1 B From 1 A, we can marginalize out Z, $f_{X,Y}(x,y) = 2\int_0^{\sqrt{1-x^2-y^2}} \frac{3}{4\pi}\, dz = \boxed{\frac{3\sqrt{1-x^2-y^2}}{2\pi}}$ when $x^2+y^2 \le 1$

Ravi Dayabhai

1 C From 1 B, we can marginalize out Y, $f_X(x) = \boxed{2\int_0^{\sqrt{1-x^2}} \frac{3}{2\pi}\sqrt{1-x^2-y^2}\, dy}$, for $-1 \le x \le 1$

Stat 110 Strategic Practice 7, Fall 2011

Prof. Joe Blitzstein (Department of Statistics, Harvard University)

1 Joint, Conditional, and Marginal Distributions

1. A random point (X, Y, Z) is chosen uniformly in the ball $B = \{(x, y, z) : x^2 + y^2 + z^2 \le 1\}$.

   (a) Find the joint PDF of X, Y, Z. $f_{X,Y,Z}(x,y,z) = \frac{1}{\frac{4}{3}\pi r^3}$, $r = 1 \Rightarrow \boxed{\frac{3}{4\pi}}$ when $x^2+y^2+z^2 \le 1$, 0 otherwise

   (b) Find the joint PDF of X, Y. (SEE ABOVE)

   (c) Find an expression for the marginal PDF of X, as an integral. (SEE ABOVE)

2. Let X and Y be i.i.d. Unif(0, 1). Find the expected value and the standard deviation of the distance between X and Y. (SEE BACK).

Needed hint, but conceptually understood!

3. Let U_1, U_2, U_3 be i.i.d. Unif(0, 1), and let $L = \min(U_1, U_2, U_3)$, $M = \max(U_1, U_2, U_3)$.

   (a) Find the marginal CDF and marginal PDF of M, and the joint CDF and joint PDF of L, M. (SEE BACK)

   Hint: for the latter, start by considering $P(L \ge l, M \le m)$.

   DUMB MISTAKE!

   (b) Find the conditional PDF of M given L. (SEE BELOW)

4. A group of $n \ge 2$ people decide to play an exciting game of Rock-Paper-Scissors. As you may recall, Rock smashes Scissors, Scissors cuts Paper, and Paper covers Rock (despite Bart Simpson saying "Good old rock, nothing beats that!").

   Usually this game is played with 2 players, but it can be extended to more players as follows. If exactly 2 of the 3 choices appear when everyone reveals their choice, say $a, b \in \{Rock, Paper, Scissors\}$ where a beats b, the game is decisive: the players who chose a win, and the players who chose b lose. Otherwise, the game is indecisive and the players play again.

   For example, with 5 players, if one player picks Rock, two pick Scissors, and two pick Paper, the round is indecisive and they play again. But if 3 pick Rock and 2 pick Scissors, then the Rock players win and the Scissors players lose the game.

3 B $f_{M|L}(m|l) = \frac{f_{M,L}(m,l)}{f_L(l)} = \frac{\cancel{6}(m-l)}{\cancel{3}(1-2l+l^2)} = \boxed{\frac{2(m-l)}{\cancel{m(m-2l)}}}$ for $m, l \in [0,1]$, $m \ge l$

$\int_0^1 f_{M,L}(m,l)\, dm = f_L(l)$

$\int_0^1 6m - 6l\, dm = 3m^2 - 6lm \Big|_0^m = \cancel{3m(m-2l)}$ for $l \in [0,1]$

right idea... but wrong limits of integration: $l \to m$ because marginalization yields full, unconditional range of L!

② $X, Y \overset{iid}{\sim}$ Unif$(0,1)$. $E(|X-Y|)$? $\sqrt{Var(|X-Y|)}$?

$E(|X-Y|) = E(\max(X,Y) - \min(X,Y)) = E(\max(X,Y)) - E(\min(X,Y))$

Because X, Y are i.i.d continuous r.vs, $P(X>Y) = P(X<Y)$ with $P(X=Y)=0$. Then, $\min(X,Y) + \max(X,Y) = X+Y$, so $E(X+Y) = E(\min(X,Y)) + E(\max(X,Y)) = 1$ from $E(X)=E(Y)=\frac{1}{2}$. Taken together, $E(\max(X,Y)) - (1 - E(\max(X,Y))) = 2E(\max(X,Y)) - 1$. So now, we just need to consider $E(\max(X,Y))$: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \max(x,y)\, f_{X,Y}(x,y)\, dx\, dy$.

