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① Define events A_i, $i \in \{1,2,3,4\}$ where A_i is the event that 7 people have birthdays in $\leq i$ seasons. We are interested in $1-(P(A_3)-P(A_2)+P(A_1))$ because $|A_3| \equiv$ # of ways to have birthdays for all 7 people where 1,2,3 seasons are represented. Subtract $|A_2|$ ways of having all seasons but 2 represented since $|A_3|$ overcounts the $\binom{4}{2}$ pairwise ways to have only 2 seasons represented. Add back $P(A_1)$ since we now don't have $\binom{4}{1}$ ways to have only 1 season represented.

Stat 110 Strategic Practice 2, Fall 2011

Prof. Joe Blitzstein (Department of Statistics, Harvard University)

1 Inclusion-Exclusion

ALMOST 1. For a group of 7 people, find the probability that all 4 seasons (winter, spring, summer, fall) occur at least once each among their birthdays, assuming that all seasons are equally likely.

$1-\left(\binom{4}{3}\left(\frac{3}{4}\right)^7-\binom{4}{2}\left(\frac{2}{4}\right)^7+\binom{4}{1}\left(\frac{1}{4}\right)^7\right)$

$P(A_3)=\binom{4}{3}\frac{3^7}{4^7}$

$P(A_2)=\binom{4}{2}\frac{2^7}{4^7}$

$P(A_1)=\binom{4}{1}\frac{1^7}{4^7}$

(SEE BACK) 2. Alice attends a small college in which each class meets only once a week. She is deciding between 30 non-overlapping classes. There are 6 classes to choose from for each day of the week, Monday through Friday. Trusting in the benevolence of randomness, Alice decides to register for 7 randomly selected classes out of the 30, with all choices equally likely. What is the probability that she will have classes every day, Monday through Friday? (This problem can be done either directly using the naive definition of probability, or using inclusion-exclusion.)

2 Independence

1. Is it possible that an event is independent of itself? If so, when?

Yes, if $P(A)=1$ or 0 by definition of independence. $P(A\cap A) \overset{\text{ind}}{=} P(A|A)P(A) = P(A)^2$. $P(A|A)$ is 1 so $P(A)=P(A)^2$ for A to be independent of itself

(SEE BACK) 2. Is it always true that if A and B are independent events, then A^c and B^c are independent events? Show that it is, or give a counterexample.

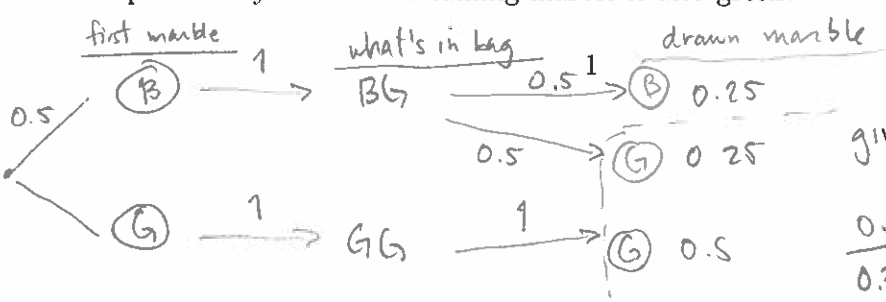
3. Give an example of 3 events A, B, C which are pairwise independent but not independent. Hint: find an example where whether C occurs is completely determined if we know whether A occurred and whether B occurred, but completely undetermined if we know only one of these things.

Consider a fair coin flipped twice: A = first flip is H, B = second flip is T, C = two flips are different (coin tosses are ind.)

4. Give an example of 3 events A, B, C which are not independent, yet satisfy $P(A \cap B \cap C) = P(A)P(B)P(C)$. Hint: consider simple and extreme cases. (SEE BACK)

3 Thinking Conditionally

1. A bag contains one marble which is either green or blue, with equal probabilities. A green marble is put in the bag (so there are 2 marbles now), and then a random marble is taken out. The marble taken out is green. What is the probability that the remaining marble is also green?



given drawn is green, $\frac{0.5}{0.75} = \frac{2}{3}$ probability remaining marble is green

(1.2) (Using inclusion-exclusion.)

