Code verification process of DNWR and NNWR code

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This document was created in the verification process for the DNWR and NNWR code, it was never intended to be super tidy and/or rigorous. There may be some copy paste errors and some plots might be slightly outdated or missing.

We first describe the tests to be conducted and then report the results for each combination of splitting scheme and time-integration method.

For the used methods, see the current paper in progress or the NNWR paper or directly the code

Contents

Todo list

1 Shared parameters and configurations

Default geometry is two coupled unit squares (lines) and the interface is positioned at $x = 0$.

All errors/updates are measured using the discrete L2 norm:

- 0 D (scalar case): $\|u\|_{L2} = |u|$
- 1 D: $\|\boldsymbol{u}\|_{L2} = \|\boldsymbol{u}\|_2 \sqrt{\Delta x}$
- 2 D: $\|\boldsymbol{u}\|_{L2} = \|\boldsymbol{u}\|_2 \Delta x$

Default space discretization given by internal points per unit length (N, $\Delta x = 1/(N+1)$) are 1D: 50, 2D: 32.

Unless mentioned otherwise, we use θ_{opt} for the relaxation parameter.

Maximum number of iterations is set to be 100.

1.1 Initial conditions

- 1D: $u_0(x) = 500 \sin(\pi/2(x+1))$
- 2D: $u_0(x, y) = 500 \sin(\pi/2(x+1)) \sin(\pi y)$

1.2 Material parameters

Default testing parameters: $\alpha_1 = \alpha_2 = 1$, $\lambda_1 = \lambda_2 = 0.1$, unless specified otherwise. Here we use $t_f = 1$.

- Air: $\alpha = 1.293 \cdot 1005$, $\lambda = 0.0243$
- Water: $\alpha = 999.7 \cdot 4192.1$, $\lambda = 0.58$
- Steel: $\alpha = 7836 \cdot 443$, $\lambda = 48.9$

Possible couplings here are Air-Water, Air-Steel, Water-Steel (ordering!). For non-equal material parameters we use $t_f = 1000$.

2 Verification tests

2.1 Basic verification

First step is to verify the correctness of the underlying discretizations, both in space and time.

Summary: It works.

2.1.1 Time-integration

We verify the correctness of the time-integration methods (**Implicit Euler** and **SDIRK2**) by looking at $\|\boldsymbol{u}(\Delta t) - \boldsymbol{u}^*\|$ for $\Delta t \to 0$ where $\boldsymbol{u}^* = \boldsymbol{u}(\Delta t_{\min}/2)$. See Figure 1 for 1D and 2 for 2D.

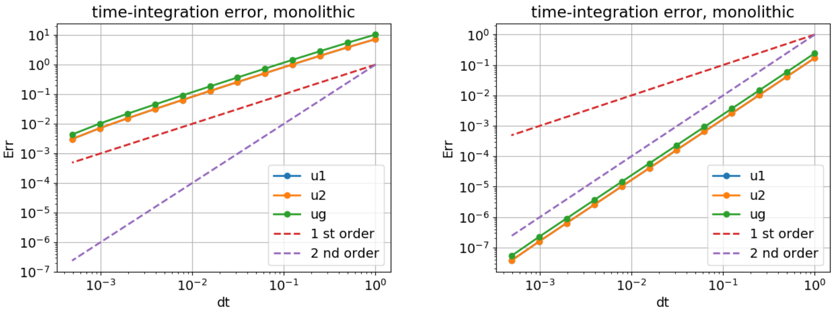


Figure 1: Time-integration order verification of monolithic system, 1D. See Section 2.1.1. Left: Implicit Euler; Right: SDIRK2

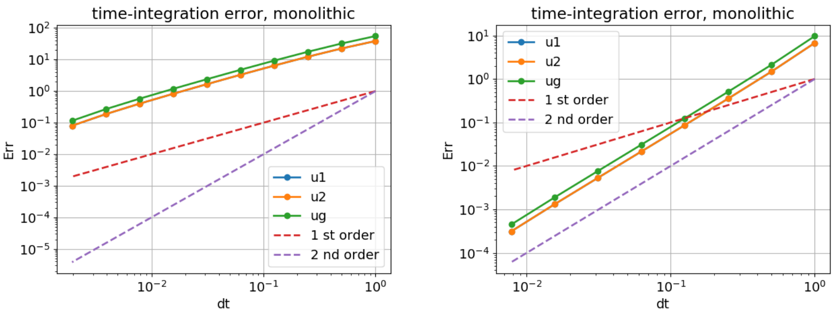


Figure 2: Time-integration order verification of monolithic system, 2D. See Section 2.1.1. Left: Implicit Euler; Right: SDIRK2

2.1.2 Space-discretization

Next we verify the error in space by looking at $\|u(\Delta t, \Delta x) - u_{ex}\|$ for $\Delta x \to 0$, a sufficiently small Δt (1/100 for SDIRK 2) and an exact reference solution for u_{ex}.

- 1D: $u_{ex}(t, x) = \exp(-(\pi^2 \lambda (t - t_0))/(4\alpha))u(t_0, x)$
- 2D: $u_{ex}(t, x, y) = \exp(-(5\pi^2 \lambda (t - t_0))/(4\alpha))u(t_0, x, y)$

The results are seen in Figure 3.

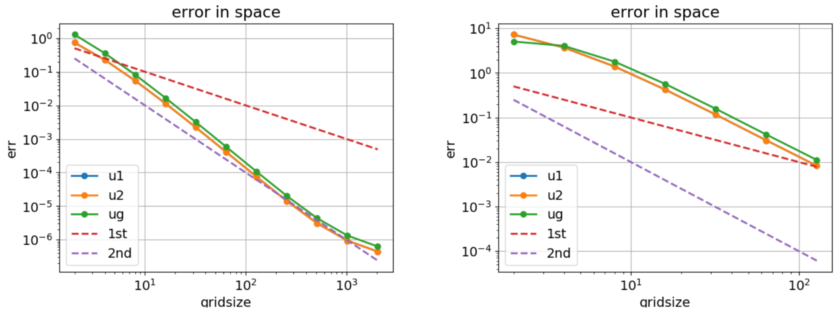


Figure 3: Spatial error order verification of monolithic system. Left: 1D; Right: 2D. See Section 2.1.2.

2.2 Method verification

With the DNWR or NNWR method, the goal is to perform time-integration of the two sub problems separately. The iterations introduced from these methods yield an additional "splitting error", which is to vanish for a sufficiently high number of iterations.

In later sections testing the methods, we refer to tests by their section.

2.2.1 Verification with monolithic solution

Let $u_h(\Delta t, TOL)$ be the solution from our iterative method for a given Δt and where the iteration converged, i.e., $\|u_\Gamma(t_e)^{(k)} - u_\Gamma(t_e)^{(k-1)}\| \leq TOL\|u_\Gamma(t_0)\|$ (relative update smaller than tolerance).

We want to show that $\|u_h(\Delta t, TOL) - u(\Delta t)\| \to 0$ for $TOL \to 0$, i.e. the splitting error given by the difference of the fully converged solution of our iterative method and the monolithic solution should tend to zero for tolerance going to zero. If this does not go to zero, it means the monolithic solution is not a fixed point of our iteration, for a given Δt.

