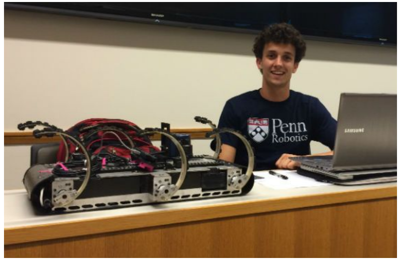
Hybrid Dynamical Type Theory

Paul Gustafson (Wright State University)
jww Jared Culbertson (AFRL), Dan Koditschek (Penn), Peter Stiller (TAMU)

Can we make behaviors modular?

Robot Whisperer:



English	Type Theory
True	$\mathbf{1}$
False	$\mathbf{0}$
A and B	$A \times B$
A or B	$A + B$
If A then B	$A \to B$
A if and only if B	$(A \to B) \times (B \to A)$
Not A	$A \to \mathbf{0}$

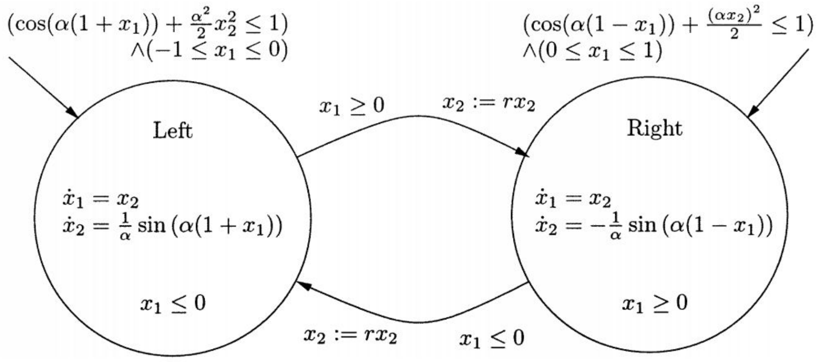
Big Picture

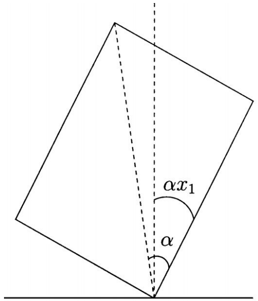
- Composition invariably leads to categories (either explicit or implicit)
  - Interfaces <-> types <-> objects
  - Controllers <-> terms <-> morphisms
- What is the right category of hybrid systems?
- How can we incorporate safety and liveness constraints into this categorical framework?
- How can we develop interoperability with the very successful LTL-based synthesis approaches?

Hybrid systems

A **hybrid system** H consists of

- a directed graph $G = (V, E, \mathfrak{s}, \mathfrak{t})$;
- for each **mode** $v \in V$,
  - an **ambient smooth system** (M_v, X_v)
  - an **active set** $I_v \subset M_v$
  - a **flow set** $F_v \subset I_v$
- for each **reset** $e \in E$, a **guard set** $Z_e \subset I_{\mathfrak{s}(e)}$ and an associated **reset map** $r_e \colon Z_e \to I_{\mathfrak{t}(e)}$.