OBSERVE We want to select events where there is at least 1 class every day. The complement of this set are events where at least 1 day does not have a class.

DEFINE Events A_i, $i \in \{x \mid 0 < x \leq 5\}$ where A_i is event where she does not have class on day i.

We want $1 - P(\bigcup_{i=1}^{5} A_i)$. $\bigcap_{i=1}^{5} A_i$ & $A_i \cap A_j \cap A_k \cap A_l$ (for $i<j<k<l$) are both $\emptyset$ because if there are only 6 classes per day, you need at least 2 days' worth of classes to choose from.

$$P\left(\bigcup_{i=1}^{5} A_i\right) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

(by symmetry)

$$\frac{\binom{5}{1}\binom{24}{7}}{\binom{30}{7}} - \frac{\binom{5}{2}\binom{18}{7}}{\binom{30}{7}} + \frac{\binom{5}{3}\binom{12}{7}}{\binom{30}{7}},$$

the rest follows.

(2.3) Prove: $P(A \cap B) = P(A)P(B) \iff P(A^c \cap B^c) = P(A^c)P(B^c)$

$$P(A \cap B) = (1 - P(A^c))(1 - P(B^c))$$
$$\text{"} \quad \text{"} = 1 - (P(A^c) + P(B^c)) + P(A^c)P(B^c)$$
$$\text{"} \quad \text{"} = 1 - (P(A^c \cup B^c) + P(A^c \cap B^c)) + P(A^c)P(B^c)$$
$$P(A \cap B) + P(A^c \cup B^c) \cancel{- 1} + P(A^c \cap B^c) = P(A^c)P(B^c) \quad \blacksquare$$

(2.4) Show a case where A, B, C are not independent but $P(A)P(B)P(C) = P(A \cap B \cap C)$

Consider when $C = \{\} \Rightarrow P(C) = 0$ and $P(A), P(B) \neq 0$.

$$P(A)P(B)\,0 \stackrel{?}{=} P(A \cap B \cap C)$$
$$0 \stackrel{\checkmark}{=} 0$$

S = event email is spam
F = event email says "free money"

$$P(S|F) = \frac{P(F|S)\,P(S)}{P(F)} = \boxed{\frac{(0.1)(0.8)}{(0.1)(0.8)+(.01)(0.2)}}$$

$P(F) = P(F|S)P(S) + P(F|\neg S)P(\neg S)$ by LOTP

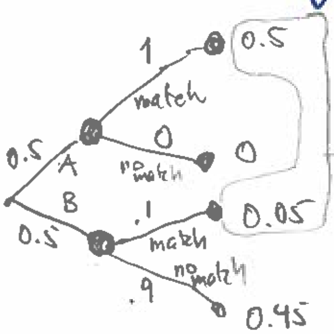
Historical note: this problem was first posed by Lewis Carroll in 1893.

2. A spam filter is designed by looking at commonly occurring phrases in spam. Suppose that 80% of email is spam. In 10% of the spam emails, the phrase "free money" is used, whereas this phrase is only used in 1% of non-spam emails. A new email has just arrived, which does mention "free money". What is the probability that it is spam?

3. Let G be the event that a certain individual is guilty of a certain robbery. In gathering evidence, it is learned that an event E_1 occurred, and a little later it is also learned that another event E_2 also occurred. (SEE BACK)

   (a) Is it possible that individually, these pieces of evidence increase the chance of guilt (so $P(G|E_1) > P(G)$ and $P(G|E_2) > P(G)$), but together they decrease the chance of guilt (so $P(G|E_1, E_2) < P(G)$)? Yes

   (b) Show that the probability of guilt given the evidence is the same regardless of whether we update our probabilities all at once, or in two steps (after getting the first piece of evidence, and again after getting the second piece of evidence). That is, we can either update all at once (computing $P(G|E_1, E_2)$ in one step), or we can first update based on E_1, so that our new probability function is $P_{\text{new}}(A) = P(A|E_1)$, and then update based on E_2 by computing $P_{\text{new}}(G|E_2)$.