This test passes, if $\|u_h(\Delta t, TOL) - u(\Delta t)\| = \mathcal{O}(TOL)$ and fails, if the error reaches a plateau. Default number of timesteps for this test is 20.

Note: For schemes that provide convergence within a single iteration for θ_{opt}, we also try a $\theta \neq \theta_{opt}$.

2.2.2 Convergence of splitting error

The previous test had two possible parameters, TOL and Δt. We also want to verify the above test holds for $\Delta t \to 0$. For this we use the sufficiently small tolerance of $TOL = 10^{-12}$.

This test passes if $\|u_h(\Delta t, TOL) - u(\Delta t)\| \lesssim TOL$ for all Δt.

2.2.3 Convergence of combined error

Ideally, both of the previous tests should pass, but not all is lost if they don't. That is, the splitting error should be seen in the context of the time-integration error. Namely, the splitting error is acceptable, even if not vanishing for $TOL \to 0$, as long as it is smaller, or of comparable size, to the time-integration error.

We test this by looking at $\|u_h(\Delta t, TOL) - u(\Delta t^*)\|$ for $\Delta t \to 0$, where $TOL = 10^{-12}$ and $\Delta t^* = \Delta t_{min}/2$, i.e. Δt^* very small. This test passes if $\|u_h(\Delta t, TOL) - u(\Delta t^*)\| = \mathcal{O}(\Delta t^p)$, where p is the time-integration order. A prerequisite of this is that the test of Section 2.2.2 at least shows the order equal to the time-integration error.

2.2.4 Convergence of adaptive method

Passing the previous test is the logical basis for the adaptive method, in which both the time-integration error and the splitting error are controlled by a single tolerance. What we should test is to see $\|u_h(TOL) - u(\Delta t^*)\| = \mathcal{O}(TOL)$ for $TOL \to 0$ and a sufficiently small Δt^*. However, this might require an extremely small Δt^*, such that we rather look at $\|u_h(TOL) - u_h(TOL^*)\|$, for a sufficiently small TOL^*, to verify the order.

2.2.5 Multirate convergence

We further want to test if the methods actually work in the multi-rate setting. To this end we do test 2.2.3 for different ratios of timesteps on both domains. We test the following combinations:

c-c Coarse-Coarse, $N1 = N2$, same timesteps

c-f $N2 = C \cdot N1$, C times more timesteps on Ω_2

f-c $N1 = C \cdot N2$

The coarse-coarse test is already done with test 2.2.3, it serves as reference here. As reference solution we use a monolithic solution with sufficiently many timesteps. We want to see the orders for coarse-fine and fine-coarse to be identical to the coarse-coarse case. We use $C = 10$.

3 DNWR

3.1 Implicit Euler

Figures: 4, 5, 6, 7 and 8. See Sections 3.1.1, 3.1.3 and 3.1.3 for non equal material parameters.

Summary: Passes all tests perfectly. For test 2.2.1 in 1D the method achieves convergence to the exact solution within 2 iterations. For a water-steel coupling the number of iterations increases for decreasing timestepsizes.

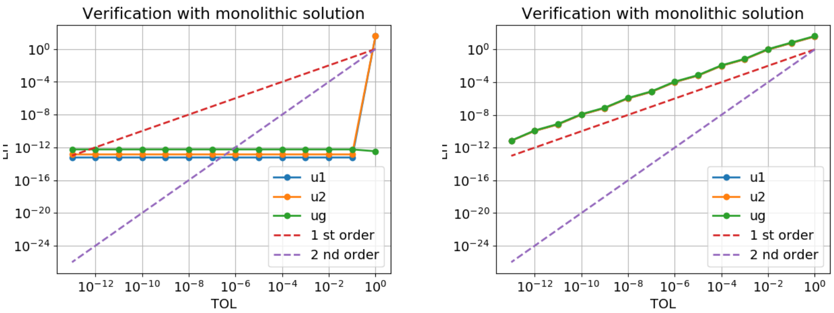


Figure 4: Test 2.2.1, DNWR, Implicit Euler, 1D for: Left: $\theta = \theta_{opt}$; Right: $\theta = 0.7$.

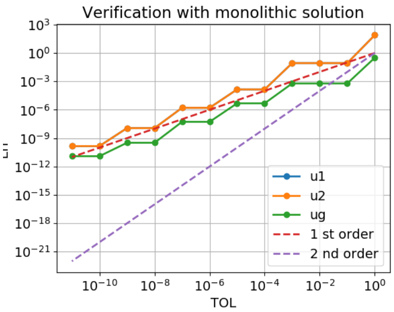


Figure 5: Test 2.2.1, DNWR, Implicit Euler, 2D.

