

Probabilistic Machine learning

Introduction to probabilistic programming languages (PPLs)

Andrés Masegosa and Thomas Dyhre Nielsen

[Material partly made by Antonio Salmeron and Helge Langseth]

- **Day 1: Probabilistic Modeling**

- Introduction to probabilistic modeling and inference
- Introduction to probabilistic programming
- Probabilistic programming in Pyro

- **Day 2 : Variational Inference for Probabilistic Machine Learning**

- Introduction to Variational inference
- Variational inference in Pyro
- Variational AutoEncoders

To get credits for the course, you need to hand in a deliverable/assignment with the following contents:

- Describe a problem/dataset for which a probabilistic model can be used for analysis. Examples include:
 - Based on your own PhD research (preferred!)
 - Taken from a public repository such as www.kaggle.com or <https://archive.ics.uci.edu>
- Design a probabilistic model for solving the problem.
- Implement your model in *Pyro*.
- Use the implementation to perform an analysis, for example:
 - Learn and apply a classifier
 - Perform Bayesian inference to compute distributions over quantities of interest (e.g., parameters, latent variables, etc.)
- Document your approach and findings in a short report (3-5 pages).

Deadline: 30th December 2023



"I always wondered how it would be if a Superior species landed on Earth and showed us How they played chess. Now I know it."

Peter Heine Nielsen.
Chess Grand Master and Magnus Carlsen's coach



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The question is,

- Can we (**humans**) learn (**interpret**) anything from it?

Examples: Predictive medicine



What should I do to prevent players from being injured?

Examples: Self-driving cars



Can I predict an event in advance so that I can avoid it?

Examples: Image generation

A majestic oil painting of a raccoon Queen wearing red French royal gown. The painting is hanging on an ornate wall decorated with wallpaper.



