Macroeconomics 2 Presentation
Equations description of
"A Behavioral New Keynesian Model" by Xavier Gabaix

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This file aims to summarise and give some all the equations of the presented article. Its goal is to serve as a complement to the presentation of May 3, 2024 in the Macroeconomics 2 class. We use the same numerisation of the equations as in the article. This file can also be considered as a complementary helper to the reading of the article.

Contents

1 A Behavioral Model

The first two equations are the same as (28) and (29) and will be explained later.

1.1 Introduction

Equation 1

$$x_t = M \cdot \mathbb{E}_t \left[xt + 1 - \sigma(i_t - \mathbb{E}_t \left[\pi_{t+1} \right] - r_t^n) \right] \tag{1}$$

Equation 2

$$\pi_t = \beta \cdot M^f \mathbb{E}_t \left[\pi_{t+1} \right] + \kappa \cdot x_t \tag{2}$$

1.2 Basic Setup and the Household's Problem

Equation 3

$$U = \mathbb{E}_t \left[\sum t = 0^{\infty} \beta^t \left(\frac{c_t^{1-\gamma} - 1}{1-\gamma} - \frac{N_t^{1+\phi}}{1+\phi} \right) \right] \tag{3}$$

Equation (3) is the flow utility of the Household, with :

- β the discount factor
- c_t the consumption of the houshold at time t
- N_t the work of the household at time t
- γ determines the concavity of the utility function with respect to the consumption, i.e. the importance of consumption in the utility function
- ϕ determines the concavity of the utility function with respect to work, i.e. the importance of work in the utility function

Equation 4

$$k_{t+1} = (1 + r_t)(k_t - c_t + y_t) \tag{4}$$

Equation (4) is the law of motion of the real financial wealth of the household, where :

- k_t is the real financial welath of the household at time t
- r_t is the real interest rate
- w_t is the real wage
- y_t is the agent's real income, defined as $y_t = w_t \cdot N_t + y_t^f$, with y_t^f the profit income (or the income from firms) at time t

Equation 5

$$\boldsymbol{X}_{t+1} = \boldsymbol{G}^{\boldsymbol{X}}\left(\boldsymbol{X}_t, \boldsymbol{\epsilon}_{t+1}\right) \tag{5}$$

Equation (5) describes the evolution of macroeconomic variables, where :

- $\boldsymbol{X}_t$ is the state vector, including several macroeconomic variables of time t, like ζ_t the aggregate TFP, and the announced actions in monetary and fiscal policy
- $\boldsymbol{G}^X$ the equilibrium transition function, i.e. the function that gives the macroeconomic variables at time $t+1$ from the macroeconomic variables at the previous period
- ϵ_t is the innovation in the economy at time t, with $\mathbb{E}_t\left[\epsilon t + 1\right] = 0$, that depends on the equilibrium policies of the agent and of the government

Equation 6

$$\begin{aligned} k_{t+1} &= G^k(c_t, N_t, k_t, \boldsymbol{X}_t) \\ &:= (1 + \bar{r} + \hat{r}(\boldsymbol{X}_t))(k_t + \bar{y} + \hat{y}(N_t, \boldsymbol{X}_t) - c_t) \end{aligned} \tag{6}$$

Equation (6) is the application of the consideration of a set of macroeconomic variables on the law of motion of real financial wealth k_t, where :

- $\bar{r}$ is the steady state value of the real interest rate, that does not depend on time
- $\hat{r}(\boldsymbol{X}_t)$ is the value of the deviation from the steady state of the real interest rate, that depends on the state vector $\boldsymbol{X}_t$ at time t
- $\bar{y}$ is the steady state value of the agent's real income, that does not depend on time
- $\hat{y}(N_t, \boldsymbol{X}_t)$ is the deviation from the steady state of the agent's real income, that depends on the number of hours worked at time t and on the state vector at time t
- c_t is the aggregate consumption level at time t of the agent

Equation 7

$$\boldsymbol{X}_{t+1} = \boldsymbol{\Gamma}\boldsymbol{X}_t + \boldsymbol{\epsilon}_{t+1} \tag{7}$$

Equation (7) describes the linear version of the equilibrium transition function, it is the linearization of the law of motion, where :

- Γ is a squared matrix that multiplies the state vector
- $\boldsymbol{X}_t$ is the state vector at time t
- ϵ_t is the innovation shock

Equation 8 (Assumption 1)

$$\boldsymbol{X}_{t+1} = \bar{m} \cdot \boldsymbol{G}^{\boldsymbol{X}} (\boldsymbol{X}_t, \boldsymbol{\epsilon}_{t+1}) \tag{8}$$

Equation (8) describes the Cognitive Discounting of the State Vector, i.e. the perception by behavioural agents of the law of motion of the macroeconomic variables, where :

- $\bar{m} \in [0n1]$ is the cognitive discount factor measuring the attention to the future

Equation 9

$$\boldsymbol{X}_{t+1} = \bar{m}(\boldsymbol{\Gamma} \boldsymbol{X}_t + \boldsymbol{\epsilon}_{t+1}) \tag{9}$$

Equation (9) is the linearized version of the perception by behavioral agents of the law of motion of the state vector.