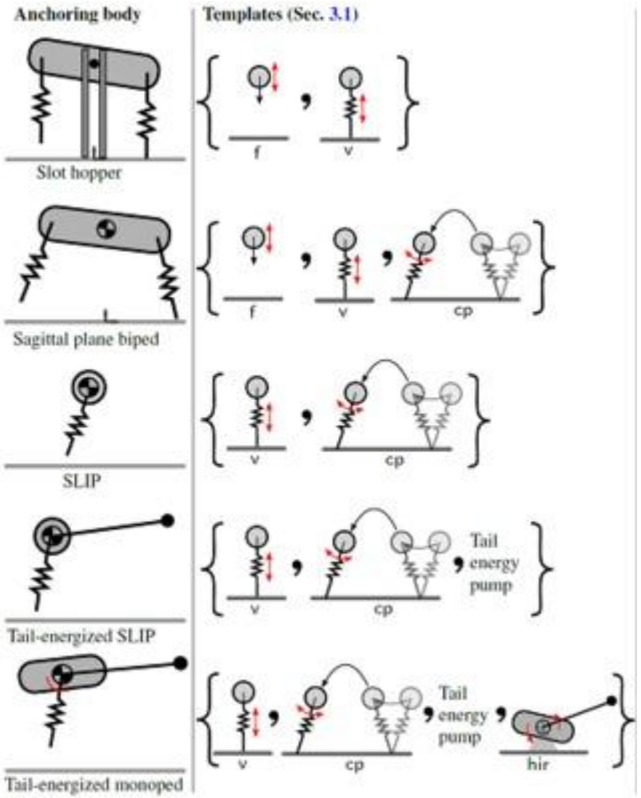
Morphisms: hybrid semiconjugacies

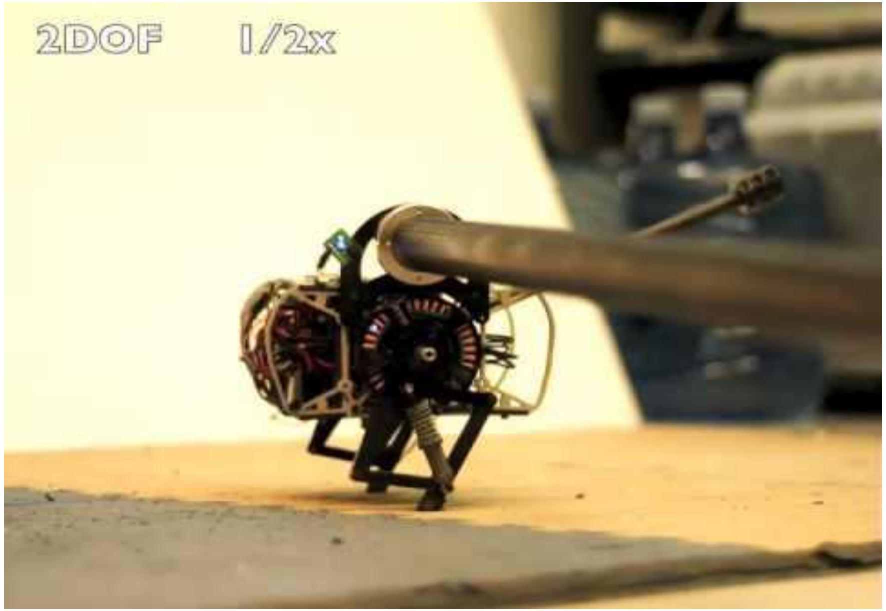
- “execution-preserving maps”

Cf. Lerman. “A category of hybrid systems.” arXiv:1612.01950.

Image source: Lygeros et al., ”Dynamical properties of hybrid automata.” IEEE Transactions on automatic control, 2003.

