1 A Behavioral Model

The first two equations are the same as (28) and (29) and will be explained later.

1.1 Introduction

Equation 1

$$x_t = M \cdot \mathbb{E}_t \left[xt + 1 - \sigma(i_t - \mathbb{E}_t \left[\pi_{t+1} \right] - r_t^n) \right] \tag{1}$$

Equation 2

$$\pi_t = \beta \cdot M^f \mathbb{E}_t \left[\pi_{t+1} \right] + \kappa \cdot x_t \tag{2}$$

1.2 Basic Setup and the Household's Problem

Equation 3

$$U = \mathbb{E}_t \left[\sum t = 0^\infty \beta^t \left(\frac{c_t^{1-\gamma} - 1}{1-\gamma} - \frac{N_t^{1+\phi}}{1+\phi} \right) \right] \tag{3}$$

Equation (3) is the flow utility of the Household, with :

- β the discount factor
- c_t the consumption of the houshold at time t
- N_t the work of the household at time t
- γ determines the concavity of the utility function with respect to the consumption, i.e. the importance of consumption in the utility function
- ϕ determines the concavity of the utility function with respect to work, i.e. the importance of work in the utility function

Equation 4

$$k_{t+1} = (1 + r_t)(k_t - c_t + y_t) \tag{4}$$

Equation (4) is the law of motion of the real financial wealth of the household, where :

- k_t is the real financial welath of the household at time t
- r_t is the real interest rate
- w_t is the real wage
- y_t is the agent's real income, defined as $y_t = w_t \cdot N_t + y_t^f$, with y_t^f the profit income (or the income from firms) at time t

Equation 5

$$\boldsymbol{X}_{t+1} = \boldsymbol{G}^{\boldsymbol{X}} \left(\boldsymbol{X}_t, \boldsymbol{\epsilon}_{t+1} \right) \tag{5}$$

Equation (5) describes the evolution of macroeconomic variables, where :

- $\boldsymbol{X}_t$ is the state vector, including several macroeconomic variables of time t, like ζ_t the aggregate TFP, and the announced actions in monetary and fiscal policy
- $\boldsymbol{G}^X$ the equilibrium transition function, i.e. the function that gives the macroeconomic variables at time $t+1$ from the macroeconomic variables at the previous period
- ϵ_t is the innovation in the economy at time t, with $\mathbb{E}_t \left[\epsilon t + 1 \right] = 0$, that depends on the equilibrium policies of the agent and of the government

Equation 6

$$\begin{aligned} k_{t+1} &= G^k(c_t, N_t, k_t, \boldsymbol{X}_t) \\ &:= (1 + \bar{r} + \hat{r}(\boldsymbol{X}_t))(k_t + \bar{y} + \hat{y}(N_t, \boldsymbol{X}t) - c_t) \end{aligned} \tag{6}$$

Equation (6) is the application of the consideration of a set of macroeconomic variables on the law of motion of real financial wealth k_t, where :

- $\bar{r}$ is the steady state value of the real interest rate, that does not depend on time
- $\hat{r}(\boldsymbol{X}_t)$ is the value of the deviation from the steady state of the real interest rate, that depends on the state vector $\boldsymbol{X}_t$ at time t
- $\bar{y}$ is the steady state value of the agent's real income, that does not depend on time
- $\hat{y}(N_t, \boldsymbol{X}_t)$ is the deviation from the steady state of the agent's real income, that depends on the number of hours worked at time t and on the state vector at time t
- c_t is the aggregate consumption level at time t of the agent

Equation 7

$$\boldsymbol{X}_{t+1} = \boldsymbol{\Gamma} \boldsymbol{X}_t + \boldsymbol{\epsilon}_{t+1} \tag{7}$$

Equation (7) describes the linear version of the equilibrium transition function, it is the linearization of the law of motion, where :

- Γ is a squared matrix that multiplies the state vector
- $\boldsymbol{X}_t$ is the state vector at time t
- ϵ_t is the innovation shock

Equation 8 (Assumption 1)

$$\boldsymbol{X}_{t+1} = \bar{m} \cdot \boldsymbol{G}^{\boldsymbol{X}}(\boldsymbol{X}_t, \boldsymbol{\epsilon}_{t+1}) \tag{8}$$

Equation (8) describes the Cognitive Discounting of the State Vector, i.e. the perception by behavioural agents of the law of motion of the macroeconomic variables, where :

- $\bar{m} \in [0n1]$ is the cognitive discount factor measuring the attention to the future

Equation 9

$$\boldsymbol{X}_{t+1} = \bar{m}(\boldsymbol{\Gamma} \boldsymbol{X}_t + \boldsymbol{\epsilon}_{t+1}) \tag{9}$$

Equation (9) is the linearized version of the perception by behavioral agents of the law of motion of the state vector.