4. A crime is committed by one of two suspects, A and B. Initially, there is equal evidence against both of them. In further investigation at the crime scene, it is found that the guilty party had a blood type found in 10% of the population. Suspect A does match this blood type, whereas the blood type of Suspect B is unknown.

$$\frac{0.5}{0.5+0.05} = \boxed{\frac{10}{11}} = \frac{P(M\cap A)}{P(M\cap A)+P(M\cap B)} \cdot P(A|M)$$

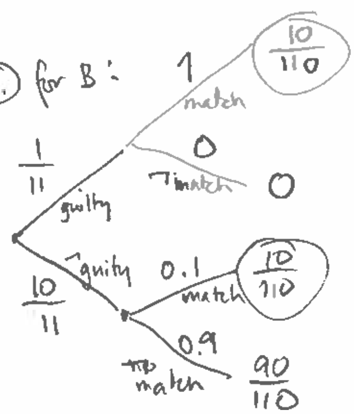
   (a) Given this new information, what is the probability that A is the guilty party? (SEE ABOVE)

   (b) Given this new information, what is the probability that B's blood type matches that found at the crime scene? (SEE BELOW) $\frac{20}{110} = \boxed{\frac{2}{11}}$

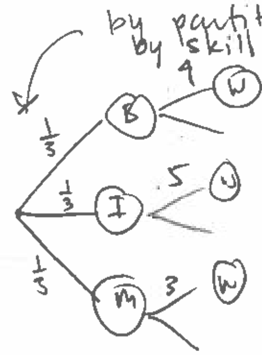
5. You are going to play 2 games of chess with an opponent whom you have never played against before (for the sake of this problem). Your opponent is equally likely to be a beginner, intermediate, or a master. Depending on which, your chances of winning an individual game are 90%, 50%, or 30%, respectively.

   (a) What is your probability of winning the first game?

4b. for B:



by partitioning by skill



$$P(W_1) = P(W_1|B)P(B) + P(W_1|I)P(I) + P(W_1|M)P(M) =$$

$$= \left(\frac{9}{10}\cdot\frac{1}{3}\right) + \left(\frac{1}{2}\cdot\frac{1}{3}\right) + \left(\frac{3}{10}\cdot\frac{1}{3}\right)$$

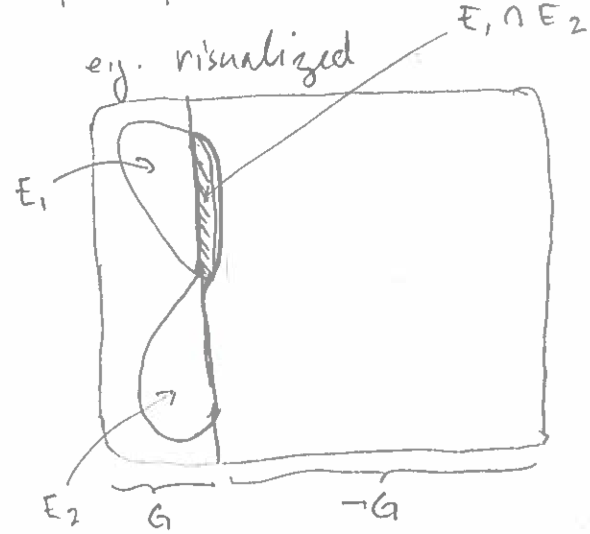
$$= \left(\frac{1}{3}\right)\left(\frac{17}{10}\right) = \boxed{\frac{17}{30}}$$

3A. To have scenario where $P(G|E_1) > P(G)$
& $P(G|E_2) > P(G)$
& $P(G|E_1 \cap E_2) < P(G)$

you need: $P(E_1|G) > P(E_1)$
$P(E_2|G) > P(E_2)$
$P(E_1 \cap E_2|G) < P(E_1 \cap E_2)$

by Baye's rule

e.g. visualized



This example shows that E_1, E_2 individually increase probability of guilt, but when taken together, only occur when suspect is $\neg G$!

3B. Show coherency of Bayes' Rule.

