# Martingale Pricing Theory

These notes develop the modern theory of martingale pricing in a discrete-time, discrete-space framework. This theory is also important for the modern theory of portfolio optimization as the problems of pricing and portfolio optimization are now recognized as being intimately related. We choose to work in a discrete-time and discrete-space environment as this will allow us to quickly develop results using a minimal amount of mathematics: we will use only the basics of linear programming duality and martingale theory. Despite this restriction, the results we obtain hold more generally for continuous-time and continuous-space models once various technical conditions are satisfied. This is not too surprising as one can imagine approximating these latter models using our discrete-time, discrete-space models by simply keeping the time horizon fixed and letting the number of periods and states go to infinity in an appropriate manner.

Notation and Definitions for Single-Period Models

We first consider a one-period model and introduce the necessary definitions and concepts in this context. We will then extend these definitions to multi-period models.
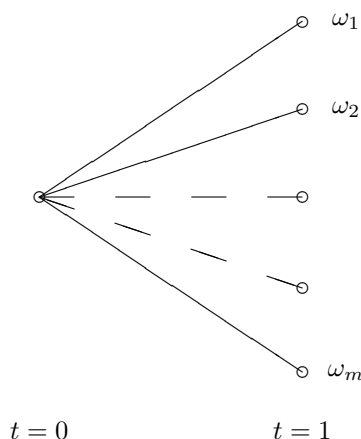
Let $t = 0$ and $t = 1$ denote the beginning and end, respectively, of the period. At $t = 0$ we assume that there are $N + 1$ securities available for trading, and at $t = 1$ one of m possible states will have occurred. Let $S_0^{(i)}$ denote the time $t = 0$ value of the ith security for $0 \leq i \leq N$, and let $S_1^{(i)}(\omega_j)$ denote its payoff at date $t = 1$ in the event that ω_j occurs. Let $P = (p_1, \ldots, p_m)$ be the *true* probability distribution describing the likelihood of each state occurring. We assume that $p_k > 0$ for each k.

Arbitrage

A **type A arbitrage** is an investment that produces immediate positive reward at $t = 0$ and has no future cost at $t = 1$. An example of a type A arbitrage would be somebody walking up to you on the street, giving you a

positive amount of cash, and asking for nothing in return, either then or in the future.

A **type** B **arbitrage** is an investment that has a non-positive cost at $t = 0$ but has a positive probability of yielding a positive payoff at $t = 1$ and zero probability of producing a negative payoff then. An example of a type B arbitrage would be a stock that costs nothing, but that will possibly generate dividend income in the future.

In finance we always assume that arbitrage opportunities do not exist[1] since if they did, market forces would quickly act to dispel them.

Linear Pricing

Definition 1 *Let $S_0^{(1)}$ and $S_0^{(2)}$ be the date $t = 0$ prices of two securities whose payoffs at date $t = 1$ are d_1 and d_2, respectively[2]. We say that **linear pricing** holds if for all α_1 and α_2, $\alpha_1 S_0^{(1)} + \alpha_2 S_0^{(2)}$ is the value of the security that pays $\alpha_1 d_1 + \alpha_2 d_2$ at date $t = 1$.*

It is easy to see that absence of type A arbitrage implies that linear pricing holds. As we always assume that arbitrage opportunities do not exist, we also assume that linear pricing always holds.

Elementary Securities, Attainability and State Prices

Definition 2 *An **elementary security** is a security that has date $t = 1$ payoff of the form $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the payoff of 1 occurs in state j.*

As there are m possible states at $t = 1$, there are at most m elementary securities.

Definition 3 *A security or contingent claim, X, is said to be **attainable** if there exists a trading strategy, $\theta = [\theta_0 \ \theta_1 \ \ldots \ \theta_{N+1}]^T$, such that*

$$
\begin{bmatrix} X(\omega_1) \\ \vdots \\ X(\omega_m) \end{bmatrix} = \begin{bmatrix} S_1^{(0)}(\omega_1) & \ldots & S_1^{(N)}(\omega_1) \\ \vdots & \vdots & \vdots \\ S_1^{(0)}(\omega_m) & \ldots & S_1^{(N)}(\omega_m) \end{bmatrix} \begin{bmatrix} \theta_0 \\ \vdots \\ \theta_N \end{bmatrix}. \tag{1}
$$

In shorthand we write $X = S_1 \theta$ where S_1 is the $m \times (N + 1)$ matrix of date 1 security payoffs. Note that θ_j represents the number of units of the jth security purchased at date 0. We call θ the **replicating portfolio**.

Example 1 (An Attainable Claim)

Consider the one-period model below where there are 4 possible states of nature and 3 securities, i.e. $m = 4$ and $N = 2$. At $t = 1$ and state ω_3, for example, the values of the 3 securities are 1.03 , 2 and 4, respectively.
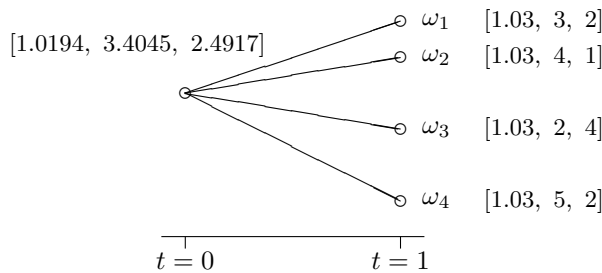
The claim $X = [7.47 \ \ 6.97 \ \ 9.97 \ \ 10.47]^T$ is an attainable claim since $X = S_1 \theta$ where $\theta = [-1 \ \ 1.5 \ \ 2]^T$ is a replicating portfolio for X.

---

[1] This is often stated as assuming that "there is no free lunch".
[2] d_1 and d_2 are therefore $m \times 1$ vectors.

Note that the date $t = 0$ cost of the three securities has nothing to do with whether or not a claim is attainable. We can now give a more formal definition of arbitrage in our one-period models.

Definition 4 *A type A **arbitrage** is a trading strategy, θ, such that $S_0\theta < 0$ and $S_1\theta = 0$. A type B* **arbitrage** *is a trading strategy, θ, such that $S_0\theta \le 0$, $S_1\theta \ge 0$ and $S_1\theta \ne 0$.*

Note for example, that if $S_0\theta < 0$ then θ has *negative* cost and therefore produces an immediate positive reward if purchased at $t = 0$.

Definition 5 *We say that a vector $\pi = (\pi_1, \ldots, \pi_m) > 0$ is a vector of **state prices** if the date $t = 0$ price, P, of any attainable security, X, satisfies*

$$P = \sum_{k=1}^{m} \pi_k X(\omega_k). \tag{2}$$

We call π_k the kth state[3] price.

Remark 1 *It is important to note that in principle there might be many state price vectors. If the kth elementary security is attainable, then it's price must be π_k and the kth component of all possible state price vectors must therefore coincide. Otherwise an arbitrage opportunity would exist.*

Example 2 (State Prices)

Returning to the model of Example 1 we can easily check[4] that
$[\pi_1 \ \pi_2 \ \pi_3 \ \pi_4]^T = [0.0737 \ 0.1910 \ 0.3705 \ 0.3357]^T$ is a vector of state prices. More generally, however, we can check that

$$
\begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.3102 \\ 0.4113 \\ 0.2682 \end{bmatrix} + \epsilon \begin{bmatrix} 0.7372 \\ -0.5898 \\ -0.2949 \\ 0.1474 \end{bmatrix}
$$

is also a vector of state prices for any ϵ such $\pi_i > 0$ for $1 \le i \le 4$. ∎

Deflating by the Numeraire Security

Let us recall that there are $N + 1$ securities and that $S_1^{(i)}(\omega_j)$ denotes the date $t = 1$ price of the ith security in state ω_j. The date $t = 0$ price of the ith security is denoted by $S_0^{(i)}$.