Equation 10

$$\mathbb{E}_t^{BR} [\boldsymbol{X}_{t+k}] = \bar{m}^k \mathbb{E}_t [\boldsymbol{X}_{t+k}] \tag{10}$$

Equation (10) defines the expectation of behavioral agents in function of the rational perception of the law of motion of the state vector, where :

- $k \geq 0$ a time period in discrete context
- $\mathbb{E}_t^{BR} [\boldsymbol{X}_{t+k}]$ is the expected value of the state vector at time $t+k$ by behavioral agents (or subjective/behavioral expectation operator)
- $\bar{m}^k$ is the cognitive discounting effect at period $t+k$
- $\mathbb{E}_t [\boldsymbol{X}_{t+k}]$ is the rational expectation of the state vector at time $t+k$

Equation 11 (Lemma 1)

$$\mathbb{E}_t^{BR} [z (\boldsymbol{X}_{t+k})] = \bar{m}^k \cdot \mathbb{E}_t [z (\boldsymbol{X}_{t+k})] \tag{11}$$

Equation (11) defines in the general case the behavioral expectation operator, for any function of the state vector, where :

- $k \geq 0$ a time period in discrete context
- $z(\cdot)$ is a function, such that $z(0) = 0$
- $\mathbb{E}_t^{BR} [z (\boldsymbol{X}_{t+k})]$ is the expected value of the image of the state vector by the function $z(\cdot)$ at time $t+k$ by behavioural agents
- $\bar{m}^k$ is the cognitive discounting effect at period $t+k$
- $\mathbb{E}_t [z (\boldsymbol{X}_{t+k})]$ is the rational expectation of the image of the state vector by the function $z(\cdot)$ at time $t+k$

Equation 12

$$\mathbb{E}_t^{BR}\left[\bar{r} + \hat{r}\left(\boldsymbol{X}_{t+k}\right)\right] = \bar{r} + \bar{m}^k \mathbb{E}_t\left[\hat{r}(\boldsymbol{X}_{t+k})\right] \tag{12}$$

Equation (12) is an example of the Lemma 1 applied to the interest rate, where :

- $k \geq 0$ a time period in discrete context
- $\bar{r}$ the steady state level of the real interest rate, that does not depend on time,
- $\hat{r}(\boldsymbol{X}_{t+k})$ is the equilibrium transition function defining the value of the deviation from the steady state of the real interest rate in the function of the state vector at time $t+k$
- $\bar{r} + \hat{r}(\boldsymbol{X}_t) = rt(\boldsymbol{X}_t)$ is the value of the real interest rate at time t
- $\mathbb{E}_t^{BR}\left[\bar{r} + \hat{r}(\boldsymbol{X}_{t+k})\right]$ is the expected value of the real interest at time $t+k$ by behavioural agents
- $\mathbb{E}_t\left[\hat{r}(\boldsymbol{X}_{t+k})\right]$ is the rational expectation of the value of the deviation of the real interest rate from the steady state at time $t+k$

1.3 The Firm's problem

Equation 13

$$P_t = \left(\int_0^1 P_{it}^{1-\varepsilon}\, di\right)^{\frac{1}{1-\varepsilon}} \tag{13}$$

Equation (13) describes the aggregate price level, where :

- P_t is the aggregate price level of the economy at time t
- $i \in [0,1]$ is the firm index
- ε is the elasticity of substitution between goods

Equation 14

$$\nu^0(q_{i\tau}, \mu_\tau, c_\tau) := \left(e^{q_{i\tau}} - (1-\tau_f)e^{-\mu_\tau}\right)e^{-\varepsilon q_{i\tau}} c_\tau \tag{14}$$

Equation (14) describes the profit of the firm before the lump sum tax of the government, where :

- ν is the real profit of the firm
- $q_{i\tau} = \ln\left(\frac{P_{i\tau}}{P_\tau}\right) = p_{i\tau} - p_\tau$ is the real log price at time τ
- $\tau_f = \frac{1}{\varepsilon}$ it the corrective wage subsidy from the government, funded by the lump sum tax

- $\mu_\tau = \zeta_t - \ln(\omega_t)$ is the labor wedge, which is zero at efficiency
- ε is the elasticity of subsititution between goods
- c_τ is the aggregate level of consumption

Equation 15

$$v\left(q_{it}, \boldsymbol{X}\tau\right) := v^0\left(q_{it} - \Pi(\boldsymbol{X}\tau), \mu(\boldsymbol{X}\tau), c(\boldsymbol{X}_\tau)\right) \tag{15}$$

Equation 15 describes the flow profit of the firm in function of the real log price and of the extended macro state vector, where :

- $q_{it} = \ln\left(\frac{P_{it}}{P_t}\right) = p_{it} - p_t$ is the real log price
- $\boldsymbol{X}\tau = (\boldsymbol{X}^{\mathcal{M}}\tau, \Pi_\tau)$ is the extended macro state vector, with $\boldsymbol{X}^{\mathcal{M}\tau}$ the vector of macro variables, including $\zeta\tau$ and possible announcements about future policy
- $\Pi\left(\boldsymbol{X}\tau\right) := p\tau - p_t = \pi_{t+1} + ... + \pi_\tau$ is the inflation between times t and τ
- $q_{it} - \Pi(\boldsymbol{X}\tau) = qi\tau$ is the real price of the firm if they didn't change its price between t and τ
- $\mu\left(\boldsymbol{X}_\tau\right)$ is the aggregate labor wedge in function of the extended state vector at time t
- $c(\boldsymbol{X}_\tau)$ is the aggregate consumption level in function of the extended state vector at time t

Equation 16

$$\max_{q_{it}} \mathbb{E}_t\left[\sum_{\tau=t}^{\infty}(\beta\theta)^{\tau-t}\frac{c\left(\boldsymbol{X_\tau}\right)^{-\gamma}}{\left(\boldsymbol{X_t}\right)^{-\gamma}} v\left(q_{it}, \boldsymbol{X}\tau\right)\right] \tag{16}$$

Equation (16) describes the maximisation program of the firm given that they have a Calvo-like probability of θ of being able to change their price at each period, where :

- t is the initial period
- τ is the time period index
- q_{it} is the real log price of the firm at time t
- β is the discount factor
- θ is the Calvo like probability that the firm can change its price at any period
- $\frac{c\left(\boldsymbol{X}\tau^{-\gamma}\right)}{c\left(\boldsymbol{X}_t^{-\gamma}\right)}$ is the adjustment in the stochastic discount factor/pricing kernel between times t and τ

Equation 17

$$\max_{q_{it}} \mathbb{E}_t^{BR} \left[\sum \tau = t^{\infty} (\beta\theta)^{\tau - t} \frac{c\left(\boldsymbol{X}\tau^{-\gamma}\right)}{c\left(\boldsymbol{X}_t^{-\gamma}\right)} v\left(q_{it,\boldsymbol{X}_\tau}\right) \right] \tag{17}$$

Equation 17 describes the maximisation program of the behavioral firm, i.e. such that it is maximisation the behavioral expectation operator of the flow profit, where :

- $\mathbb{E}_t^{BR}$ is the behavioral/subjective expected value operator
- t is the initial period
- τ is the time period index
- q_{it} is the real log price of the firm at time t
- β is the discount factor
- θ is the Calvo like probability that the firm can change its price at any period
- $\frac{c\left(\boldsymbol{X}\tau^{-\gamma}\right)}{c\left(\boldsymbol{X}_t^{-\gamma}\right)}$ is the adjustment in the stochastic discount factor between times t and τ and it is approximately 1 when linearised around deterministic steady state.