CASE 1 Update all at once:

$$P(G|E_1 \cap E_2) = \frac{P(E_1 \cap E_2|G)\,P(G)}{P(E_1 \cap E_2)} = \frac{P(E_2|G \cap E_1)\,P(G|E_1)\,\cancel{P(E_1)}}{P(E_2|E_1)\,\cancel{P(E_1)}}$$

equivalent

CASE 2 Update sequentially:

$$P_{new}(G) = P(G|E_1) = \frac{P(E_1|G)\,P(G)}{P(E_1)}$$ } incorporate E_1

$$P_{new}(G|E_2) = \frac{P_{new}(E_2|G)\,P_{new}(G)}{P_{new}(E_2)}$$ } incorporate E_2

(= by def. of cond. prob)

$$= \frac{P(E_2|G \cap E_1)\,P(E_1|G)\,P(G)}{P(E_2|E_1)\,P(E_1)} = \frac{P(E_2|G \cap E_1)\,P(G|E_1)\,\cancel{P(E_1)}}{P(E_2|E_1)\,\cancel{P(E_1)}}$$

→ (case of extra conditioning)

(b) Congratulations: you won the first game! Given this information, what is the probability that you will also win the second game (assume that, given the skill level of your opponent, the outcomes of the games are independent)?

(c) Explain the distinction between assuming that the outcomes of the games are independent and assuming that they are conditionally independent given the opponent's skill level. Which of these assumptions seems more reasonable and why?

$P(W|B) = 0.9$, $P(W|I) = 0.5$, $P(W|M) = 0.3$

5b) $P(W_2 \cap W_1) = P(W_2 | W_1)\, P(W_1)$, $P(W_1) = \frac{17}{30}$

$$P(W_2|W_1) = P(W_2 | B, W_1)\,P(B|W_1) + P(W_2|I,W_1)\,P(I|W_1) + P(W_2|M,W_1)\,P(M|W_1)$$

with conditional independence assumptions.

$\Downarrow$

$P(W_2|B) = P(W_2|B,W_1)$

$P(W_2|I) = P(W_2|I,W_1)$

$P(W_2|M) = P(W_2|M,W_1)$

$P(B|W_1) = \frac{P(W_1|B)}{P(W_1)} = \frac{\frac{9}{10}\cdot\frac{1}{3}}{\frac{17}{30}} = \frac{9}{17}$

$P(I|W_1) = \frac{5}{17}$

$P(M|W_1) = \frac{3}{17}$

$$P(W_2|W_1) = \left[\left(\frac{9}{10^2}\right)\left(\frac{9}{\cancel{17}}\right) + \left(\frac{5}{10^2}\right)\left(\frac{5}{\cancel{17}}\right) + \left(\frac{3}{10^2}\right)\left(\frac{3}{\cancel{17}}\right)\right]$$

$P(W_1) = \frac{\cancel{17}\,1}{\cancel{30}\,3}$

$$\boxed{P(W_2 \cap W_1) = \frac{\left(\frac{9}{10}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{10}\right)^2}{3}}$$

c) Conditionally independent outcomes seems more reasonable than absolute outcomes for games b/c it's more likely the case your win rate against an opponent of a particular skill is constant over many games, NOT that your win rate against any opponent is constant over many games!

HW 2) Let A_i be the hands where suit i is missing.

$$P\left(\bigcup_{i=1}^{4} A_i\right) = \sum_{i=1}^{4} P(A_i) - \sum_{i=1}^{4}\sum_{j>i} P(A_i \cap A_j) + \sum_{i=1}^{4}\sum_{j>i}\sum_{k>j} P(A_i \cap A_j \cap A_k).$$

Note that $\bigcap_{i=1}^{4} A_i = \{\}$. By symmetry:

$$\sum_{i=1}^{4} P(A_i) = \frac{\binom{4}{1}\binom{39}{13}}{\binom{52}{13}}$$

$$\sum_{i=1}^{4}\sum_{j>i} P(A_i \cap A_j) = \frac{\binom{4}{2}\binom{26}{13}}{\binom{52}{13}}$$

$$\sum_{i=1}^{4}\sum_{j>i}\sum_{k>j} P(A_i \cap A_j \cap A_k) = \left(\frac{\binom{4}{3}\binom{13}{13}}{\binom{52}{13}}\right)$$

$$\boxed{\frac{\binom{4}{1}\binom{39}{13} - \binom{4}{2}\binom{26}{13} + \binom{4}{3}\binom{13}{13}}{\binom{52}{13}}}$$

If Arby under-values $(A \cup B)$ certificates: Buy $(A \cup B)$ certs from Arby and sell him A + B certs.