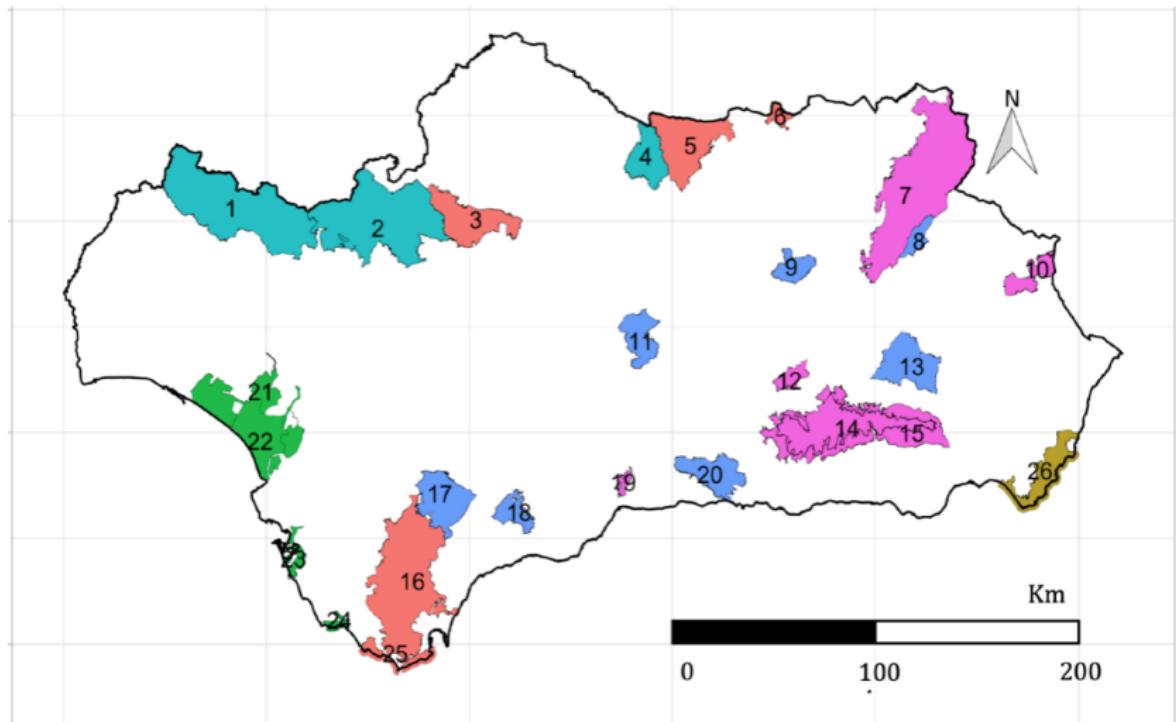
Imagen

unprecedented photorealism × deep level of language understanding

Google Research, Brain Team

Generating images from written descriptions

Examples: Land use



Monitoring protected areas

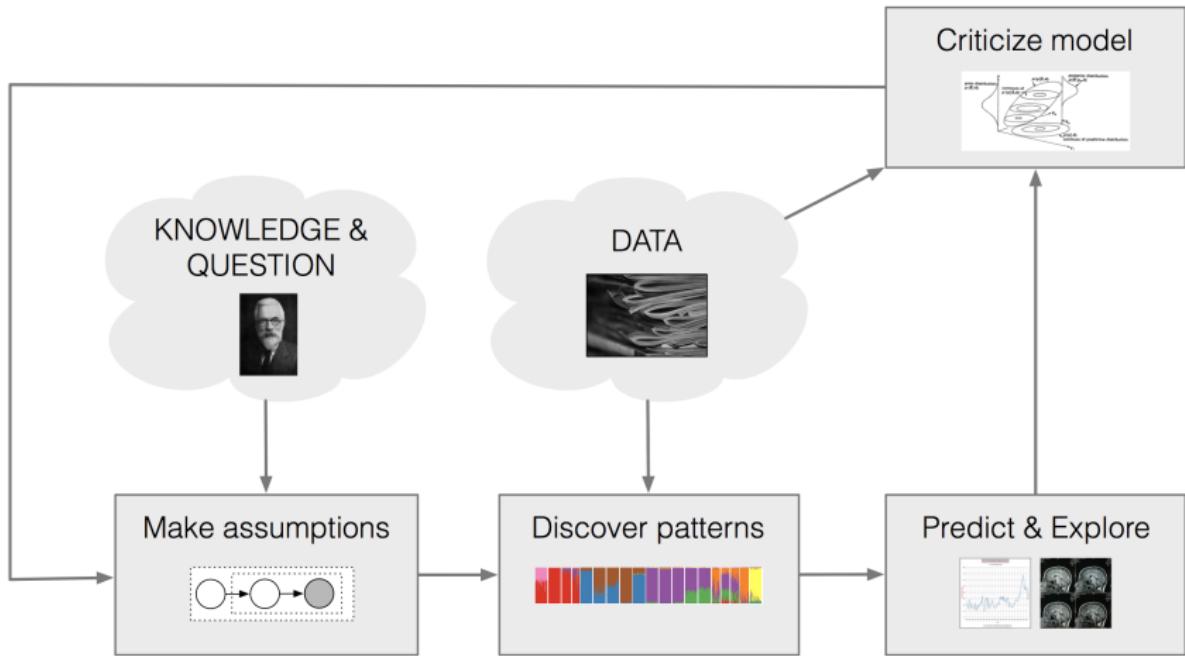
All the previous examples:

- Operate in environments where large amounts of data are available
- However, data don't cover all the possible scenarios \Rightarrow **UNCERTAINTY**
- Use a probabilistic model learnt from data
- Use inference algorithms to carry out prediction and structure analysis

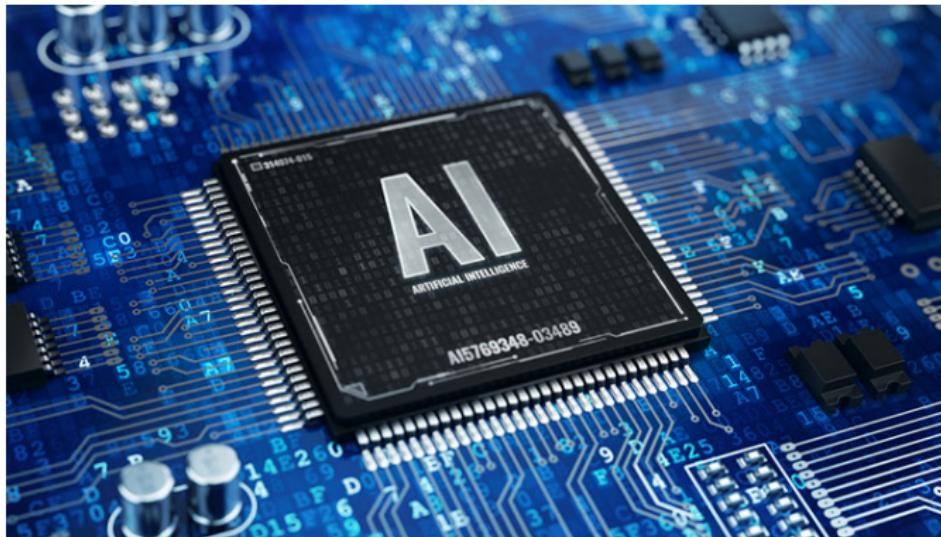
Probabilistic models offer:

- Principled quantification of uncertainty
- Natural way of dealing with missing data
- Interpretability

The probabilistic modeling cycle



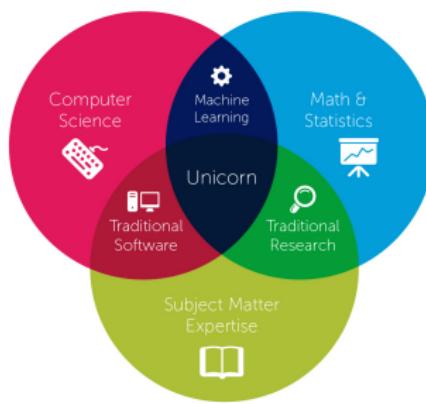
[Box, 1980; Rubin, 1984; Gelman+ 1996; Blei, 2014]



The development of **machine learning systems** requires enormous efforts.