1.4 Model solution

Equation 18

$$\hat{c}_t = \mathbb{E}_t \left[\hat{c}_{t+1} - \frac{1}{\gamma R} \hat{r}_t \right] \tag{18}$$

Equation (18) is the linearized version of the Euler equation obtained from the presented model. It is also called the investment-savings (IS) curve, where :

- $\hat{c}_t$ is the value of the deviation from the steady state of the aggregate consumption at time t
- $\mathbb{E}_t\left[\hat{c}_{t+1}\right]$ is the rational expectation of the value of the deviation from the steady state of the aggregate consumption at time $t+1$
- γ is the factor of the importance of consumption
- $R := 1 + \bar{r}$ is defined from the real interest rate at the steady state (cf. page 7 of the article)

Equation 19

$$\hat{c}_t = M \cdot \mathbb{E}_t \left[\hat{c}_{t+1}\right] - \sigma \hat{r}_t \tag{19}$$

Equation (19) is the application of Lemma 1 (equation (11)) on the previous Euler equation, i.e. a cognitively discounted aggregate Euler equation, where :

- M is the macro parameter of attention, such that $M = \bar{m}$ here
- $\sigma = \frac{1}{\gamma R}$

Equation 20

$$N_t^{\phi} = \omega_t c_t^{\gamma} \tag{20}$$

Equation (20) the result of the static First Order Condition for labor supply, where :

- N_t is the quantity of labor provided at time t
- ω_t is the real wage at time t
- c_t is the aggregate quantity of consumption at time t
- γ is the consumption importance in the utility

Equation 21

$$\hat{c}_t^n = \frac{1+\phi}{\gamma+\phi} \zeta t \tag{21}$$

Equation (21) is the equation giving us the consumption in a natural economy where there are no frictions in prices i.e. flexible price economy. where :

- $\hat{c}_t^n$ is the flexible price/natural economy consumption
- γ is the the elasticity of substitution between goods.
- ϕ is the Frisch elasticity of labour
- ζ_t is the Total Factor prodcutivity

  We calculate this natural economy consumption the same way we calculated the flexible price economy consumption using the log-linearised version of the natural economy price, first-order condition for labour supply, and market clearing condition for consumption and income.

is the flexible price/natural economy consumption

Equation 22

$$\hat{c}_t^n = M \cdot \mathbb{E}_t \left[\hat{c}_{t+1}^n\right] - \sigma \hat{r}_t^n \tag{22}$$

Equation (22) is the application of Lemma 1 (equation (11)) on the previous Euler equation, i.e. a cognitively discounted aggregate Euler equation for the natural economy where there are no price frictions. Here :

- M is the macro parameter of attention, such that $M = \bar{m}$ here
- $\sigma = \frac{1}{\gamma R}$
- $\hat{r}_t^n$ is the real interest rate for the natural economy

Equation 23

$$r_t^{n0} = \bar{r} + \frac{1+\phi}{\sigma(\gamma+\phi)} \left(M \cdot \mathbb{E}_t \left[\zeta_{t+1}\right] - \zeta_t\right) \tag{23}$$

Equation (23) is the equation for the pure natural rate of interest: this is the interest rate that prevails in an economy without pricing frictions, and undisturbed by government policy (in particular, budget deficits). In this equation :

- r_t^{n0} is pure natural rate of interest
- γ is the the elasticity of substitution between goods.
- ϕ is the Frisch elasticity of labour
- $\sigma = \frac{1}{\gamma R}$
- M is the macro parameter of attention, such that $M = \bar{m}$ here
- ζ_t and ζ_{t+1} is the Total Factor productivity in time t and $t+1$ respectively

This pure natural interest rate is calculated by isolating the $\hat{r}_t^n$ in Equation (22) and then replacing $\hat{c}_t^n = \frac{1+\phi}{\gamma+\phi}\zeta t$, $\hat{c}_{t+1}^n = \frac{1+\phi}{\gamma+\phi}\zeta t + 1$ and $\hat{r}_t^n = r_t^n - \bar{r}$ and by assuming $r_t^n = r_t^{n0}$ which holds when there are no budget deficits.

Equation 24

$$x_t = M \cdot \mathbb{E}_t \left[xt+1\right] - \sigma(\hat{r}_t - \hat{r}_t^n) \tag{24}$$

Equation (24) is the equation for the behavioural discounted Euler equation where :

- x_t is the output gap at period t
- x_{t+1} is the output gap at period $t+1$
- M is the macro parameter of attention, such that $M = \bar{m}$ here

- $\sigma = \frac{1}{\gamma R}$
- $\hat{r}_t$ is the deviation of the real interest rate from the steady state.
- $\hat{r}_t^n$ is the deviation of the real interest rate from the steady state for the natural economy.

