Introduction to

Algorithm Design and Analysis

[17] Dynamic Programming 2

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In the last class…

- **Basic idea of DP**
- **Least cost matrix multiplication**
  - BF1, BF2
  - A DP solution
- **Weighted binary search tree**
  - The same DP solution

DP - II

- **From the DP perspective**
  - All-pairs shortest paths; SSSP over DAG
- **More DP problems**
  - Edit distance
  - Highway restaurants; Separating sequence of words
  - Changing coins
- **Elements of DP**

All-pairs Shortest Paths

- **BF2**
  - Path length k
    - k in [1,n]
- **Floyd algorithm**
  - Index range k
    - k in [1,n]

BF2

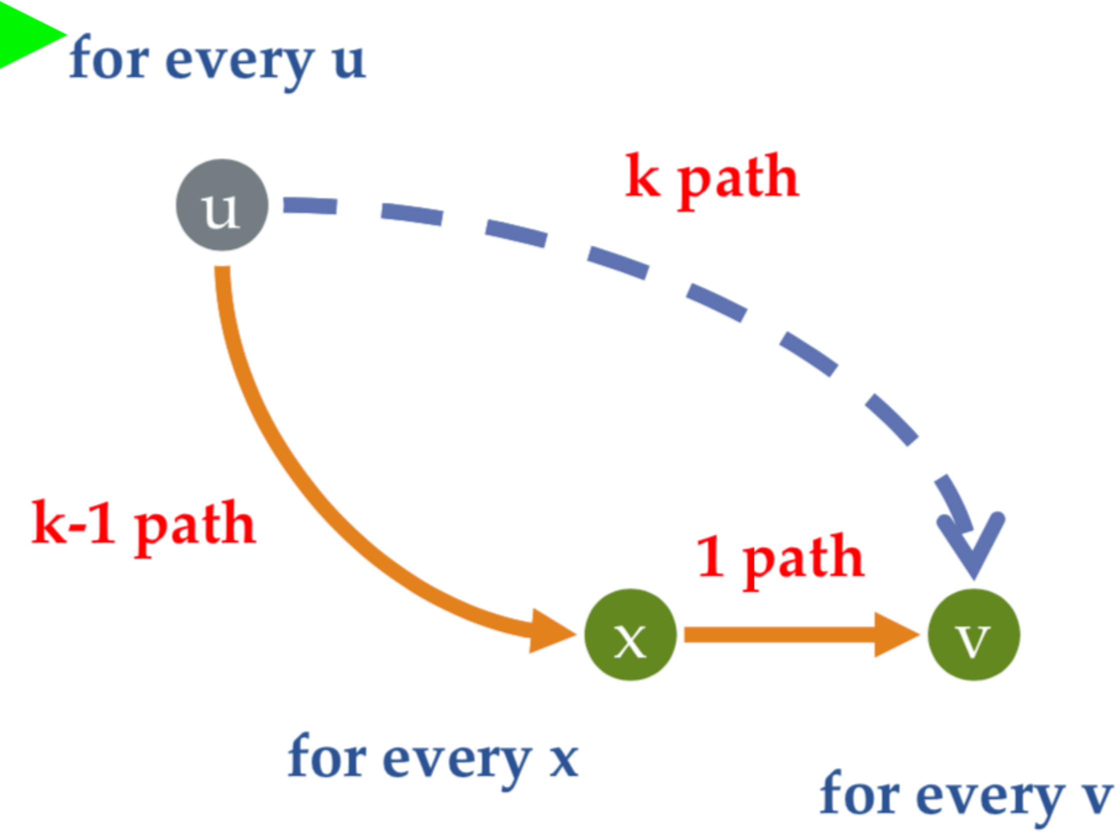
Length of the shortest path of at most k edges

$$dist(u,v,k) = \begin{cases} 0 & \text{if } u = v \\ \infty & \text{if } k = 0 \text{ and } u \neq v \\ \min\limits_{x} \left(dist(u,x,k-1) + w(x\to v) \right) & \text{otherwise} \end{cases}$$