Equation 10

$$\mathbb{E}_t^{BR}\left[\boldsymbol{X}_{t+k}\right] = \bar{m}^k \mathbb{E}_t\left[\boldsymbol{X}_{t+k}\right] \tag{10}$$

Equation (10) defines the expectation of behavioral agents in function of the rational perception of the law of motion of the state vector, where :

- $k \geq 0$ a time period in discrete context
- $\mathbb{E}_t^{BR}\left[\boldsymbol{X}_{t+k}\right]$ is the expected value of the state vector at time $t+k$ by behavioral agents (or subjective/behavioral expectation operator)
- $\bar{m}^k$ is the cognitive discounting effect at period $t+k$
- $\mathbb{E}_t\left[\boldsymbol{X}_{t+k}\right]$ is the rational expectation of the state vector at time $t+k$

Equation 11 (Lemma 1)

$$\mathbb{E}_t^{BR}\left[z\left(\boldsymbol{X}_{t+k}\right)\right] = \bar{m}^k \cdot \mathbb{E}_t\left[z\left(\boldsymbol{X}_{t+k}\right)\right] \tag{11}$$

Equation (11) defines in the general case the behavioral expectation operator, for any function of the state vector, where :

- $k \geq 0$ a time period in discrete context
- $z(\cdot)$ is a function, such that $z(0) = 0$
- $\mathbb{E}_t^{BR}\left[z\left(\boldsymbol{X}_{t+k}\right)\right]$ is the expected value of the image of the state vector by the function $z(\cdot)$ at time $t+k$ by behavioural agents
- $\bar{m}^k$ is the cognitive discounting effect at period $t+k$
- $\mathbb{E}_t\left[z\left(\boldsymbol{X}_{t+k}\right)\right]$ is the rational expectation of the image of the state vector by the function $z(\cdot)$ at time $t+k$

Equation 12

$$\mathbb{E}_t^{BR}\left[\bar{r} + \hat{r}\left(\boldsymbol{X}_{t+k}\right)\right] = \bar{r} + \bar{m}^k \mathbb{E}_t\left[\hat{r}(\boldsymbol{X}_{t+k})\right] \tag{12}$$

Equation (12) is an example of the Lemma 1 applied to the interest rate, where :

- $k \geq 0$ a time period in discrete context
- $\bar{r}$ the steady state level of the real interest rate, that does not depend on time,
- $\hat{r}(\boldsymbol{X}_{t+k})$ is the equilibrium transition function defining the value of the deviation from the steady state of the real interest rate in the function of the state vector at time $t+k$
- $\bar{r} + \hat{r}(\boldsymbol{X}_t) = rt(\boldsymbol{X}_t)$ is the value of the real interest rate at time t
- $\mathbb{E}_t^{BR}\left[\bar{r} + \hat{r}(\boldsymbol{X}_{t+k})\right]$ is the expected value of the real interest at time $t+k$ by behavioural agents
- $\mathbb{E}_t\left[\hat{r}(\boldsymbol{X}_{t+k})\right]$ is the rational expectation of the value of the deviation of the real interest rate from the steady state at time $t+k$

1.3 The Firm's problem

Equation 13

$$P_t = \left(\int_0^1 P_{it}^{1-\varepsilon}\, di\right)^{\frac{1}{1-\varepsilon}} \tag{13}$$

Equation (13) describes the aggregate price level, where :

- P_t is the aggregate price level of the economy at time t
- $i \in [0, 1]$ is the firm index
- ε is the elasticity of substitution between goods