This is calculated by subtracting the expression $\hat{c}_t = M \cdot \mathbb{E}_t [\hat{c}_{t+1}] - \sigma \hat{r}_t$ i.e. Equation (19) and $\hat{c}_t^n = M \cdot \mathbb{E}_t \left[\hat{c}_{t+1}^n\right] - \sigma \hat{r}_t^n$ i.e. Equation (22)

Equation 25

$$x_t = M \cdot \mathbb{E}_t [xt + 1] - \sigma(i_t - \mathbb{E}_t [\pi_{t+1}] - r_t^n) \tag{25}$$

Equation (25) is the behavioural discounted Euler equation after replacing the equation for Fisher equation where $\hat{r}_t = r_t - \bar{r} = (i_t - \mathbb{E}_t [\pi_{t+1}] - r_t^n)$

- i_t is the nominal interest rate
- $\mathbb{E}_t [\pi_{t+1}]$ is the expected inflation in the future
- $M = \bar{m}$
- $\sigma = \frac{1}{\gamma R}$

Equation 26

$$x_t = -\sigma \sum_{k \geq 0} M \cdot \mathbb{E}_t \left[\hat{r}_{t+k} - \hat{r}_{t+k}^n\right] \tag{26}$$

Equation (26) iteratively using Equation (24) reduces to this equation. It indicates that changes in the interest rate in the 1000^{th} period will have a discounted impact on the output gap. Therefore, the effect of changes in interest rates on the output gap diminishes over time, with changes in the 1000^{th} period having a smaller impact compared to changes in the 1^{st} period.

Equation 27

$$p_t^* = p_t + (1 - \beta\theta) \sum_{k=0}^{\infty} (\beta\theta\bar{m})^k \cdot \mathbb{E}_t [\pi_{t+1} + ... + \pi_{t+k} - \mu_{t+k}] \tag{27}$$

Equation (27) is the optimal price for a behaviourial firm resetting its price

- p_t^* is the the price a behaviourial firm will reset its price to.
- $\bar{m}$ is the cognitive discounting factor.
- $p_t^* = qit + p_t$ where q_{it} is the linearised version of the maximisation solution to problem 17. q_{it} is the price for the firms that can adjust their prices and p_t is the price of the firms that can't adjust their prices.

Equation 28 - Proposition 2, the first equation

$$x_t = M \cdot \mathbb{E}_t\left[x_{t+1}\right] - \sigma(i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^n) \tag{28}$$

Equation (28) is the Behavioural IS Curve. Essentially we are representing the Euler equation in terms of the output gap and also using the Fisher Equation.

Equation 29 - Proposition 2, second equation

$$\pi_t = \beta \cdot M^f \mathbb{E}_t\left[\pi_{t+1}\right] + \kappa \cdot x_t \tag{29}$$

Equation (29) is the Behavioural Phillips curve where:

- M^f is the aggregate attention parameter for firms and $M^f = \bar{m}\left(\theta + \frac{1-\beta\theta}{1-\beta\theta\bar{m}}(1-\theta)\right)$
- $\kappa = \widetilde{\kappa}$ is the slope of the Phillips curve and $\widetilde{\kappa} = \frac{(1-\theta)(1-\beta\theta)}{\theta}(\gamma + \phi)$ which is the slope obtained from fully rational firms.

Equation 30

$$\begin{cases} M = \bar{m} \\ \sigma = \frac{1}{\gamma R} \\ M^f = \bar{m}\left(\theta + \frac{1-\beta\theta}{1-\beta\theta\bar{m}}(1-\theta)\right) \end{cases} \tag{30}$$

Equation (30) defines the parameters in Proposition 2.

2 Consequences of the Model

2.1 The Taylor Principle Reconsidered: Equilibria Are Determinate Even with a Fixed Interest Rate

Equation 31

$$i_t = \phi_\pi \pi_t + \phi_x x_t + j_t \tag{31}$$

The equation (31) sets the nominal interest i_t in a Taylor rule fashion. j_t is just a constant.

Equation 32

$$\mathbf{z}_t = \mathbf{A}\mathbb{E}_t\left[z_{t+1}\right] + \mathbf{b}a_t \tag{32}$$

The system of Proposition 2 can be represented as equation (32), where:

- $\mathbf{z}_t := (x_t, \pi_t)'$
- $\mathbf{A}$: see equation (33)
- $\mathbf{b} = \frac{-\sigma}{1+\sigma(\phi_x + \kappa\phi_\pi)}(1, \kappa)'$
- $a_t := j_t - r_t^n$

Equation 33

$$\mathbf{A} = \frac{1}{1+\sigma(\phi_x + \kappa\phi_\pi)} \begin{pmatrix} M & \sigma(1-\beta^f\phi_\pi) \\ \kappa M & \beta^f(1+\sigma\phi_x) + \kappa\sigma \end{pmatrix} \tag{33}$$

where:

- $\beta^f := \beta M_f$

Equation 34 - Proposition 3

Equilibrium Determinacy with Behavioral Agents: There is a determinate equilibrium (i.e., all of **A**'s eigenvalues are less than 1 in modulus) if and only if

$$\phi_\pi + \frac{(1-\beta M^f)}{\kappa}\phi_x + \frac{(1-\beta M^f)(1-M)}{\kappa\sigma} > 1 \tag{34}$$

Equation 35

In particular, when monetary policy is passive (i.e., when $\phi_\pi = \phi_x = 0$), we have a determinate equilibrium if and only if bounded rationality is strong enough, in the sense that:

$$\frac{(1-\beta M^f)(1-M)}{\kappa\sigma} > 1 \tag{35}$$

This is **Strong enough bounded rationality condition**. Condition (35) does not hold in the traditional model, where M = 1. The condition means that agents are boundedly rational enough (i.e., M is sufficiently less than 1) and the firm-level pricing or cognitive frictions are large enough.

Equation 36

Permanent Interest Rate Peg. Take the (admittedly extreme) case of a permanent peg. Then, in the traditional model, there is always a continuum of bounded equilibria, technically, because the matrix A has a root greater than 1 (in modulus) when M = 1.In the behavioral model, we do get a definite non-explosive equilibrium:

$$\mathbf{z}_t = \mathbb{E}_t \left[\sum_{\tau \geq t} \mathbf{A}^{\tau - t} \mathbf{b} a_\tau \right] \tag{36}$$

Cochrane (2018) made the point that we'd expect an economy such as Japan's to be quite volatile, if the ZLB is expected to last forever: conceivably, the economy could jump from one equilibrium to the next at each period. This is a problem for the rational model, which is solved if agents are behavioral enough, so that (35) holds.