- It can be a **highly technical** task.

Data Science

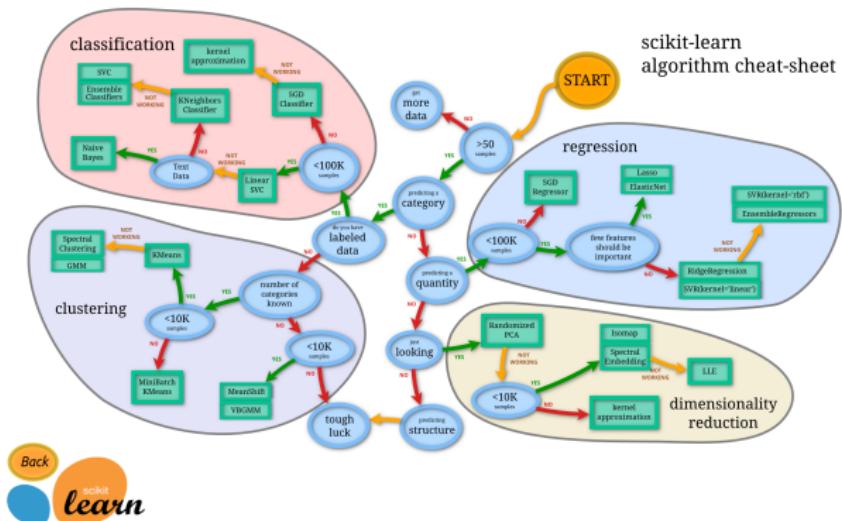


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The development of **machine learning systems** requires enormous efforts.

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- It requires of **highly qualified experts**.

Machine Learning Systems

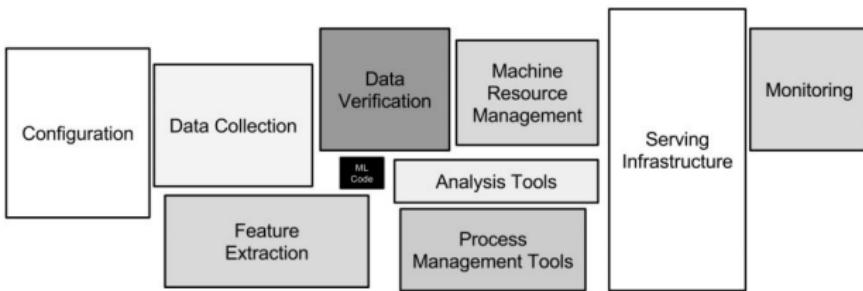


The development of **machine learning systems** requires enormous efforts.

- It can be a **highly technical** task.
- It requires of **highly qualified experts**.
- It is difficult to find the **ML model most suitable** for an application.

Hidden Technical Debt in Machine Learning Systems

D. Sculley, Gary Holt, Daniel Golovin, Eugene Davydov, Todd Phillips
{dsculley, gholt, dg, edavydov, toddphillips}@google.com
Google, Inc.



The development of **machine learning systems** requires enormous efforts.

- It can be a **highly technical** task.
- It requires of **highly qualified experts**.
- It is difficult to find the **ML model most suitable** for an application.
- Programming a ML model is a **complex task**; many problems are intermingled.

Wanted: Artificial intelligence experts

In artificial intelligence, job openings are rising faster than job seekers.

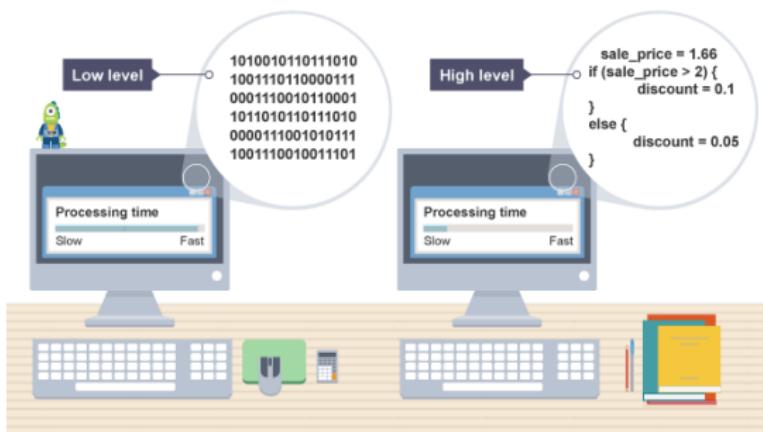


[https://www.ml.cmu.edu/\[...\]/good-news-for-job-seekers-with-ai-skills-there-is-a-shortage-of-qualified-workers.html](https://www.ml.cmu.edu/[...]/good-news-for-job-seekers-with-ai-skills-there-is-a-shortage-of-qualified-workers.html)