Templates and anchors



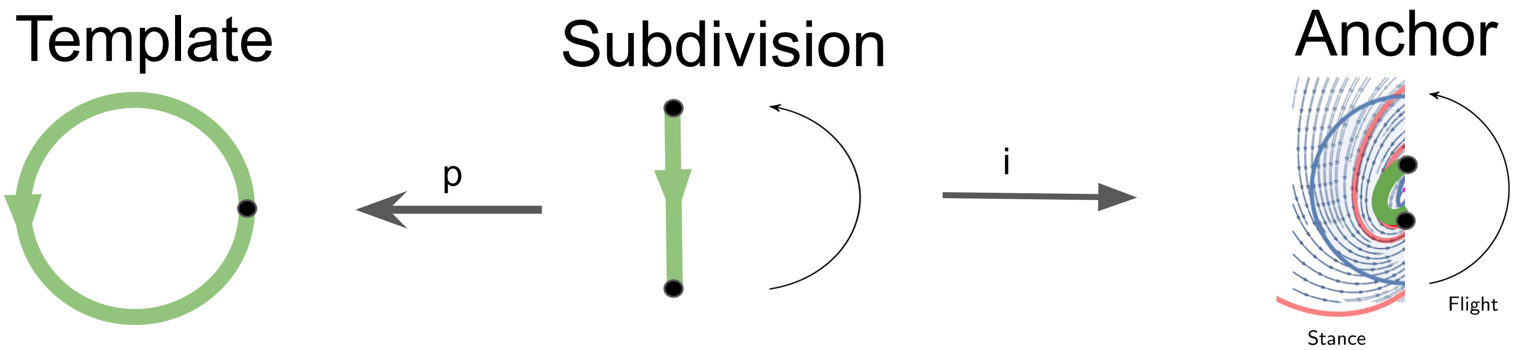


De, Avik, and Daniel E. Koditschek. "Parallel composition of templates for tail-energized planar hopping." 2015 IEEE International Conference on Robotics and Automation (ICRA). IEEE, 2015.

Anchoring a limit cycle in a vertical hopper

A **template-anchor pair** is a span $T \xleftarrow{p} S \xrightarrow{i} A$ such that

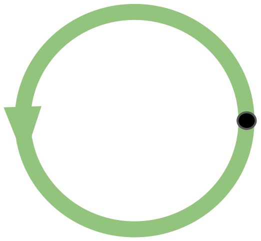
- p is a hybrid subdivision;
- i is a hybrid embedding;
- $i(S)$ is attracting in A.

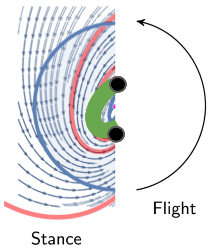


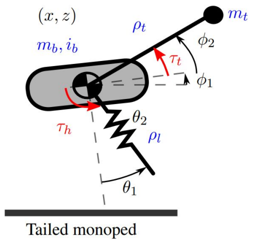
Hierarchical composition

Theorem (CGKS). Template-anchor pairs are weakly associatively composable.

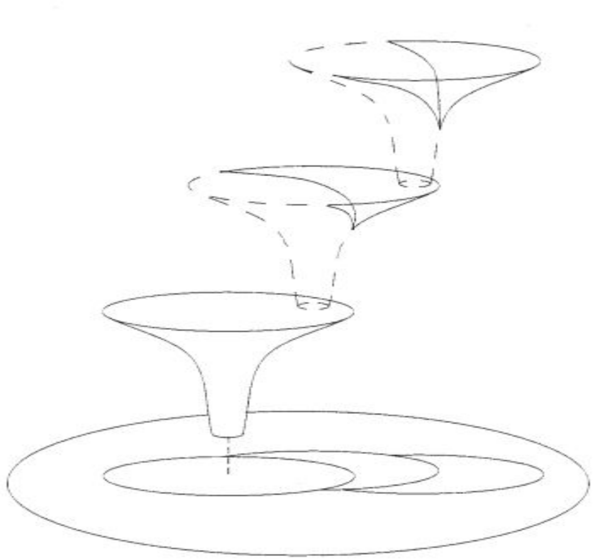
$$
\begin{tikzcd}
 & & S = S_1 \times_{A_1} S_2 \arrow[dl, two heads] \arrow[dr, hook] \arrow[dll, bend right, two heads, "p"'] \arrow[drr, bend left, hook, "i"] \arrow[dd, phantom, "\lrcorner" rotate=-45] & & \\
 & S_1 \arrow[dl, two heads] \arrow[dr, hook] & & S_2 \arrow[dl, two heads] \arrow[dr, hook] & \\
T_1 & & A_1 & & A_2
\end{tikzcd}
$$