Equation 37

Long-Lasting Interest Rate Peg. The economy is still very volatile (in the rational model) in the less extreme case of a peg lasting for a long but finite duration. We can get:

$$\mathbf{z}_0(T) = \left(\mathbf{I} + \mathbf{A}_{ZLB} + ... + \mathbf{A}_{ZLB}^{T-1}\right) \underline{\mathbf{b}} + \mathbf{A}_{ZLB}^{T} \mathbb{E}_0 \left[\mathbf{z}_T\right] \tag{37}$$

where:

- $\mathbf{A}_{ZLB}$ is the value of matrix $\mathbf{A}$ in equation (33) when $\phi_\pi = \phi_x = j = 0$ in the Taylor rule.

The system (32) is, at the ZLB(t≤T): $\mathbf{z}_t = \mathbb{E}_t \mathbf{A}_{ZLB} \mathbf{z}_{t+1} + \underline{\mathbf{b}}$ with $\underline{\mathbf{b}} := (1, \kappa)\, \sigma \underline{r}$ where $\underline{r} \leq 0$ is the real interest rate that prevails during the ZLB. Iterating forward, we can get equation (37).

2.2 The ZLB Is Less Costly with Behavioral Agents

Equation 38 - Propostion 4

Call $x_0\,(T)$ the output gap at time 0, given the ZLB will lasts for T periods. In the traditional rational case, we obtain an unboundedly intense recession as the length of the ZLB increases: $\lim_{n\to\infty} x_0\,(T) = -\infty$.This also holds when myopia is mild, i.e., (35) fails. However, suppose cognitive myopia is strong enough, i.e., (35) holds. Then, we obtain a boundedly intense recession:

$$\lim_{T\to\infty} x_0\,(T) = \frac{\sigma\left(1-\beta M^f\right)}{(1-M)\left(1-\beta M^f\right) - \kappa\sigma} \underline{r} < 0 \tag{38}$$

Myopia has to be stronger when agents are highly sensitive to the interest rate (high σ) and price flexibility is high (high κ). High price flexibility makes the system very reactive, and a high myopia is useful to counterbalance that.

3 Implications for Monetary Policy

3.1 Welfare with Behavioral Agents and the Central Bank's Objective

Equation 39 - Lemma 3

The welfare loss from inflation and output gap is:

$$W = -K\mathbb{E}_0 \sum_{t=0}^{\infty} \frac{1}{2} \beta^t \left(\pi_t^2 + \vartheta x_t^2\right) + W_- \tag{39}$$

where:

- $\vartheta = \frac{\overline{\kappa}}{\epsilon}$
- $K = u_c c\left(\gamma + \phi\right)\left(\epsilon / \overline{\kappa}\right)$
- W_{-} is a constant (made explicit in equation (200) in the online Appendix).
- $\overline{\kappa}$ is the Phillips curve coefficient with rational firms (derived in Proposition 2).
- ϵ is the elasticity of demand.

3.2 Optimal Policy with Complex Trade-Offs: Reaction to a Cost-Push Shock

Equation 40 & 41 - Proposition 5

Optimal Policy with Commitment: Suboptimality of Price-Level Targeting: To fight a time-0 cost-push shock, the optimal commitment policy entails, at time $t \geq 0$:

$$\pi_t = \frac{-\vartheta}{\kappa}\left(x_t - M^f x_{t-1} \mathbf{1}_{t>0}\right) \tag{40}$$

so that the (log) price level ($p_t = \sum_{\tau=0}^{t} \pi_\tau$, normalizing the initial log price level to p_{-1}=0) satisfies:

$$p_t = \frac{-\vartheta}{\kappa}\left(x_t + \left(1 - M^f\right)\sum_{\tau=0}^{t-1} x_\tau\right) \tag{41}$$

With rational firms ($M^f = 1$) , the optimal policy involves "price-level targeting". it ensures that the price level mean- reverts to a fixed target ($p_t = (-\nu/\kappa)x_t \rightarrow 0$ in the long run). However, with behavioral firms, the price level is higher (even in the long run) after a positive cost-push shock: the optimal policy does not seek to bring the price level back to baseline.

Equation 42 - Propostion 6

Optimal Discretionary Policy: The optimal discretionary policy entails:

$$\pi_t = \frac{-\vartheta}{\kappa} x_t \tag{42}$$

so that on the equilibrium path: $i_t = K\nu_t + r_t^n$. where:

- $K = \frac{\kappa\sigma^{-1}(1-M\rho_\nu)+\vartheta\rho_\nu}{\kappa^2+\vartheta(1-\beta M^f\rho_\nu)}$

For persistent shocks (ρ_ν>0), the optimal policy is less aggressive (K is lower) when firms are more behavioral.

4 Implications for Fiscal Policy

4.1 Cognitive Discounting Generates a Failure of Ricardian Equivalence

Equation 43

The public debt evolves as:

$$B_{t+1} = B_t + Rd_t \tag{43}$$

where:

- B_t is the real value of government debt in period t, before period- t taxes.
- $d_t := \mathcal{T}_t + (r/R)B_t$. d_t is the budget deficit (after the payment of the interest rate on debt) in period t.
- $\mathcal{T}_t$ is the lump-sum transfer given by the government to the agent (so that $-\mathcal{T}_t$ is a tax).

No-Ponzi condition is the usual one, $\lim\limits_{t\to\infty} \beta^t B_t = 0$, which here takes the form $\lim\limits_{t\to\infty} \beta^t \left(\sum_{s=0}^{t-1} d_s\right) = 0$. Hence, debt does not necessarily mean-revert, and can follow a random walk.