We can simplify this to $\int_0^1\int_0^1 \max(x,y)\, dx\, dy$ because the joint distribution is uniform over the area bounded by square with area $= 1$, $\forall x,y \in \{0,1\}$. We split this up by cases:

$\underbrace{\int_0^1\int_0^x x\, dy dx}_{\text{when } x \text{ is max}} + \underbrace{\int_0^1\int_0^y y\, dx\, dy}_{\text{when } y \text{ is max}} = \int_0^1 x^2\, dx + \int_0^1 y^2\, dx = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. So,

$E(|X-Y|) = 2 \cdot \frac{2}{3} - 1 = \boxed{\frac{1}{3}}$. For $Var(|X-Y|)$, let's start with definition of variance: $E(|X-Y|^2) - \left(\frac{1}{3}\right)^2$. We can drop the absolute value: $E((X-Y)^2) - \frac{1}{9}$.

$E((X-Y)^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)^2 f_{X,Y}(x,y)\, dx\, dy = \int_0^1\int_0^1 x^2 - 2xy + y^2\, dx\, dy =$

$\int_0^1 \frac{1}{3} - y + y^2\, dy = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$. Therefore, $Var(|X-Y|) = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$, so standard deviation of distance between X & Y is $\boxed{\sqrt{\frac{1}{18}}}$

③ A Because U_1, U_2, U_3 are i.i.d, $P(L>l, M\le m) = (m-l)^3$. Also, $\boxed{P(M\le m) = m^3}$ (for same reason, $P(U_1, U_2, U_3 \le m)$), which is $F_M(m)$. $\boxed{f_M(m) = 3m^2}$ for $m \in [0,1]$. By LOTP, $P(M\le m) = m^3 = P(L\le l, M\le m) + P(L>l, M\le m)$, so $F_{M,L}(m,l)$ is $\boxed{m^3 - (m-l)^3}$ for $m, l \in [0,1]$ and $m \ge l \Rightarrow f_{M,L}(m,l) = \frac{\partial^2}{\partial m \partial l} F_{M,L}(m,l)$.

$f_{M,L}(m,l) = \frac{\partial}{\partial l}\left(3m^2 - 3(m-l)^2\right) = \boxed{6(m-l)}$ for $m \in [0,1]$ and $m \ge l$. Your original method was correct, too! $\int_0^m\int_l^1 f_{L,M}(l,m)\, dl\, dm = (m-l)^3 \Rightarrow f_{L,M}(l,m) = 6(m-l)$

④ⓒ $P(\text{decisive} \mid n=5) = \frac{2^5-2}{3^4} = \frac{23}{81}$. Letting $n \to \infty$, $\frac{2^n-2}{3^{n-1}} = \frac{3(2^n-2)}{3^n} \to 0$ ✓ because in the limit $\left(\frac{2}{3}\right)^n \to 0$. This makes sense b/c with more players, it's much likelier k players, $k \geq 1$, will pick the third choice even when $n-k$ players choose between other two options. $\sum_{k=1}^{n-1} \binom{n}{k} \xrightarrow{n\to\infty} \infty$!

Assume that the n players independently and randomly choose between Rock, Scissors, and Paper, with equal probabilities. Let X, Y, Z be the number of players who pick Rock, Scissors, Paper, respectively in one game.

ALMOST (a) Find the joint PMF of X, Y, Z. Multinomial (right instincts!) (SEE BELOW)

ALMOST! DUMB MISTAKE! (b) Find the probability that the game is decisive. Simplify your answer (it should not involve a sum of many terms). (SEE BELOW)

~~ALMOST~~ ACTUALLY 100% CORRECT! (c) What is the probability that the game is decisive for $n = 5$? What is the limiting probability that a game is decisive as $n \to \infty$? Explain briefly why your answer makes sense. (SEE ABOVE)

5. A chicken lays n eggs. Each egg independently does or doesn't hatch, with probability p of hatching. For each egg that hatches, the chick does or doesn't survive (independently of the other eggs), with probability s of survival. Let $N \sim \text{Bin}(n, p)$ be the number of eggs which hatch, X be the number of chicks which survive, and Y be the number of chicks which hatch but don't survive (so $X + Y = N$). Find the marginal PMF of X, and the joint PMF of X and Y. Are they independent? (SEE BACK)

④ⓐ $f_{X,Y,Z}(x,y,z) = P(X=x \mid Y=y, Z=z)\,P(Y=y \mid Z=z)\,P(Z=z)$ ~~where~~ $n = X+Y+Z$

correct approach

$= 1 \cdot \binom{n-z}{y}\left(\frac{1}{2}\right)^y\left(\frac{1}{2}\right)^{(n-z-y)} \cdot \binom{n}{z}\left(\frac{1}{3}\right)^z\left(\frac{2}{3}\right)^{n-z}$

$\frac{1}{2}$ because only two choices left!