Equation 14

$$v^0(q_{i\tau}, \mu_\tau, c_\tau) := \left(e^{q_{i\tau}} - (1-\tau_f)e^{-\mu_\tau}\right) e^{-\varepsilon q_{i\tau}} c_\tau \tag{14}$$

Equation (14) describes the profit of the firm before the lump sum tax of the government, where :

- v is the real profit of the firm
- $q_{i\tau} = \ln\left(\frac{P_{i\tau}}{P_\tau}\right) = p_{i\tau} - p_\tau$ is the real log price at time τ
- $\tau_f = \frac{1}{\varepsilon}$ it the corrective wage subsidy from the government, funded by the lump sum tax

- $\mu_\tau = \zeta_t - \ln(\omega_t)$ is the labor wedge, which is zero at efficiency
- ε is the elasticity of subsititution between goods
- c_τ is the aggregate level of consumption

Equation 15

$$v\left(q_{it}, \boldsymbol{X}\tau\right) := v^0\left(q_{it} - \Pi(\boldsymbol{X}\tau), \mu(\boldsymbol{X}\tau), c(\boldsymbol{X}_\tau)\right) \tag{15}$$

Equation 15 describes the flow profit of the firm in function of the real log price and of the extended macro state vector, where :

- $q_{it} = \ln\left(\frac{P_{it}}{P_t}\right) = p_{it} - p_t$ is the real log price
- $\boldsymbol{X}\tau = (\boldsymbol{X}^{\mathcal{M}}\tau, \Pi_\tau)$ is the extended macro state vector, with $\boldsymbol{X}^{\mathcal{M}\tau}$ the vector of macro variables, including $\zeta\tau$ and possible announcements about future policy
- $\Pi\left(\boldsymbol{X}\tau\right) := p\tau - p_t = \pi_{t+1} + ... + \pi_\tau$ is the inflation between times t and τ
- $q_{it} - \Pi(\boldsymbol{X}\tau) = qi\tau$ is the real price of the firm if they didn't change its price between t and τ
- $\mu\left(\boldsymbol{X}_\tau\right)$ is the aggregate labor wedge in function of the extended state vector at time t
- $c(\boldsymbol{X}_\tau)$ is the aggregate consumption level in function of the extended state vector at time t

Equation 16

$$\max_{q_{it}} \mathbb{E}_t\left[\sum_{\tau=t}^{\infty}(\beta\theta)^{\tau-t}\frac{c\left(\boldsymbol{X_\tau}\right)^{-\gamma}}{\left(\boldsymbol{X_t}\right)^{-\gamma}} v\left(q_{it}, \boldsymbol{X}\tau\right)\right] \tag{16}$$

Equation (16) describes the maximisation program of the firm given that they have a Calvo-like probability of θ of being able to change their price at each period, where :

- t is the initial period
- τ is the time period index
- q_{it} is the real log price of the firm at time t
- β is the discount factor
- θ is the Calvo like probability that the firm can change its price at any period
- $\frac{c\left(\boldsymbol{X}\tau^{-\gamma}\right)}{c\left(\boldsymbol{X}_t^{-\gamma}\right)}$ is the adjustment in the stochastic discount factor/pricing kernel between times t and τ

Equation 17

$$\max_{q_{it}} \mathbb{E}_t^{BR}\left[\sum \tau = t^{\infty} (\beta\theta)^{\tau - t} \frac{c\left(\boldsymbol{X}\tau^{-\gamma}\right)}{c\left(\boldsymbol{X}_t^{-\gamma}\right)} v\left(q_{it,\boldsymbol{X}_\tau}\right)\right] \tag{17}$$

Equation 17 describes the maximisation program of the behavioral firm, i.e. such that it is maximisation the behavioral expectation operator of the flow profit, where :

- $\mathbb{E}_t^{BR}$ is the behavioral/subjective expected value operator
- t is the initial period
- τ is the time period index
- q_{it} is the real log price of the firm at time t
- β is the discount factor
- θ is the Calvo like probability that the firm can change its price at any period
- $\frac{c\left(\boldsymbol{X}\tau^{-\gamma}\right)}{c\left(\boldsymbol{X}_t^{-\gamma}\right)}$ is the adjustment in the stochastic discount factor between times t and τ and it is approximately 1 when linearised around deterministic steady state.