$\Updownarrow$

$P_{Arby}(A \cup B) < P(A) + P(B)$

Stat 110 Homework 2, Fall 2011

If Arby over-values $(A \cup B)$ certs, do the opposite

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1. Arby has a belief system assigning a number $P_{\text{Arby}}(A)$ between 0 and 1 to every event A (for some sample space). This represents Arby's subjective degree of belief about how likely A is to occur. For any event A, Arby is willing to pay a price of $1000 \cdot P_{\text{Arby}}(A)$ dollars to buy a certificate such as the one shown below:

> **Certificate**
>
> The owner of this certificate can redeem it for \$1000 if A occurs. No value if A does not occur, except as required by federal, state, or local law. No expiration date.

Likewise, Arby is willing to sell such a certificate at the same price. Indeed, Arby is willing to buy or sell any number of certificates at this price, as Arby considers it the "fair" price.

Arby, not having taken Stat 110, stubbornly refuses to accept the axioms of probability. In particular, suppose that there are two *disjoint* events A and B with

Arby's [wrong] beliefs

$$P_{\text{Arby}}(A \cup B) \neq P_{\text{Arby}}(A) + P_{\text{Arby}}(B).$$

$P(A) + P(B) - \cancel{P(A \cap B)}^{0} = P(A \cup B)$

Show how to make Arby go bankrupt, by giving a list of transactions Arby is willing to make that will *guarantee* that Arby will lose money (you can assume it will be known whether A occurred and whether B occurred the day after any certificates are bought/sold).

2. A card player is dealt a 13 card hand from a well-shuffled, standard deck of cards. What is the probability that the hand is void in at least one suit ("void in a suit" means having no cards of that suit)? (SEE BACK OF PREVIOUS PAGE)

3. A family has 3 children, creatively named A, B, and C.

(a) Discuss intuitively (but clearly) whether the event "A is older than B" is independent of the event "A is older than C."

(b) Find the probability that A is older than B, given that A is older than C.

(Events A, B, C ≡ age of child, resp.)

3a) Prior to knowing $A > C$, A, B, and C could be arranged in 3! orders of birth. But knowing $A > C$, we know that A cannot be the youngest, which makes the number of age arrangements smaller $\Rightarrow P(A>B) < P(A>B \mid A>C)$

$\Rightarrow P((A>B) \cap (A>C)) \neq P(A>B)\,P(A>C) < P(A>B \mid A>C)\,P(A>C)$

$\Updownarrow$ not independent

3b) $P(A>B \mid A>C) = \dfrac{P(A>B \cap A>C)}{P(A>C)}$. Assume equally likely birthorders:

$P(A>B \cap A>C) = \dfrac{2}{3!} = \dfrac{\cancel{2}}{3 \cdot \cancel{2} \cdot 1} = \dfrac{1}{3}$

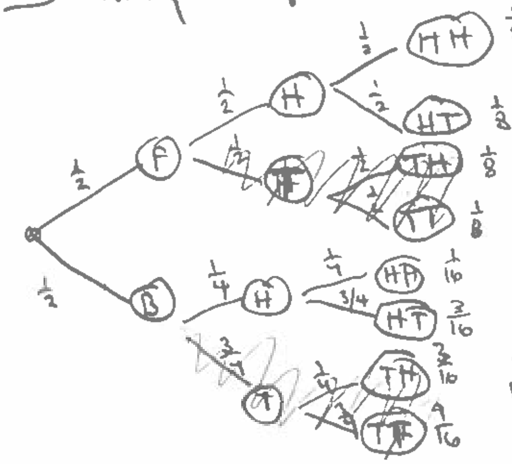
$P(A>C) = \dfrac{3}{3!} = \dfrac{\cancel{3}}{\cancel{3} \cdot 2 \cdot 1} = \dfrac{1}{2}$