$= \boxed{\frac{n!}{x!\,y!\,z!}\left(\frac{1}{3}\right)^{\cancel{x+y+z}}}$

$= \frac{n!}{x!\,y!\,z!}\left(\frac{1}{2}\right)^{n-z}\left(\frac{1}{3}\right)^z\left(\frac{2}{3}\right)^{n-z}$ for $x+y+z = n$, 0 otherwise.

④ⓑ. Game is decisive iff 2 options (of 3) are chosen by n players.

By PIE, $P(\text{decisive}) = P(X=0, Y>0, Z>0) + P(Y=0, X>0, Z>0) + P(Z=0, X>0, Y>0)$ since the pairwise, triplewise events are all impossible. By SYMMETRY, $P(\text{decisive}) = 3P(X=0, Y>0, Z>0)$. ✓

$3P(X=0, Y>0, Z>0) = 3P(Y>0, Z>0 \mid X=0)\,P(X=0) = 3\left[1 - P(Y \leq 0 \cup Z \leq 0 \mid X=0)\right]P(X=0)$.

But since $P(Y \leq 0 \cup Z \leq 0 \mid X=0) = P(Y=0 \mid X=0) + P(Z=0 \mid X=0)$ since X, Y, Z can never be negative and $n \geq 2$ and $n = X+Y+Z$, we can simplify $P(Y=0 \mid X=0) + P(Z=0 \mid X=0) = \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n$. $P(X=0) = \binom{n}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{y+z}$, so $P(\text{decisive})$

$\boxed{3 \cdot \left(\frac{2}{3}\right)^{y+z}\left[1 - \left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^n\right]}$ b/c $z=n$ and $y=n$

(here, $n = z+y$, so further simplification

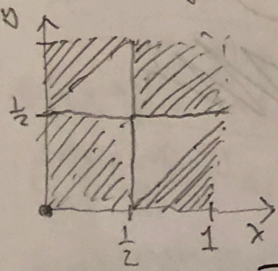
$\frac{2^n}{3^{n-1}}\left[1 - \left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^n\right] = \boxed{\frac{2^n-2}{3^{n-1}}}$

⑤ This also follows multinomial form, where the 3 categories are hatch + survive, hatch + die, not hatch with each "egg" "placed" in a category with probabilities ps, $p(1-s)$, $1-p$, respectively. So joint PMF of X, Y is: $\boxed{\frac{n!}{x!\,y!\,(n-x-y)!}(ps)^x(p(1-s))^y(1-p)^{n-x-y}}$ ✓ for $x+y=n$. To get the marginal PMF of X, we ~~our~~ marginalize out Y (which is the same as "lumping" Y and $n-X-Y$ together $\Rightarrow$ 2 categories, hatch + survive (X), not hatch or not survive $(n-X)$). $P(X=x)=\sum_{h=0}^{n}\sum_{y=0}^{h}P(X=x, Y=y, N=h)$, which is $\boxed{f_X(x)=\sum_{h=0}^{n}\sum_{y=0}^{h}\frac{n!}{x!\,y!\,(n-x-y)!}(ps)^x((1-s)p)^y(1-p)^{n-x-y}}$ while correct it's hard to work with. Using "lumping" approach, $\boxed{P(X=x)=\binom{n}{x}(ps)^x(1-ps)^{n-x}}$ ✓.

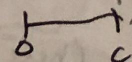
Because $P(Y=y)=\binom{n}{y}((1-s)p)^y(1-p+sp)^{n-y}$ by "lumping" ✓ we see that $P(X=x)P(Y=y)\neq f_{X,Y}(x,y) \Rightarrow$ NOT independent ✓

[HW 7]

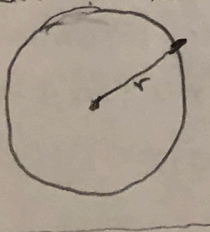
①(A) Let $X, Y \overset{iid}{\sim} \text{Unif}(0,1)$; region of interest that would yield triangle is where $\max(X,Y)>\frac12 \cap \min(X,Y)<\frac12 \cap \max(X,Y)-\min(X,Y)<\frac12$. This corresponds to white region below. By symmetry, this region is $\frac14$ of total area which is $1 \Rightarrow \frac{1/4}{1} \Rightarrow \boxed{P(\text{triangle})=\frac14}$ ✓ (uniformly spread)

