Xi'an Jiaotong-Liverpool University

西交利物浦大学

Paper CODE	EXAMINER	DEPARTMENT	TEL
CSE 107		Computer Science and Software Engineering	

1st SEMESTER 2019/20 FINAL EXAMINATION

Undergraduate – Year 2

DISCRETE MATHEMATICS AND STATISTICS

TIME ALLOWED: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. **Total marks available are 100, accounting for 80% of the overall module marks.**
2. **The numbers within square brackets or parentheses on the right indicate the marks for each question.**
3. **Answer all questions.**
4. **Answers should be written in English in the answer script provided.**
5. **Relevant and clear steps should be included in the answers.**

Notes:

- **To obtain full marks for each question, relevant and clear steps must be included in the answers.**
- **Partial marks may be awarded depending on the degree of completeness and clarity.**

Question 1: Proof Techniques **[10 marks]**

(a) Let a be a positive real number, and the Statement S be: "if a is not rational number, then $\sqrt[6]{a}$ is also not a rational number".

1. Prove the Statement S by *contradiction*.
2. If you think the converse of the Statement S is true, prove it. If not, give a counter-example.

(7 marks)

(b) For all integer $m \geq 1$, use proof by *induction* to show that:

$$2m(\frac{4m+1}{3})(m+\frac{1}{2}) = (1^2 + 2^2 + 3^2 + \cdots + (2m)^2).$$

(3 marks)

Question 2: Set Theory **[10 marks]**

(a) Let the universal set be $\mathbb{Z}$ and $K = \{1,2,3,4,5,6,7,8,9\}$, $A = \{x \in \mathbb{Z} \mid x \text{ is odd and } 2 < x < 6\}$, $B = \{x \in \mathbb{Z} \mid x \text{ is odd and } 5 < x < 11\}$ and $C = \{x \in \mathbb{Z} \mid x \text{ is even and } 5 < x < 9\}$.
Find the characteristic vector of:
1. $(B \cap C) \cup A$ on K
2. $(B - C) \cup A$ on K
3. $(A \mathbin{\Delta} C) \cap (B \cap \sim A)$ on K

(4 marks)

(b) Consider the following statement: "for any sets X and Y, we have $X \subseteq Y$ if and only if $(Y - X) \cup X = Y$". If you think that it is true, prove it. Otherwise, give a counterexample to show that it is false.

(6 marks)

Question 3: Relations **[16 marks]**

(a) Let $A = \{a, b, c, d\}$ and R ={(d,b),(a,d),(c,c),(a,b),(b,a)}. Represent R over A using a digraph and determine the corresponding adjacency matrix of the digraph.

(4 marks)

(b) Let R and S be two partial orders on a set X, and T is a relation on X such that aTb (where $a,b \in X$) if and only if both aRb and aSb hold. Is T also a partial order on X? Justify your answer with proofs and/or counterexamples

(6 marks)

(c) Let $A = \{a, b, c\}$ and R be a relation on A. Prove or disprove that both R and R^2 must have the same transitive closure.

(6 marks)

Question 4: Functions **[18 marks]**

(a) Let $g: \mathbb{Q} \to \mathbb{Q}$, check whether $g\,(x) = \frac{x+7}{2x-5}$ and $g^{-1}(x) = \frac{5x+2}{2x-1}$ are inverses.

(2 marks)

(b) Let $f: \mathbb{Z}^{+} \text{x}\, \mathbb{Z} \to \mathbb{Q}$ be given by $f(x, y) = \frac{2x}{y}$. Prove or disprove that $f(x, y)$ is a bijective function.

(2 marks)

(c) Let A, B and C be non-empty finite sets, and $f: A \to B$, $g: B \to C$ and $h: A \to C$. Prove or disprove the following:

1. If f is surjective and g is not injective, then $g \circ f$ is not injective.
2. If f and g are bijective, and h is defined by h(x)=f(g(x)), then h is bijective.
3. Let $D \subseteq C$, then $g(g^{-1}(D)) \subseteq D$.
4. If g is surjective, then $g(g^{-1}(D)) = D$.

(14 marks)

Question 5: Logic **[22 marks]**

(a) Without completing the truth table for the formula, show that the formula $(a \to ((b \to (c \to d)) \to e)) \land (\sim((f \land g) \land (g \to \neg f)))$ is neither a contradiction nor a tautology.

(6 marks)

(b) State whether $(p \land q)$ and $(p \lor q) \land (\sim(p \land \sim q) \land (\sim q \lor p))$ are logically equivalent. If yes, prove it. If not, explain why.

(4 marks)

(c) Consider the signature *S = {Student, Male, Female, mary, helen, peter}* consisting of a binary predicate symbol *Student*, two unary predicate symbols *Male* and *Female*, and three constant symbols *mary*, *helen and peter*. Assume that these symbols have the following meaning:

- *Student* means "is a student of" (i.e., *Student (a, b)* states *a* is a student of *b*).
- *Male* means "is male" (i.e., *Male(a)* states *a* is male).
- *Female* means "is female" (i.e., *Female(a)* states *a* is female).
- *mary, helen and peter* refer to "Mary", "Helen" and "Peter", respectively.

Translate the following sentences into *S*-formulae; that is, for each of the following sentences provide an *S*-formula for each one of the six sentences:

1. Mary is a student of Helen.
2. Peter has a female student.
3. All students of Peter are also students of Helen.
4. Mary has at least 2 male students.
5. Everybody has a student.
6. Nobody is a student of everybody else.

(12 marks)

Question 6: Combinatorics and Probability **[24 marks]**

(a) For all integers $m \geq 1$, let A be set of m elements. Answer the following questions with justifications:

1. How many distinct relations on A are there?
2. How many of such relations on A are anti-symmetric?
3. How many of such relations on A are reflexive and anti-symmetric?

(9 marks)

(b) Explain what the "*Theory of Pigeonhole Principle*" means

(3 marks)

(c) For all integers $m \geq 3$, $Q_1, \ldots, Qm$ are propositions that can be true or false independently with the probability of 0.5. Answer the following questions with justifications:

1. What is the probability that the proposition $(Q_1 \vee \ldots \vee Q_{m-1}) \rightarrow Q_m$ is true?
2. What is the probability that the proposition $(Q_1 \wedge \ldots \wedge Q_{m-1}) \rightarrow Q_m$ is true?
3. What is the expectation of the number of the propositions Q_i that are true?

(6 marks)

(d) There are 200 lockers along with 200 distinct locker numbers for 50 workers. Any worker can take any of the 200 locker numbers. Workers can also share the same locker. Suppose you are an officer and are recording the locker number for all 50 workers, one at a time until you have found a match (this means that a locker number has already been recorded). Answer the following questions with justifications:

1. What is the probability that it takes more than 13 workers for this to happen?
2. What is the probability that it takes exactly 15 workers for this to happen?

(6 marks)

END OF EXAM PAPER