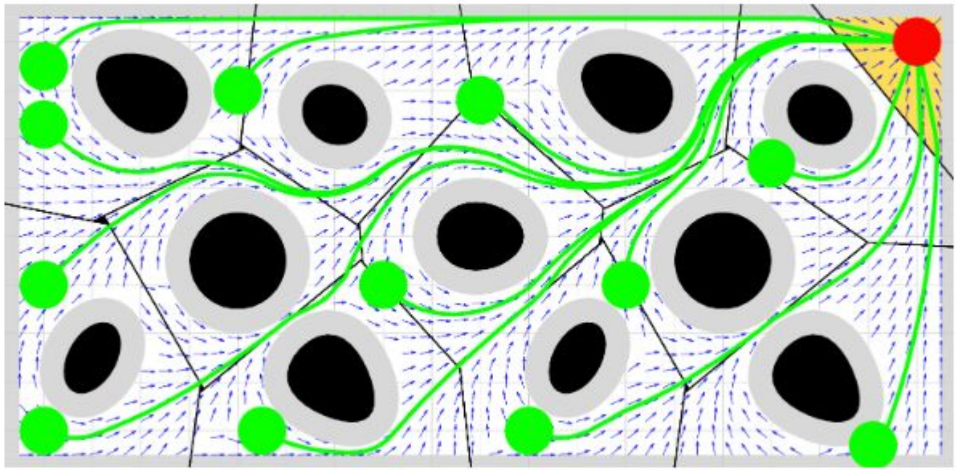
Sequential composition



Burridge, Robert R., Alfred A. Rizzi, and Daniel E. Koditschek. "Sequential composition of dynamically dexterous robot behaviors." *The International Journal of Robotics Research* 18.6 (1999): 534-555.

Goal: define a class of “funnel-like” hybrid systems closed under sequentially composition

A “navigate-to-goal” funnel

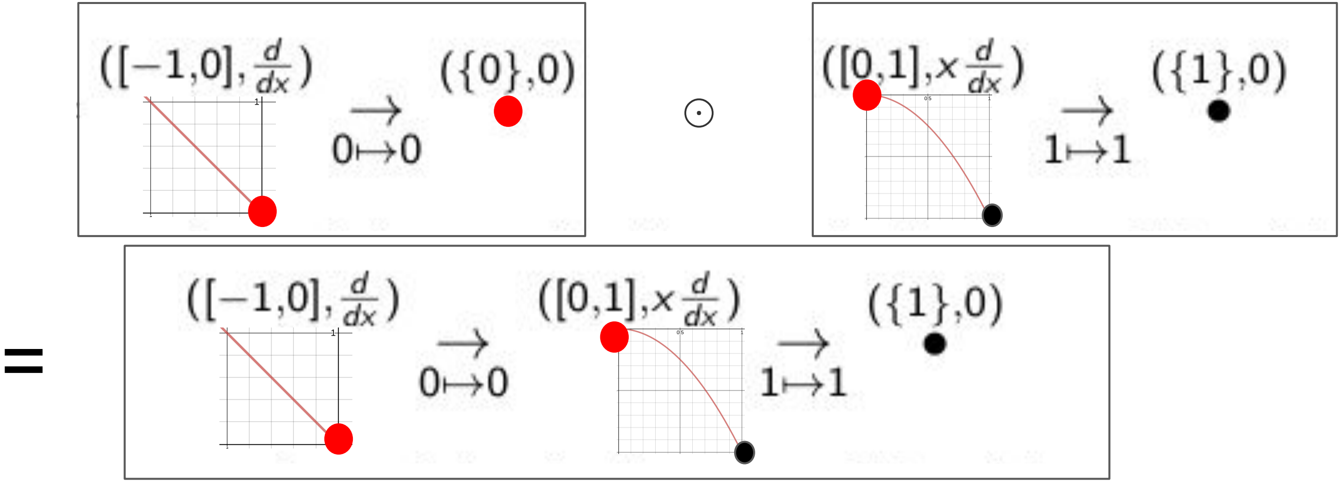


Theorem 3. *The piecewise continuously differentiable “move-to-projected-goal” law in (11) leaves the robot’s free space* $\mathcal{F}$ *(1) positively invariant; and if Assumption 2 holds, then its unique continuously differentiable flow, starting at almost[1] any configuration* $\mathrm{x} \in \mathcal{F}$*, asymptotically reaches the goal location* x^**, while strictly decreasing the squared Euclidean distance to the goal,* $\|\mathrm{x} - \mathrm{x}^*\|^2$*, along the way.*

