
9
−3
−2
−1

The picture manifold – $\mathbb{E}_{p_\omega}\left[\mathbf{X} \mid \mathbf{z}\right]$ for different values of $\mathbf{z}$

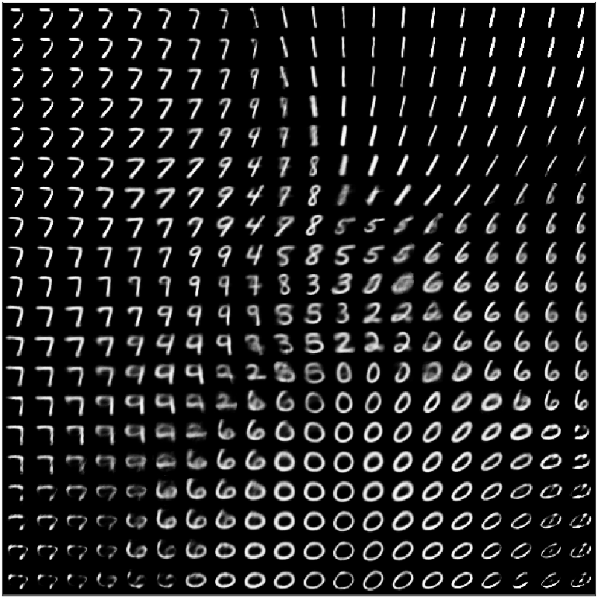


Manifold after 1 epoch



Manifold after 250 epochs

The picture manifold – $\mathbb{E}_{p_\omega}[\mathbf{X} \mid \mathbf{z}]$ for different values of $\mathbf{z}$

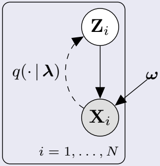


Manifold after 500 epochs

Variational Auto-Encoders in Pyro

Notes

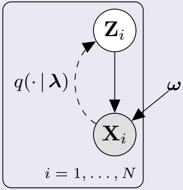
- The PYRO.MODULE call registers the parameters in the decoder network with Pyro.
- The decoder network is a subclass of NN.MODULE; the class inherits methods such as PARAMETERS() and BACKWARD for calculating gradients.



Pyro specification of a decoder

Notes

- The encoder and guide follow the same structure as the encoder and model



Code Task: VAEs in Pyro

- Learn how a VAE is coded in Pryo.
- We provide a VAE with a **linear decoder**.
- **Exercise 1: Define a Non-Linear Decoder**
  - A MLP with a hidden layer with non-linearities (e.g. Relu).
- **Exercise 2: Explore the latent space**
  - Moving from linear to non-linear decoders with different capacity.
- Notebook:

  `Day2-Evening/students_VAE.ipynb`.

Conclusions

- **Bayesian Machine Learning**
  - Represents unobserved quantities using **distributions**
  - Models **epistemic** uncertainty using $p(\boldsymbol{\theta} \mid \mathcal{D})$

- **Bayesian Machine Learning**

- **Variational inference**
  - **Provides** $q(\boldsymbol{\theta} \mid \boldsymbol{\lambda})$**:** A distributional approximation to $p(\boldsymbol{\theta} \mid \mathcal{D})$
  - **Objective:** $\arg\min_{\boldsymbol{\lambda}} \mathrm{KL}\left(q(\boldsymbol{\theta} \mid \boldsymbol{\lambda}) || p(\boldsymbol{\theta} \mid \mathcal{D})\right) \Leftrightarrow \arg\max_{\boldsymbol{\lambda}} \mathcal{L}\left(q(\boldsymbol{\theta} \mid \boldsymbol{\lambda})\right)$
  - **Mean-field:** Divide and conquer strategy for high-dimensional posteriors
  - **Main caveat:** $q(\boldsymbol{\theta} \mid \boldsymbol{\lambda})$ underestimates the uncertainty of $p(\boldsymbol{\theta} \mid \mathcal{D})$

Recap & Conclusions

- **Bayesian Machine Learning**
- **Variational inference**
- **Coordinate Ascent Variational Inference**
  - Analytic expressions for some models (i.e., conjugate exponential family)
  - CAVI is very **efficient and stable** if it can be used
  - In principle requires **manual derivation** of updating equations
    - There are **tools** to help (using *variational message passing*)

Recap & Conclusions

- **Bayesian Machine Learning**
- **Variational inference**
- **Coordinate Ascent Variational Inference**
- **Gradient-based Variational Inference**
  - Provides the tools for VI over **arbitrary** probabilistic models
  - Directly integrates with the tools of deep learning
    - Automatic differentiation, sampling from standard distributions, and SGD
  - Sampling to approximate expectations: **Beware of the variance!**

Recap & Conclusions

- **Bayesian Machine Learning**
- **Variational inference**
- **Coordinate Ascent Variational Inference**
- **Gradient-based Variational Inference**
- **Probabilistic programming languages**
  - PPLs fuel the “build – compute – critique – repeat” - cycle through
    - ease and flexibility of modelling
    - powerful inference engines
    - efficient model evaluations
  - Many available tools (Pyro, TF Probability, Infer.net, Turing.jl, . . .)

Recap & Conclusions

- **Bayesian Machine Learning**
- **Variational inference**
- **Coordinate Ascent Variational Inference**
- **Gradient-based Variational Inference**
- **Probabilistic programming languages**
- **What's next?**
  - The "VI toolbox" is reaching maturity
    - From *only* a research area to almost a *prerequisite* for Probabilistic AI
    - . . . yet there are still things to explore further!
  - Today's material should suffice to read (and write!) Prob-AI papers