Definition 6 *A **numeraire security** is a security with a strictly positive price at all times, t.*

It is often convenient to express the price of a security in units of a chosen numeraire. For example, if the nth security is the numeraire security, then we define

$$\overline{S}_t^{(i)}(\omega_j) := \frac{S_t^{(i)}(\omega_j)}{S_t^{(n)}(\omega_j)}$$

to be the date t, state ω_j price (in units of the numeraire security) of the ith security. We say that we are *deflating* by the nth or numeraire security. Note that the deflated price of the numeraire security is always constant and equal to 1.

---

[3] We insist that each π_j is strictly positive as later we will want to prove that the existence of state prices is equivalent to the absence of arbitrage.

[4] See Appendix A for a brief description of how to find such a vector of state prices.

Definition 7 *The **cash account** is a particular security that earns interest at the risk-free rate of interest. In a single period model, the date $t = 1$ value of the cash account is $1 + r$ (assuming that \$1 had been deposited at date $t = 0$), regardless of the terminal state and where r is the one-period interest rate that prevailed at $t = 0$.*

In practice, we often deflate by the cash account *if* it is available. Note that deflating by the cash account is then equivalent to the usual process of discounting. We will use the zeroth security with price process, $S_t^{(0)}$, to denote the cash account whenever it is available.

Example 3 (**Numeraire and Cash Account**)

Note that any security in Example 1 could serve as a numeraire security since each of the 3 securities has a strictly positive price process. It is also clear that the zeroth security in that example is actually the cash account. ∎

Equivalent Martingale Measure (EMM)

We assume that we have chosen a specific numeraire security with price process, $S_t^{(n)}$.

Definition 8 *An **equivalent martingale measure** (EMM) or **risk-neutral probability measure** is a set of probabilities, $Q = (q_1, \ldots, q_m)$ such that*

1. *$q_k > 0$ for all k.*

2. *The deflated security prices are martingales. That is*

$$\overline{S}_0^{(i)} \; := \; \frac{S_0^{(i)}}{S_0^{(n)}} \; = \; \mathrm{E}_0^Q \left[\frac{S_1^{(i)}}{S_1^{(n)}} \right] \; =: \; \mathrm{E}_0^Q \left[\overline{S}_1^{(i)} \right]$$

for all i where $\mathrm{E}_0^Q[.]$ denotes expectation with respect to the risk-neutral probability measure, Q.

Remark 2 *Note that a set of risk-neutral probabilities, or EMM, is specific to the chosen numeraire security, $S_t^{(n)}$. In fact it would be more accurate to speak of an EMM-numeraire pair.*

Complete Markets

We now assume that there are no arbitrage opportunities. If there is a full set of m elementary securities available (i.e. they are all attainable), then we can use the state prices to compute the date $t = 0$ price, S_0, of *any* security. To see this, let $x = (x_1, \ldots, x_m)$ be the vector of possible date $t = 1$ payoffs of a particular security. We may then write

$$x \; = \; \sum_{i=1}^{m} x_i e_i$$

and use linear pricing to obtain $S_0 = \sum_{i=1}^{m} x_i \pi_i$.

If a full set of elementary securities exists, then as we have just seen, we can construct and price every possible security. We have the following definition.

Definition 9 *If every random variable X is attainable, then we say that we have a **complete market**. Otherwise we have an **incomplete market**.*

Note that if a full set of elementary securities is available, then the market is complete.

Exercise 1 *Is the model of Example 1 complete or incomplete?*

Martingale Pricing Theory: Single-Period Models

We are now ready to derive the main results of martingale pricing theory for single period models.

Proposition 1 *If an equivalent martingale measure, Q, exists, then there can be no arbitrage opportunities.*

Exercise 2 *Prove Proposition 1.*

Exercise 3 *Convince yourself that if we did not insist on each q_k being strictly positive in Definition 8 then Proposition 1 would not hold.*

Theorem 2 *Assume there is a security with a strictly positive price process[5], $S_t^{(n)}$. If there is a set of positive state prices, then a risk-neutral probability measure, Q, exists with $S_t^{(n)}$ as the numeraire security. Moreover, there is a one-to-one correspondence between sets of positive state prices and risk-neutral probability measures.*

Proof: Suppose a set of positive state prices, $\pi = (\pi_1, \ldots, \pi_m)$, exists. For all j we then have (by linear pricing)

$$
\begin{aligned}
S_0^{(j)} &= \sum_{k=1}^m \pi_k S_1^{(j)}(\omega_k) \\
&= \left(\sum_{l=1}^m \pi_l S_1^{(n)}(\omega_l) \right) \sum_{k=1}^m \frac{\pi_k S_1^{(n)}(\omega_k)}{\sum_{l=1}^m \pi_l S_1^{(n)}(\omega_l)} \frac{S_1^{(j)}(\omega_k)}{S_1^{(n)}(\omega_k)}.
\end{aligned}
\tag{3}
$$

Now observe that $\sum_{l=1}^m \pi_l S_1^{(n)}(\omega_l) = S_0^{(n)}$ and that if we define

$$
q_k := \frac{\pi_k S_1^{(n)}(\omega_k)}{\sum_{l=1}^m \pi_l S_1^{(n)}(\omega_l)},
\tag{4}
$$

then $Q := (q_1, \ldots, q_m)$ defines a probability measure. Equation (3) then implies

$$
\frac{S_0^{(j)}}{S_0^{(n)}} = \sum_{k=1}^m q_k \frac{S_1^{(j)}(\omega_k)}{S_1^{(n)}(\omega_k)} = \mathrm{E}_0^Q \left[\frac{S_1^{(j)}}{S_1^{(n)}} \right]
\tag{5}
$$

and so Q is a risk-neutral probability measure, as desired.

The one-to-one correspondence between sets of positive state prices and risk-neutral probability measures is clear from (4). ∎

Remark 3 *The true real-world probabilities, $P = (p_1, \ldots, p_m)$, are almost irrelevant here. The only connection between P and Q is that they must be* equivalent. *That is $p_k > 0 \Leftrightarrow q_k > 0$. Note that in the statement of Theorem 2 we assumed that the set of state prices was positive. This and equation (4) implied that each $q_k > 0$ so that Q is indeed equivalent to P. (Recall it was assumed at the beginning that each $p_k > 0$.)*

Absence of Arbitrage ≡ Existence of Positive State Prices ≡ Existence of EMM

Before we establish the main result, we first need the following theorem which we will prove using the theory of linear programming.

Theorem 3 *Let A be an $m \times n$ matrix and suppose that the matrix equation $Ax = p$ for $p \geq 0$ cannot be solved except for the case $p = 0$. Then there exists a vector $y > 0$ such that $A^T y = 0$.*

Proof: We will use the following result from the theory of linear programming:

---

[5]If a cash account is available then this assumption is automatically satisfied.

> *If a primal linear program, P, is infeasible then its dual linear program, D, is either also infeasible, or it has an unbounded objective function.*

Now consider the following sequence of linear programs, $\mathbf{P}_i$, for $i = 1, \ldots m$:

$$\min \quad 0^T x \quad (\mathbf{P}_i)$$
$$\text{subject to} \quad Ax \geq \epsilon_i$$

where ϵ_i has a 1 in the ith position and 0 everywhere else. The dual, $\mathbf{D}_i$, of each primal problem, $\mathbf{P}_i$ is

$$\max \quad y_i \quad (\mathbf{D}_i)$$
$$\text{subject to} \quad A^T y = 0$$
$$y \geq 0$$

By assumption, each of the primal problems, $\mathbf{P}_i$, is infeasible. It is also clear that each of the dual problems are feasible (take y equal to the zero vector). By the LP result above, it is therefore the case that each $\mathbf{D}_i$ has an unbounded objective function. This implies, in particular, that corresponding to each $\mathbf{D}_i$, there exists a vector $y^i \geq 0$ with $A^T y^i = 0$ and $y_i^i > 0$, i.e. the ith component of y^i is strictly positive. Now taking

$$y^* \; = \; \sum_{i=1}^{m} y^i,$$

we clearly see that $A^T y^* = 0$ and y^* is strictly positive. ■

We now prove the following important result regarding absence of arbitrage and existence of positive state prices.