$\dfrac{\frac{1}{3}}{\frac{1}{2}} = \boxed{\dfrac{2}{3}}$ vs. prior of $\frac{1}{2}$ (by symmetry)

(4A) $P(\text{fair} \mid H_1, H_2) = ? = \dfrac{P(\text{fair} \cap H_1 \cap H_2)}{P(H_1 \cap H_2)} = \dfrac{P(H_1, H_2 \mid \text{fair})\, P(\text{fair})}{P(H_1, H_2 \mid \text{fair})\, P(\text{fair}) + P(H_1, H_2 \mid \neg\text{fair})\, P(\neg\text{fair})}$

$$\boxed{= \dfrac{\left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right) + \left(\frac{1}{4}\right)^2 \left(\frac{1}{2}\right)}}$$

(4B) Events "first toss of C is Heads" & "second toss of C is Heads" are <u>NOT</u> independent. Knowing the first toss is heads updates the probability that second toss is heads,



$P(\text{second toss of C is Heads} \mid \text{first toss of C is Heads})$.

$\neq$

$P(\text{second toss of C is Heads})$ x

$= P(x \mid \text{fair})\, P(\text{fair}) + P(x \mid \neg\text{fair})\, P(\neg\text{fair})$

$$\dfrac{\frac{2}{16} + \frac{1}{16}}{\frac{2}{16} + \frac{2}{16} + \frac{1}{16} + \frac{3}{16}} = \frac{3}{8} \neq \left(\frac{1}{8} + \frac{1}{8}\right)\frac{1}{2} + \left(\frac{1}{16} + \frac{3}{16}\right)\frac{1}{2} = \left(\frac{1}{4}\right)\frac{1}{2} + \left(\frac{1}{4}\right)\frac{1}{2} = \frac{1}{4}$$

(4C) Let A be event exactly 3 Heads in 10 flips.

$P(A) = P(A \mid \text{fair})\, P(\text{fair}) + P(A \mid \neg\text{fair})\, P(\neg\text{fair})$

$= P(A \mid \text{fair})\, \frac{1}{2} + P(A \mid \neg\text{fair})\, \frac{1}{2} = \boxed{\frac{1}{2}\binom{10}{3}\left[\left(\frac{1}{2}\right)^{10} + \left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^7\right]}$

$\binom{10}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^7$ $\qquad$ $\binom{10}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^7$

4. Two coins are in a hat. The coins look alike, but one coin is fair (with probability 1/2 of Heads), while the other coin is biased, with probability 1/4 of Heads. One of the coins is randomly pulled from the hat, without knowing which of the two it is. Call the chosen coin "Coin C".

(a) Coin C is tossed twice, showing Heads both times. Given this information, what is the probability that Coin C is the fair coin? (SEE BACK OF PREVIOUS PAGE)

(b) Are the events "first toss of Coin C is Heads" and "second toss of Coin C is Heads" independent? Explain. (SEE BACK OF PREVIOUS PAGE)

(c) Find the probability that in 10 flips of Coin C, there will be exactly 3 Heads. (The coin is equally likely to be either of the 2 coins; do not assume it already landed Heads twice as in (a).) (SEE BACK OF PREVIOUS PAGE)

5. A woman has been murdered, and her husband is accused of having committed the murder. It is known that the man abused his wife repeatedly in the past, and the prosecution argues that this is important evidence pointing towards the man's guilt. The defense attorney says that the history of abuse is irrelevant, as only 1 in 1000 men who beat their wives end up murdering them.

Assume that the defense attorney's 1 in 1000 figure is correct, and that half of men who murder their wives previously abused them. Also assume that 20% of murdered women were killed by their husbands, and that if a woman is murdered and the husband is *not* guilty, then there is only a 10% chance that the husband abused her. What is the probability that the man is guilty? Is the prosecution right that the abuse is important evidence in favor of guilt? (SEE BACK)

6. A family has two children. Assume that birth month is independent of gender, with boys and girls equally likely and all months equally likely, and assume that the elder child's characteristics are independent of the younger child's characteristics).

(a) Find the probability that both are girls, given that the elder child is a girl who was born in March.