Consequences:

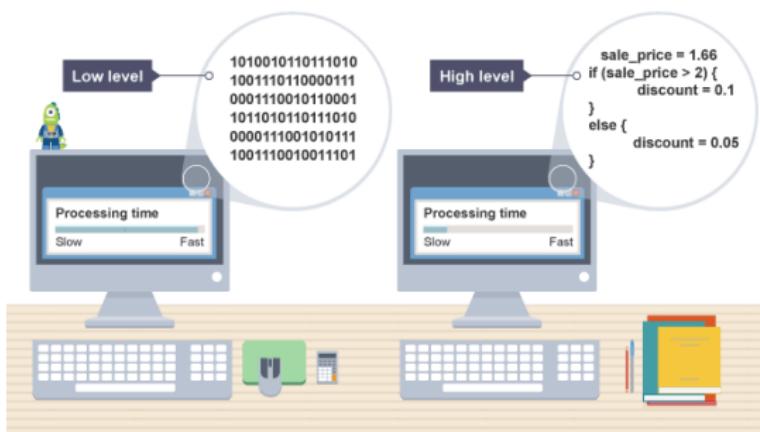
- Shortage of AI experts (and high salaries).
- Only big corporations have the resources for developing ML systems.

Programming Languages



Similar situation than 50 years ago:

- People used to program in low-level programming languages.
- Programming was complex and demand high-expertise.
- Focus on application and low-level hardware details.



High-level programming languages brought many advantages:

- Programmers focused on the applications.
- Hardware Experts focused on compilers.
- High gains in productivity.
- “Democratization” of the software development.

Big Data and Machine Learning Ecosystem



Claire D. Costa. Best Python Libraries for Machine Learning and Deep Learning.

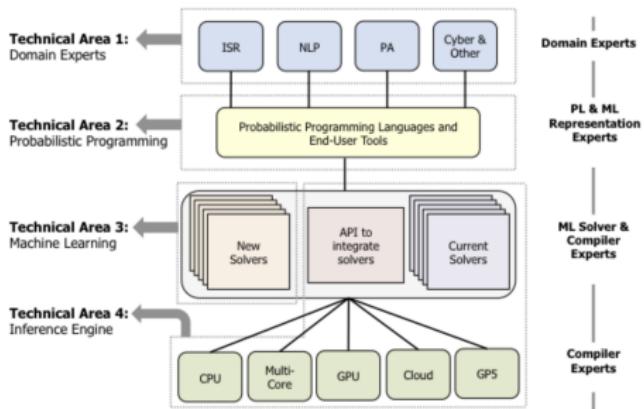
<https://towardsdatascience.com/best-python-libraries-for-machine-learning-and-deep-learning-b0bd40c7e8c>

Big Data and Machine Learning Libraries:

- **High-quality**, well-maintained, and open-source libraries
- Provide **high-level abstractions**.
- Hide **low level details** under the hood.
- Increase the **adoption** of the technologies.

What are the "high-level libraries" in Probabilistic AI?

Why probabilistic programming languages (PPLs)?

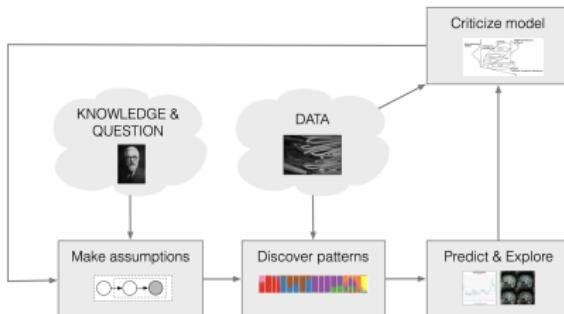


PPLs as high-level programming languages for **probabilistic ML systems**:

- Stacked architecture.
- Different Domain Experts can code their models using the same language.
- ML experts can focus on the development of ML methods/algorithms (ML solvers).
- Compiler experts can focus on running these ML solvers on specialized hardware.

Why probabilistic programming languages (PPLs)?

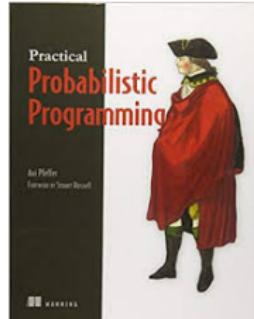
Box's Loop



[Box, 1980; Rubin, 1984; Gelman+ 1996; Blei, 2014]

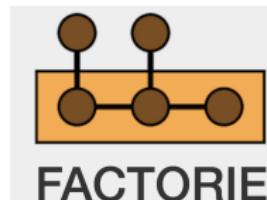
Benefits of PPLs for developing probabilistic machine learning systems:

- Simplify probabilistic machine learning model code.
- Reduce development time and cost to encourage experimentation.
- Reduce the necessary level of expertise.
- “Democratization” of the development of probabilistic ML systems.