Theorem 4 *In the one-period model there is no arbitrage if and only if there exists a set of positive state prices.*

Proof: (i) Suppose first that there is a set of positive state prices, $\pi := (\pi_1, \ldots, \pi_m)$. If $x \geq 0$ is the date $t = 1$ payoff of an attainable security, then the price, S, of the security is given by

$$S \; = \; \sum_{j=1}^{m} \pi_j x_j \; \geq \; 0.$$

If some $x_j > 0$ then $S > 0$, and if $x = 0$ then $S = 0$. Therefore[6] there is no arbitrage opportunity.

(ii) Suppose now that there is no arbitrage. Consider the $(m+1) \times (N+1)$ matrix, A, defined by

$$A = \begin{bmatrix} S_1^{(0)}(\omega_1) & \ldots & S_1^{(N)}(\omega_1) \\ \vdots & \vdots & \vdots \\ S_1^{(0)}(\omega_m) & \ldots & S_1^{(N)}(\omega_m) \\ -S_0^{(0)} & \ldots & -S_0^{(N)} \end{bmatrix}$$

and observe (convince yourself) that the absence of arbitrage opportunities implies the non-existence of an N-vector, x, with

$$Ax \geq 0 \quad \text{and} \quad Ax \neq 0.$$

In this context, the ith component of x represents the number of units of the ith security that was purchased or sold at $t = 0$. Theorem 3 then assures us of the existence of a strictly positive vector π that satisfies $A^T \pi = 0$. We can normalize π so that $\pi_{m+1} = 1$ and we then obtain

$$S_0^{(i)} \; = \; \sum_{j=1}^{m} \pi_j S_1^{(i)}(\omega_j).$$

---

[6]This result is really the same as Proposition 1 in light of the equivalence of positive state prices and risk-neutral probabilities that was shown in Theorem 2.

That is, $\pi := (\pi_1, \ldots, \pi_m)$ is a vector of positive state prices. ∎

Theorems 2 and 4 imply the following theorem which encapsulates the principal results for our single-period model.

Theorem 5 *Assume there exists a security with strictly positive price process. Then the absence of arbitrage, the existence of positive state prices and the existence of an EMM, Q, are all equivalent.*

Example 4 (An Arbitrage-Free Market)

The model in Example 1 is arbitrage-free since we saw in Example 2 that a vector of positive state prices exists for this market ∎

Example 5 (A Market with Arbitrage Opportunities)

Consider the one-period, 2-state model below.
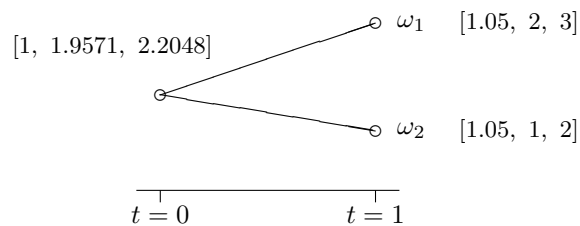
No positive state price vector exists for this model so there must be an arbitrage opportunity. ∎

Exercise 4 *Find an arbitrage strategy, θ, in the model of Example 5.*

Complete Markets ≡ Unique EMM

We now turn to the important question of completeness and we have the following formulation that is equivalent to Definition 9. We state this as a theorem but the proof is immediate given our original definition.

Theorem 6 *Assume that there are no arbitrage opportunities. Then the market is complete if and only if the matrix of date $t = 1$ payoffs, S_1, has rank m.*

Example 6 (An Incomplete Market)

The model of Example 1 is arbitrage-free by Theorem 4 since we saw in Example 2 that a vector of positive state prices exists for this model. However the model is incomplete since the rank of the payoff matrix, S_1, can be at most 3 which is less than the number of possible states, 4. ∎

Example 7 (A Complete Market)

Consider the one-period model below where there are 4 possible states of nature and 4 securities, i.e. $m = 4$ and $N = 3$.
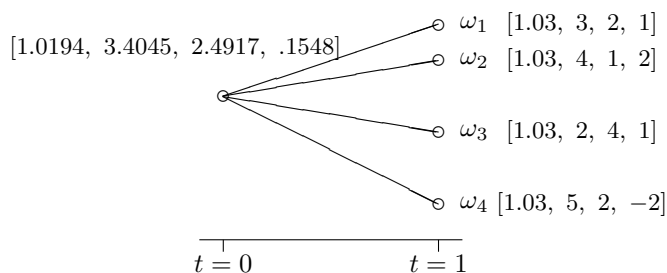
We can easily check[7] that $\text{rank}(S_1) = 4 = m$, so that this model is indeed complete. (We can also confirm that this model is arbitrage free by Theorem 4 and noting that the state price vector of Example 4 is a (unique) state price vector here.) ▌

Exercise 5 *Show that if a market is incomplete, then at least one elementary security is not attainable.*

Suppose now that the market is incomplete so that at least one elementary security, say e_j, is not attainable. By Theorem 4, however, we can still define a set of positive state prices if there are no arbitrage opportunities. In particular, we can define the state price $\pi_j > 0$ even though the jth elementary security is not attainable. A number of interesting questions arise regarding the uniqueness of state price and risk-neutral probability measures, and whether or not markets are complete. The following theorem addresses these questions.

Theorem 7 *Assume there exists a security with strictly positive price process and there are no arbitrage opportunities. Then the market is complete if and only if there exists exactly one equivalent martingale measure (or equivalently, one vector of positive state prices).*

Proof: (i) Suppose first that the market is complete. Then there exists a unique set of positive state prices, and therefore by Theorem 2, a unique risk-neutral probability measure.

(ii) Suppose now that there exists exactly one risk-neutral probability measure. We will derive a contradiction by assuming that the market is *not* complete. Suppose then that the random variable $X = (X_1, \ldots, X_m)$ is not attainable. This implies that there does not exist an $(N+1)$-vector, θ, such that $S_1\theta = X$.

Therefore, using a technique[8] similar to that in the proof of Theorem 3, we can show there exists a vector, h, such that $h^T S_1 = 0$ and $h^T X > 0$. Let Q be some risk-neutral probability measure[9] and define $\widehat{Q}$ by

$$\widehat{Q}(\omega_j) = Q(\omega_j) + \lambda h_j S_1^{(n)}(\omega_j)$$

where $S_1^{(n)}$ is the date $t = 1$ price of the numeraire security and $\lambda > 0$ is chosen so that $\widehat{Q}(\omega_j) > 0$ for all j. Note $\widehat{Q}$ is a probability measure since $h^T S_1 = 0$ implies $\sum_j h_j S_1^{(n)}(\omega_j) = 0$. It is also easy to see (check) that since Q is an equivalent probability measure, so too is $\widehat{Q}$ and $Q \neq \widehat{Q}$. Therefore we have a contradiction and so the market must be complete. ∎

Remark: It is easy to check that the price of X under Q is different to the price of X under $\widehat{Q}$ in Theorem 7. This could not be the case if X was attainable. Why?

[7]This is a trivial task using *Matlab* or other suitable software.
[8]Consider the linear program: $\min_\theta 0$ subject to $S_1\theta = X$ and formulate its dual.
[9]We know such a Q exists since there are no arbitrage opportunities.

