Polarization phase-contrast imaging

In this document we will introduce the technique of polarization phase-contrast imaging (PPCI), which is used in various stages of our experiment to measure the in-situ column density of of the dilute gas. The general goal of atom cloud imaging is to obtain a quantitative measure of the density distribution of the sample by looking at the intensity, phase and polarization profiles of light that has gone through it. The widely used absorption imaging technique looks at the intensity attenuation profile for a resonant, or nearly resonant laser beam. On the other hand, light with a larger detuning is used for PPCI; the attenuation of the intensity can be neglected and the information about the density of the gas can be extracted from the phase profile of the transmitted beam. The treatment is separated into two parts, first we calculate the relationship between the phase shift of the transmitted light and the column density of the cloud. Then we examine the experimental setup which allows one to convert the phase shift profile into an intensity profile that can be recorded on a CCD camera.

1 Dielectric properties of the atom cloud

In the following we want to find what happens with the electric field of a laser beam as it goes through a cloud of atoms. We treat the atom cloud as a continous dielectric medium in which the electric field induces a polarization $\boldsymbol{\mathcal{P}}(\boldsymbol{r}, t)$. This polarization corresponds to the electric dipole moment per unit volume in the cloud. If we can calculate the electric dipole moment induced on one atom, $\langle \boldsymbol{d} \rangle$, and if we know the density of the cloud, $n(\vec{r})$, then we can simply write the induced polarization as

$$\boldsymbol{\mathcal{P}}(\boldsymbol{r}, t) = \langle \boldsymbol{d} \rangle n(\boldsymbol{r}) \tag{1}$$

The induced dipole moment on an atom is proportional to the electric field[1] $\boldsymbol{\mathcal{E}}$, and in the general case the proportionality is given by the electric susceptibility tensor, $\boldsymbol{\chi}$:

$$\langle \boldsymbol{d} \rangle = \epsilon_0 \boldsymbol{\chi} \boldsymbol{\mathcal{E}} \tag{2}$$

An ensemble of atoms (in which each atom may be in a different quantum state) is best described using the density matrix, which is defined as

$$\boldsymbol{\rho} = \frac{1}{N} \sum_{i=1}^{N} |\psi_i\rangle\langle\psi_i| \tag{3}$$

where the i^{th} atom in the ensemble is in state $|\psi_i\rangle$. The expectation value of the electric dipole moment is then given by

$$\langle \boldsymbol{d} \rangle = \mathrm{Tr}(\boldsymbol{\rho} \boldsymbol{d}) \tag{4}$$

The equation of motion for the density matrix follows from the Schrodinger equation for the $|\psi_i\rangle$, it is known as the Liouville equation, which in the presence of relaxation from any excited states is [1]

$$i\hbar \dot{\boldsymbol{\rho}} = [\hat{H}, \boldsymbol{\rho}] - \frac{i\hbar}{2} (\boldsymbol{\Gamma} \boldsymbol{\rho} + \boldsymbol{\rho} \boldsymbol{\Gamma}) \tag{5}$$

where $\boldsymbol{\Gamma}$ is a diagonal matrix with the decay rate of each state on the diagonal, which is included to account for spontaneous emmission.

The Liouville equation for the density matrix will be solved by performing a unitary transformation to a rotating frame, as is shown in Chapter 10 of [1], but before we can proceed to define the transformation

