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MASTER'S THESIS

Semidefinite Programming for Geometric Problems in Computer Vision

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Author’s declaration

I declare that I have work out the presented thesis independently and that I have listed all information sources used in accordance with the Methodical Guidelines about Maintaining Ethical Principles for Writing Academic Theses.

Prohlášení autora práce

Prohlašuji, že jsem předloženou práci vypracoval samostatně a že jsem uvedl veškeré použité informační zdroje v souladu s Metodickým pokynem o dodržování etických principů při přípravě vysokoškolských závěrečných prací.

V Praze dne ....

Podpis autora práce

Abstract

Keywords:

Abstrakt

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List of Algorithms

1. Introduction

[1]

2. Semidefinite programming

2.1. Preliminaries on semidefinite programs

We introduce here some notation and preliminaries about symmetric matrices and semidefinite programs. We will introduce further notation and preliminaries later on in the text when needed.

At the beginning, let us denote the inner product for two vectors x, $y \in \mathbb{R}^n$

$$\langle x, y\rangle \quad = \quad \sum_{i=1}^{n} x_i y_i \tag{2.1}$$

and the Frobenius inner product for two matrices X, $Y \in \mathbb{R}^{n\times m}$.

$$\langle X, Y\rangle \quad = \quad \sum_{i=1}^{n}\sum_{j=1}^{m} X_{ij} Y_{ij} \tag{2.2}$$

2.1.1. Symmetric matrices

Let Sym_n denotes the space of $n \times n$ real symmetric matrices.

For matrix $M \in \mathrm{Sym}_n$, the notation $M \succeq 0$ means that M is positive semidefinite. $M \succeq 0$ if and only if any of the following equivalent properties holds.

1. $x^\top M x \geq 0$ for all $x \in \mathbb{R}^n$.
2. All eigenvalues of M are nonnegative.

For matrix $M \in \mathrm{Sym}_n$, the notation $M \succ 0$ means that M is positive definite. $M \succ 0$ if and only if any of the following equivalent properties holds.

1. $M \succeq 0$ and $\operatorname{rank} M = n$.
2. $x^\top M x > 0$ for all $x \in \mathbb{R}^n$.
3. All eigenvalues of M are positive.

2.1.2. Semidefinite programs

The standard (primal) form of a semidefinite program in variable $X \in \mathrm{Sym}_n$ is defined as follows:

$$\begin{aligned} p^* \quad = \quad & \sup_{X\in\mathrm{Sym}_n} \quad \langle C, X\rangle \\ & \text{s.t.} \qquad \langle A_i, X\rangle \;=\; b_i \quad (i = 1, \ldots, m) \\ & \qquad\qquad\qquad X \;\succeq\; 0 \end{aligned} \tag{2.3}$$

where C, A_1, $\ldots$, $A_m \in \mathrm{Sym}_n$ and $b \in \mathbb{R}^m$ are given.

The dual form of the primal form is the following program in variable $y \in \mathbb{R}^m$.

$$\begin{aligned} d^* \quad = \quad & \inf_{y\in\mathbb{R}^m} \quad b^\top y \\ & \text{s.t.} \quad \sum_{i=1}^{m} A_i y_i - C \;\succeq\; 0 \end{aligned} \tag{2.4}$$

2.2. State of the art review

2.3. Theoretical background

2.4. Nesterov's approach

In this section, we will follow Chapter 4 of [2] by Y. Nesterov, which is devoted to convex minimization problems. We will extract from it only the minimum, just to be able to introduce a algorithm for SDP programs solving. We will present some basic definitions and theorems, but we will not prove them. For the proofs look into [2].

2.4.1. Self-concordant functions

Definition 2.1 (Self-concordant function in $\mathbb{R}$). A closed convex function $f : \mathbb{R} \mapsto \mathbb{R}$ is self-concordant if there exist a constant $M_f \geq 0$ such that the inequality

$$|f'''(x)| \leq M_f f''(x)^{3/2} \tag{2.5}$$

holds for all $x \in \operatorname{dom} f$.

For better understanding of self-concordant functions we provide several examples.

Example 2.1.

1. Linear and convex quadratic functions.

$$f'''(x) = 0 \text{ for all } x \tag{2.6}$$

   Linear and convex quadratic functions are self-concordant with constant $M_f = 0$.

2. Negative logarithms.

$$\begin{aligned} f(x) &= -\ln(x) \text{ for } x > 0 && (2.7)\\ f'(x) &= -\frac{1}{x} && (2.8)\\ f''(x) &= \frac{1}{x^2} && (2.9)\\ f'''(x) &= -\frac{2}{x^3} && (2.10)\\ \frac{|f'''(x)|}{f''(x)^{3/2}} &= 2 && (2.11) \end{aligned}$$

   Negative logarithms are self-concordant functions with constant $M_f = 2$.

3. Exponential functions.

$$\begin{aligned} f(x) &= e^x && (2.12)\\ f''(x) = f'''(x) &= e^x && (2.13)\\ \frac{|f'''(x)|}{f''(x)^{3/2}} &= e^{-x/2} \to \infty \text{ as } x \to -\infty && (2.14) \end{aligned}$$

   Exponential functions are not self-concordant functions.

Definition 2.2 (Self-concordant function in $\mathbb{R}^n$). A closed convex function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is self-concordant if function

$$g(t) = f(x + tv) \tag{2.15}$$

is self-concordant for all $x \in \operatorname{dom} f$ and all $v \in \mathbb{R}^n$.

Now, let us focus on the main properties of self-concordant functions.

Theorem 2.1. Let functions f_i be self-concordant with constants M_i and let $\alpha_i > 0$, $i = 1, 2$. Then the function

$$f(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x) \tag{2.16}$$

is self-concordant with constant

$$M_f = \max\left\{\frac{1}{\sqrt{\alpha_1}} M_1, \frac{1}{\sqrt{\alpha_2}} M_2\right\} \tag{2.17}$$

and

$$\operatorname{dom} f = \operatorname{dom} f_1 \cap \operatorname{dom} f_2. \tag{2.18}$$

Corollary 2.1. Let function f be self-concordant with some constant M_f and $\alpha > 0$. Then the function $\phi(x) = \alpha f(x)$ is also self-concordant with the constant $M_\phi = \frac{1}{\sqrt{\alpha}} M_f$.

We call function $f(x)$ as the standard self-concordant function if $f(x)$ is some self-concordant function with the constant $M_f = 2$. Using Corollary 2.1, we can see that any self-concordant function can be transformed into the standard self-concordant function by scaling.

Theorem 2.2. Let function f be self-concordant. If $\operatorname{dom} f$ contains no straight line, then the Hessian $f''(x)$ is nondegenerate at any x from $\operatorname{dom} f$.

For some self-concordant function $f(x)$, for which we assume, that $\operatorname{dom} f$ contains no straight line (which implies that all $f''(x)$ are nondegenerate, see Theorem 2.2), we denote two local norms as

$$\|u\|_x = \sqrt{u^\top f''(x) u} \tag{2.19}$$

$$\|u\|_x^* = \sqrt{u^\top f''(x)^{-1} u}. \tag{2.20}$$

2.5. Implementation details

2.6. Comparison with the state of the art methods

3. Conclusion

A. Contents of the enclosed CD

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Bibliography

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