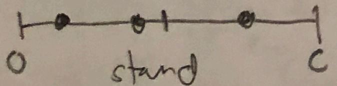


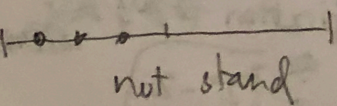
~~①(B)~~. Let $X, Y, Z \overset{iid}{\sim} \text{Unif}(0, C)$ where $C=2\pi r$ and r is radius of table. For table to stand we need at most 2 legs on one half of perimeter, or never 3 legs on on half of perimeter. This is ALMOST

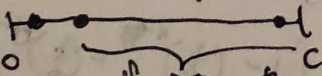
$1-P((X,Y,Z<\frac12 C)\cup(X,Y,Z>\frac12 C)) = 1-[P(X,Y,Z<\pi r)+P(X,Y,Z>\pi r)]$ because of disjointedness. By symmetry, this is equivalent to $1-2P(X,Y,Z<\pi r)$. $P(X,Y,Z<\pi r)=\left(\pi r\cdot\frac{1}{2\pi r}\right)^3=\frac18$ by independence. Therefore, probability of picking 3 legs randomly/uniformly such that the table stands is $1-2\left(\frac18\right)=\boxed{\frac34}$ ✗ $\boxed{\frac14}$ Didn't consider arc lengths (since  wraps around) — from ①(A) probability of one leg/arc NOT meeting triangle conditions is $\frac34$, so $P(\text{table})=1-\frac34=\frac14$.

C=2πr









if >0, then table falls!

Stat 110 Homework 7, Fall 2011

Prof. Joe Blitzstein (Department of Statistics, Harvard University)

1. (a) A stick is broken into three pieces by picking two points independently and uniformly along the stick, and breaking the stick at those two points. What is the probability that the three pieces can be assembled into a triangle? (SEE BACK OF PREVIOUS PAGE)

Hint: a triangle can be formed from 3 line segments of lengths a, b, c if and only if $a, b, c \in (0, 1/2)$. The probability can be interpreted geometrically as proportional to an area in the plane, avoiding all calculus, but make sure for that approach that the distribution of the random point in the plane is Uniform over some region.

ALMOST; didn't consider all orientations / trying to ...

(b) Three legs are positioned uniformly and independently on the perimeter of a round table. What is the probability that the table will stand? (SEE BACK OF PREVIOUS PAGE)

MOSTLY RIGHT; DUMB MISTAKE

2. Let (X, Y) be a uniformly random point in the triangle in the plane with vertices $(0,0), (0,1), (1,0)$. Find the joint PDF of X and Y, the marginal PDF of X, and the conditional PDF of X given Y. (SEE BACK)

3. Let X, Y be i.i.d. Expo(λ). Find $E|X - Y|$ in two ways: (a) using a 2-D LOTUS and (b) using the memoryless property without any calculus. (SEE BACK). needed help w/ strategy of evaluating ∬

4. Two students, A and B, are working independently on homework (not necessarily for the same class). Student A takes $Y_1 \sim$ Expo(λ_1) hours to finish his or her homework, while B takes $Y_2 \sim$ Expo(λ_2) hours. (SEE ATTACHED)

(a) Find the CDF and PDF of $\frac{Y_1}{Y_2}$, the ratio of their problem-solving times.

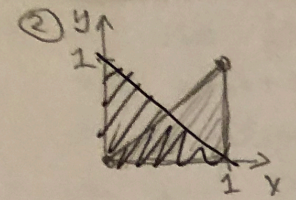
(b) Find the probability that A finishes his or her homework before B does.

5. The bus company from Blissville decides to start service in Blotchville, sensing a promising business opportunity. Meanwhile, Fred has moved back to Blotchville, inspired by a close reading of *I Had Trouble in Getting to Solla Sollew*. Now when Fred arrives at the bus stop, either of two independent bus lines may come by (both of which take him home). The Blissville company's bus arrival times are exactly 10 minutes apart, whereas the time from one Blotchville company bus to the next is Expo($\frac{1}{10}$). Fred arrives at a uniformly random time on a certain day.

(a) What is the probability that the Blotchville company bus arrives first? (SEE ATTACHED)

Hint: one good way is to use the continuous Law of Total Probability.

(b) What is the CDF of Fred's waiting time for a bus?



② Probability is uniformly distributed across area of triangle, Area $= \frac{1}{2}(1)(1) = \frac{1}{2}$. So the joint PDF of X, Y is $f_{X,Y}(x,y) = \frac{1}{\frac{1}{2}} = \boxed{2}$ for $x \in (0,1)$ and $0<y<x$. The marginal PDF of X is $f_X(x)$ and is found by marginalizing out Y: $\int_0^x f_{X,Y}(x,y)\,dy = f_X(x)$.