<u>APSP(V, E, w):</u>

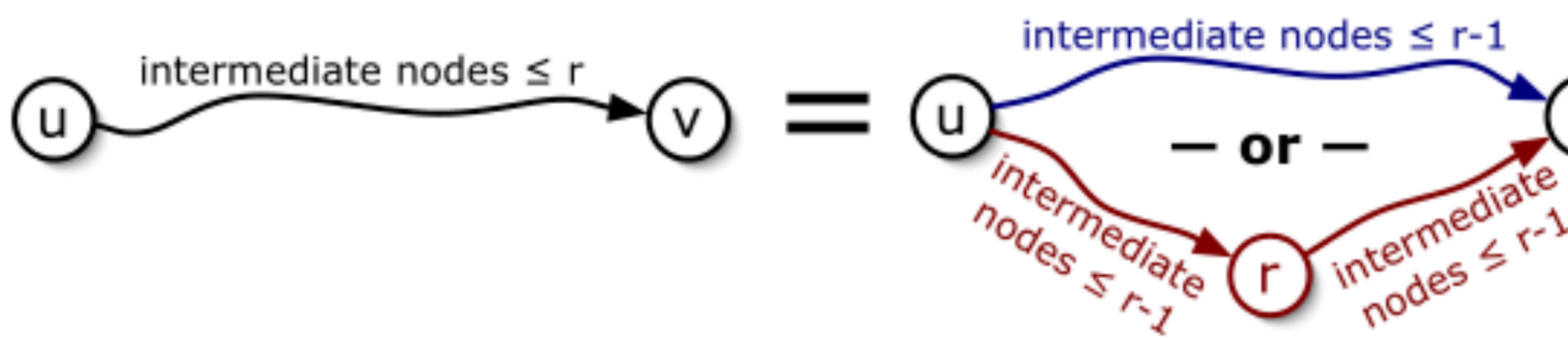
```
for all vertices u
    for all vertices v
        if u = v
            dist[u, v, 0] ← 0
        else
            dist[u, v, 0] ← ∞
for k ← 1 to V − 1
    for all vertices u
        for all vertices v
            dist[u, v, k] ← ∞
            for all vertices x
                if dist[u, v, k] > dist[u, x, k − 1] + w(x→v)
                    dist[u, v, k] ← dist[u, x, k − 1] + w(x→v)
```

$O(n^4)$



Floyd Algorithm

- Basic idea



- Smart recursion

$$dist(u,v,r) = \begin{cases} w(u \to v) & \text{if } r = 0 \\ \min\left\{dist(u,v,r-1),\ dist(u,r,r-1) + dist(r,v,r-1)\right\} & \text{otherwise} \end{cases}$$

Floyd Algorithm

- Basic DP (3-dimensional)

```
FloydWarshall(V, E, w):
  for all vertices u
      for all vertices v
          dist[u, v, 0] ← w(u→v)
  for r ← 1 to V
      for all vertices u
          for all vertices v
              if dist[u, v, r − 1] < dist[u, r, r − 1] + dist[r, v, r − 1]
                  dist[u, v, r] ← dist[u, v, r − 1]
              else
                  dist[u, v, r] ← dist[u, r, r − 1] + dist[r, v, r − 1]
```

$O(n^3)$

- Improved DP (2-dimensional)