Notation and Definitions for Multi-Period Models

Before extending our single-period results to multi-period models, we first need to extend some of our single-period definitions and introduce the concept of *trading strategies* and *self-financing trading strategies*. We will assume[10] for now that none of the securities in our multi-period models pay dividends. (We will return to the case where they do pay dividends at the end of these notes.)

As before we will assume that there are $N+1$ securities, m possible states of nature and that the true probability measure is denoted by $P = (p_1, \ldots, p_m)$. We assume that the investment horizon is $[0, T]$ and that there are a total of T trading periods. Securities may therefore be purchased or sold at any date t for $t = 0, 1, \ldots, T-1$. Figure 1 below shows a typical multi-period model with $T = 2$ and $m = 9$ possible states. The manner in which information is revealed as time elapses is clear from this model. For example, at node $I_1^{4,5}$ the available information tells us that the true state of the world is either ω_4 or ω_5. In particular, no other state is possible at $I_1^{4,5}$.
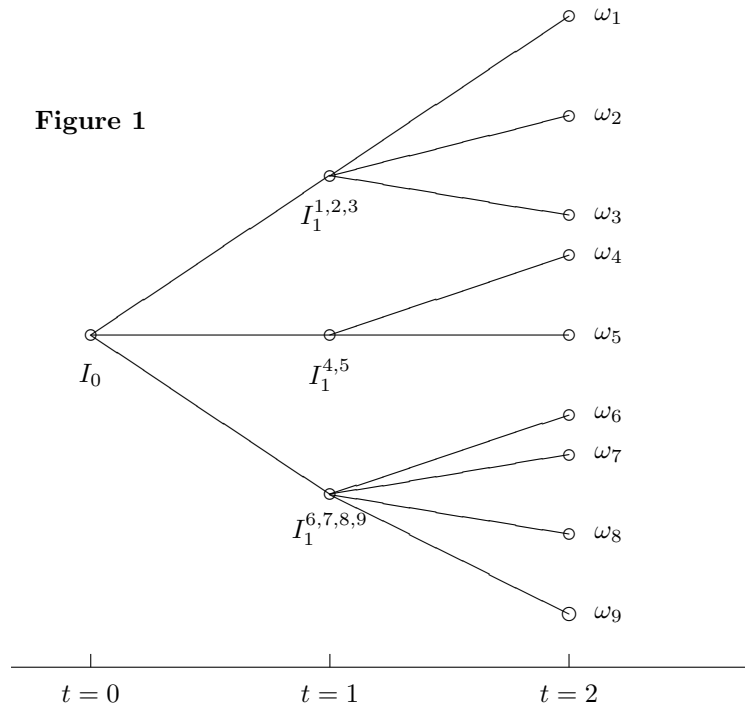
Figure 1

Note that the multi-period model is composed of a series of single-period models. At date $t = 0$ in Figure 1, for example, there is a single one-period model corresponding to node I_0. Similarly at date $t = 1$ there are three possible one-period models corresponding to nodes $I_1^{1,2,3}$, $I_1^{4,5}$ and $I_1^{6,7,8,9}$, respectively. The particular one-period model that prevails at $t = 1$ will depend on the true state of nature. Given a probability measure, $P = (p_1, \ldots, p_m)$, we can easily compute the conditional probabilities of each state. In Figure 1, for example, $\mathbf{P}(\omega_1 | I_1^{1,2,3}) = p_1/(p_1 + p_2 + p_3)$. These conditional probabilities can be interpreted as probabilities in the corresponding single-period models. For example, $p_1 = \mathbf{P}(I_1^{1,2,3} | I_0) \ \mathbf{P}(\omega_1 | I_1^{1,2,3})$. This observation (applied to risk-neutral probabilities) will allow us to easily generalize our single-period results to multi-period models.

---

[10]This assumption can easily be relaxed at the expense of extra book-keeping. The existence of an equivalent martingale measure, Q, under no arbitrage assumptions will still hold as will the results regarding market completeness. Deflated security prices will no longer be Q-martingales, however, if they pay dividends. Instead, deflated gains processes will be Q-martingales where the gains process of a security at time t is the time t value of the security plus dividends that have been paid up to time t.

Trading Strategies and Self-Financing Trading Strategies

Definition 10 *A* **predictable** *stochastic process is a process whose time t value, X_t say, is known at time $t - 1$ given all the information that is available at time $t - 1$.*

Definition 11 *A* **trading strategy** *is a vector, $\theta_t = (\theta_t^{(0)}(\omega), \ldots, \theta_t^{(N)}(\omega))$, of predictable stochastic processes that describes the number of units of each security held just before trading at time t, as a function of t and ω.*

For example, $\theta_t^{(i)}(\omega)$ is the number of units of the ith security held[11] between times $t - 1$ and t in state ω. We will sometimes write $\theta_t^{(i)}$, omitting the explicit dependence on ω. Note that θ_t is known at date $t - 1$ as we insisted in Definition 11 that θ_t be predictable. In our financial context, 'predictable' means that θ_t cannot depend on information that is not yet available at time $t - 1$.

Example 8 (**Constraints Imposed by Predictability of Trading Strategies**)
In Figure 1 it must be the case that for all $i = 0, \ldots, N$,

$$
\begin{aligned}
\theta_2^{(i)}(\omega_1) &= \theta_2^{(i)}(\omega_2) = \theta_2^{(i)}(\omega_3) \\
\theta_2^{(i)}(\omega_4) &= \theta_2^{(i)}(\omega_5) \\
\theta_2^{(i)}(\omega_6) &= \theta_2^{(i)}(\omega_7) = \theta_2^{(i)}(\omega_8) = \theta_2^{(i)}(\omega_9).
\end{aligned}
$$

∎

Exercise 6 *What can you say about the relationship between the $\theta_1^{(i)}(\omega_j)$'s for $j = 1, \ldots, m$?*

Definition 12 *The* **value process**, *$V_t(\theta)$, associated with a trading strategy, θ_t, is defined by*

$$
V_t = \begin{cases}
\sum_{i=0}^{N} \theta_1^{(i)} S_0^{(i)} & \text{for } t = 0 \\[2mm]
\sum_{i=0}^{N} \theta_t^{(i)} S_t^{(i)} & \text{for } t \geq 1.
\end{cases}
$$

Definition 13 *A* **self-financing** *trading strategy is a strategy, θ_t, where changes in V_t are due entirely to trading gains or losses, rather than the addition or withdrawal of cash funds. In particular, a self-financing strategy satisfies*

$$
V_t = \sum_{i=0}^{N} \theta_{t+1}^{(i)} S_t^{(i)} \quad \text{for } t = 1, \ldots, T - 1.
$$

Definition 13 states that the value of a self-financing portfolio just before trading or *re-balancing* is equal to the value of the portfolio just after trading, i.e., no additional funds have been deposited or withdrawn.

Exercise 7 *Show that if a trading strategy, θ_t, is self-financing then the corresponding value process, V_t, satisfies*

$$
V_{t+1} - V_t = \sum_{i=0}^{N} \theta_{t+1}^{(i)} \left(S_{t+1}^{(i)} - S_t^{(i)} \right).
$$

Clearly then changes in the value of the portfolio are due to capital gains or losses and are not due to the injection or withdrawal of funds.

We can now extend the one-period definitions of arbitrage opportunities, attainable claims and completeness.

---

[11]If $\theta_t^{(i)}$ is negative then it corresponds to the number of units sold short.