(b) Find the probability that both are girls, given that at least one is a girl who was born in March.

(6A) Intuitively, $\boxed{\frac{1}{2}}$ since younger sibling can be boy or girl

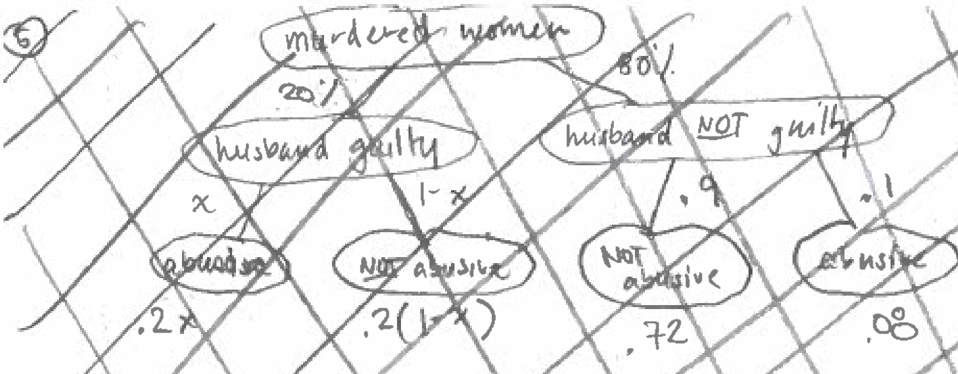
Also, because $P(G_1 \cap G_2 \mid G_2 \cap M_2) = \frac{P(G_1 \cap G_2 \cap M_2)}{P(G_2 \cap M_2)} = \frac{(\frac{1}{2})\cancel{(\frac{1}{2})}\cancel{(\frac{1}{12})}}{\cancel{(\frac{1}{2})}\cancel{(\frac{1}{12})}}$

(6B) $P(G_1 \cap G_2 \mid (G_1 \cap M_1) \cup (G_2 \cap M_2)) =$

$$\frac{P((G_1 \cap G_2 \cap G_1 \cap M_1) \cup (G_1 \cap G_2 \cap G_2 \cap M_2))}{P((G_1 \cap M_1) \cup (G_2 \cap M_2))}^2 = \frac{\cancel{2(\frac{1}{2})}(\frac{1}{2})(\frac{1}{12}) - \frac{1}{2 \cdot 12 \cdot 2 \cdot 12}}{\cancel{2(\frac{1}{2})}(\frac{1}{12}) - \frac{1}{2 \cdot 12 \cdot 2 \cdot 12}} =$$

$$\boxed{\frac{12 \cdot 2 - 1}{12 \cdot 2^2 - 1} = \frac{23}{47}}$$

(5)



We know that:

$$\frac{1}{1000} = \frac{(.2)x}{(.2)x + .08}$$

$$(.001)[(.2)x + .08] = (0.2)x$$

$$(.0002)x + .08 = (.2)x$$

$$\frac{.08}{(.2 - .0002)} = x = .400$$

where x is probability husband is abusive given he is guilty. So, $P(G|A)$ where G is event of guilt and A is event husband is abusive:

$$P(G|A) = \frac{P(A|G)\,P(G)}{P(A|G)P(G) + P(A|\neg G)P(\neg G)} = \frac{x(.2)}{x(.2) + (.1)(.8)}$$

(5) Yes, prosecution is correct that evidence of abuse is important vis-a-vis guilt: prior probability of guilt $= 0.2$ rises to posterior of $0.\overline{5}$!

GIVEN THE WIFE IS MURDERED:

$$P(G|A) = \frac{P(A|G)\,P(G)}{P(A|G)\,P(G) + P(A|\neg G)\,P(\neg G)} = \frac{(.5)(.2)}{(.5)(.2)+(.1)(.8)} = \boxed{\frac{5}{9}} \ \left(\text{vs. } \frac{1}{5}\right)$$

The defense attorney is incorrectly "changes base" — we don't care about the % of abusive husbands that murder their wives — we care about the % of abusive husbands that murder their wives taken from the group of abusive husbands <u>whose wives were murdered</u>!