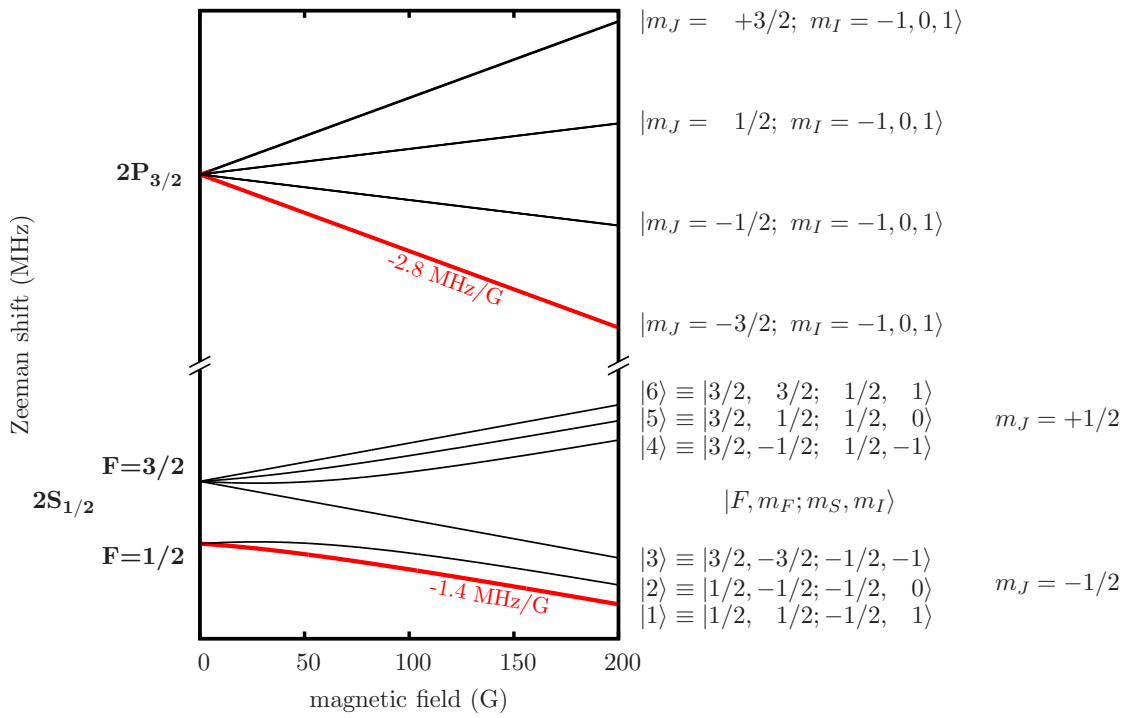

Figure 1: Level diagram relevant for imaging transitions

we must introduce the atomic states of interest in our problem, in other words the basis in which we will represent the atomic density matrix.

Figure 1 shows the energy levels of the $2S_{1/2}$ and $2P_{3/2}$ states of a ^{6}Li atom in the presence of a magnetic field. In our experiment we tipically have atoms in an incoherent spin mixure of states $|1\rangle$ and $|2\rangle$. The fields where we operate are typically > 300 G, where good quantum numbers are J, m_J and I, m_I. In other words, the field is high enough that the angular momentum of the valence electron is decuopled from the angular momentum of the nucleus, and the energy of the atom in the magnetic field is approximately $(g_J \mu_B m_J + g_I \mu_B m_I)B$.

The angular monetum projection selection rule for a one photon transition is $\Delta m_J = 0, \pm 1$, which means that we can drive transitions from states $|1\rangle$ and $|2\rangle$ (both of which have $m_J = -1/2$) to the $2P_{3/2}$ excited states with $m_J = -3/2, -1/2, +1/2$. We typically use σ_- polarized light, so we drive transitions to $m_J = -3/2$, which is advantageous because this is a cycling transition, which means it can only decay back to $m_J = -1/2$. In phase-contrast imaging the probe light is detuned far enough from the atomic state that the population in the excited state can in most cases be neglected. The fact that the transition closer to the probe laser is a cycling transition is thus inconseqiential. A basic analysis considers only the $m_J = -3/2$ excited states, but if the detuning is sufficiently large, such as it is sometimes necessary for in-situ imaging of very dense clouds, then we must also take into account the $m_J = -1/2$ and $m_J = +1/2$ states.

The basis set of states thus should consist of eight states. Four of them are state $|1\rangle$, which has a nuclear spin projection $m_I = +1$ plus the three corresponding excited states $m_J = -3/2, -1/2, +1/2$ with $m_I = +1$. The other four correspon to the $m_I = 0$ subspace, which contains $|2\rangle$. Since the electric dipole does not couple the nuclear spins, we will solve the problem in one of the m_I substate, and the solution for the other one will be identical. So the basis set of states that we will consider is $\{|g\rangle, |e_-\rangle, |e_0\rangle, |e_+\rangle\}$, where the first is the ground state and the last three are excited states.