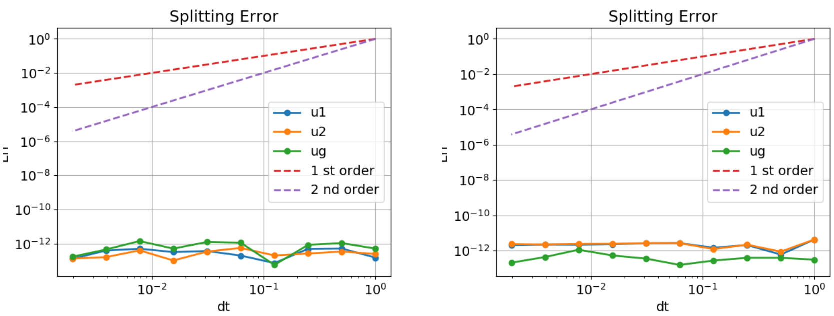


Figure 6: Test 2.2.2, DNWR, Implicit Euler. Left: 1D, Right: 2D

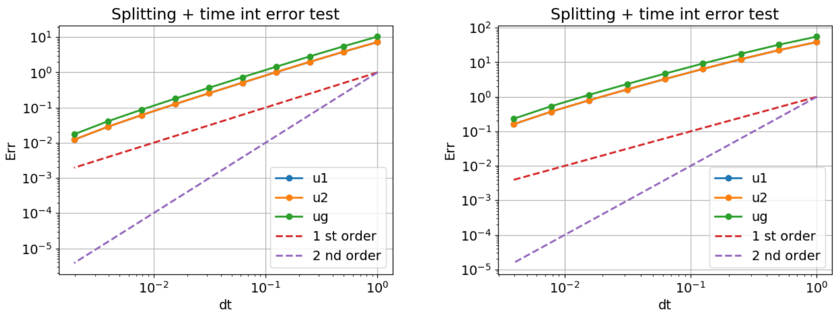


Figure 7: Test 2.2.3, DNWR, Implicit Euler. Left: 1D, Right: 2D

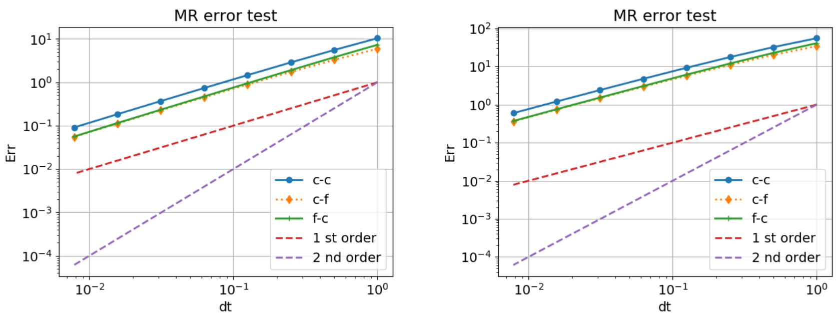


Figure 8: Test 2.2.5, DNWR, Implicit Euler. Left: 1D, Right: 2D

3.1.1 Air-Steel

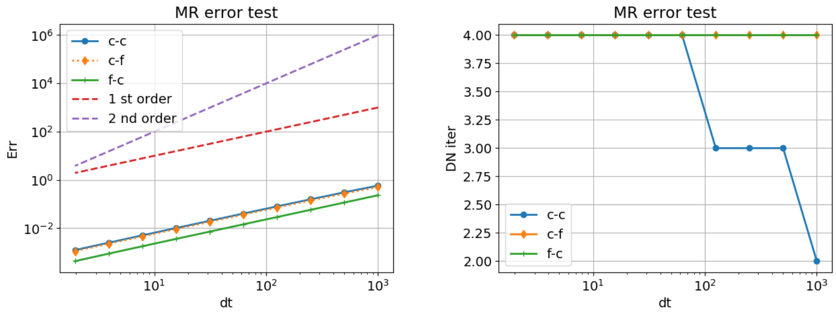


Figure 9: Test 2.2.5 for Air-Steel coupling, DNWR, Implicit Euler. 1D

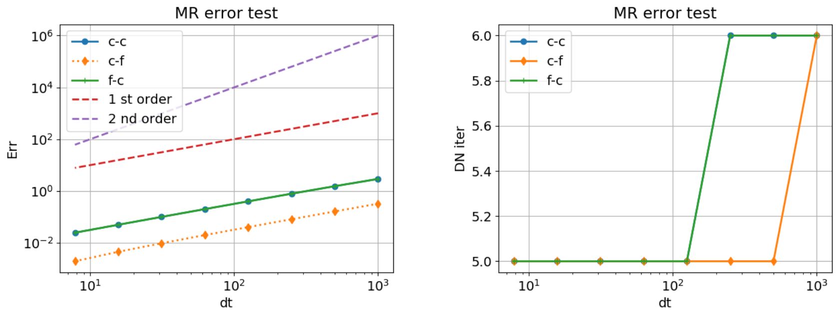


Figure 10: Test 2.2.5 for Air-Steel coupling, DNWR, Implicit Euler. 2D

3.1.2 Air-Water

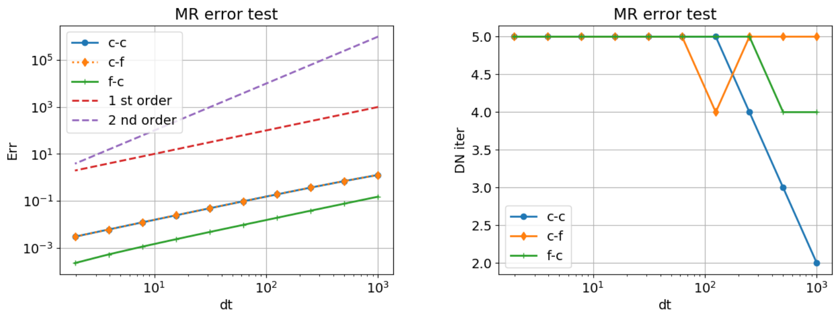


Figure 11: Test 2.2.5 for Air-Water coupling, DNWR, Implicit Euler. 1D

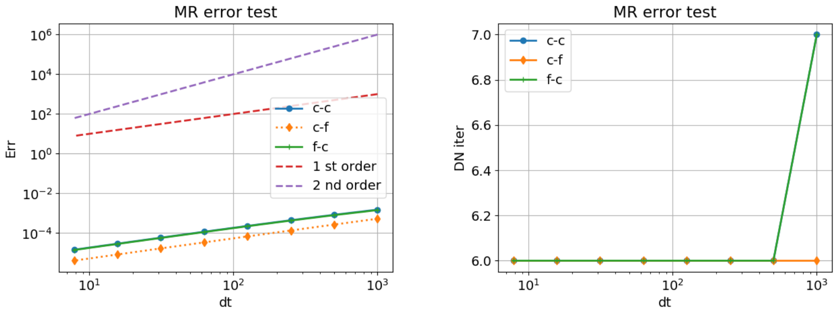


Figure 12: Test 2.2.5 for Air-Water coupling, DNWR, Implicit Euler. 2D

3.1.3 Water-Steel

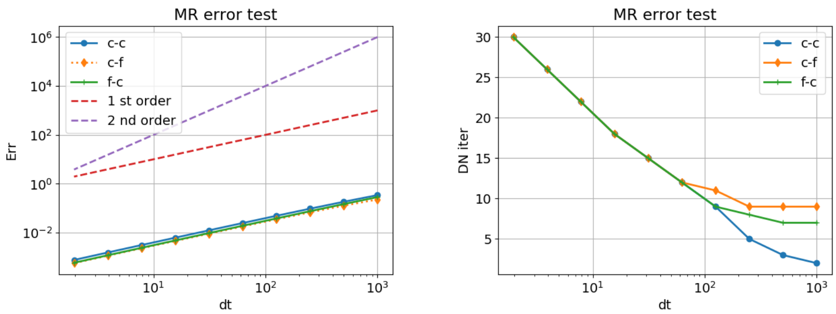


Figure 13: Test 2.2.5 for Water-Steel coupling, DNWR, Implicit Euler. 1D

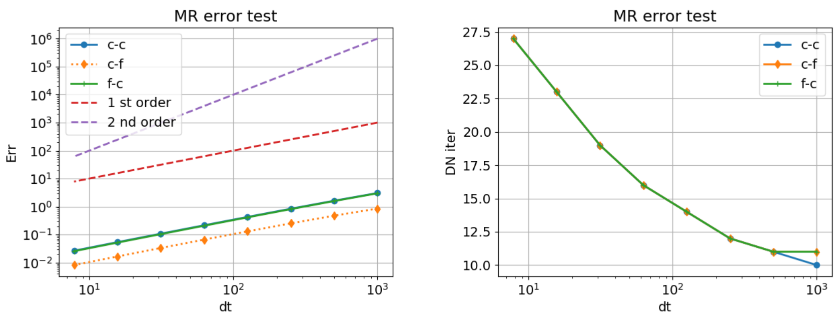


Figure 14: Test 2.2.5 for Water-Steel coupling, DNWR, Implicit Euler. 2D

3.2 SDIRK2

Figures: 15, 16, 17, 18 and 19. See Sections 3.2.1, 3.2.3 and 3.2.3 for non equal material parameters.

Summary: Method works fine.

Longer version: The monolithic solution is not the fixed point (except for 1D, there we get $\boldsymbol{u}_\Gamma$ correctly). The splitting error, likely due to the approximations of $\dot{\boldsymbol{u}}_\Gamma$ in the Dirichlet solver, is of order $\mathcal{O}(\Delta t^2)$. As such, the combined time-integration and splitting error still converges with $\mathcal{O}(\Delta t^2)$.