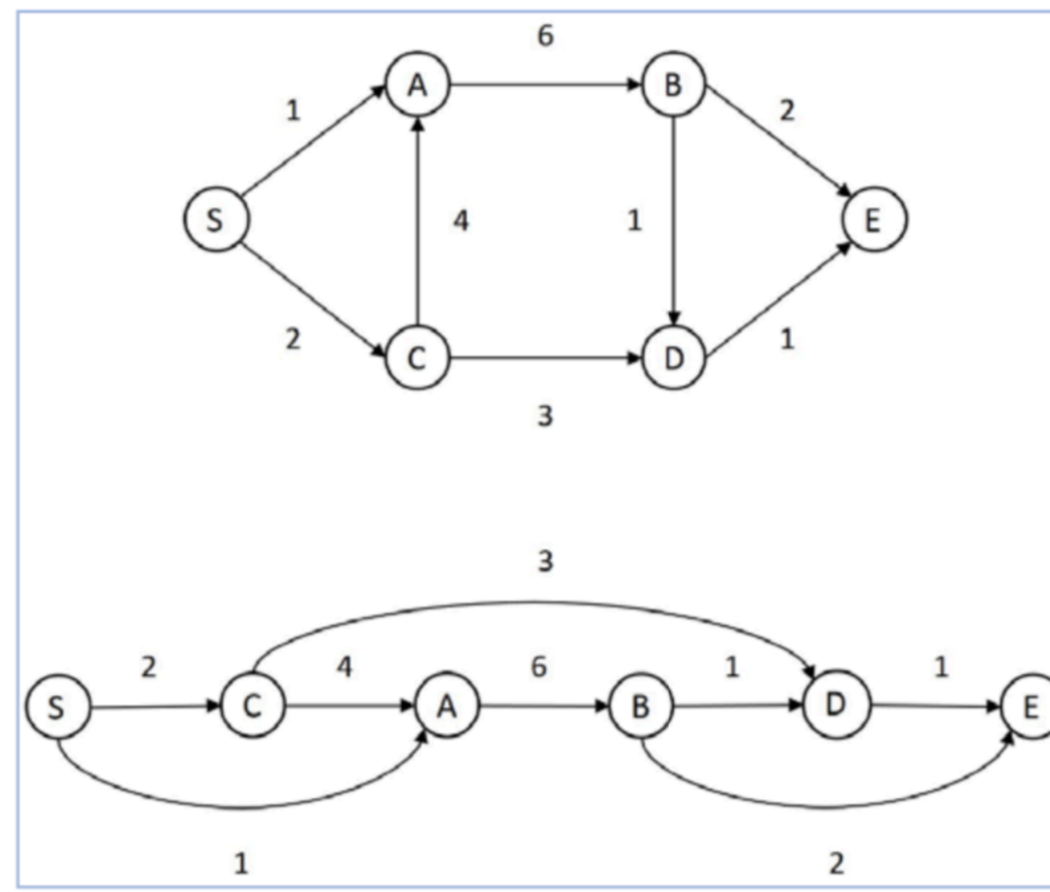
```
FloydWarshall2(V, E, w):
  for all vertices u
      for all vertices v
          dist[u, v] ← w(u→v)
  for all vertices r
      for all vertices u
          for all vertices v
              if dist[u, v] > dist[u, r] + dist[r, v]
                  dist[u, v] ← dist[u, r] + dist[r, v]
```