[1] As long as we keep the probe laser power below the nonlinear regime

The density matrix is then

$$\boldsymbol{\rho} = \begin{bmatrix} \rho_{gg} & \rho_{+g} & \rho_{0g} & \rho_{-g} \\ \rho_{g+} & \rho_{++} & \rho_{0+} & \rho_{-+} \\ \rho_{g0} & \rho_{+0} & \rho_{0+} & \rho_{-0} \\ \rho_{g-} & \rho_{+-} & \rho_{0-} & \rho_{--} \end{bmatrix} \tag{6}$$

The unperturbed Hamiltonian of the atom is

$$H_0 = \hbar \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \omega_- & 0 & 0 \\ 0 & 0 & \omega_0 & 0 \\ 0 & 0 & 0 & \omega_+ \end{bmatrix} \tag{7}$$

For the interaction between the atoms and the oscillating electric field of the probe laser, we use the electric dipole Hamiltonian $H_I = -\boldsymbol{d} \cdot \boldsymbol{\mathcal{E}} \cos(\omega t)$, we can expand the vectors in the spherical basis $\{\boldsymbol{e}_\pm, \boldsymbol{e}_0\}$, which results in

$$H_I = -\left(\mathcal{E}_x d_x + \mathcal{E}_y d_y + \mathcal{E}_z d_z\right) \cos(\omega t) \tag{8}$$

To evaluate the matrix for H_I we need to find the matrix elements of the cartesian components of $\boldsymbol{d}$. We evaluate the the components of $\boldsymbol{d}$ in the spherical basis, d_q, where $q \in -1, 0, 1$ and then later we can evaluate the cartesian components by projecting them onto the spherical basis vectors.

To evaluate the matrix elements of the d_q, the Wigner-Eckart theorem[2] is used, in order to make the angular momentum selection rules explicit in the 3-j symbol. This defines the reduced matrix element $\langle e \| d \| g \rangle$,

$$\langle e | d_q | g \rangle = (-1)^{J_e - m_e} \begin{pmatrix} J_e & 1 & J_g \\ -m_{Je} & q & m_{Jg} \end{pmatrix} \langle e \| d \| g \rangle \tag{9}$$

The reduced matrix element satisfies $\langle e \| d \| g \rangle = -\langle g \| d \| e \rangle$, so the reader may verify that (in units of the reduced matrix element)

$$d_{-1} = \frac{\sqrt{3}}{6} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad d_0 = \frac{\sqrt{6}}{6} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad d_1 = \frac{\sqrt{3}}{6} \begin{bmatrix} 0 & 0 & 0 & 1 \\ -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

and upon using $d_x = \frac{1}{\sqrt{2}}(d_- - d_+)$, $d_y = \frac{i}{\sqrt{2}}(d_- + d_+)$, and $d_z = d_0$ gives

$$d_x = \frac{1}{2\sqrt{6}} \begin{bmatrix} 0 & \sqrt{3} & 0 & -1 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad d_y = \frac{i}{2\sqrt{6}} \begin{bmatrix} 0 & \sqrt{3} & 0 & 1 \\ -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad d_z = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

We know define three different Rabi frequencies as

$$\hbar\Omega_x = \frac{\langle e \| d \| g \rangle \mathcal{E}_x}{2\sqrt{6}} \quad \hbar\Omega_y = \frac{\langle e \| d \| g \rangle \mathcal{E}_y}{2\sqrt{6}} \quad \hbar\Omega_z = \frac{\langle e \| d \| g \rangle \mathcal{E}_z}{\sqrt{6}} \tag{12}$$

and we can write

$$H_I = \hbar \cos(\omega t) \begin{bmatrix} 0 & -\sqrt{3}(\Omega_x + i\Omega_y) & -\Omega_z & (\Omega_x - i\Omega_y) \\ -\sqrt{3}(\Omega_x - i\Omega_y) & 0 & 0 & 0 \\ -\Omega_z & 0 & 0 & 0 \\ (\Omega x + i\Omega_y) & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