Equation 44 & 45 - Propostion 7

Discounted Euler Equation with Sensitivity to Budget Deficits: Because agents are not Ricardian, budget deficits temporarily increase economic activity. The IS curve (24) becomes

$$x_t = M\mathbb{E}_t\left[x_{t+1}\right] + b_d d_t - \sigma\left(i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^{n0}\right) \tag{44}$$

where:

- r_t^{n0} is the "pure" natural rate with zero deficits (derived in (23)).
- d_t is the budget deficit.
- $b_d = \frac{\phi r R(1-\bar(m)}{(\phi+\gamma)(R-\bar(m)}$ is the sensitivity to deficits. When agents are rational, $b_d = 0$, but with behavioral agents, $b_d>0$.

In the sequel, we will write this equation by saying that the behavioral IS curve (25) holds, but with the following modified natural rate, which captures the stimulative action of deficits:

$$r_t^n = r_t^{n0} + \frac{b_d}{\sigma} d_t \tag{45}$$

Hence, bounded rationality gives both a discounted IS curve and an impact of fiscal policy: $b_d>0$. Deficit-financed (lump-sum) tax cuts have a “stimulative” impact on the economy.

4.2 Consequences for Fiscal Policy

Equation 46 - Lemma 4

First Best: When there are shocks to the natural rate of interest, the first best is achieved if and only if at all dates:

$$i_t = r_t^n \equiv r_t^{n0} + \frac{b_d}{\sigma} d_t \tag{46}$$

where:

- r_t^{n0} is the "pure" natural rate of interest given in (23) and is independent of fiscal and monetary policy.

This condition pins down the optimal sum of monetary and fiscal policy (i.e., the value of $i_t - (b_d/\sigma)d_t$), but not their precise values, as the two policies are perfect substitutes.

Equation 47

With behavioral agents, there is an easy first best policy:

$$\text{First best at the ZLB: } i_t = 0 \text{ and deficit: } d_t = \frac{-\sigma}{b_d r_t^{n0}} \tag{47}$$

i.e., fiscal policy runs deficits to stimulate demand.

Equation 48

Suppose that the government purchases at time 0 an amount G_0, financed by a deficit $d_0 = G_0$, and the central bank does not change the nominal rate at time 0 (we keep future deficits at 0 for t ¿ 0, so that debt is permanently higher). Then the fiscal multiplier is:

$$\frac{dY_0}{dG_0} = 1 + b_d \tag{48}$$

reflecting the fact that government spending has a "direct" effect of increasing GDP one-for-one, and then an "indirect" effect of making people feel richer.

5 Behavioral Enrichments to the Model

5.1 Term Structure of Consumer Attention

Equation 49

$$\begin{aligned} k_{t+1} &= \mathbf{G}^{k,BR}(c_t, N_t, k_t, \mathbf{X}_t) \\ &:= (1 + \bar{r} + \hat{r}^{BR}(\mathbf{X}_t))(k_t + \bar{y} + \hat{y}^{BR}(N_t, \mathbf{X}_t) - c_t) \end{aligned} \tag{49}$$

Equation (49) translates the perception of the law of motion of the personal wealth k. It is a rewriting of equation (6) with the addition of an attention discount factor to each of the perceived variables, where :

- k_{t+1} is the personal wealth at time $t+1$
- $\mathbf{G}^{k,BR}$ is the transition function of personal wealth under bounded rationality, i.e. the perception of the transition function under bounded rationality
- c_t is the consumption at time t
- N_t is work at time t
- $\bar{r}$ the interest rate at the steady state
- $\hat{r}^{BR}$ the perception under bounded rationality of the deviation of the interest rate from the steady state
- $\mathbf{X}_t$ the state vector
- $\bar{y}$ the income at the steady state
- $\hat{y}^{BR}(N_t, \mathbf{X}_t)$ the perception under bounded rationality of the deviation of the income from the steady state

Equation 50

$$\begin{cases} \hat{r}^{BR} = m_r \cdot \hat{r}(\mathbf{X}_t) \\ \hat{y}^{BR}(N_t, \mathbf{X}_t) = m_y \cdot \hat{y}(\mathbf{X}_t) + \omega(\mathbf{X}_t)(N_t - N_t(\mathbf{X}_t)) \end{cases} \tag{50}$$

The equation (50) defines the perceived values under bounded rationality of the interest rate and of the income, where :

- $\hat{r}^{BR}$ is the perception of the deviation of the interest rate from the steady state under bounded rationality
- $m_r \in [0,1]$ is the attention discount factor for the interest rate
- $\hat{r}_t(\mathbf{X}_t)$ is the objective value of the deviation of the interest rate from the steady state
- $\hat{y}^{BR}(N_t, \mathbf{X}_t)$ is the perception of the deviation of the personal income from the steady state under bounded rationality t
- $\hat{y}^{BR}(\mathbf{X}_t) = \hat{y}^{BR}(N(\mathbf{X}_t), \mathbf{X}_t) = m_y \cdot \hat{y}(\mathbf{X}_t)$ is the perceived deviation value of the aggregate income
- $m_y \in [0,1]$ is the attention discount factor for the personal income
- $\omega(\mathbf{X}_t)$ is the real wage
- $N(\mathbf{X}_t)$ is the aggregate labor supply

The perceptions of the laws of motion are thus further refined by differentiating the attention level in function of the economic variable considered. Taken together, equations (49) and (50) can be taken as more comple budget constraints, that are to take into account in the maximisation process of the utility function. Note that both discount factors m_r and m_y apply to the period t, meaning they are contemporaneous discount factor.