Arslan, Omur, and Daniel E. Koditschek. "Sensor-based reactive navigation in unknown convex sphere worlds." *The International Journal of Robotics Research* (2019).

How to define ``funnel-like'' systems?

- **Problem:** the naive measure-theoretic and topologically notions of "almost all" are incompatible with fully general sequential composition
- Example:

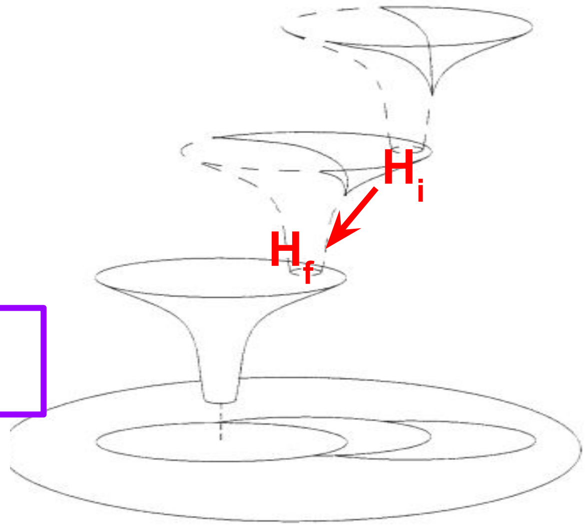


Directed systems

A **directed hybrid system** $H \colon H_i \rightsquigarrow H_f$ is a tuple (H, η_i, η_f) consisting of

- a metric hybrid system H,
- embeddings $\eta_i \colon H_i \to H$ and
- a hybrid embedding $\eta_f \colon H_f \to H$ such that each component $(\eta_f)_v$ is a diffeomorphism, and $G(H_f)$ is a sink in $G(H)$

such that for all $\varepsilon, T > 0$ and $x \in H$, there exists an (ε, T)-**chain** from x to some $y \in H_f$.



Composable!

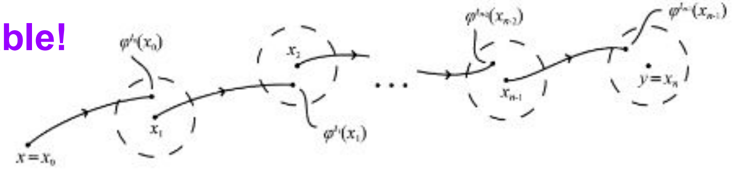
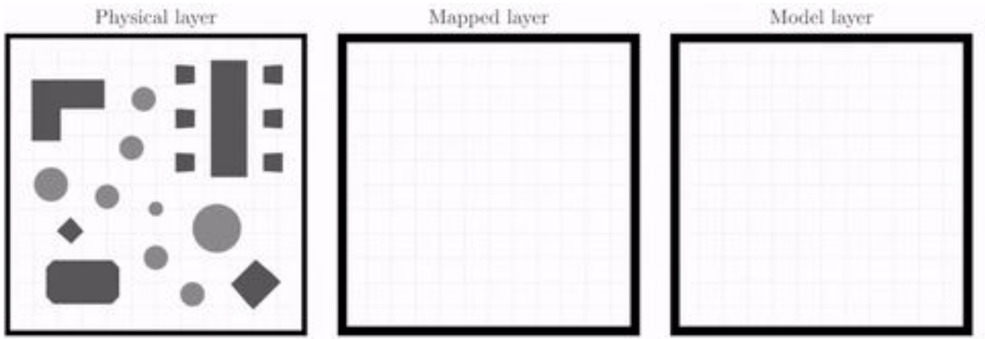


Image source: Alongi and Nelson, *Recurrence and Topology*. AMS, 2007.