Arbitrage

Definition 14 *We define a* **type A arbitrage** *opportunity to be a self-financing trading strategy, θ_t, such that $V_0(\theta) < 0$ and $V_T(\theta) = 0$. Similarly, a* **type B arbitrage** *opportunity is defined to be a self-financing trading strategy, θ_t, such that $V_0(\theta) = 0$, $V_T(\theta) \geq 0$ and $\mathrm{E}_0^P[V_T(\theta)] > 0$.*

Attainability and Complete Markets

Definition 15 *A* **contingent claim**, *C, is a random variable whose value at time T is known at that time given the information available then. It can be interpreted as the time T value of a security (or, depending on the context, the time t value if this value is known by time $t < T$).*

Definition 16 *We say that the contingent claim C is* **attainable** *if there exists a self-financing trading strategy, θ_t, whose value process, V_T, satisfies $V_T = C$.*

Note that the value of the claim, C, in Definition 16 must equal the initial value of the replicating portfolio, V_0, if there are no arbitrage opportunities available. We can now extend our definition of completeness.

Definition 17 *We say that the market is* **complete** *if every contingent claim is attainable. Otherwise the market is said to be* **incomplete***.*

Note that the above definitions of attainability and (in)completeness are consistent with our definitions for single-period models. With our definitions of a numeraire security and the cash account remaining unchanged, we can now define what we mean by an **equivalent martingale measure** (EMM), or set of **risk-neutral probabilities**.

Equivalent Martingale Measures (EMMs)

We assume again that we have in mind a specific numeraire security with price process, $S_t^{(n)}$.

Definition 18 *An* **equivalent martingale measure** *(EMM), $Q = (q_1, \ldots, q_m)$, is a set of probabilities such that*

1. *$q_i > 0$ for all $i = 1, \ldots, m$.*

2. *The deflated security prices are martingales. That is*

$$\overline{S}_t^{(i)} \; := \; \frac{S_t^{(i)}}{S_t^{(n)}} \; = \; \mathrm{E}_t^Q \left[\frac{S_{t+s}^{(i)}}{S_{t+s}^{(n)}} \right] \; =: \; \mathrm{E}_t^Q \left[\overline{S}_{t+s}^{(i)} \right]$$

for $s, t \geq 0$, for all $i = 0, \ldots, N$, and where $\mathrm{E}_t^Q[.]$ denotes the expectation under Q conditional on information available at time t. (We also refer to Q as a set of **risk-neutral probabilities***.)*

Martingale Pricing Theory: Multi-Period Models

We will now generalize the results for single-period models to multi-period models. This is easily done using our single-period results and requires very little extra work.

Absence of Arbitrage ≡ Existence of EMM

We begin with two propositions that enable us to generalize Proposition 1.

Proposition 8 *If an equivalent martingale measure, Q, exists, then the* deflated *value process, V_t, of any self-financing trading strategy is a Q-martingale.*

Proof: Let θ_t be the self-financing trading strategy and let $\overline{V}_{t+1} := V_{t+1}/S_{t+1}^{(n)}$ denote the deflated value process. We then have

$$
\begin{aligned}
\mathrm{E}_t^Q\left[\overline{V}_{t+1}\right] &= \mathrm{E}_t^Q\left[\sum_{i=0}^N \theta_{t+1}^{(i)} \overline{S}_{t+1}^{(i)}\right] \\
&= \sum_{i=0}^N \theta_{t+1}^{(i)} \mathrm{E}_t^Q\left[\overline{S}_{t+1}^{(i)}\right] \\
&= \sum_{i=0}^N \theta_{t+1}^{(i)} \overline{S}_t^{(i)} \\
&= \overline{V}_t
\end{aligned}
$$

demonstrating that $\overline{V}_t$ is indeed a martingale, as claimed. ∎

Remark 4 *Note that Proposition 8 implies that the deflated price of any attainable security can be computed as the Q-expectation of the terminal deflated value of the security.*

Proposition 9 *If an equivalent martingale measure, Q, exists, then there can be no arbitrage opportunities.*

Proof: The proof follows almost immediately from Proposition 8. ∎

We can now now state our principal result for multi-period models, assuming as usual that a numeraire security exists.

Theorem 10 *In the multi-period model there is no arbitrage if and only if there exists an EMM, Q.*

Proof: (i) Suppose first that there is no arbitrage as defined by Definition 14. Then we can easily argue there is no arbitrage (as defined by Definition 4) in any of the embedded one-period models. Theorem 5 then implies that each of the the embedded one-period models has a set of risk-neutral probabilities. By multiplying these probabilities as described in the paragraph immediately following Figure 1, we can construct an EMM, Q, as defined by Definition 18.

(ii) Suppose there exists an EMM, Q. Then Proposition 9 gives the result. ∎

Complete Markets ≡ Unique EMM

As was the case with single-period models, it is also true that multi-period models are complete if and only if the EMM is unique. (We are assuming here that there is no arbitrage so that an EMM is guaranteed to exist.)

Proposition 11 *The market is complete if and only if every embedded one-period model is complete.*

Exercise 8 *Prove Proposition 11*

We have the following theorem.

Theorem 12 *Assume there exists a security with strictly positive price process and that there are no arbitrage opportunities. Then the market is complete if and only if there exists exactly one risk-neutral martingale measure, Q.*

Proof: (i) Suppose the market is complete. Then by Proposition 11 every embedded one-period model is complete so we can apply Theorem 7 to show that the EMM, Q, (which must exist since there is no arbitrage) is unique.

(ii) Suppose now Q is unique. Then the risk-neutral probability measure corresponding to each one-period model is also unique. Now apply Theorem 7 again to obtain that the multi-period model is complete. ∎

State Prices

As in the single-period models, we also have an equivalence between equivalent martingale measures, Q, and sets of state prices. We will use $\pi_t^{\{t+s\}}(\Lambda)$ to denote the time t price of a security that pays \$1 at time $t+s$ in the event that $\omega \in \Lambda$. We are implicity assuming that we can tell at time $t+s$ whether or not $\omega \in \Lambda$. In Figure 1, for example, $\pi_0^{\{1\}}(\omega_4, \omega_5)$ is a valid expression whereas $\pi_0^{\{1\}}(\omega_4)$ is not.

Why is Absence of Arbitrage ≡ Existence of an EMM ?

Let us now develop some intuition for why the discounted price process, S_j^t/S_i^t, should be a Q-martingale if there are no arbitrage opportunities. First, it is clear that we should not expect a non-deflated price process to be a martingale under the true probability measure, P. After all, if a cash account is available, then it will always grow in value as long as the risk-free rate of interest is positive. It cannot therefore be a P-martingale.

It makes sense then that we should compare the price processes of securities *relative* to one another rather than examine them on an absolute basis. This is why we deflate by some positive security. Even after deflating, however, it is still not reasonable to expect deflated price processes to be P-martingales. After all, some securities are riskier than others and since investors are generally risk averse it makes sense that riskier securities should have higher expected rates of return. However, if we change to an equivalent martingale measure, Q, where probabilities are adjusted to reflect the riskiness of the various states, then we can expect deflated price processes to be martingales under Q. The vital point here is that each q_k must be strictly positive since we have assumed that each p_k is strictly positive.

As a further aid to developing some intuition, we might consider the following three scenarios:

Scenario 1

Imagine a multi-period model with two assets, both of whose price processes, X_t and Y_t say, are *deterministic* and positive. Convince yourself that in this model it must be the case that X_t/Y_t is a martingale if there are to be no arbitrage opportunities. (A martingale in a deterministic model must be a constant process. Moreover, in a deterministic model a risk-neutral measure, Q, must coincide with the true probability measure, P.)

Scenario 2

Generalize scenario 1 to a deterministic model with n assets, each of which has a positive price process. Note that you can choose to deflate by any process you choose. Again it should be clear that deflated security price processes are (deterministic) martingales.