$f_X(x) = \int_0^{\cancel{1-}x} 2\,dy = 2y\Big|_0^{\cancel{1-}x} = \boxed{2\cancel{(1-}x\cancel{)}}$ for $x \in (0,1)$. The conditional PDF of $X|Y$, $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$. First, we need $f_Y(y)$ which we get by marginalizing out X

$f_Y(y) = \int_y^1 2\,dx = 2x\Big|_y^1 = 2-2y$ for $0 \le y \le 1$. $f_{X|Y}(x|y) = \frac{2}{2(1-y)} = \boxed{\frac{1}{1-y}}$ for $y \in (0,1)$.

constant w.r.t x, so $X|Y \sim \text{unif}(0, 1-Y)$

③A Using scale transform, start with $\lambda = 1$, because then $X', Y' \overset{iid}{\sim} \text{Expo}(1)$. Later we can transform back into general form: $\frac{X'}{\lambda} = X$, $\frac{Y'}{\lambda} = Y \sim \text{Expo}(\lambda)$

By LOTUS, $E|X'-Y'| = 2\int_0^\infty \int_0^x (x-y)e^{-x}e^{-y}\,dy\,dx = 2\int_0^\infty e^{-x}\left(\int_0^x xe^{-y} - ye^{-y}\,dy\right)dx =$

$2\int_0^\infty e^{-x}\left[\int_0^x xe^{-y}\,dy - \int_0^x ye^{-y}\,dy\right]dx = 2\int_0^\infty e^{-x}\left[\int_0^x xe^{-y}\,dy - 1 + e^{-x}(x+1)\right]dx =$

$2\int_0^\infty e^{-x}\left[x(1-e^{-x}) - 1 + e^{-x}(x+1)\right]dx = 2\int_0^\infty (e^{-x} + x - 1)e^{-x}\,dx = 2\int_0^\infty e^{-2x} + xe^{-x} - e^{-x}\,dx$

$= 2\left[\int_0^\infty e^{-2x}\,dx + \int_0^\infty xe^{-x}\,dx - \int_0^\infty e^{-x}\,dx\right] = 2\left(\frac{1}{2}+1-1\right) = 1 = E|X'-Y'| \Rightarrow$

$\frac{1}{\lambda} = E\left|\frac{X'}{\lambda} - \frac{Y'}{\lambda}\right|$, since $\lambda > 0$. $\Rightarrow \boxed{E|X-Y| = \frac{1}{\lambda}}$. Needed general strategy to solve. ..

③B. $E|X-Y|$ is the same as $E(\max(X,Y) - \min(X,Y)) = E(\max(X,Y)) - E(\min(X,Y))$. We know $E(\min(X,Y)) = \frac{1}{2\lambda}$. $\min(X,Y) \sim \text{Expo}(2\lambda)$ since $X, Y \overset{iid}{\sim} \text{Expo}(\lambda)$.

CDF of $\max(X,Y)$ is $F_M(m) = F_X(m)F_Y(m) = (1-e^{-\lambda m})^2$ by INDEPENDENCE. We could use LOTUS to get $E(\max(X,Y)) = \int_0^\infty m f_M(m)\,dm = \int_0^\infty m\,2\lambda e^{-\lambda m}(\lambda e^{-\lambda m})\,dm$

I can't evaluate this, so relying on memorylessness: $E(X|X \ge Y)$ is ? $E(X|X\ge Y) = E(Y) + E(X)$ by MEMORYLESSNESS. Y is

$E(\max(X,Y))$ when $X > Y$. $E(X|X\ge Y) = E(Y) + E(X)$ by MEMORYLESSNESS. Y is $E(\min(X,Y))$ when $X>Y$, so $E(Y) + E(X) - E(Y) = E(\max(X,Y)) - E(\min(X,Y)) =$ $\min(X,Y)$ when $X > Y$, so $E(Y)+E(X)-E(Y) = E(\max(X,Y)) - E(\min(X,Y)) =$

$E(X) = \frac{1}{\lambda}$. This result holds when $Y > X$, too, and by symmetry of iid cont. RVs, $\frac{1}{2}\left(\frac{1}{\lambda}\right) + \frac{1}{2}\left(\frac{1}{\lambda}\right) = \boxed{\frac{1}{\lambda}}$.

should this be $E(Y|Y<X)$?