Tests with different materials work fine, the number of iterations for the Water-Steel coupling increases for $\Delta t \to 0$.

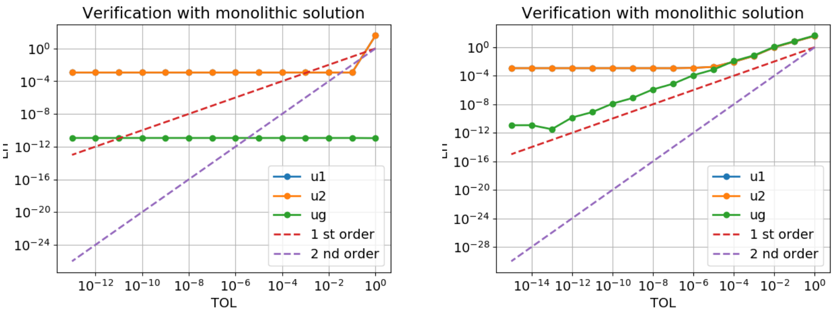


Figure 15: Test 2.2.1, DNWR, SDIRK2, 1D for: Left: $\theta = \theta_{opt}$; Right: $\theta = 0.7$.

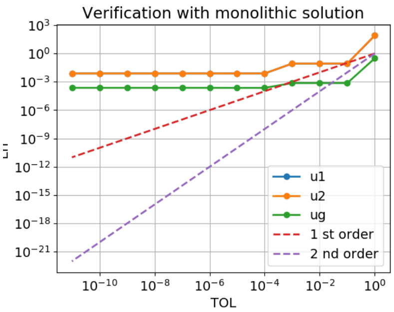


Figure 16: Test 2.2.1, DNWR, SDIRK2, 2D.