$O(n^3)$

SSSP over a DAG

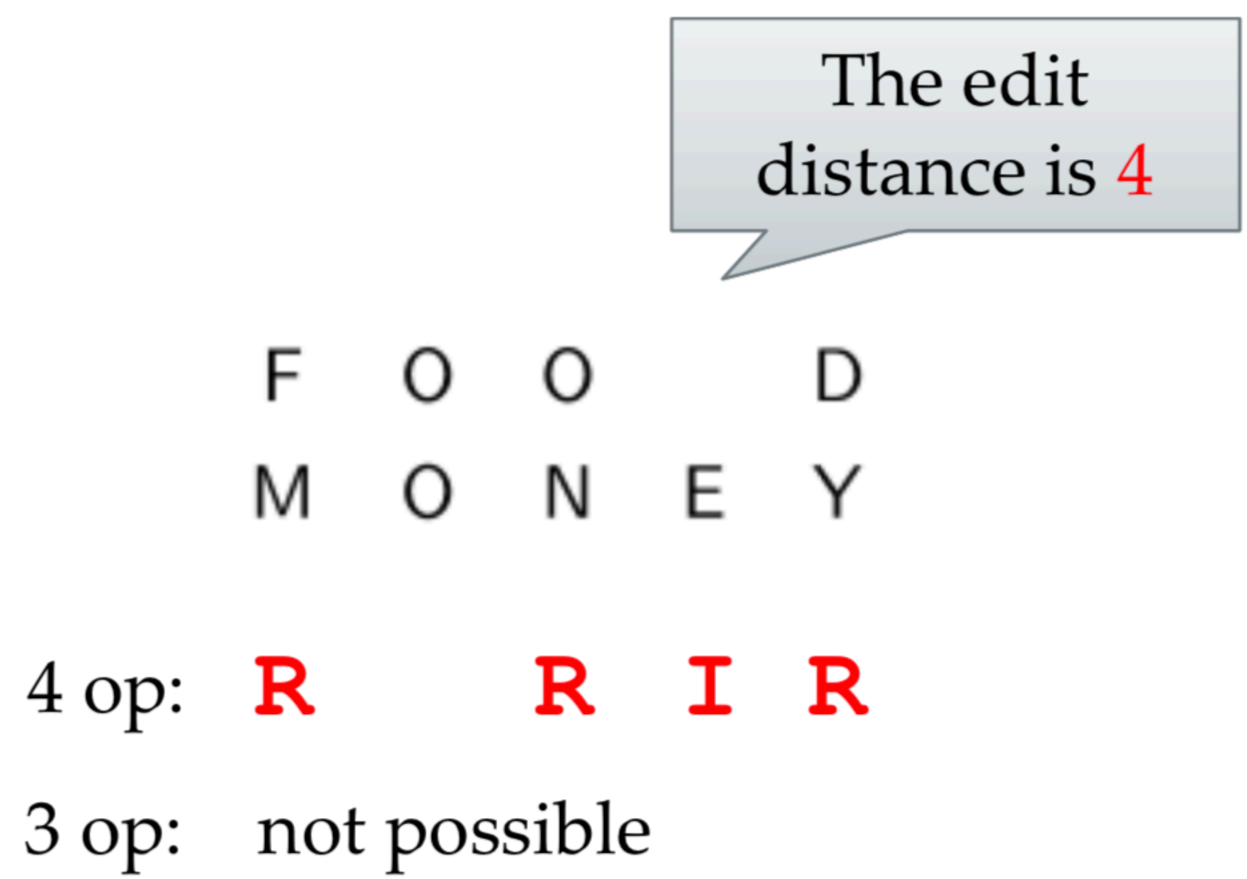
- **Subprogblems**
  - One problem for each node
    - dis[1..n]
- **Dynamic programming**
  - Topological ordering of nodes in a DAG
- **More than SSSP**
  - As long as the recursion succeeds