1st Generation of PPLs :

- Bugs, WinBugs, Jags, Figaro, etc.
- Turing-complete probabilistic programming languages. (i.e. they can represent any computable probability distribution).
- Inference engine based on Monte Carlo methods.
- Not able to scale to large data samples/high-dimensional models.



2nd Generation of PPLs :

- Infer.net, Factorie, Amidst, etc.
- Inference engine based on message passage algorithms and/or variational inference methods.
- Scale to large data samples/high-dimensional models.
- Restricted probabilistic model families (i.e. factor graphs, conjugate exponential family, etc.)



3rd Generation of PPLs :

- Pyro, Stan, PyMC, InferPy, etc.
- Black Box Variational Inference and Hamiltonian Monte-Carlo.
- Scale to large data samples/high-dimensional models.
- Turing-complete probabilistic programming languages.
 - Strong focus on probabilistic models with **deep neural networks**.
- Most rely on deep learning frameworks (Pytorch, JAX, TensorFlow, etc).
 - Specialized hardware like GPUs, TPUs, etc.
 - Automatic differentiation methods.

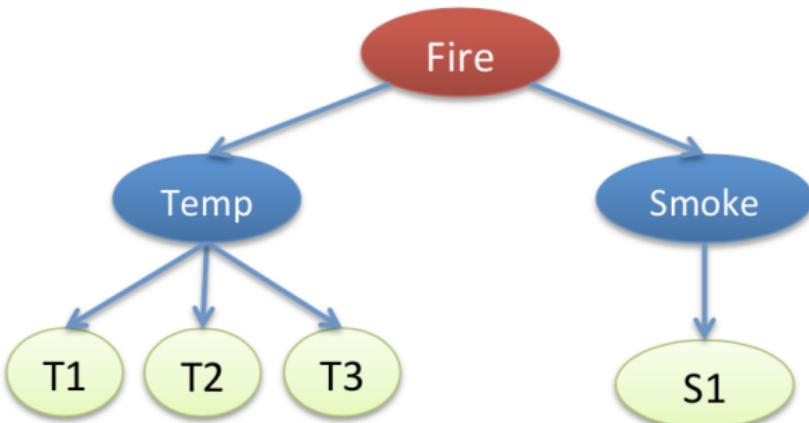


Pyro's main features (www.pyro.ai) :

- Initially developed by UBER (the car riding company).
- Community of contributors and a dedicated team at Broad Institute (US).
- Rely on Pytorch (Deep Learning Framework).
- Enable GPU acceleration and distributed learning.

<https://github.com/PGM-Lab/2023-PhD-Course-PML>

Probabilistic Graphical Models



$$p(f, t, s, t_1, t_2, t_3, s_1) = p(t_1|t)p(t_2|t)p(t_3|t)p(s_1|s)p(t|f)p(s|f)p(f)$$

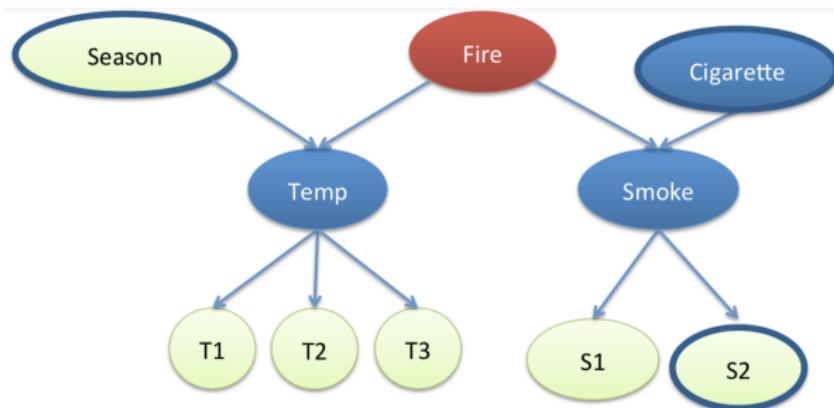
A Bayesian network over random variables X_1, \dots, X_n consists of

- A DAG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{X_1, \dots, X_n\}$
- A set of local conditional distributions $\mathcal{P} = \{p(X_i|pa(X_i)), X_i \in \mathcal{V}\}$ where $pa(X_i)$ denotes the parents of X_i according to \mathcal{E}

Every Bayesian network encodes a joint distribution factorized as

$$p(X_1, \dots, X_n) = \prod_{i=1}^n p(X_i|pa(X_i))$$

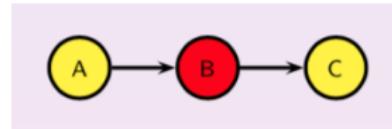
Bayesian networks: modular structure



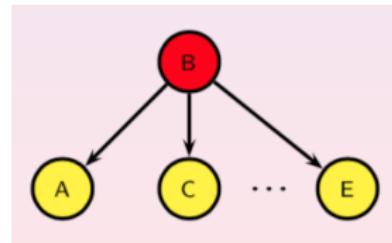
$$p(\textcolor{red}{se}, f, \textcolor{red}{c}, t, s, t_1, t_2, t_3, s_1) = p(t_1|t)p(t_2|t)p(t_3|t)p(s_1|s)\textcolor{red}{p(s_2|s)} \\ \textcolor{red}{p(t|se, f)p(s|f, c)p(se)p(f)p(c)}$$