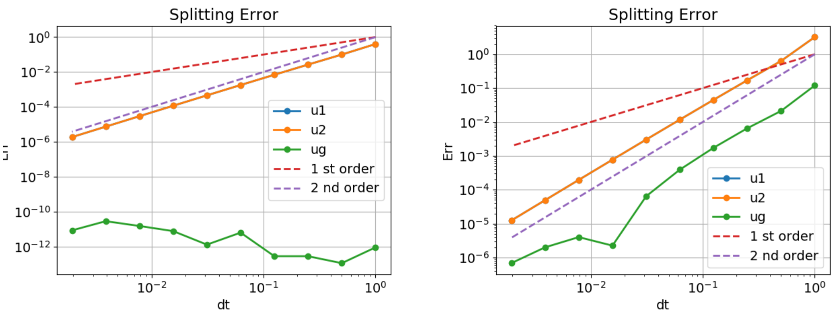


Figure 17: Test 2.2.2, DNWR, SDIRK2. Left: 1D, Right: 2D

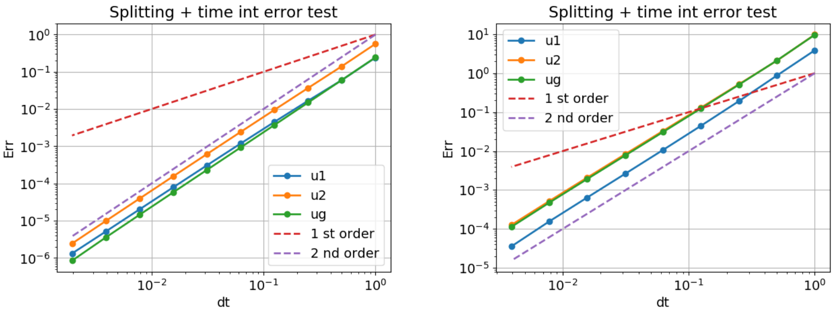


Figure 18: Test 2.2.3, DNWR, SDIRK2. Left: 1D, Right: 2D

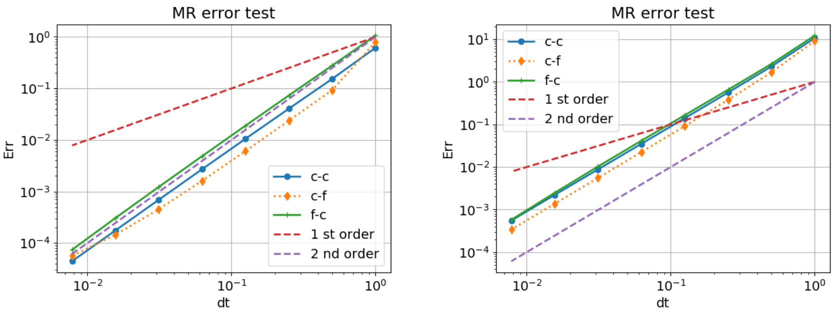


Figure 19: Test 2.2.5, DNWR, SDIRK2. Left: 1D, Right: 2D

3.2.1 Air-Steel

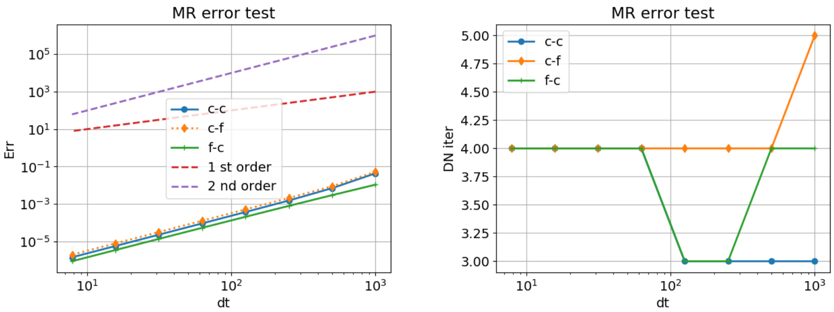


Figure 20: Test 2.2.5 for Air-Steel coupling, DNWR, SDIRK2. 1D

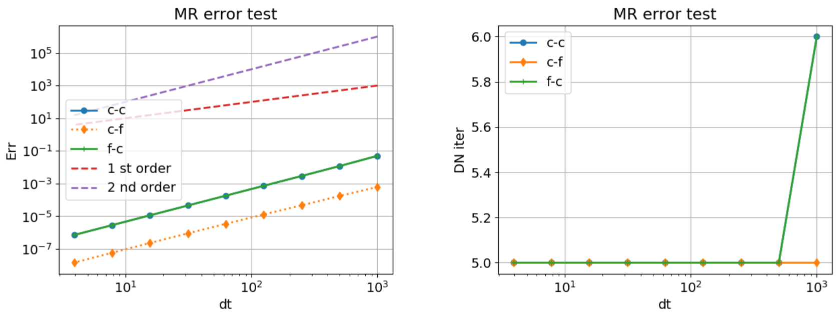


Figure 21: Test 2.2.5 for Air-Steel coupling, DNWR, SDIRK2. 2D

3.2.2 Air-Water

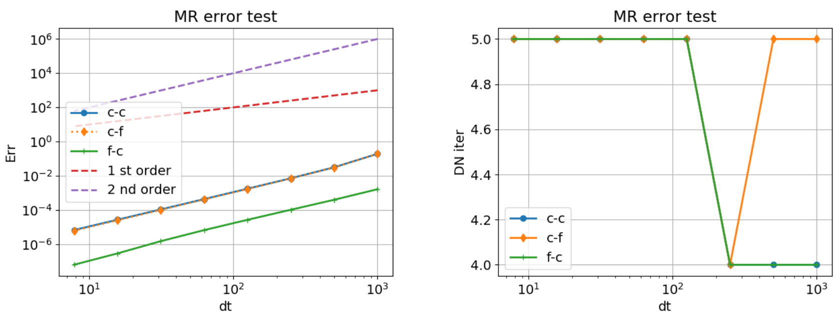


Figure 22: Test 2.2.5 for Air-Water coupling, DNWR, SDIRK2. 1D

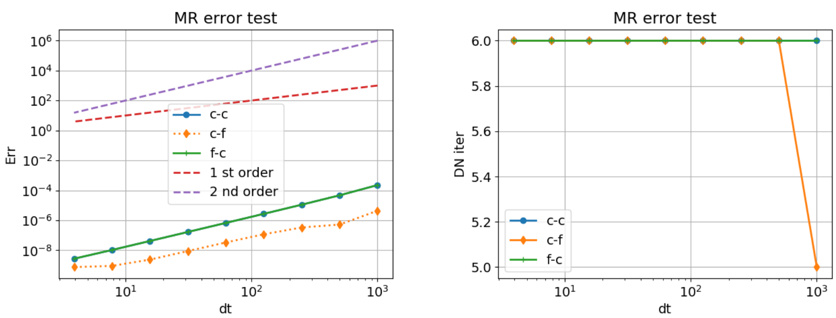


Figure 23: Test 2.2.5 for Air-Water coupling, DNWR, SDIRK2. 2D

3.2.3 Water-Steel

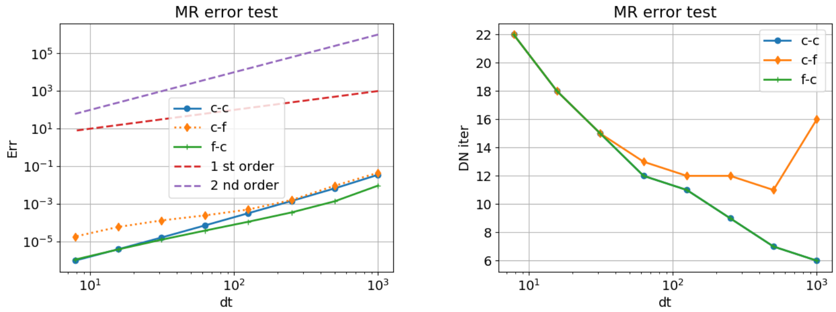


Figure 24: Test 2.2.5 for Water-Steel coupling, DNWR, SDIRK2. 1D

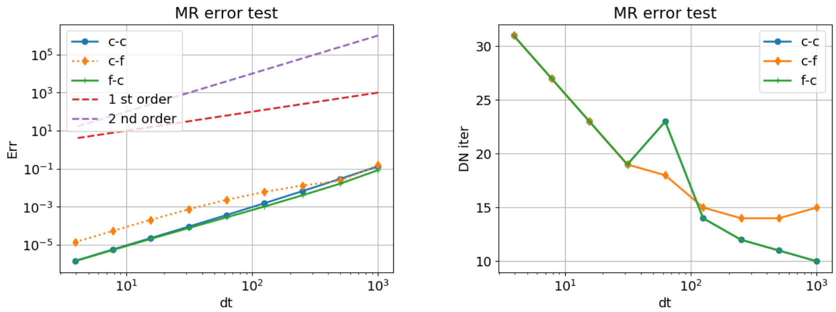


Figure 25: Test 2.2.5 for Water-Steel coupling, DNWR, SDIRK2. 2D

3.3 SDIRK2 time adaptive, single adaptive

Double adaptive: Independent adaptivity for both subdomains.
Figures: 26. Figures 27, 28 and 29 for different material parameters.
Summary: Everything works well!

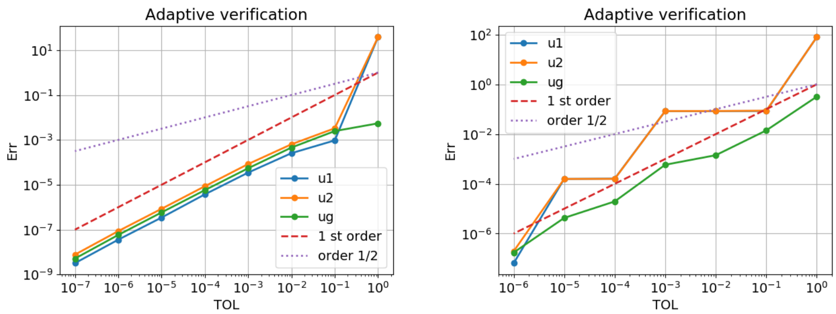


Figure 26: Test 2.2.3, DNWR, Time adaptive SDIRK2, single adaptive. Left: 1D, Right: 2D

3.3.1 Air-Steel

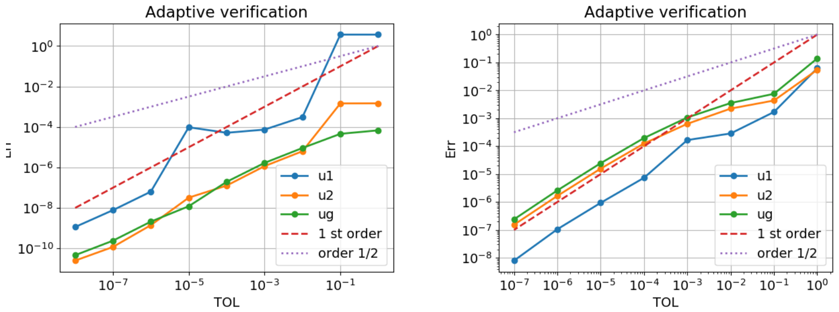


Figure 27: Test 2.2.5 for Air-Steel coupling, DNWR, SDIRK2 TA single. Left: 1D. Right: 2D

3.3.2 Air-Water

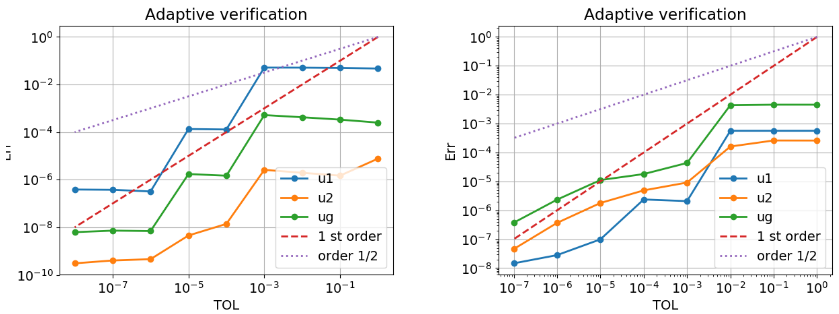


Figure 28: Test 2.2.5 for Air-Water coupling, DNWR, SDIRK2 TA single. Left: 1D. Right: 2D