A double category of hybrid systems



V. Vasilopoulos, D.E. Koditschek (2018). Reactive Navigation in Partially Known Non-Convex Environments. In WAFR 2018.

$$
\begin{tikzcd}
H_i \arrow[rr, "\alpha_i"] \arrow[d, squiggly, "H"'] & & K_i \arrow[d, squiggly, "K"] \\
{} \arrow[rr, phantom, "\overset{\alpha}{\Longrightarrow}"] & & {} \\
H_f \arrow[rr, "\alpha_f"'] & & K_f
\end{tikzcd}
$$

Linear dependent type theory

1. Assign dynamic input and output conditions + safety specs to sensor-parametrized subcontrollers
2. Linear fragment
    a. Manages resources and liveness
    b. Types and terms correspond to directed systems under sequential composition
3. Nonlinear fragment
    a. Manages with sensor-dependent parameters and safety
    b. Internal language of presheaves over the sensorium
        i. Example: in this open set of sensor readings, d(robot, O_i) > ε

Navigation example types

$$
\begin{aligned}
go &: (g : X, n : \mathbb{N}) \to Free \otimes (s : See(n)) \multimap (At(g) \otimes See(n)) \oplus Interrupt(s) \\
Interrupt &: See(n) \multimap Free \otimes (NewObs(See(n+1)) \oplus LoseObs(See(n-1)) \oplus TimeStep)) \\
detect &: See(n) \multimap See(n-1) \oplus See(n) \oplus See(n+1) \\
nearestObs &: See(n) \to List(X) \\
projGoal &: ConvHull(n) \to X \to X \\
voronoi &: See(n) \to ConvHull \\
ConvHull &= List(X) \\
Safe &= (s : See(n)) \to d(x, nearestObs(s)) > R \\
controller &: (g : X) \to (c : Free \otimes See(n) \multimap At(g) \otimes (m : \mathbb{N}, See(m)),\ Safe(c))
\end{aligned}
$$

Semantics of simple types

Type	Template	Presheaf (evaluated at $U \subset B$)
$See(n)$	$(X^n \times \mathbb{R}^n, 0)$	$\lvert\pi_0(f^{-1}([0, M]))\rvert = n$ for all $f \in \pi_{C(S^1, \overline{\mathbb{R}}_{\geq 0})}(U)$
$Free$	$(*, *)$	$\top$
$At(g)$	$(X, \nabla \lVert x - g \rVert^2)$	$\sup_{x \in U} d(x, g) < \epsilon$
$Safe$	$(X, -\sum_i \nabla \lVert x - o_i \rVert^2)$	$\sup_{x \in U, o \in \bigcup_i O_i} d(x, o) > r$

Integration with LTL-based controller synthesis

1. What LTL buys you
    a. Automatic synthesis
    b. Provable safety/finite-time task completion for particular control systems
2. What dependent LL buys you
    a. Correct-by-construction composition of subcontrollers
    b. Physical grounding
        i. Extend safe/unsafe sets with dynamic interfaces between behaviors
3. Complementary -- embed LTL specs into dependent linear types
    a. Example: "Eventually(Always(g))" becomes "(A $\multimap$ B) and g(supp(B))"
    b. Use synthesized controllers in correct-by-construction composite controllers

Operational semantics

1. No simple notion of abstract machine/lambda calculus for operational semantics
2. Can we define a “gradual” version of operational semantics based on successive template embeddings?
    a. Examples
        i. Anchor At(g) point attractor template in a differential drive robot
        ii. Anchor See(n) template inside navigation + sensing product corresponds to stabilizing sensor readings