Equation 51, Lemma 5 (Term Structure of Attention)

$$\begin{cases} \mathbb{E}_t^{BR}\left[\hat{r}^{BR}(\mathbf{X}_{t+k})\right] = m_r \cdot \bar{m}^k \cdot \mathbb{E}_t\left[\hat{r}(\mathbf{X}_{t+k})\right] \\ \mathbb{E}_t^{BR}\left[\hat{y}^{BR}(\mathbf{X}_{t+k})\right] = m_y \cdot \bar{m}^k \cdot \mathbb{E}_t\left[\hat{y}(\mathbf{X}_{t+k})\right] \end{cases} \tag{51}$$

Equation (51) is the application of Lemma 1 (equation (11)) on the perceived values just defined in equations (49) and (50), where :

- $\mathbb{E}_t^{BR}$ is the operator of the expected value under bounded rationality, i.e. the behavioral expectation defined in equation (11)
- $\hat{y}^{BR}(N_t, \mathbf{X}_t)$ is the perception of the deviation of the personal from the steady state under bounded rationality, which is function of the number of hours worked, and of the macroeconomics state vector at time t
- $\hat{r}^{BR}$ is the perception of the deviation of the interest rate from the steady state under bounded rationality
- $m_r \in [0, 1]$ is the attention discount factor for the interest rate
- $m_y \in [0, 1]$ is the attention discount factor for the personal income
- $\bar{m}$ is the general discount factor applicable for all future variables

Equation (51) is the main consequence of the refinment previously done in equations (49) and (50) : not only are the agents generally myopic to the future, but they are also not fully attentive to present variables, and their attention level can vary depending on which one they consider. It can indeed seem plausible to say that they pay more attention to the income than to the interest rate, even when they have the information about the current period. In this case it would mean that $m_r < my$. This equation, coupled with the next one, argues that one is not fully rational even when they have acces to the information of their present income and personal Euler equation, because they have this present attention deficiency to income m_y.

Equation 52

$$\hat{c}_t = \mathbb{E}_t\left[\sum_{\tau \geq t} \frac{\bar{m}^{\tau-t}}{R^{\tau-t}} \left(b_r m_r \hat{r}(\mathbf{X}_\tau) + m_Y \frac{\bar{r}}{R} \hat{y}(\mathbf{X}_\tau)\right)\right] \tag{52}$$

This equation can be better understood with the precisions :

$$\begin{cases} c_t = c_t^d + \hat{c}_t \\ c_t^d = \bar{y} + b_k \cdot k_t \\ b_k := \frac{\bar{r}}{R} \cdot \frac{\phi}{\phi+\gamma} \end{cases}$$

$$\Longleftrightarrow$$

$$\hat{c}_t = c_t - c_t^d = c_t - \bar{y} - \frac{\bar{r}}{R} \cdot \frac{\phi}{\phi+\gamma} \cdot k_t$$

To get to the final form of equation (52), we should then plug the expression of (49), and take into account equations (50) and (51). Globally, equation (52) is the solution for consumption in a model with a structured attention of consumer, that relates to equations (18)-(19) in the baseline model, where :

- $\hat{c}_t$ is the value of the deviation from the steady state of consumption
- τ is a time period in the future
- $b_r := -\frac{1}{\gamma \cdot R^2}$ is the coefficient associated to the interest rate
- $m_Y = \frac{\phi \cdot m_y + \gamma}{\phi + \gamma}$ is the coefficient associated to the income

Even though the equation is more complex than in the baseline model, we should retain that the solution of the consumer here is only affected by this variable specific discount dampening.

Equation 52.5 : Discounted Euler Equation, with Term Structure of Attention (Proposition 9)

Equation 52.5 is not directly defined in the article, but is described as a refinment of equation (24).

$$x_t = M \cdot \mathbb{E}_t\left[x_{t+1}\right] - \sigma(\hat{r}_t - \hat{r}_t^n) \tag{52.5}$$

Where :

- $M = \frac{\bar{m}}{(R - r \cdot m_Y)} \in [0, 1]$ is the "macro", or global parameter of attention
- $\sigma := \frac{m_r}{\gamma \cdot R \cdot (R - r \cdot m_Y)} \in \left[0, \frac{1}{\gamma \cdot R}\right]$ is the coefficient of the effect of the interest rate gap on the output gap

Basically, this equation is the result of the refinment of equation (24) with the given attention structure.

Equation 53

$$\frac{\Delta^{GE}}{\Delta^{\text{direct}}} = R^{\tau+1} \tag{53}$$

To understand this equation, we have to add :

$$\begin{cases} \Delta^{\text{direct}} := \frac{\partial \hat{c}_0}{\partial \hat{r}_\tau}\Big|_{(y_t)_{t\geq 0} \text{ held constant}} = -\alpha \cdot \frac{1}{R^\tau} \\ \Delta^{GE} := \frac{\partial \hat{c}_0}{\partial \hat{r}_\tau} = -\alpha R \end{cases}$$

Equation (53) describes the effect of a future variation of the interest rate on consumption on the present consumption of **of the rational agent**, where :

- Δ^{direct} is the direct effect of the variation of interest rate on present consumption, with no retroaction effect on the income
- Δ^{GE} is the indirect effect of the variation of interest rate on present consumption, with retroaction effect on the income

Equation (53) refers to the Keynesian coefficient multiplicator in the rational case. Dividing the indirect impact by the direct impact allows to see the magnitude of the general impact of the variation of the interest rate.

Equation 54

$$\frac{\Delta^{GE}}{\Delta^{\text{direct}}} = \left(\frac{R}{R - rm_Y}\right)^{\tau+1} \in \left[1, R^{\tau+1}\right] \tag{54}$$

To understand this equation, we have to add :

$$\begin{cases} \Delta^{\text{direct}} := \frac{\partial \hat{c}_0}{\partial \hat{r}_\tau}\Big|_{(y_t)_{t\geq 0} \text{ held constant}} = -\alpha \cdot m_r \cdot \bar{m}^\tau \frac{1}{R^\tau} \\ \Delta^{GE} := \frac{\partial \hat{c}_0}{\partial \hat{r}_\tau} = -\alpha m_r \cdot M^\tau \frac{R}{R - r \cdot m_Y} R \end{cases}$$

Where :

- $M = \frac{\bar{m}}{R - r \cdot m_Y}$ is the attention discount factor
- Δ^{direct} is the direct effect of a change of real interest rate
- Δ^{GE} is the indirect effect of a change of real interest rate

Equation (54) is the same as equation (53), but withing the beahvioral framework, with a structured attention term. We see that within this framework, with a attention dampening for future periods, the magnitude of the interest rate variation is lessened by the attention coefficients. This explains why the Keynesian multiplication factor is not as strong as predicted in real life. Equation (53) only works if consumers believe that other consumers are fully rational, equation (54) does not impose this condition.