3.3.3 Water-Steel

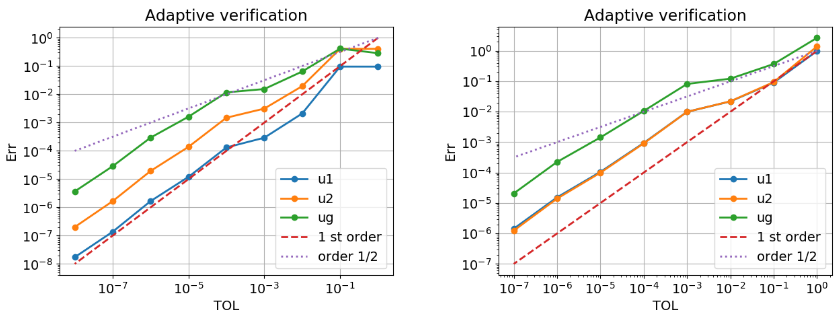


Figure 29: Test 2.2.5 for Water-Steel coupling, DNWR, SDIRK2 TA single. Left: 1D. Right: 2D

3.4 SDIRK2 time adaptive, double adaptive

Single adaptive: Do adaptive time-integration for Dirichlet problem and use **same** timesteps for Neumann problem.

Figures: 30. Figures 31, 32 and 34 for different material parameters.

Summary: Everything works well! Air-water 2D order looks a bit off, but the correct order is assumed eventually (tested via larger Δx and smaller tolerances, see Figure 33)

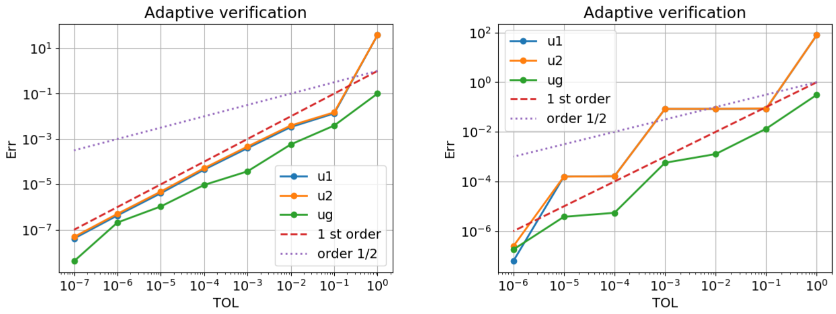


Figure 30: Test 2.2.3, DNWR, Time adaptive SDIRK2, double adaptive. Left: 1D, Right: 2D

3.4.1 Air-Steel

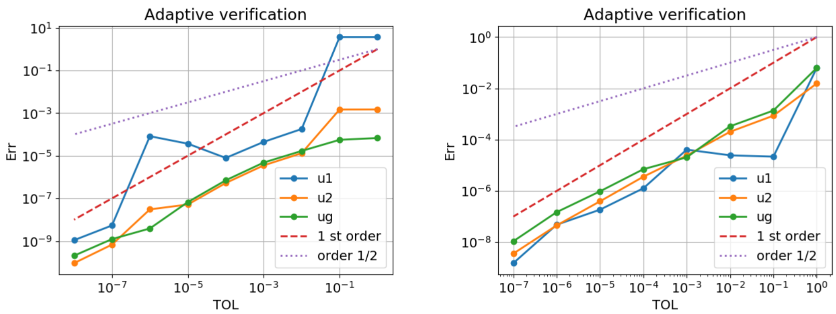


Figure 31: Test 2.2.5 for Air-Steel coupling, DNWR, SDIRK2 TA double. Left: 1D. Right: 2D

3.4.2 Air-Water

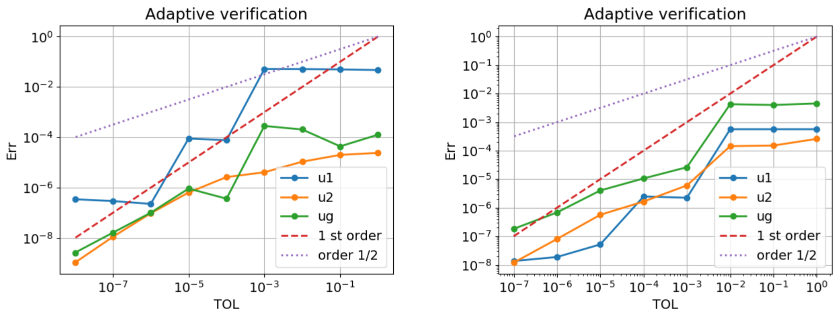


Figure 32: Test 2.2.5 for Air-Water coupling, DNWR, SDIRK2 TA double. Left: 1D. Right: 2D

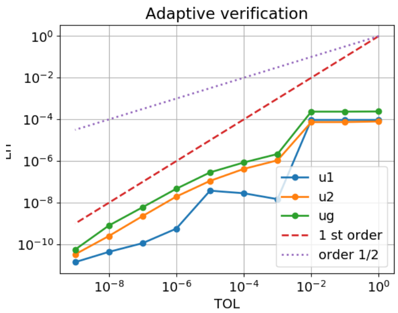


Figure 33: Test 2.2.5 for Air-Water coupling, DNWR, SDIRK2 TA double. 2D

3.4.3 Water-Steel

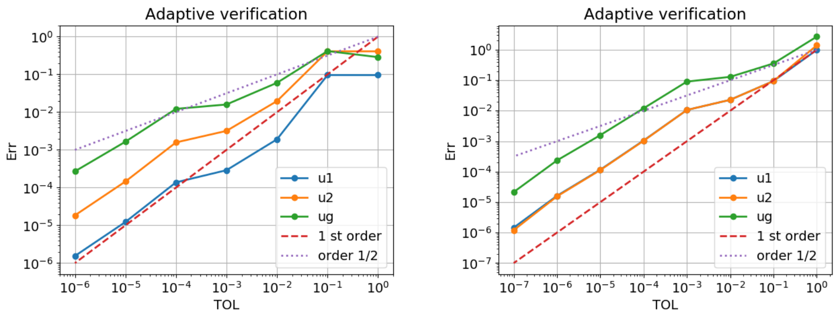


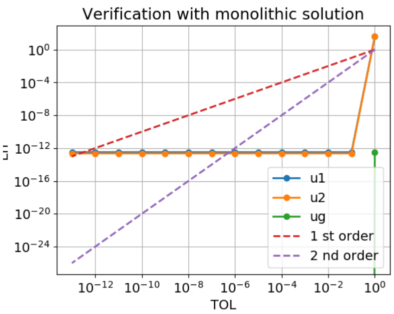
Figure 34: Test 2.2.5 for Water-Steel coupling, DNWR, SDIRK2 TA double. Left: 1D. Right: 2D

4 NNWR

4.1 Implicit Euler

Figures: 35, 36, 37, 38 and 39. See Sections 4.1.1, 4.1.3 and 4.1.3 for non equal material parameters.

Summary: Passes all tests for identical material parameters, number of iterations in MR case get quite large, see 40. For non identical materials, the correct orders are obtained, but at a very high cost. The number of iterations are very high and often hit the pre-set maximum. Not 100% sure if the iterations are in fact divergent or just converging very very slowly, since the errors are still reasonably small. Either way, the iterations numbers are too large to be feasible in practice.