[HW 7]

4 $T = Y_1/Y_2$ where $Y_1 \perp\!\!\!\perp Y_2$ and $Y_1 \sim \text{Expo}(\lambda_1)$, $Y_2 \sim \text{Expo}(\lambda_2)$. The CDF of T is the same thing as $P(Y_1 \le tY_2) = \int_0^\infty \int_0^{ty_2} \lambda_1 e^{-\lambda_1 y_1} \lambda_2 e^{-\lambda_2 y_2} \, dy_1 \, dy_2$. Simplifying, we get

$P(T \le t) = F_T(t) = \int_0^\infty \lambda_2 e^{-\lambda_2 y_2} \left(\int_0^{ty_2} \lambda_1 e^{-\lambda_1 y_1} dy_1 \right) dy_2 = \int_0^\infty \lambda_2 e^{-\lambda_2 y_2} \left(1 - e^{-\lambda_1 t y_2}\right) dy_2 =$

$(t<0)$

$= \underbrace{\int_0^\infty \lambda_2 e^{-\lambda_2 y_2} dy_2}_{1} - \int_0^\infty \lambda_2 e^{-y_2(\lambda_2 + \lambda_1 t)} dy_2 = 1 + \lambda_2 \int_0^\infty -e^{-y_2 k} dy_2$ where

$k = (\lambda_2 + \lambda_1 t)$. This yields $\boxed{F_T(t) = 1 - \frac{\lambda_2}{(\lambda_2 + \lambda_1 t)} = \frac{\lambda_1 t}{\lambda_2 + \lambda_1 t}}$ The PDF of T, $f_T(t)$, is the derivative of the CDF with respect to t: $F_T'(t) = \frac{\lambda_1((\lambda_2 + \cancel{\lambda_1 t}) - \cancel{\lambda_1 t})}{(\lambda_2 + \lambda_1 t)^2} =$

$\boxed{\frac{\lambda_2 \lambda_1}{(\lambda_2 + \lambda_1 t)} = f_T(t)}$. The probability A finishes before B is equivalent to $P(\frac{Y_1}{Y_2} < 1)$ which is equivalent to $P(T < 1) =$ $\boxed{F_T(1) = \frac{\lambda_1}{\lambda_2 + \lambda_1}}$.

5 A. Let $X \sim \text{Unif}(0,10)$ be waiting time for Bliss, bus and $Y \sim \text{Expo}(\frac{1}{10})$ be waiting time for Blotch bus (from MEMORYLESSNESS), we are interested in $P(Y < X) = \int_0^{10} \int_0^x f_{X,Y}(x,y) \, dy \, dx = \int_0^{10} f_X(x) \left(\int_0^x f_{Y|X}(y|x) \, dy \right) dx =$

$\int_0^{10} \frac{1}{10}(1 - e^{-\frac{x}{10}}) \, dx$ by INDEPENDENCE & LOTP. $P(Y<X) = 1 - \frac{1}{10}\left(-10e^{-\frac{x}{10}} \Big|_0^{10}\right) =$

$= 1 + e^{-1} - 1 = \boxed{e^{-1}}$

B. ~~If~~ $T = \min(X,Y)$, then we are interested in CDF of T, $P(T \le t) =$

$F_T(t) = P(\min(X,Y) \le t) \iff P(\text{at least one of } X \text{ or } Y \le t) =$

$1 - P(\text{both } X, Y > t) = 1 - P(X > t \cap Y > t) = 1 - P(X>t)P(Y>t)$ by INDEP.

$1 - (1 - F_X(t))(1 - F_Y(t)) = 1 - \left[\left(1 - \int_0^t \frac{1}{10} dx\right)\left(\cancel{1 - 1} + e^{-\frac{t}{10}}\right)\right] = 1 - \left[\left(1 - \frac{t}{10}\right) e^{-\frac{t}{10}}\right] = F_T(t)$

Therefore, $\boxed{F_T(t) = 1 - e^{-\frac{t}{10}} + \frac{t}{10} e^{-\frac{t}{10}}}$ for $t \in [0, 10]$; $\left\{ \begin{array}{l} F_T(t) = 0 \text{ for } t < 0 \\ F_T(t) = 1 \text{ for } t \ge 10 \end{array} \right\}$

6 A. $f_{X,Y,Z}(x,y,z) = \frac{e^{-3\lambda}(3\lambda)^x}{x!} \cdot \frac{e^{-6\lambda}(6\lambda)^y}{y!} \cdot \frac{e^{-6\lambda}(6\lambda)^z}{z!} = \boxed{\frac{e^{-15\lambda}(3\lambda)^{x+y+z}(2)^{y+z}}{x!\,y!\,z!}}$

for $x, y, z > 0$

~~B~~ $f_{X,Y,Z}(x,y,z \mid X+Y+Z = 36) = \frac{e^{-15\lambda}(3\lambda)^{36} 2^{36-x}}{x!\,y!\,z!} = \boxed{\frac{e^{-15\lambda}(6\lambda)^{36}}{x!\,y!\,z!\,2^x}}$

LAZY, didn't think!