Interpreting Bayesian network structures: d -separation

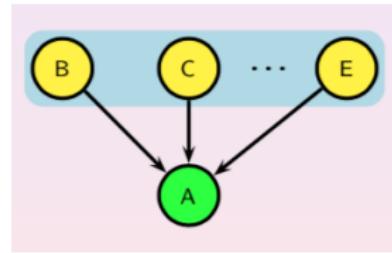
- Serial connection



- Diverging connection



- Converging connection



Inference

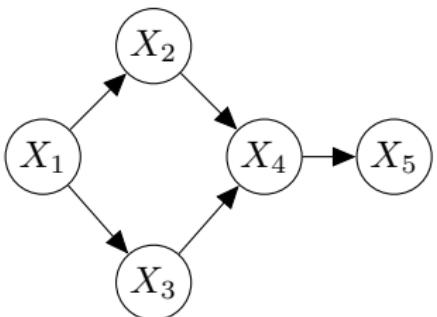
Assume a Bayesian network over variables $\mathbf{X} = \{X_1, \dots, X_n\}$

$$\left. \begin{array}{c} \text{Bayesian network} \\ + \\ \text{Evidence } (\mathbf{X}_E) \end{array} \right\} \Rightarrow P(\mathbf{X}_I | \mathbf{X}_E)?$$

Inference methods

- Exact
 - Brute force: compute $P(\mathbf{X}, \mathbf{X}_E)$ and marginalize out $\mathbf{X} \setminus \mathbf{X}_I$
 - Take advantage of the network structure
- Approximate
 - Sampling
 - Deterministic

Exact inference: Variable elimination



- We are interested in X_5
- All variables are discrete
- $E = \emptyset$

$$\begin{aligned} p(x_5) &= \sum_{x_1 x_4} p(x_1, x_2, x_3, x_4, x_5) \\ &= \sum_{x_1 x_4} p(x_1)p(x_2|x_1)p(x_3|x_1)p(x_4|x_2, x_3)p(x_5|x_4) \\ &= \sum_{x_2 x_4} \sum_{x_1} p(x_1)p(x_2|x_1)p(x_3|x_1)p(x_4|x_2, x_3)p(x_5|x_4) \\ &= \sum_{x_2 x_4} p(x_4|x_2, x_3)p(x_5|x_4) \boxed{\sum_{x_1} p(x_1)p(x_2|x_1)p(x_3|x_1)} \\ &= \sum_{x_2 x_4} p(x_4|x_2, x_3)p(x_5|x_4) \textcolor{red}{h(x_2, x_3)} \end{aligned}$$

We have reached a similar problem as initially, but with one variable less.

Input

\mathcal{P} : Conditional distributions in the network

\mathbf{X} : Variables in the network

X_I : Target variable

\mathbf{X}_E : Observed variables

\mathbf{Y} : All variables in \mathbf{X} except X_I ($\mathbf{Y} = \mathbf{X} \setminus \{X_I\}$).

- ① Restrict all the distributions in \mathcal{P} to the evidence $\mathbf{X}_E = \mathbf{x}_E$
- ② For each $Y \in \mathbf{Y}$,
 - ① Let \mathcal{P}_Y be the set of distributions in \mathcal{P} that contain Y .
 - ② $q := \prod_{p \in \mathcal{P}_Y} p$.
 - ③ $r := \sum_y q$.
 - ④ $\mathcal{P} := \mathcal{P} \setminus \mathcal{P}_Y \cup \{r\}$.
- ③ $p(x_I) = \prod_{p \in \mathcal{P}} p$.
- ④ Normalize p

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- ③ $p(x_I) = \prod_{p \in \mathcal{P}} p$
- ④ Normalize p

- Product of functions raises complexity
 - Exponentially in the case of **discrete** variables
- Complexity also depends on the **elimination order**
- Representation of densities turns out to be relevant
 - **Closed-form** solutions to product and marginalization are preferable

- A Bayesian network is a representation of a joint probability distribution over $\mathbf{X} \Rightarrow$ it describes some **population** consisting of all the possible configurations of \mathbf{X}
- If the entire population was available, the **inference problem** could be solved exactly, basically by **counting cases**
- **Problem:** Population size can be huge or even infinite.
- **Monte Carlo** methods operate by drawing an artificial **sample** from it using some random mechanism
- The sample (**much smaller than the population**), is used to estimate the distribution of each variable of interest.