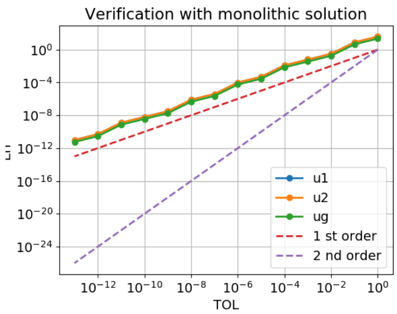


Figure 35: Test 2.2.1, NNWR, Implicit Euler, 1D for: Left: $\theta = \theta_{opt}$; Right: $\theta = 0.7$.

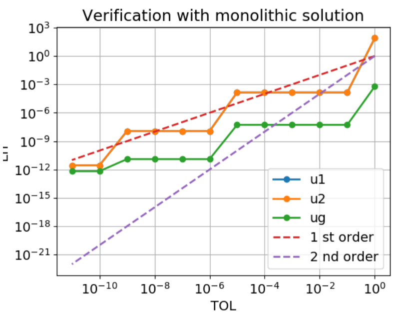


Figure 36: Test 2.2.1, NNWR, Implicit Euler, 2D.

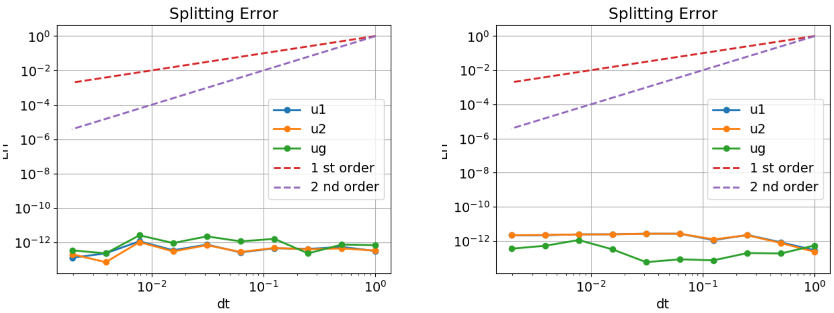


Figure 37: Test 2.2.2, NNWR, Implicit Euler. Left: 1D, Right: 2D

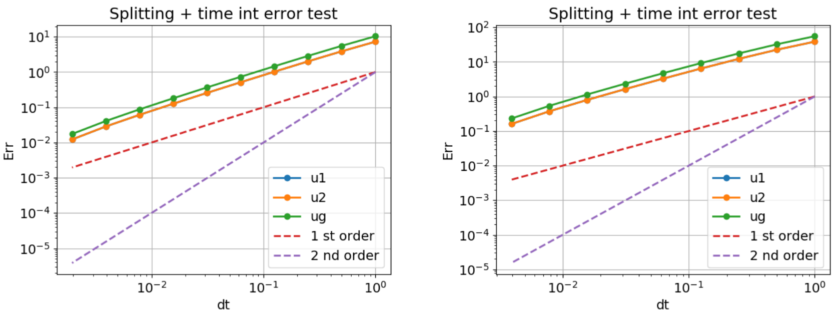


Figure 38: Test 2.2.3, NNWR, Implicit Euler. Left: 1D, Right: 2D

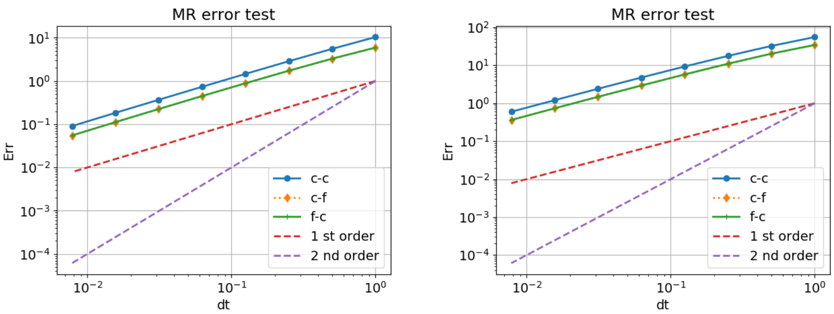


Figure 39: Test 2.2.5, NNWR, Implicit Euler. Left: 1D, Right: 2D

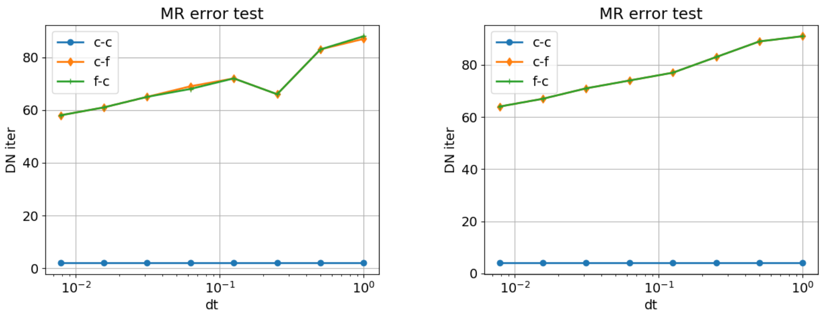


Figure 40: Test 2.2.5, iterations, NNWR, Implicit Euler. Left: 1D, Right: 2D

4.1.1 Air-Steel

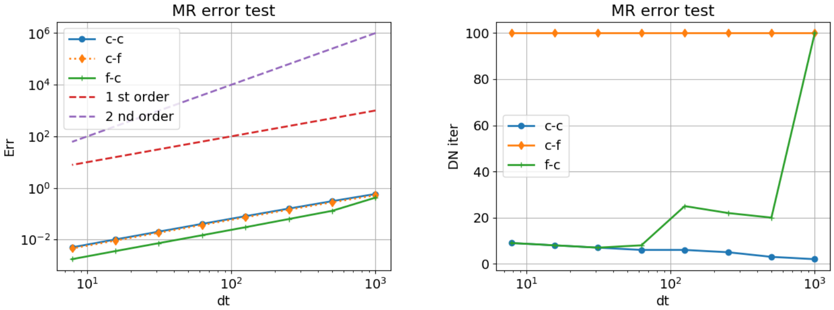


Figure 41: Test 2.2.5 for Air-Steel coupling, NNWR, Implicit Euler. 1D

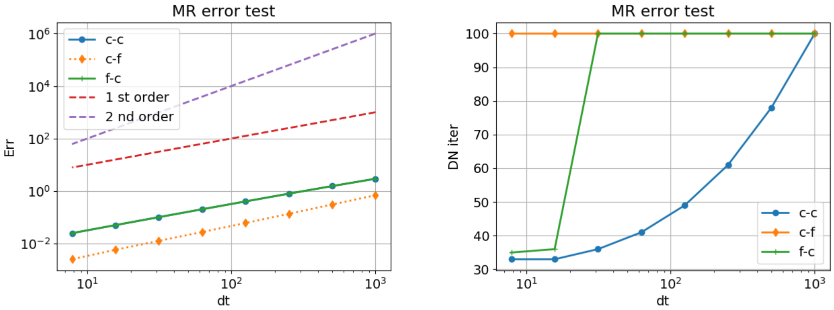


Figure 42: Test 2.2.5 for Air-Steel coupling, NNWR, Implicit Euler. 2D

4.1.2 Air-Water

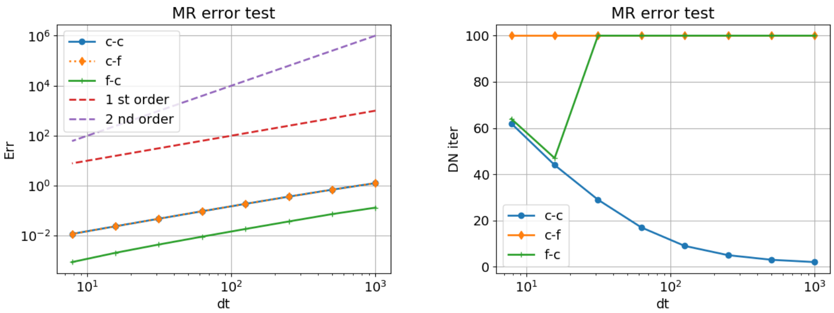


Figure 43: Test 2.2.5 for Air-Water coupling, NNWR, Implicit Euler. 1D

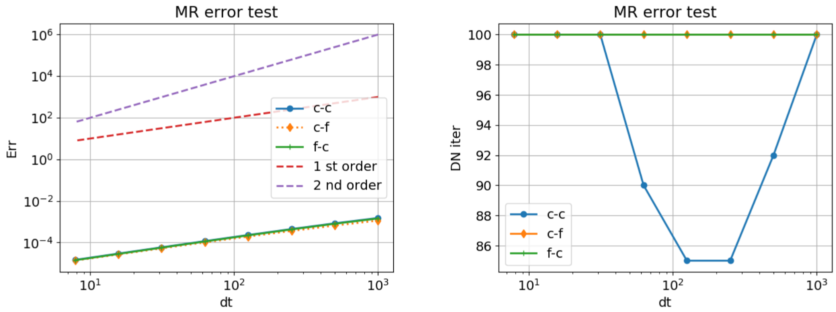