(SEE BACK FOR WORKED SOL'N)

(SEE BACK)

6 B $f_{X,Y,Z}(x,y,z \mid X+Y+Z=36) = P(X=x, Y=y, Z=z \mid T=t) = \dfrac{P(T=t \mid X=x, Y=y, Z=z)\,P(X=x,Y=y,Z=z)}{P(T=t)}$

Let $T = X+Y+Z \sim \text{Pois}(15\lambda)$. (for $x+y+z=t$); (0 otherwise)

X,Y,Z is conditionally Multinomial given T=t

$$= \frac{(1)\dfrac{e^{-3\lambda}(3\lambda)^x e^{-6\lambda}(6\lambda)^y e^{-6\lambda}(6\lambda)^z}{x!\,y!\,z!}}{\dfrac{e^{-15\lambda}(15\lambda)^t}{t!}}$$

$$= \frac{t!}{x!\,y!\,z!}\left(\frac{3}{15}\right)^x\left(\frac{6}{15}\right)^y\left(\frac{6}{15}\right)^z \quad \text{(multinomial!)}$$

7 C $P(X=x, Y=y \mid X+Y+Z=36)$ is 0 when $Z \neq 36-X-Y$. Let T be defined as it was from 6 B. $P(X=x, Y=y \mid T=t) = \dfrac{P(T=t \mid X=x, Y=y)\,P(X=x, Y=y)}{P(T=t)}$ for when $x+y+z=t$, 0 otherwise. This gets us back to 6 B. We want $P(W=w \mid T=t)$ where $W = X+Y$. Using LUMPING property of Multinomial, $P(W=w \mid T=t) = \dfrac{t!}{w!\,(t-w)!}\left(\frac{9}{15}\right)^w\left(\frac{6}{15}\right)^{n-t}$.

This means $W \mid T=36 \sim \text{Bin}\left(36, \frac{9}{15}\right)$. Expectation of Binomial is $\boxed{np = 36\left(\frac{9}{15}\right) = \frac{108}{5}}$. Variance of Binomial = $\boxed{np(1-p) = \frac{108}{5}\left(\frac{6}{15}\right)}$.

6. Emails arrive in an inbox according to a Poisson process with rate λ (so the number of emails in a time interval of length t is distributed as $\text{Pois}(\lambda t)$, and the numbers of emails arriving in disjoint time intervals are independent). Let X, Y, Z be the numbers of emails that arrive from 9 am to noon, noon to 6 pm, and 6 pm to midnight (respectively) on a certain day.

(SEE BACK)

(a) Find the joint PMF of X, Y, Z.

(b) Find the conditional joint PMF of X, Y, Z given that $X + Y + Z = 36$.

(needed hint, but well understood)

(c) Find the conditional PMF of $X + Y$ given that $X + Y + Z = 36$, and find $E(X+Y|X+Y+Z = 36)$ and $\text{Var}(X+Y|X+Y+Z = 36)$ (conditional expectation and conditional variance given an event are defined in the same way as expectation and variance, using the conditional distribution given the event in place of the unconditional distribution).

7. Shakespeare wrote a total of 884647 words in his known works. Of course, many words are used more than once, and the number of distinct words in Shakespeare's known writings is 31534 (according to one computation). This puts a lower bound on the size of Shakespeare's vocabulary, but it is likely that Shakespeare knew words which he did not use in these known writings.

More specifically, suppose that a new poem of Shakespeare were uncovered, and consider the following (seemingly impossible) problem: give a good prediction of the number of words in the new poem that do not appear anywhere in Shakespeare's previously known works.

The statisticians Ronald Thisted and Bradley Efron studied this problem in a paper called "Did Shakespeare write a newly-discovered poem?", which performed statistical tests to try to determine whether Shakespeare was the author of a poem discovered by a Shakespearean scholar in 1985. A simplified version of their method is developed in the problem below. The method was originally invented by Alan Turing (the founder of computer science) and I.J. Good as part of the effort to break the German Enigma code during World War II.

Let N be the number of distinct words that Shakespeare knew, and assume these words are numbered from 1 to N. Suppose for simplicity that Shakespeare wrote only two plays, A and B. The plays are reasonably long and they are of the same length. Let X_j be the number of times that word j appears in play A, and Y_j be the number of times it appears in play B, for $1 \leq j \leq N$.