5.2 Flattening of the Phillips Curve via Imperfect Firm Attention

Equation 55

$$v^{BR}(q_{it}, (\mathbf{X}_\tau)) := v^0 \left(q_{it} - m_\pi^f \Pi(\mathbf{X}_\tau), m_x^f \mu(\mathbf{X}_\tau), c(\mathbf{X}_\tau)\right) \tag{55}$$

Equation (55) introduces specific attention deficiency for firms, where :

- $v^0(q_{i\tau}, \mu_\tau, c_\tau) := \left(e^{q_{i\tau}} - (1-\tau_f)e^{-\mu_\tau}\right) e^{-\varepsilon q_{i\tau}} c_\tau$ is the current real profit of the firm, as in equation (14) of the baseline model
- m_π^f is the attention deficit to inflation
- m_x^f is the attention deficit to marginal cost

This equation takes the main elements of equation (15), but introduces a variable specific attention deficiency factor for the firms. It allows to have a new maximisation program, for firms with bounded rationality, described in the next equation.

Equation 56

$$\max_{q_{it}} \mathbb{E}_t^{BR} \left[\sum_{\tau=t}^{\infty} (\beta\theta)^{\tau-t} \frac{c(\mathbf{X}_\tau)^{-\gamma}}{c(\mathbf{X}_t)^{-\gamma}} v^{BR}(q_{it}, \mathbf{X}_\tau) \right] \tag{56}$$

Equation (56) describes the maximisation program of a firm under bounded rationality, it is the same as equation (16), where :

- v^{BR} is the perception of the real profit at time t, described in equation (55)

Equation 57

$$p_t^* = p_t + (1-\beta\theta) \sum_{k=0}^{\infty} (\beta\theta\bar{m})^k \, \mathbb{E}_t \left[m_\pi^f (\pi_{t+1} + ... + \pi_{t+k}) - m_x^f \mu_{t+k} \right] \tag{57}$$

Equation (57) is the solution to the maximisation program described in equation (56), but with the introduced variable specific attention discount factors, m_π^f and m_x^f. It is the equivalent of equation (27) in the baseline model.

Equation 57.5 - Proposition 10

Equation (57.5) is not directly mentioned in the article, but is described as a refined version of equation (29) with different parameters.

$$\pi_t = \beta \cdot M^f \cdot \mathbb{E}t \left[\pi_{t+1}\right] + \kappa \cdot x_t \tag{57.5}$$

$$\Longleftrightarrow$$

$$\pi_t = \beta \cdot \bar{m} \left(\theta + m_\pi^f \cdot (1-\theta) \cdot \frac{1 - \beta \cdot \theta}{1 - \beta \cdot \theta \cdot \bar{m}} \right) \cdot \mathbb{E}t \left[\pi_{t+1}\right] + m_x^f \cdot \bar{\kappa} \cdot x_t \tag{57.5}$$

The two differences are given by :

- $M^f = \bar{m}\left(\theta + m_\pi^f \cdot (1-\theta) \cdot \frac{1-\beta\cdot\theta}{1-\beta\cdot\theta\cdot\bar{m}}\right) \in [0,1]$ for the general attention factor of the firm
- $\kappa = m_x^f \cdot \bar{\kappa}$, as detailed in the next equation

Equation 58

$$\kappa = m_x^f \bar{\kappa} \tag{58}$$

Equation (58) describes the attention by the firm under bounded rationality to current macroeconomic production, where :

- m_x^f is the attention deficiency to the output gap
- $\bar{\kappa}$ is the slope of the traditional Phillips curve, i.e. the effect coefficient of the output gap on inflation

Equation (58) depicts the fact that the firms attention deficiency to the output gap affects the way they perceive their potential profit.

5.3 Nonconstant Trend Inflation and Neo-Fisherian Paradoxes

Equation 59

$$\pi_t^d = (1-\zeta)\bar{\pi}_t + \zeta\bar{\pi}_t^{CB} \tag{59}$$

Where :

- π_t^d is the "default" inflation value, perceived by the firms
- $\bar{\pi}_t$ is the moving average of past inflation
- $\bar{\pi}_t^{CB}$ is the inflation guidance, i.e. the inflation target declared by the Central Bank
- $\zeta \in [0,1]$ is a weight factor on past inflation, which is not the same as the Total Factor productivity defined in the baseline model

Equation (59) describes the fact that firms predict a default value of inflation that is given by a weighted average of what they observed in the past regarding the actual inflation and the central bank policy.

Equation 60

$$x_t = M \cdot \mathbb{E}_t\left[x_{t+1}\right] - \sigma\left(i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^n\right) \tag{60}$$

Equation (60) is exactly the same as the IS curve from equation (28).

Equation 61

$$\hat{\pi}_t = \beta \cdot M^f \cdot \mathbb{E}_t \left[\hat{\pi}_{t+1}\right] + \kappa \cdot x_t \tag{61}$$

Equation (61) is obtained by taking into account the nonzero trend inflation in the formulation of inflation. This equation is thus the same as (29), but with inflation π_t replace by $\hat{\pi}_t$ the deviation from the default value, where :

- $\pi_t = \pi_t^d + \hat{\pi}_t$, with π_t^d the default inflation and $\hat{\pi}_t$ the deviation from the default value

Together, equations (60) and (61) constitute Proposition 11 of the paper, and describe the behavioral new keynesian model augmented by a nonzero trend inflation.

Equation 62, Proposition 12

$$\phi_\pi + \zeta \frac{(1-\beta M^f)}{\kappa} \phi_x + \zeta \frac{(1-\beta M^f)(1-M)}{\kappa\sigma} > 1 \tag{62}$$

Equation (62) describes the Equilibrium Determinacy of the model with Behavioral Agents. It is the condition for the model to be determinate in this refined framework, i.e. the equivalent of equation (34) in the baseline model. All the terms have been defined previously.