$$D.dis = min\{\, B.dis + 1, C.dis + 3\,\}$$



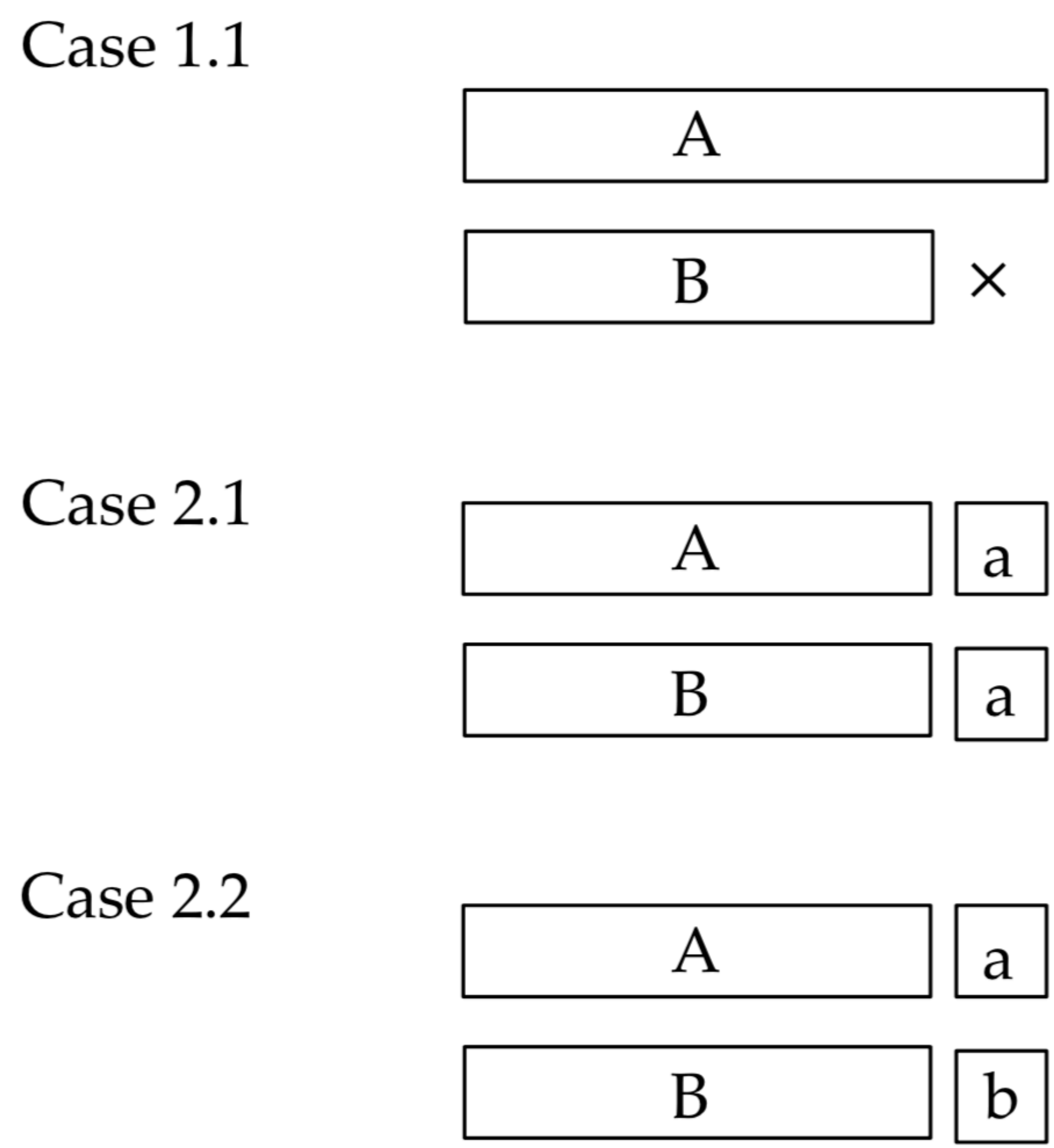
Edit Distance

- You can edit a word by
  - **I**nsert, **D**elete, **R**eplace
- Edit distance
  - Minimum number of edit operations
- Problem
  - Given two strings, compute the edit distance



“BF” Recursion

- Case 1
  - 1.1 Insert
  - 1.2 dual of case 1.1
- Case 2
  - 2.1 a=a
  - 2.2 a!=b