1.4 Model solution

Equation 18

$$\hat{c}_t = \mathbb{E}_t\left[\hat{c}_{t+1} - \frac{1}{\gamma R}\hat{r}_t\right] \tag{18}$$

Equation (18) is the linearized version of the Euler equation obtained from the presented model. It is also called the investment-savings (IS) curve, where :

- $\hat{c}_t$ is the value of the deviation from the steady state of the aggregate consumption at time t
- $\mathbb{E}_t\left[\hat{c}_{t+1}\right]$ is the rational expectation of the value of the deviation from the steady state of the aggregate consumption at time $t+1$
- γ is the factor of the importance of consumption
- $R := 1 + \bar{r}$ is defined from the real interest rate at the steady state (cf. page 7 of the article)

Equation 19

$$\hat{c}_t = M \cdot \mathbb{E}_t \left[\hat{c}_{t+1} \right] - \sigma \hat{r}_t \tag{19}$$

Equation (19) is the application of Lemma 1 (equation (11)) on the previous Euler equation, i.e. a cognitively discounted aggregate Euler equation, where :

- M is the macro parameter of attention, such that $M = \bar{m}$ here
- $\sigma = \frac{1}{\gamma R}$

Equation 20

$$N_t^{\phi} = \omega_t c_t^{\gamma} \tag{20}$$

Equation (20) the result of the static First Order Condition for labor supply, where :

- N_t is the quantity of labor provided at time t
- ω_t is the real wage at time t
- c_t is the aggregate quantity of consumption at time t
- γ is the consumption importance in the utility

Equation 21

$$\hat{c}_t^n = \frac{1+\phi}{\gamma+\phi} \zeta t \tag{21}$$

Equation (21) is the equation giving us the consumption in a natural economy where there are no frictions in prices i.e. flexible price economy. where :

- $\hat{c}_t^n$ is the flexible price/natural economy consumption
- γ is the the elasticity of substitution between goods.
- ϕ is the Frisch elasticity of labour
- ζ_t is the Total Factor prodcutivity

  We calculate this natural economy consumption the same way we calculated the flexible price economy consumption using the log-linearised version of the natural economy price, first-order condition for labour supply, and market clearing condition for consumption and income.

is the flexible price/natural economy consumption

Equation 22

$$\hat{c}_t^n = M \cdot \mathbb{E}_t \left[\hat{c}_{t+1}^n\right] - \sigma \hat{r}_t^n \tag{22}$$

Equation (22) is the application of Lemma 1 (equation (11)) on the previous Euler equation, i.e. a cognitively discounted aggregate Euler equation for the natural economy where there are no price frictions. Here :

- M is the macro parameter of attention, such that $M = \bar{m}$ here
- $\sigma = \frac{1}{\gamma R}$
- $\hat{r}_t^n$ is the real interest rate for the natural economy

Equation 23

$$r_t^{n0} = \bar{r} + \frac{1+\phi}{\sigma(\gamma+\phi)} \left(M \cdot \mathbb{E}_t \left[\zeta_{t+1}\right] - \zeta_t\right) \tag{23}$$

Equation (23) is the equation for the pure natural rate of interest: this is the interest rate that prevails in an economy without pricing frictions, and undisturbed by government policy (in particular, budget deficits). In this equation :

- r_t^{n0} is pure natural rate of interest
- γ is the the elasticity of substitution between goods.
- ϕ is the Frisch elasticity of labour
- $\sigma = \frac{1}{\gamma R}$
- M is the macro parameter of attention, such that $M = \bar{m}$ here
- ζ_t and ζ_{t+1} is the Total Factor productivity in time t and $t+1$ respectively

This pure natural interest rate is calculated by isolating the $\hat{r}_t^n$ in Equation (22) and then replacing $\hat{c}_t^n = \frac{1+\phi}{\gamma+\phi}\zeta t$, $\hat{c}_{t+1}^n = \frac{1+\phi}{\gamma+\phi}\zeta t + 1$ and $\hat{r}_t^n = r_t^n - \bar{r}$ and by assuming $r_t^n = r_t^{n0}$ which holds when there are no budget deficits.