(a) Explain why it is reasonable to model X_j as being Poisson, and Y_j as being Poisson with the same parameter as X_j.

This is a reasonable modelling assumption because the Poisson is discrete and has a support ≥ 0 which makes sense for counting frequency of occurrences of word. By POISSON PARADIGM, Poisson counts # of success over some interval (here, full length play) when we know use of word, on average, happens λ times per play.

[HW 7]

7 ~~a~~. $P(Y_{eyeball} > 0 \cap X_{eyeball} = 0) = \left(1 - \frac{e^{-\lambda}\cancel{\lambda^0}}{\cancel{0!}}\right)\left(\frac{e^{-\lambda}\cancel{\lambda^0}}{\cancel{0!}}\right) =$
$= (1-e^{-\lambda})e^{-\lambda}$. The Taylor series approximation of e^x is $\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$. Substituting $-\lambda$ for x we get
$e^{-\lambda} \approx 1 + \frac{-\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{-\lambda^3}{3!} + \dots$ $1-e^{-\lambda} = 1-\left(1-\lambda+\frac{\lambda^2}{2!}-\frac{\lambda^3}{3!}+\dots\right)$,
so $(1-e^{-\lambda})e^{-\lambda} = \boxed{e^{-\lambda}\left(\lambda - \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} - \frac{\lambda^4}{4!} + \dots\right)}$.

7 b. $\int_0^\infty P(X=x, Y=y \mid \lambda) f_0(\lambda)\, d\lambda = P(X=x, Y=y)$ is the general strategy:
From 7a, $\int_0^\infty e^{-\lambda}\left(\lambda - \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} \dots\right) f_0(\lambda)\, d\lambda = \underbrace{\int_0^\infty \frac{e^{-\lambda}\lambda}{1!} f_0(\lambda)\, d\lambda}_{P(X=1)} - \underbrace{\int_0^\infty \frac{e^{-\lambda}\lambda^2}{2!} f_0(\lambda)\, d\lambda}_{P(X=2)} + \dots$
by LOTP.

7 c. Let I_j be indicator that word j is used in play B but not play A. We are interested in $E(I_1 + I_2 + \dots + I_N)$. By LINEARITY, and b we get $E(I_1) + E(I_2) + \dots = \sum_{j=1}^{N} \left(P(X_j=1) - P(X_j=2) + P(X_j=3) - \dots\right)$. By POISSON PARADIGM, $\underbrace{\sum_{j=1}^{N} P(X_j=1)}_{E(W_1)} - \underbrace{\sum_{j=1}^{N} P(X_j=2)}_{E(W_2)} + \underbrace{\sum_{j=1}^{N} P(X_j=3)}_{E(W_3)} - \dots$

$E(W_1) - E(W_2) + E(W_3) - \dots$

(b) Let the numbers of occurrences of the word "eyeball" (which was coined by Shakespare) in the two plays be independent Pois(λ) r.v.s. Show that the probability that "eyeball" is used in play B but not in play A is (SEE BACK OF PREVIOUS PAGE)

$$e^{-\lambda}(\lambda - \lambda^2/2! + \lambda^3/3! - \lambda^4/4! + \dots).$$

(c) Now assume that λ from (b) is unknown and is itself taken to be a random variable to reflect this uncertainty. So let λ have a PDF f_0. Let X be the number of times the word "eyeball" appears in play A and Y be the corresponding value for play B. Assume that the conditional distribution of X, Y given λ is that they are independent Pois(λ) r.v.s. Show that the probability that "eyeball" is used in play B but not in play A is the alternating series (SEE BACK OF PREVIOUS PAGE)

$$P(X = 1) - P(X = 2) + P(X = 3) - P(X = 4) + \dots.$$

Hint: condition on λ and use (b).

(d) Assume that every word's numbers of occurrences in A and B are distributed as in (c), where λ may be different for different words but f_0 is fixed. Let W_j be the number of words that appear exactly ~~j times in play A~~. Show that the expected number of distinct words appearing in play B but not in play A is (SEE BACK OF PREVIOUS PAGE)

$$E(W_1) - E(W_2) + E(W_3) - E(W_4) + \dots.$$

(This shows that $W_1 - W_2 + W_3 - W_4 + \dots$ is an *unbiased* predictor of the number of distinct words appearing in play B but not in play A: on average it is correct. Moreover, it can be computed just from having seen play A, without needing to know f_0 or any of the λ_j. This method can be extended in various ways to give predictions for unobserved plays based on observed plays.)