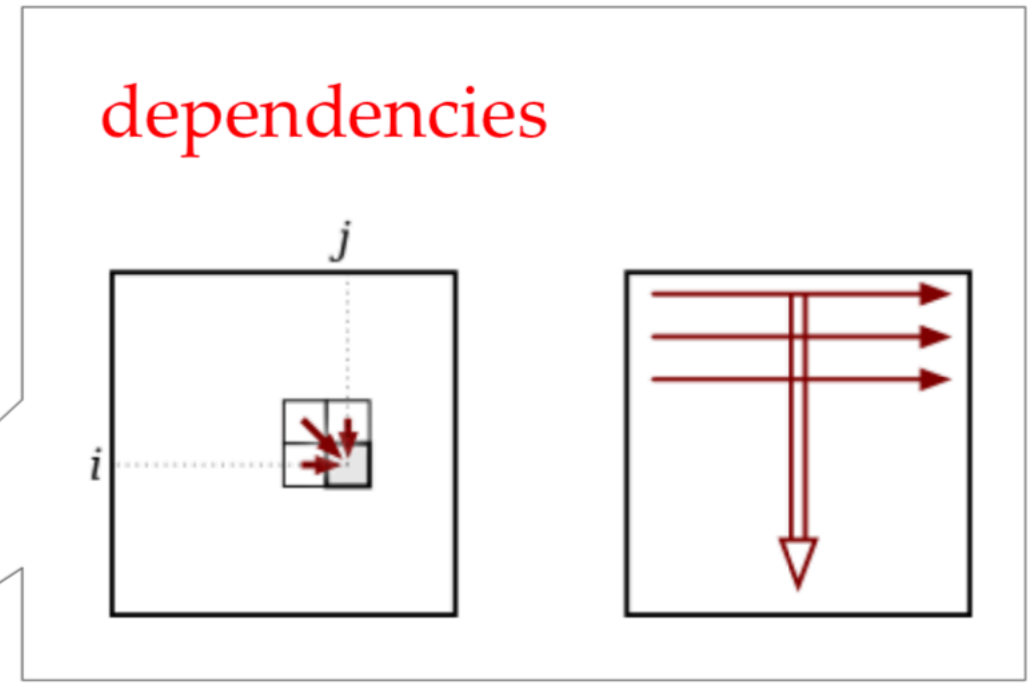
“BF” Recursion

- **EditDis(i,j)**
  - Base case:
    - If i=0, EditDis(i,j)=j
    - If j=0, EditDis(i,j)=i
  - Recursion:

$$EditDis(A[1..m], B[1..n]) = min \begin{cases} EditDis(A[1..m-1], B[1..n]) + 1 \\ EditDis(A[1..m], B[1..n-1]) + 1 \\ EditDis(A[1..m-1], B[1..n-1]) + I\{A[m] \neq B[n]\} \end{cases}$$

Smart Programming

- DP dict
    - EditDis[1..m, 1..n]
- DP algorithm




```
EditDistance(A[1..m], B[1..n]):
  for j ← 1 to n
      Edit[0, j] ← j
  for i ← 1 to m
      Edit[i, 0] ← i
      for j ← 1 to n
          if A[i] = B[j]
              Edit[i, j] ← min {Edit[i − 1, j] + 1, Edit[i, j − 1] + 1, Edit[i − 1, j − 1]}
          else
              Edit[i, j] ← min {Edit[i − 1, j] + 1, Edit[i, j − 1] + 1, Edit[i − 1, j − 1] + 1}
  return Edit[m, n]
```

Example

algorithm

vs.

altruistic

		A	L	G	O	R	I	T	H	M
	0	1	2	3	4	5	6	7	8	9
A	1	0	1	2	3	4	5	6	7	8
L	2	1	0	1	2	3	4	5	6	7
T	3	2	1	1	2	3	4	4	5	6
R	4	3	2	2	2	2	3	4	5	6
U	5	4	3	3	3	3	3	4	5	6
I	6	5	4	4	4	4	3	4	5	6
S	7	6	5	5	5	5	4	4	5	6
T	8	7	6	6	6	6	5	4	5	6
I	9	8	7	7	7	7	6	5	5	6
C	10	9	8	8	8	8	7	6	6	6

DP in One Dimension

- **Highway restaurants**
  - n possible locations on a straight line
    - $m_1, m_2, m_3, \ldots, m_n$
  - At most one restaurant at one location
    - Expected profit for location i is p_i
  - Any two restaurants should be at least k miles apart
- **How to arrange the restaurants**
  - To obtain the maximum expected profit

Highway Restaurants

- **The recursion**
  - P(j): the max profit achievable using only first j locations
    - P(0)=0
  - prev[j]: largest index before j and k miles away

$$P(j) = \max(p_j + P(prev[j]), P(j-1))$$

Highway Restaurants

- **One dimension DP algorithm**
  - Fill in P[0], P[1], …, P[n]

(First compute the prev$[\cdot]$ array)
$i = 0$
for $j = 1$ to n:
    while $m_{i+1} \leq m_j - k$:
        $i = i + 1$
    prev$[j] = i$
(Now the dynamic programming begins)
$P[0] = 0$
for $j = 1$ to n:
    $P[j] = \max(p_j + P[\texttt{prev}[j]], P[j-1])$
return $P[n]$