Equation 24

$$x_t = M \cdot \mathbb{E}_t \left[xt + 1\right] - \sigma(\hat{r}_t - \hat{r}_t^n) \tag{24}$$

Equation (24) is the equation for the behavioural discounted Euler equation where :

- x_t is the output gap at period t
- x_{t+1} is the output gap at period $t+1$
- M is the macro parameter of attention, such that $M = \bar{m}$ here

- $\sigma = \frac{1}{\gamma R}$
- $\hat{r}_t$ is the deviation of the real interest rate from the steady state.
- $\hat{r}_t^n$ is the deviation of the real interest rate from the steady state for the natural economy.

This is calculated by subtracting the expression $\hat{c}_t = M \cdot \mathbb{E}_t\left[\hat{c}_{t+1}\right] - \sigma \hat{r}_t$ i.e. Equation (19) and $\hat{c}_t^n = M \cdot \mathbb{E}_t\left[\hat{c}_{t+1}^n\right] - \sigma \hat{r}_t^n$ i.e. Equation (22)

Equation 25

$$x_t = M \cdot \mathbb{E}_t\left[xt+1\right] - \sigma(i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^n) \tag{25}$$

Equation (25) is the behavioural discounted Euler equation after replacing the equation for Fisher equation where $\hat{r}_t = r_t - \bar{r} = (i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^n)$

- i_t is the nominal interest rate
- $\mathbb{E}_t\left[\pi_{t+1}\right]$ is the expected inflation in the future
- $M = \bar{m}$
- $\sigma = \frac{1}{\gamma R}$

Equation 26

$$x_t = -\sigma \sum_{k \geq 0} M \cdot \mathbb{E}_t\left[\hat{r}_{t+k} - \hat{r}_{t+k}^n\right] \tag{26}$$

Equation (26) iteratively using Equation (24) reduces to this equation. It indicates that changes in the interest rate in the 1000^{th} period will have a discounted impact on the output gap. Therefore, the effect of changes in interest rates on the output gap diminishes over time, with changes in the 1000^{th} period having a smaller impact compared to changes in the 1^{st} period.

Equation 27

$$p_t^* = p_t + (1-\beta\theta) \sum_{k=0}^{\infty} (\beta\theta\bar{m})^k \cdot \mathbb{E}_t\left[\pi_{t+1} + ... + \pi_{t+k} - \mu_{t+k}\right] \tag{27}$$

Equation (27) is the optimal price for a behaviourial firm resetting its price

- p_t^* is the the price a behaviourial firm will reset its price to.
- $\bar{m}$ is the cognitive discounting factor.
- $p_t^* = qit + p_t$ where q_{it} is the linearised version of the maximisation solution to problem 17. q_{it} is the price for the firms that can adjust their prices and p_t is the price of the firms that can't adjust their prices.

Equation 28 - Proposition 2, the first equation

$$x_t = M \cdot \mathbb{E}_t [xt + 1] - \sigma(i_t - \mathbb{E}_t [\pi_{t+1}] - r_t^n) \tag{28}$$

Equation (28) is the Behavioural IS Curve. Essentially we are representing the Euler equation in terms of the output gap and also using the Fisher Equation.

Equation 29 - Proposition 2, second equation

$$\pi_t = \beta \cdot M^f \mathbb{E}_t [\pi_{t+1}] + \kappa \cdot x_t \tag{29}$$

Equation (29) is the Behavioural Phillips curve where:

- M^f is the aggregate attention parameter for firms and $M^f = \bar{m} \left(\theta + \frac{1-\beta\theta}{1-\beta\theta\bar{m}}(1-\theta) \right)$
- $\kappa = \widetilde{\kappa}$ is the slope of the Phillips curve and $\widetilde{\kappa} = \frac{(1-\theta)(1-\beta\theta)}{\theta}(\gamma + \phi)$ which is the slope obtained from fully rational firms.

Equation 30

$$\begin{cases} M = \bar{m} \\ \sigma = \frac{1}{\gamma R} \\ M^f = \bar{m} \left(\theta + \frac{1-\beta\theta}{1-\beta\theta\bar{m}}(1-\theta) \right) \end{cases} \tag{30}$$

Equation (30) defines the parameters in Proposition 2.