[2] Sec. 5.4.1 on [2]

$|\langle R_e\, L\, S\, J||d||R_g\, L\, S\, J\rangle|^2$,

$$|\langle e|d_q|g\rangle|^2 = \begin{pmatrix} J_e & 1 & J_g \\ -m_{Je} & q & m_{Jg} \end{pmatrix}^2 |\langle R_e\, L\, S\, J||d||R_g\, L\, S\, J\rangle|^2 \tag{14}$$

The dipole moment operator only acts on the electronic angular momentum, L, and our quantum state is expressed in the LS coupled scheme, where L and S are coupled to form $J = L + S$. One can use the formula for the expectation of a single part operator on a coupled scheme[3], in this case the dipole moment operating on the LS coupled state. One obtains

$$|\langle e|d_q|g\rangle|^2 = \begin{pmatrix} J_e & 1 & J_g \\ -m_{Je} & q & m_{Jg} \end{pmatrix}^2 (2J_e+1)(2J_g+1) \begin{Bmatrix} L_g & J_g & S \\ J_e & L_e & 1 \end{Bmatrix}^2 |\langle R_e\, L||d||R_g\, L\rangle|^2 \tag{15}$$

The fully reduced matrix element can be related to the lifetime, τ, or the natural linewidth, Γ, of the excited state [**?**],

$$|\langle R_e\, L||d||R_g\, L\rangle|^2 = \frac{1}{\tau}\frac{9\epsilon_0\hbar\lambda^3}{8\pi^2} = \Gamma\frac{9\epsilon_0\hbar\lambda^3}{8\pi^2}$$

resulting finally in

$$|\langle e|d_q|g\rangle|^2 = \begin{pmatrix} J_e & 1 & J_g \\ -m_{Je} & q & m_{Jg} \end{pmatrix}^2 (2J_e+1)(2J_g+1) \begin{Bmatrix} L_g & J_g & S \\ J_e & L_e & 1 \end{Bmatrix}^2 \Gamma\frac{9\epsilon_0\hbar\lambda^3}{8\pi^2} \tag{16}$$

At this point we plug in some of the values for our case, $L_g = 0$, $J_g = 1/2$, $m_{Jg} = -1/2$, $L_e = 1$, and $J_e = 3/2$. The $6j$ symbol can be evaluated in Mathematica. This results in,

[3] Sec. 7.1.7 on [2]

We perform a transformation to the rotating frame using the unitary matrix U,

where the electron's wave function is: and the equation of motion for the c coefficients is derived from the Schrodinger equation:

$$i\hbar\dot{c}_k(t) = \sum_m c_m(t)e^{-i(E_m-E_k)t/\hbar}\langle k|\hat{H}_{\rm int}(t)|m\rangle \tag{17}$$

The interaction between the atom and the electric field is given by the optical Bloch equations, an $\hat{H}_{\rm int}(t) = -\boldsymbol{d}\cdot\boldsymbol{E}\cos(\omega t)$. It is useful to define the density matrix $\rho_{mn} = c_n^* c_m$, which has the following equation of motion

$$\dot{\rho}_{mn} = \frac{1}{i\hbar}\sum_l \left(\rho_{ln}e^{-i(E_l-E_m)t/\hbar}\langle m|\hat{H_{\rm int}}|l\rangle - \rho_{ml}e^{i(E_l-E_n)t/\hbar}\langle l|\hat{H_{\rm int}}|n\rangle\right) \tag{18}$$

2 Experimental setup for PPCI

References

[1] M. Auzinsh, D. Budker, and S. Rochester, *Optically Polarized Atoms: Understanding Light-atom Interactions* (OUP Oxford, 2010).

[2] A. Edmonds, *Angular Momentum in Quantum Mechanics*, *Investigations in Physics Series* (Princeton University Press, 1996).