Scenario 3

Now consider a one period stochastic model that runs from date $t = 0$ to date $t = 1$. There are only two possible outcomes at date $t = 1$ and we assume there are only two assets, $S_t^{(1)}$ and $S_t^{(2)}$. Again, convince yourself that if there are to be no arbitrage opportunities, then it must be the case that there is probability measure, Q, such that $S_t^{(1)}/S_t^{(2)}$ is a Q-martingale. Of course we have already given a proof of this result (and much more), but it helps intuition to look at this very simple case and see directly why an EMM must exist if there is no arbitrage.

Once these simple cases are understood, it is no longer surprising that the result (equivalence of absence of arbitrage and existence of an EMM) extends to multiple periods, multiple assets and even continuous time. You can also see that the numeraire asset can actually be any asset with a strictly positive price process. Of course we commonly deflate by the cash account in practice as it is often very convenient to do so, but it is important to note that our results hold if we deflate by other positive price processes. We now consider some examples that will use the various concepts and results that we have developed.

Example 9 (A Complete Market)

There are two time periods and three securities. We will use $S_t^{(i)}(\omega_k)$ to denote the value of the ith security in state ω_k at date t for $i = 0, 1, 2$, for $k = 1, \ldots, 9$ and for $t = 0, 1, 2$. These values are given in the tree above, so for example, the value of the 0th security at date $t = 2$ satisfies

$$S_2^{(0)}(\omega_k) = \begin{cases} 1.1235, & \text{for} & k = 1, 2, 3 \\ 1.1025, & \text{for} & k = 4, 5, 6 \\ 1.0815, & \text{for} & k = 7, 8, 9. \end{cases}$$

Note that the zeroth security is equivalent to a cash account that earns money at the risk-free interest rate. We will use $R_t(\omega_k)$ to denote the gross risk-free rate at date t, state ω_k. The properties of the cash account imply that R satisfies $R_0(\omega_k) = 1.05$ for all i, and[12]

$$R_1(\omega_k) = \begin{cases} 1.07, & \text{for} & k = 1, 2, 3 \\ 1.05, & \text{for} & k = 4, 5, 6 \\ 1.03, & \text{for} & k = 7, 8, 9. \end{cases}$$

---

[12]Of course the value of R_2 is unknown and irrelevant as it would only apply to cash-flows between $t = 2$ and $t = 3$.

Key: $[S_t^{(0)}, S_t^{(1)}, S_t^{(2)}]$
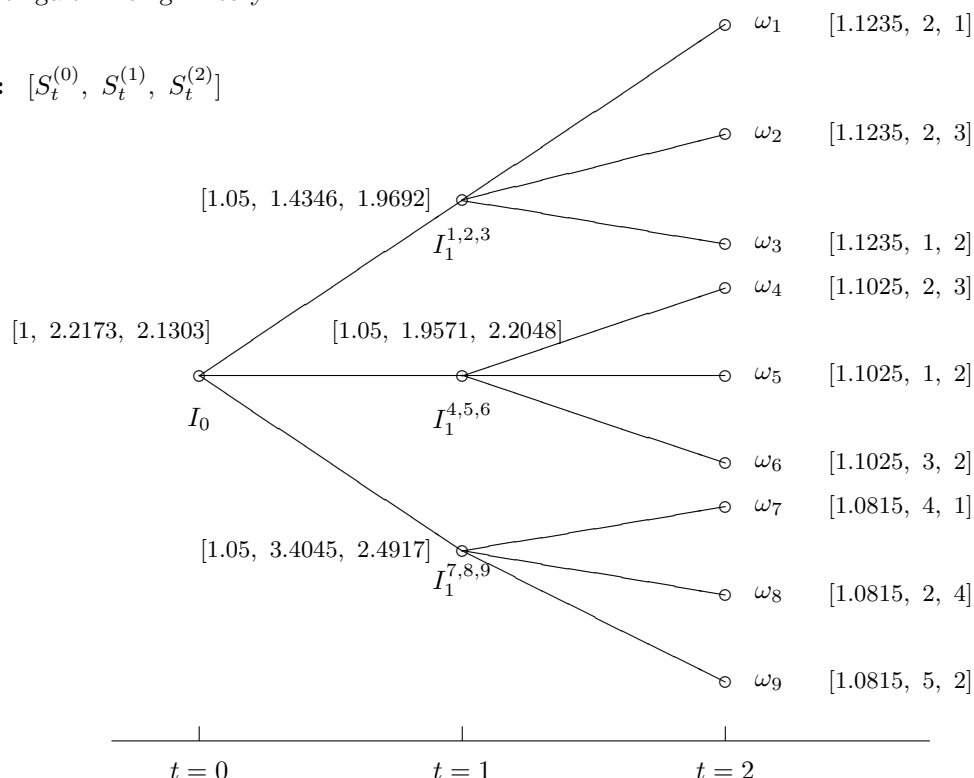
Questions

(1) Are there any arbitrage opportunities in this market?

(2) If not, is this a complete or incomplete market?

(3) Compute the state prices in this model.

(4) Compute the risk-neutral or martingale probabilities when we discount by the cash account, i.e., the zero[th] security.

(5) Compute the risk-neutral probabilities (i.e. the martingale measure) when we discount by the *second* security.

(6) Using the state prices, find the price of a call option on the the first asset with strike $k = 2$ and expiration date $t = 2$.

(7) Confirm your answer in (6) by recomputing the option price using the martingale measure of (5).

Solutions

(1) No, because there exists a set of state prices or equivalently, risk-neutral probabilities. (Recall that by definition, risk neutral probabilities are strictly positive.) We can confirm this by checking that in each embedded one-period model there is a strictly positive solution to $S_t = \pi_t S_{t+1}$ where S_t is the vector of security prices at a particular time t node and S_{t+1} is the *matrix* of date $t = 1$ prices at the successor nodes.

(2) Complete, because we have a unique set of state prices or equivalently a unique equivalent martingale measure. We can check this by confirming that each embedded one-period model has a payoff matrix of full rank.

(3) First compute (how?) the prices at date 1 of $1 to be paid in each of the terminal states at date 2. These are the state prices at date 1, $\pi_1^{\{2\}}$, and we find

$$\pi_1^{\{2\}}(\omega_1) = .2 \; , \; \pi_1^{\{2\}}(\omega_2) = .3 \; \text{ and } \; \pi_1^{\{2\}}(\omega_3) = 0.4346 \; \text{ at } \; I_1^{1,2,3}$$

$$\pi_1^{\{2\}}(\omega_4) = .3 \; , \; \pi_1^{\{2\}}(\omega_5) = .3 \; \text{ and } \; \pi_1^{\{2\}}(\omega_6) = 0.3524 \; \text{ at } \; I_1^{4,5,6}$$

$$\pi_1^{\{2\}}(\omega_7) = .25 \; , \; \pi_1^{\{2\}}(\omega_8) = .4 \; \text{ and } \; \pi_1^{\{2\}}(\omega_9 d) = 0.3209 \; \text{ at } \; I_1^{7,8,9}$$

The value at date 0 of \$1 at nodes $I_1^{1,2,3}$, $I_1^{4,5,6}$ and $I_1^{7,8,9}$, respectively, is given by

$$\pi_0^{\{1\}}(I_1^{1,2,3}) = .3 \; , \; \pi_0^{\{1\}}(I_1^{4,5,6}) = .3 \; \text{ and } \; \pi_0^{\{1\}}(I_1^{7,8,9}) = .3524.$$

Therefore the state prices at date $t = 0$ are (why?)

$$\pi_0^{\{2\}}(\omega_1) = .06, \; \pi_0^{\{2\}}(\omega_2) = .09, \; \pi_0^{\{2\}}(\omega_3) = 0.1304, \; \pi_0^{\{2\}}(\omega_4) = .09, \; \pi_0^{\{2\}}(\omega_5) = .09$$

$$\pi_0^{\{2\}}(\omega_6) = 0.1057, \; \pi_0^{\{2\}}(\omega_7) = 0.0881, \; \pi_0^{\{2\}}(\omega_8) = 0.1410, \; \pi_0^{\{2\}}(\omega_9) = 0.1131.$$

We can easily check that these state prices do indeed correctly price (subject to rounding errors) the three securities at date $t = 0$.