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Key issues in a Monte Carlo inference algorithm:

- ① The sampling mechanism
- ② The functions (estimators) which compute the probabilities from the sample

Importance sampling. General setting

- Assume we have a random variable X with density $p(x)$
- Importance sampling is a technique designed for estimating the expected value of a function $f(X)$. It is based on the following transformation:

$$\mathbb{E}_p[f(x)] = \int f(x)p(x)dx = \int \frac{p(x)}{p^*(x)}f(x)p^*(x)dx = \mathbb{E}_{p^*}\left[\frac{p(x)}{p^*(x)}f(x)\right],$$

where p^* is a density function called the sampling or proposal distribution, s.t. $p^*(x) > 0$ whenever $p(x) > 0$.

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where p^* is a density function called the sampling or proposal distribution, s.t. $p^*(x) > 0$ whenever $p(x) > 0$.

- Therefore, $\mathbb{E}_p[f(x)]$ can be estimated by drawing a sample $x^{(1)}, \dots, x^{(m)}$ from p^* and computing

$$\hat{\mathbb{E}}_p[f(x)] = \frac{1}{m} \sum_{j=1}^m \frac{p(x^{(j)})}{p^*(x^{(j)})} f(x^{(j)}),$$

which is specially convenient if p^* is easier to handle than p

- We start with an initial configuration of the variables in the network
- A new item in the sample is generated conditional on the previous configuration
- The elements of the sample are no longer independent, instead they form a **Markov chain**

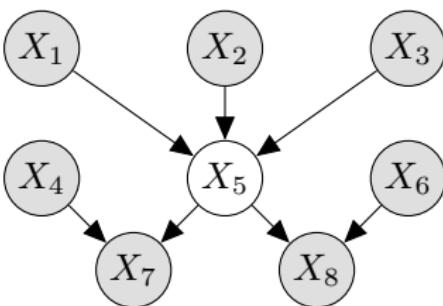
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Gibbs sampling

A new item in the sample is generated by simulating each variable at a time conditional on the value of the other variables sampled so far:

- Assume $\mathbf{X} = \{X_1, \dots, X_n\}$
- Let $\mathbf{x}^{(i)} = (x_1^{(i)}, \dots, x_n^{(i)})$ be the current configuration
- A new configuration, $\mathbf{x}^{(i+1)}$ is generated by sampling each variable X_k , $k = 1, \dots, n$ with

$$p(x_j^{(i+1)} | x_1^{(i+1)}, \dots, x_{j-1}^{(i+1)}, x_{j+1}^{(i)}, \dots, x_n^{(i)})$$



$$p^*(X_5) \propto p(X_5|X_1, X_2, X_3)p(X_7|X_4, X_5)p(X_8|X_5, X_6)$$

Problems

- Generating an initial configuration with positive probability can be hard
- The dependence between elements of the sample can make it **converge slowly**

Learning From Data

Thumbtack example

We have tossed a thumb tack 100 times. It has landed pin up 80 times, and we now look for the probability θ of pin up (\rightsquigarrow model that best fits the observations/data):



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We can measure how well a model fits the data using the likelihood:

$$\begin{aligned} p(\text{data}|\theta) &= p(\text{pin up}, \text{pin up}, \text{pin down}, \dots, \text{pin up}|\theta) \\ &= p(\text{pin up}|\theta) \cdot p(\text{pin up}|\theta) \cdot p(\text{pin down}|\theta) \cdot \dots \cdot p(\text{pin up}|\theta) \end{aligned}$$

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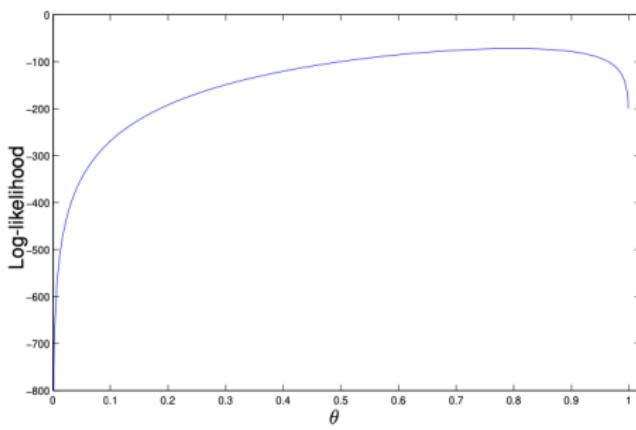


We select the parameter $\hat{\theta}$ that maximizes:

$$\hat{\theta} = \arg \max_{\theta} p(\text{data} | \theta) \quad (\text{or } \log p(\text{data} | \theta))$$

$$= \arg \max_{\theta} \prod_{i=1}^{100} p(x_i | \theta)$$

$$= \arg \max_{\theta} \theta^{80} (1 - \theta)^{20}.$$



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By setting:

$$\frac{d}{d\theta} \theta^{80} (1 - \theta)^{20} = 0 \quad \left(\text{or } \frac{d}{d\theta} \log(\theta^{80} (1 - \theta)^{20}) = 0 \right)$$

we get the *maximum likelihood estimate*:

$$\hat{\theta} = 0.8.$$

- A random variable X
- A probability density/mass function p associated with X
- A parameter θ , whose value is fixed but unknown
- We assume p is known except for parameter θ , and denote this fact as $p(x|\theta)$
- For a sample X_1, \dots, X_n drawn from $f(x|\theta)$, the likelihood of the data is

$$p(x_1, \dots, x_n | \theta) = \prod_{i=1}^n p(x_i | \theta)$$

i.e. the joint density/mass of the sample regarded as a function of the parameter