Words into Lines

- **Words into lines**
  - Word-length $w_1, w_2, \ldots, w_n$ and line-width: W
- **Basic constraint**
  - If $w_1, w_2, \ldots, w_j$ are in one line, then $w_i+w_{i+1}+\ldots+w_j \leq W$
- **Penalty for one line: some function of X. X is:**
  - 0 for the last line in a paragraph, and
  - $W-(w_i+w_{i+1}+\ldots+w_j)$ for other lines
- **The problem**
  - How to make the penalty of the paragraph, which is the sum of the penalties of individual lines, minimized

Greedy Solution

i	word	w
1	Those	6
2	who	4
3	cannot	7
4	remember	9
5	the	4
6	past	5
7	are	4
8	condemned	10
9	to	3
10	repeat	7
11	it.	4

W is 17, and penalty is X^3

Solution by greedy strategy

words	(1,2,3)	(4,5)	(6,7)	(8,9)	(10,11)
X	0	4	8	4	0
penalty	0	64	512	64	0

Total penalty is **640**

An improved solution

words	(1,2)	(3,4)	(5,6,7)	(8,9)	(10,11)
X	7	1	4	4	0
penalty	343	1	64	64	0

Total penalty is **472**

Problem Decomposition

- **Representation of subproblem: a pair of indexes (i,j), breaking words i through j into lines with minimum penalty.**
- **Two kinds of subproblem**
  - (k,n): the penalty of the last line is 0
  - all other subproblems
- **For some k, the combination of the optimal solution for (1,k) and (k+1, n) gives a optimal solution for (1,n)**
- **Subproblem graph**
  - About n^2 vertices
  - Each vertex (i,j) has an edge to about j-i other vertices, so, the number of edges is in $\Theta(n^3)$

Simpler Identification of Subproblems

- **If a subproblem concludes the paragraph, then (k,n) can be simplified as (k)**
  - About k subproblems
- **Can we eliminate the use of (i,j) with $j<n$?**
  - Put the first k words in the first line (with the basic constraint satisfied), the subproblem to be solved is (k+1, n)
  - Optimizing the solution over all k's. (k is at most W/2)

One-dimension Recursion

- **One-dimension problem space**
  - (1,n), (2,n), …, (n,n)

Subproblem (i,n)

Algorithm: lineBreak(w, W, i, n, L)

the current line

if $w_i + w_{i+1} + \cdots + w_n \leq W$ **then**

    <Put all words on line L, set penelty to 0> ;

else

    for $k = 1; w_i + \cdots + w_{i+k-1} \leq W; k++$ **do**

        $X = W - (w_i + \cdots + w_{i+k-1})$;

        $kPenalty = lineCost(X) + \text{lineBreak}(w, W, i + k, n, L + 1)$;

        <Set penalty always to the minimum *kPenalty*> ;

        <Updating k_{min}, which records the k part that produced the minimum penalty> ;

        <Put words i through $i + k_{min} - 1$ on line L> ;

return *penalty* ;

Dynamic Programming

- **Topological ordering of subproblems**
  - Penalty[n]->Penalty[n-1]->,…,->Penalty[1]

```
Algorithm: lineBreakDP
for i = n; i ≥ 1; i−− do
    if all words through w_i to w_n can be put in one line then
        Penalty[i] = 0 ;
        <put all words through i to n in one line> ;
    else
        for k = 1; w_i + ··· + w_{i+k−1} ≤ W; k++ do
            calculate the penalty Cost_cur of putting k words in this line ;
            minCost=min{minCost, Cost_cur + Penalty[i + k]} ;
            <Updating k_min, which records the k part that produced the minimum
             penalty> ;
            <Put words i through i + k_min − 1 on one line> ;
    Penalty[i] = minCost ;
```

Analysis of lineBreakDP

- **Each subproblem is identified by only one integer k, for (k,n)**
  - Number of vertex in the subproblem graph: at most n.
  - So, in DP version, the recursion is executed at most n times.
- **So, the running time is in $\Theta(Wn)$**
  - The loop is executed at most W/2 times.
  - In fact, W, the line width, is usually a constant. So, $\Theta(n)$.
  - The extra space for the dictionary is in $\Theta(n)$.

