Module Code	Examiner	Email of Examiner	Tel
CSE107			

1st SEMESTER 2019/20 Remote Open-Book Resit Exam

Undergraduate – Year 2

DISCRETE MATHEMATICS AND STATISTICS

Exam Duration: ***2 Hours***

Crash Time Allowed: ***30 Minutes***

INSTRUCTIONS TO CANDIDATES

1. **This is a remote open-book resit exam.** ***Please sign the integrity disclaimer immediately after you initiate the online open-book resit exam on ICE. Upload your answers on ICE (when ICE Assignment function is used) together with the integrity disclaimer, and complete the assessment independently and honestly.***
2. **This exam consists of six questions. There are a total of 100 marks. The numbers within square brackets or parentheses on the right indicate the marks for each question. Relevant and clear steps should be included in the answers.**
3. **Answer all questions. There is NO penalty for providing a wrong answer.**
4. **Only English solutions are accepted.** ***Answers need to be handwritten and fully and clearly scanned or photographed for submission as one single PDF document via ICE (Take-home open book exam)*****.**

5. **The duration is *2 hours*, and an additional *30-minutes* crash time beyond the exam duration will be allowed for you to report and resolve minor technical issues which may be encountered during the exam. Where there are any major problems preventing you from continuing the exam or submitting your answers in time, please do not hesitate to email the Module Examiner or Assessment Team of Registry.**

Question 1: Proof Techniques **[10 marks]**

(a) Let a and b be irrational numbers. Prove or disprove that ab is an irrational number.

(2 marks)

(b) For $a \in \mathbb{Z}$, show that a is odd if and only if a^3 is odd.

(5 marks)

(c) For all natural numbers $m \geq 3$, use proof by *induction* to show that $\frac{(m+2)!}{2} > 2^{m+1}$.

(3 marks)

Question 2: Set Theory **[12 marks]**

(a) Let the universal set be $\mathbb{Z}$, $A = \{x \in \mathbb{Z} \mid x \text{ is even and } 3 < x < 9\}$, $B = \{x \in \mathbb{Z} \mid x \text{ is odd and } 4 < x < 8\}$, and $C = \{x \in \mathbb{Z} \mid x \text{ is odd and } 3 < x < 9\}$. Find the elements of statement:

a) $(C \cap A) \cup (A \cup B) \cup \sim\mathbb{Z}$
b) $((A - B)\Delta C) \cap \mathbb{Z}$
c) $(\sim(\sim A \cap \sim B) \cup (C \cap A)) \cap ((C \,\Delta\, (A - B)) \cap \mathbb{Z})$

(6 marks)

(b) Show that $(\mathbb{R}\mathrm{x}\mathbb{Z}) \cap (\mathbb{Z}\mathrm{x}\mathbb{R}) = \mathbb{Z}\mathrm{x}\mathbb{Z}$.

(6 marks)

Question 3: Relations **[14 marks]**

(a) Let $A = \{1,2,3,4\}$ and R ={(1,3),(3,4),(2,2),(1,1),(2,4)}. Represent R over A using a digraph and determine the corresponding adjacency matrix of the digraph.

(4 marks)

(b) Let R be the relation on $\mathbb{Z}$ given by xRy such that $3x$-y=3. If you think that R is both reflexive and symmetric, prove it. If not, give a counterexample.

(4 marks)

(c) Let $X = \{p, r, s, t\}$. What is the transitive closure of the relation: {(s, p), (p, r), (r, s), (s, t)} on X?

(6 marks)

Question 4: Functions **[16 marks]**

(a) Let $f: \mathbb{Z} \to \mathbb{Z}$, check whether $f(x) = \frac{3x+7}{2x-14}$ and $f^{-1}(x) = \frac{7\text{x}-2}{14\text{x}+1}$ are inverses.

(3 marks)

(b) Let $g: \mathbb{Z}^{+} \text{x}\, \mathbb{Z}^{+} \to \mathbb{Q}$ be given by $g(a,b) = \frac{a}{2b} + 3$. Prove or disprove that $g(a,b)$ is a bijective function.

(3 marks)

(c) Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x^2 + 3$ and $g: \mathbb{R} \to \mathbb{R}$ be given by $g(x) = 5x - 1$. Find $(f \circ g)(x)$, $(g \circ f)(x)$, $(f \circ f)(x)$ and $(g \circ g)(x)$.

(4 marks)

(d) Let g: Q → S be a function and A, B ⊂ Q, then g (A ∩ B) ⊂ (g (A) ∩ g (B)). If you think that it is true, prove it. If not, explain why.

(6 marks)

Question 5: Logic **[21 marks]**

(a) Without using any truth tables, show that the proposition $(\sim(a \to \sim b) \to (c \to (\sim d \to e))) \wedge (\sim((f \wedge g) \wedge (g \to \neg f)))$ is not a tautology.

(5 marks)

(b) For each of the two cases 1. and 2. below, show whether the statement on the left-hand side is equivalent to the statement on the right-hand side.

1. $\neg((p \wedge q) \vee \neg r) \equiv \neg(p \wedge q) \wedge r$
2. $(p \wedge q) \to r \equiv (p \to r) \wedge (q \to p)$

(4 marks)

(c) Let *S* be the signature consisting of a unary predicate symbol *Film*, a binary predicate symbol *IsActorOf* and the individual constants *peter* and *spiderman*. Assume that these symbols have the following meaning:

- *Film(x)* states: *x* is a film.
- *IsActorOf(x, y)* states: *x* is an actor in *y*.
- *peter* and *spiderman* refer to "*Peter*" and "*Spiderman*", respectively.

Translate the following sentences into *S*-formulae:

1. "*Spiderman*" is not a film.
2. Peter is an actor in "*Spiderman*".
3. There is a film that Peter is an actor in.
4. Peter is an actor in at least 2 films.
5. For every film there is an actor.
6. There is a film whose only actor is Peter.

(12 marks)

Question 6: Combinatorics and Probability **[27 marks]**

(a) For all integers $m \geq 1$, let *A* be a set of m elements and then there are 2^{m^2} distinct relations on *A*. Answer the following questions with justifications:

1. How many of such distinct relations are reflexive?
2. How many of such distinct relations are symmetric?
3. How many of such distinct relations are both reflexive and symmetric?

(10 marks)

(b) John has 100 students. Prove that at least two students were born in the same month of the year.

(2 marks)

(c) For all integers $m \geq 3$, $Q_1, \ldots, Q_m$ are propositions that can be true or false independently with the probability of 0.5. Answer the following questions with justifications:

1. What is the probability that the proposition $(Q_1 \vee Q_2) \rightarrow Q_3$ is true?
2. What is the probability that the proposition $(Q_1 \vee \ldots \vee Q_{m-1}) \rightarrow Q_m$ is true?

(5 marks)

(d) Snooker players Davis and White meet in the final match of the 2019 World Snooker Championship, where two players play against each other until one player wins 4 frames. Assume that:

- the probability that Davis wins a frame is 0.4;
- the probability that White wins a frame is 0.6; and
- these probabilities do not change during the final match.

1. Determine that the probability that Davis wins the match without losing any frames.
2. Determine that the probability that the match ends in 5 frames.
3. Determine that the probability that the match ends in 8 frames.
4. Determine that the probability that White wins the match and the match ends in 7 frames.

(10 marks)

END OF EXAM PAPER