- Learning problem is to find the parameter that maximize the log-likelihood of the data:

$$\max_{\theta} \ln p(x_1, \dots, x_n | \theta)$$

Bayesian parameter learning

Basis

- Parameters are modeled as random variables and information about them can be included prior to observing data
- By Bayes' rule, the prior information is combined with the likelihood, yielding a posterior distribution that is used for making inferences about the parameter

More formally Parameter learning amounts to calculating the posterior distribution over the parameters given the data $\mathcal{D} = (\mathbf{x}_1, \dots, \mathbf{x}_N)$:

$$p(\boldsymbol{\theta} | \mathcal{D}) = \frac{p(\boldsymbol{\theta}, \mathcal{D})}{p(\mathcal{D})} = \frac{p(\boldsymbol{\theta})p(\mathcal{D} | \boldsymbol{\theta})}{p(\mathcal{D})}$$

$$p(\mathcal{D} | \boldsymbol{\theta}) = \prod_{i=1}^N p(\mathbf{x}_i | \boldsymbol{\theta}) \quad [\text{Assuming independently sampled data}]$$

Posterior predictive distribution

For making predictions about future cases we use

$$p(\mathbf{X} | \mathcal{D}) = \int_{\boldsymbol{\theta}} p(\mathbf{X} | \boldsymbol{\theta})p(\boldsymbol{\theta} | \mathcal{D})d\boldsymbol{\theta}$$

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Model and data

Consider the variables X_1, X_2, \dots, X_N with $P(X_i|\theta) = \theta$:

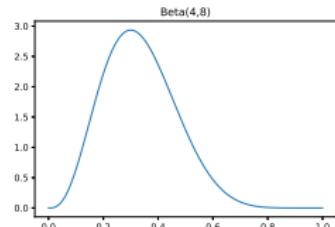
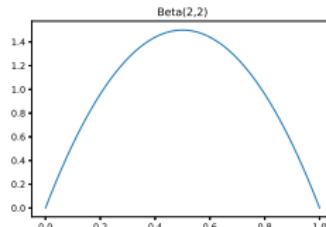
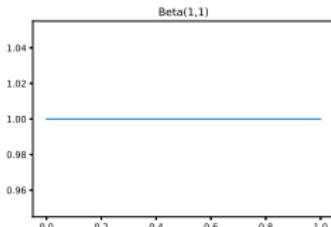
- $X_i = 1$ represents pin up and $X_i = 0$ represents pin down
- $\theta \in [0, 1]$ is the probability of pin up

Prior

The beta distribution $\text{Beta}(\theta | a, b)$ over θ :

$$p(\theta) \propto \theta^{a-1} (1 - \theta)^{b-1},$$

where a and b are hyper parameters.



Model and data

Suppose that $X_1, X_2, \dots, X_N \sim \text{Ber}(\theta)$

- N_1 are the number of pin up
- N_0 are the number of pin down ($N = N_0 + N_1$)

Model and data

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$$p(\theta | \mathcal{D}) = \text{Beta}(\theta | a + N_1, b + N_0)$$

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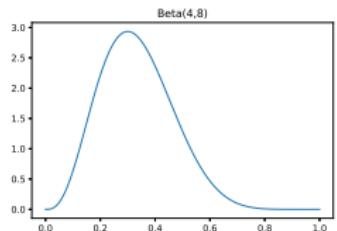
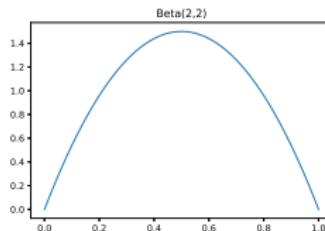
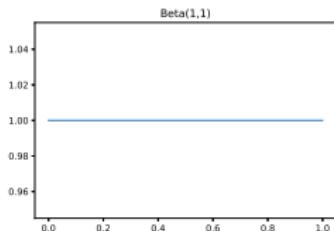
Posterior predictive distribution

$$\begin{aligned} p(X_{N+1} = 1 | \mathcal{D}) &= \int_0^1 p(X_{N+1} = 1 | \theta) p(\theta | \mathcal{D}) d\theta \\ &= \int_0^1 \theta \text{Beta}(\theta | a + N_1, b + N_0) d\theta = \frac{a + N_1}{a + b + N} \end{aligned}$$

Beta posterior: example

Assume that $N_0 = 5$ and $N_1 = 9$.

Prior



Posterior