Making Change: Revisited

- **How to pay a given amount of money?**
  - Using the smallest possible number of coins
  - With certain systems of coinage
- **We have known that the greedy strategy fails sometimes?**

Subproblems

- **Assumptions**
  - Given n different denotations
  - A coin of denomination i has d_i units
  - The amount to be paid: N
- **Subproblem [i,j]**
  - The minimum number of coins required to pay an amount of j units, using only coins of denominations 1 to i.
- **The problem**
  - Figure out subproblem [n, N] (as c[n, N])

Dependency of Subproblems

- **c[i,0] is 0 for all i**
- **When we are to pay an amount j using coins of denominations 1 to i, we have two choices:**
  - No coins of denomination i is used: c[i-1, j]
  - One coins of denomination i is used: $1+c[i,j-d_i]$
- **So, $c[i,j]=\min(c[i-1,j], 1+c[i,j-d_i])$**

Data Structure

Define a array coin$[1..n, 0..N]$ for all $c[i, j]$

an example

$$\begin{array}{cc} & \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} d_1=1 \\ d_2=4 \\ d_3=6 \end{array} & \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 2 & 3 & 1 & 2 & 3 & 4 & 2 \\ 0 & 1 & 2 & 3 & 1 & 2 & 1 & 2 & 2 \end{bmatrix} \end{array}$$

direction of computation

The Procedure

```
int coinChange(int N, int n, int[] coin)
    int denomination[]=[d_1,d_2,...,d_n];
    for (i=1; i≤n; i++)
        coin[i,0]=0;
    for (i=1; i≤n; i++)
        for (j=1; i≤N; j++)
            if (i= =1 && j<denomination[i])  coin[i,j]=+∞ ;
            else if (i= =1) coin[i,j]=1+coin[1, j-denomination[1]];
            else if (j<denomination[i]) coin[i,j]=cost[i-1, j];
            else coin[i,j]=min(coin[i-1, j], 1+coin[i, j-denomination[i]];
    return coin[n,N];
```

in $\Theta(n\boldsymbol{N})$,

n is usually a constant

Other DP Problems

- **Text string problems**
  - Longest common subsequence, …
  - Variations of standard text string problems, …
- **One dimensional problems**
  - Arrangements along a straight line, …
- **Graph problems**
  - Vertex cover, …
- **Hard problems**
  - Knapsack problems and variations, …

Principle of optimality

- **Given an optimal sequence of decisions, each subsequence must be optimal by itself.**
  - Positive example: shortest path
  - Counterexample: longest (simple) path
- **DP relies on the principle of optimality**
  - The optimal solution to any nontrivial instance of a problem is a combination of optimal solutions to some of its sub-instances.
  - It is often not obvious which sub-instances are relevant to the instance under consideration.

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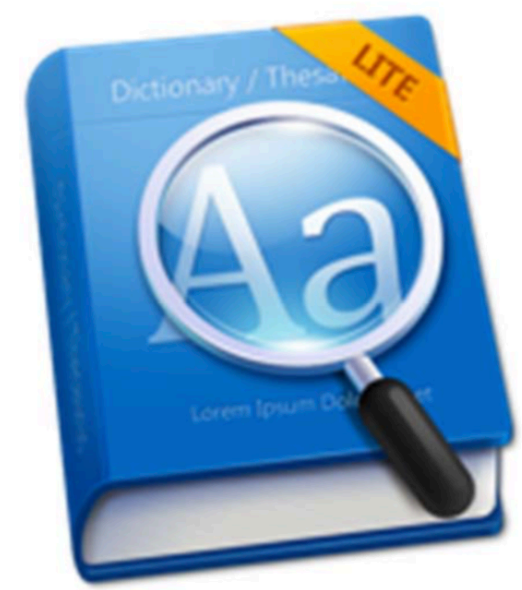
Optimal Substructure

Elements of Dynamic Programming

- **Symptoms of DP**
  - Overlapping subproblems
  - Optimal substructure
- **How to use DP**
  - “Brute force” recursion
    - Overlapping subproblems
  - “Smart” programming
    - Topological ordering of subproblems

DP Dictionary



Thank you!

Q & A