Introduction to Optimization
HW 4

## Augmented Lagrangian method for Constrained Optimization

**Task 1**

1. Find the optimal solution to the following optimization problem:

$$\min_x x^T Mx + c^T x$$

$$s.t.$$

$$Ax = b$$

$$M \succ 0$$

$$A \in \mathbb{R}^{m\times n}, x \in \mathbb{R}^n, c \in \mathbb{R}^n, b \in \mathbb{R}^m, M \in \mathbb{R}^{n\times n}$$

Assume $M, AM^{-1}A^T$ are invertible.

2. Find the optimal solution to the following optimization problem:

$$\min_x \|x - c\|_2^2$$

$$s.t.$$

$$Ax = b$$

$$x \in \mathbb{R}^n, c \in \mathbb{R}^n$$

Assume $AA^T$ is invertible.

Consider the problem:

$$\min_x x^T Ax + b^T x$$

$$s.t.$$

$$1_3^T x = 1$$

$$x_3 \le 1$$

$$1_3 \triangleq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathrm{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, x \in \mathbb{R}^3$$

a. Is the problem convex?
b. Find and solve the dual problem.

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3. Write the dual problem for the following linear programming problem:

$$\min_x c^T x$$
$$s.t.$$
$$Ax \le b$$
$$x \ge 0$$
$$A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n, c \in \mathbb{R}^n, b \in \mathbb{R}^m$$

**Task 2**

Consider the following quadratic programming problem:

$$minimize \quad 2(x_1 - 5)^2 + (x_2 - 1)^2$$
$$subject\ to:$$
$$x_2 \le 1 - \frac{x_1}{2}$$
$$x_2 \ge x_1$$
$$x_2 \ge -x_1$$

1. Draw graphically (approximately, by hand, or in computer) feasible area and the level sets of the objective function. Find out (from the graph) the active constraints.

2. Calculate the optimal solution at the intersection of the active constraints.

3. Calculate the Lagrange multipliers using KKT conditions. What is the optimal value of the objective function?

4. Write down the dual problem

5. Substitute the obtained in Q.3 Lagrange multipliers into the dual problem.
   Is the dual optimum achieved at that point?

6. Program in Matlab Augmented Lagrangian solver for general nonlinear optimization problems with inequality constraints,

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$$x \in \mathbb{R}^n$$
$$\text{minimize} \quad f(x)$$
$$\text{subject to}:$$
$$g_i(x) \le 0, \quad i = 1, 2, \ldots, m$$

Use the Newton method for inner optimization and the quadratic-logarithmic penalty function given in the lectures:

$$\varphi(t) = \begin{cases} \frac{t^2}{2} + t & t \ge -\frac{1}{2} \\ -\frac{1}{4} log(-2t) - \frac{3}{8} & t < -\frac{1}{2} \end{cases}$$

Assume that for any given x user provides function value, gradient, and Hessian of the objective function and constraints.

7. Write the problem used in Q.1-5 in general form for Quadratic programming problems:

$$\text{minimize} \quad \frac{1}{2} x^T Q x + d^T x + e$$
$$\text{subject to}:$$
$$Ax - b \le 0$$

Solve it using your Augmented Lagrangian solver.

8. Check whether your solution and Lagrange multipliers correspond to those obtained in Q. 2,3

NOTE: In your report build 4 semilogy subplots showing how the following quantities change with global Newton iterations (X-axis shows total count of Newton iterations over all unconstrained inner optimizations):

a) Gradient of the Augmented Lagrangian aggregate
b) Maximal constraint violation
c) Residual in the objective function $\left| f(x) - f(x^*) \right|$
d) Distance to the optimal point $\left\| x - x^* \right\|$ and to the optimal multipliers $\left\| \lambda - \lambda^* \right\|$