(**4**) When we deflate by the cash account the risk-neutral probabilities for the nine possible paths at time 0 may be computed using the expression

$$q_k = \frac{\pi_0^{\{2\}}(\omega_k) S_2^{(0)}(\omega_k)}{\sum \pi_0^{\{2\}}(\omega_j) S_2^{(0)}(\omega_j)}. \tag{6}$$

We have not shown that the expression in (6) is in fact correct, though note that it generalizes the one-period expression in (4).

Exercise 9 *Check that (6) is indeed correct. (You may do this by deriving it in exactly the same manner as (4). Alternatively, it may be derived by using equation (4) to compute the risk-neutral probabilities of the embedded one-period models and multiplying them appropriately to obtain the q_k's. Note that the risk-neutral probabilities in the one-period models are conditional risk-neutral probabilities of the multi-period model.)*

We therefore have

q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9
0.0674	0.1011	0.1465	0.0992	0.0992	0.1165	0.0953	0.1525	0.1223

(**5**) Similarly, when we deflate by the second asset, the risk-neutral probabilities are given by:

q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9
0.0282	0.1267	0.1224	0.1267	0.0845	0.0992	0.0414	0.2647	0.1062

(**6**) The payoffs of the call option and the state prices are given by:

State	1	2	3	4	5	6	7	8	9
Payoff	0	0	0	0	0	1	2	0	3
State Price	.06	.09	.1304	.09	.09	.1057	.0881	.1410	.1131

The price of the option is therefore (why?) given by $.1057 + (2 \times .0881) + (3 \times .1131) = .6212$.

(**7**) Using the risk-neutral probabilities when we deflate by the second asset we have:

State	1	2	3	4	5	6	7	8	9
Deflated Payoff	0	0	0	0	0	1/2	2/1	0	3/2
Risk-Neutral Probabilities	0.0282	0.1267	0.1224	0.1267	0.0845	0.0992	0.0414	0.2647	0.1062

The price of the option deflated by the initial price of the second asset is therefore given by $(.0992 \times .5) + (2 \times .0414) + (3/2) \times .1062 = .2917$. And so the option price is given by $.2917 \times 2.1303 = .6214$, which is the same answer (modulo rounding errors) as we obtained in (6). ∎

In Example 9 we computed the option price by working directly from the date $t = 2$ payoffs to the date $t = 0$ price. Another method for pricing derivative securities is to work backwards through the tree in the manner of *dynamic programming*, or *backwards iteration*. That is we first compute the price at the date $t = 1$ nodes and then use the date $t = 1$ price to compute the date $t = 0$ price. This technique of course, is also implemented using the risk-neutral probabilities or equivalently, the state prices.

Exercise 10 *Repeat Question 6 of Example 9, this time using dynamic programming to compute the option price.*

Remark 5 *While R_t was stochastic in Example 9, we still refer to it as a* risk-free *interest rate. This interpretation is valid since we know for certain at date i the date $i + 1$ value of $\$1$ invested in the cash account at date i.*

Example 10 (An Incomplete Market)

Consider the same tree as in Example 9 only now state ω_6 is a successor state to node $I_1^{6,7,8,9}$ instead of node $I_1^{4,5}$. We have also changed the payoff of the zeroth asset in this state so that our interpretation of the the zeroth asset as a cash account remains appropriate. The new tree is given on the next page.

Questions

(1) Is this model arbitrage free?

(2) Suppose the prices of the three securities were such that there were no arbitrage opportunities. Without bothering to compute such prices, do you think the model would then be a complete or incomplete model?

(3) Suppose again that the security prices were such that there were no arbitrage opportunities. Give a simple argument for why forward contracts are attainable. (We can therefore price them in this model.)

Solutions

(1) We know the absence of arbitrage is equivalent to the existence of positive state prices or, equivalently, risk-neutral probabilities. Moreover, if the model is arbitrage free then so is every one-period sub-market so all we need to do is see if we can construct positive state prices for each of the four one-period markets represented by the nodes $I_1^{1,2,3}$, $I_1^{4,5}$, $I_1^{6,7,8,9}$ and I_0.

First, it is clear that the one-period market beginning at $I_1^{1,2,3}$ is arbitrage-free since this is the same as the corresponding one-period model in Example 9. For the subproblem beginning at $I_1^{6,7,8,9}$ we can take (check) $[\pi_1^{\{2\}}(\omega_6) \ \pi_1^{\{2\}}(\omega_7) \ \pi_1^{\{2\}}(\omega_8) \ \pi_1^{\{2\}}(\omega_9)] = [0.0737 \ 0.1910 \ 0.3705 \ 0.3357]$ so this sub-market is also arbitrage free.

However, it is not possible to find a state price vector, $[\pi_1^{\{2\}}(\omega_4) \ \pi_1^{\{2\}}(\omega_5)]$, for the one-period market beginning at $I_1^{4,5}$. In particular, this implies there is an arbitrage opportunity there and so we can conclude[13] that the model is not arbitrage-free.

(2) The model is incomplete as the rank of the payoff matrix in the one-period model beginning at $I_1^{6,7,8,9}$ is less than 4.

(3) This is left as an exercise. (However we will return to this issue and the pricing of futures contracts in the *Martingale Pricing Applied to Forwards, Futures and Options* lecture notes.)

---

[13]You can check the one-period model beginning at node I_0 if you like!

Key: $[S_t^{(0)}, \; S_t^{(1)}, \; S_t^{(2)}]$
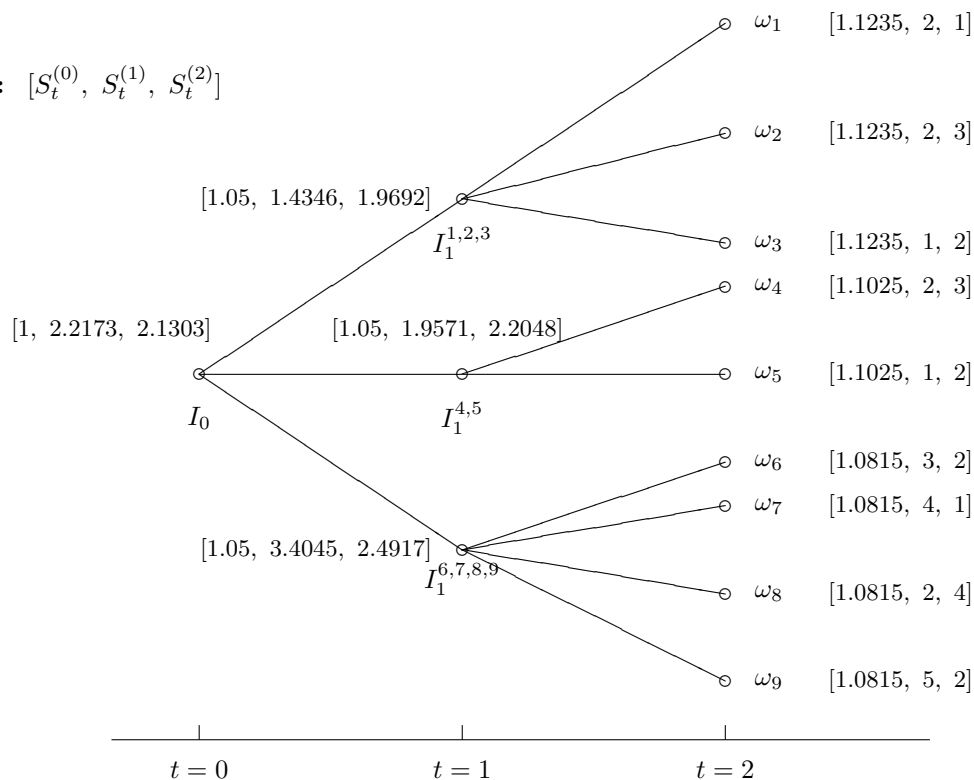
Exercise 11 *Find an arbitrage opportunity in the one-period model beginning at node $I_1^{4,5}$ in Example 10.*

Dividends and Intermediate Cash-Flows

Thus far, we have assumed that none of the securities pay intermediate cash-flows. An example of such a security is a dividend-paying stock. This is not an issue in the single period models since any such cash-flows are captured in the date $t = 1$ value of the securities. For multi-period models, however, we sometimes need to explicitly model these intermediate cash payments. All of the results that we have derived in these notes still go through, however, as long as we make suitable adjustments to our price processes and are careful with our bookkeeping. In particular, deflated *cumulative gains* processes rather than deflated security prices are now Q-martingales. The cumulative gain process, G_t, of a security at time t is equal to value of the security at time t plus accumulated cash payments that result from holding the security.

Consider our discrete-time, discrete-space framework where a particular security pays dividends. Then if the model is arbitrage-free there exists an EMM, Q, such that

$$\overline{S}_t \; = \; \mathrm{E}_t^Q \left[\sum_{j=t+1}^{t+s} \overline{D}_j \; + \; \overline{S}_{t+s} \right]$$

where D_j is the time j dividend that you receive if you hold one unit of the security, and S_t is its time t *ex-dividend* price. This result is easy to derive using our earlier results. All we have to do is view each dividend

as a separate security with S_t then interpreted as the price of the *portfolio* consisting of these individual securities as well as a security that is worth S_{t+s} at date $t + s$. The definitions of complete and incomplete markets are unchanged and the associated results we derived earlier still hold when we also account for the dividends in the various payoff matrices. For example, if θ_t is a self-financing strategy in a model with dividends then V_t, the corresponding value process, should satisfy

$$V_{t+1} - V_t \;=\; \sum_{i=0}^{N} \theta_{t+1}^{(i)} \left(S_{t+1}^{(i)} + D_{t+1}^{(i)} - S_t^{(i)} \right). \tag{7}$$

Note that the time t dividends, $D_t^{(i)}$, do not appear in (7).

Exercise 12 *Adapt the various definitions and results of earlier sections to accommodate models with dividends. This is easy but tedious and so you might prefer to just think about it instead of actually doing it!*

Martingale Pricing in Practice When Markets Are Incomplete

When markets are incomplete there are infinitely many EMM's and it is necessary to choose one when we wish to price a derivative security that is not attainable. There are numerous ways that this is done in practice but they all seem to be variations of the following algorithm:

1. A subset of EMM's, $\{Q(\theta) \,:\, \theta \in R^k\}$, is assumed to contain the 'correct' EMM, Q^*. This class is parameterized by the k-dimensional parameter vector, θ.

2. Liquid securities with prices available in the market are chosen as the 'calibration' securities.

3. The parameter vector, θ, is then chosen to make the prices of the 'calibration' securities in our model match their observed market prices. If k is small relative to the number of 'calibration' securities, then an exact match will in general not be possible and it will be necessary to solve an optimization problem to find the best possible fit. The optimization problem will typically be non-linear and will need to be performed numerically. If k is large relative to the number of 'calibration' securities, then an exact fit might be possible.

It is worth mentioning that there is no reference to the true data-generating measure, P, in the above algorithm. This is no accident: it is common in practice to go straight to the class $\{Q(\theta) \,:\, \theta \in R^k\}$ without first considering or estimating P. This approach is not without merit as the resulting model is arbitrage-free and calibrated to market prices (possibly only approximately) by construction. It does have some weaknesses, however. For example, little attention is paid to whether or not there is an equivalent probability measure, P, that is consistent with empirical data. It is also necessary in practice to frequently update the estimated parameter, θ, as market prices of the 'calibration' securities change through time. It is clear that this is unsatisfactory, as it implies that our model is 'wrong'. But unless we can find the perfect model (and this is very unlikely as market conditions are always changing), there is little we can do about this.

Appendix A

In Example 1 we saw that $[\pi_1 \ \pi_2 \ \pi_3 \ \pi_4]^T = [0.0737 \ 0.1910 \ 0.3705 \ 0.3357]^T$ was a vector of state prices for the security market of that example. In particular, it satisfied $S_1^T \pi = S_0$, that is

$$\begin{bmatrix} 1.03 & 1.03 & 1.03 & 1.03 \\ 3.00 & 4.00 & 2.00 & 5.00 \\ 2.00 & 1.00 & 4.00 & 2.00 \end{bmatrix} \begin{bmatrix} 0.0737 \\ 0.1910 \\ 0.3705 \\ 0.3357 \end{bmatrix} = \begin{bmatrix} 1.0194 \\ 3.4045 \\ 2.4917 \end{bmatrix}$$

where S_0 was the vector of date 0 security prices, and $S1$ was the matrix of date 1 security prices. How did we find such a vector π?

Consider the equation $Ax = b$ where A is a known $m \times n$ matrix and b is a known $m \times 1$ vector. We would like to solve this equation for the vector x. There may be no solution, infinitely many solutions or a unique solution. If more than one solution exists so that $Ax_1 = Ax_2 = b4$ with $x_1 \neq x_2$ then $y := x_1 - x_2$ is an element of the *nullspace* of A. That is $Ay = 0$. In particular, any solution to $Ax = b$ can be written as the sum of a particular solution, x_1, and an element, w, of the nullspace.

How do we apply this to finding a vector positive state prices, assuming such a vector exists. First try and find a particular solution. In Matlab, for example, this can be done[14] by using the matrix divide command "\":

```
>> x1=S1'\S0

x1 =
         0
    0.3102
    0.4113
    0.2682
```

We then find the nullspace[15] of $S1^T$ by using the "null" command:

```
>> w=null(S')

w =
    0.7372
   -0.5898
   -0.2949
    0.1474
```

Now every possible solution to $S_1^T \pi = S_0$ can be written as

$$\begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.3102 \\ 0.4113 \\ 0.2682 \end{bmatrix} + \epsilon \begin{bmatrix} 0.7372 \\ -0.5898 \\ -0.2949 \\ 0.1474 \end{bmatrix}$$

where ϵ is any real number. We just need to find an ϵ such that the solution is strictly positive. If no such epsilon exists, then there is no vector of state prices.

Challenge Question

A mathematically inclined fund manager has constructed a very sophisticated trading strategy. He has spent considerable time doing this and his strategy was computed without reference to any historical data. Now that the strategy is complete, he wishes to test it by applying it to historical stock market data of the past 20 years. In order to get a clean data-set, he only includes the return data for stocks of companies that existed 20 years ago and that still exist today. As a result he can test his strategy using a clean set of historical returns over 20 years for a fixed number of stocks. The fund manager is a little concerned, however, that there might be problems with his data-set. What do you think?

---

[14] The transpose of a matrix, A, in Matlab is A'.

[15] The nullspace of a matrix is a *vector space* and the "null" command in Matlab will return a *basis* for this vector space. Any element of the nullspace can then be written as a linear combination of elements in the basis. In our example, the *dimensionality* of the nullspace is 1 and so there is only one element in the basis which we have denoted by 'w'.