Figure 44: Test 2.2.5 for Air-Water coupling, NNWR, Implicit Euler. 2D

4.1.3 Water-Steel

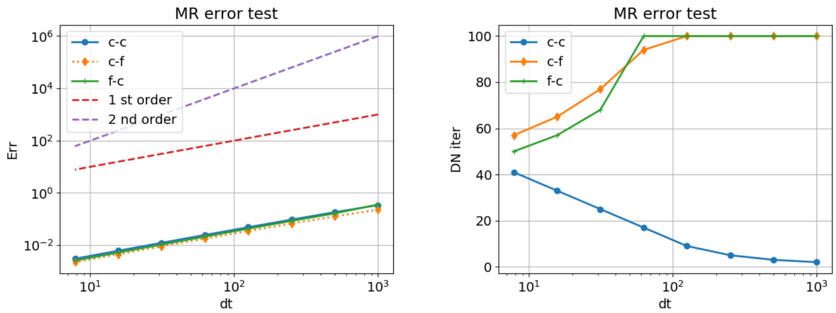


Figure 45: Test 2.2.5 for Water-Steel coupling, NNWR, Implicit Euler. 1D

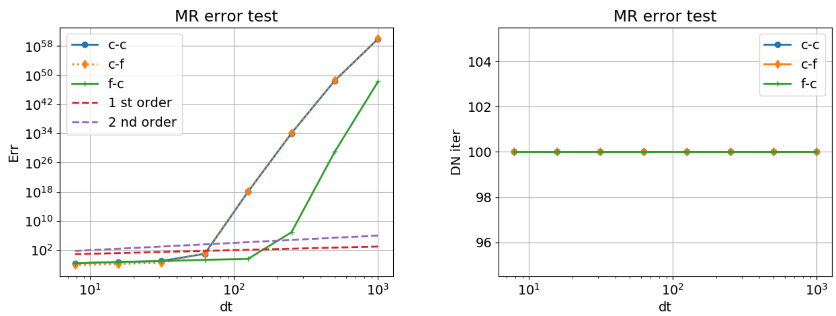


Figure 46: Test 2.2.5 for Water-Steel coupling, NNWR, Implicit Euler. 2D

4.2 SDIRK2

Figures: 47, 48, 49, 50 and 51. See Sections 4.2.1, 4.2.3 and 4.2.3 for non equal material parameters.

Summary: Passes all tests for identical material parameters, number of iterations in MR case get quite large, see 52. For non identical materials, the correct orders are mostly obtained, but the iterations diverge in some cases for too large Δt and reach the maximum number of iterations in most other cases. The method appears too sensitive with regards to Θ.

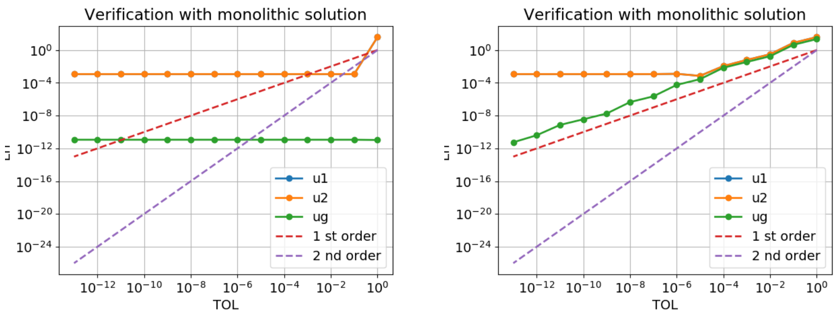


Figure 47: Test 2.2.1, NNWR, SDIRK2, 1D for: Left: $\theta = \theta_{opt}$; Right: $\theta = 0.7$.

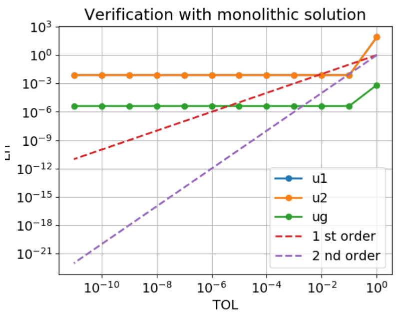


Figure 48: Test 2.2.1, NNWR, SDIRK2, 2D.

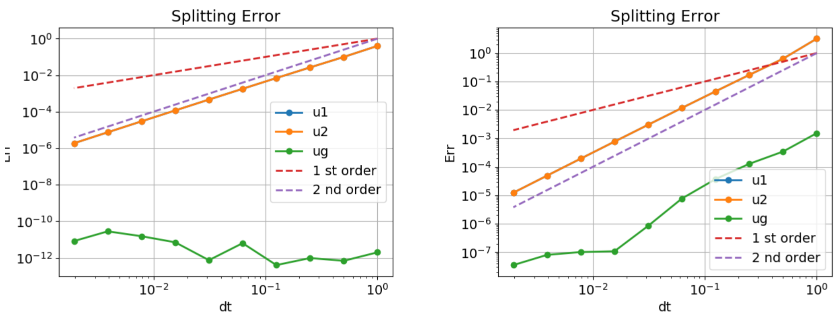


Figure 49: Test 2.2.2, NNWR, SDIRK2. Left: 1D, Right: 2D

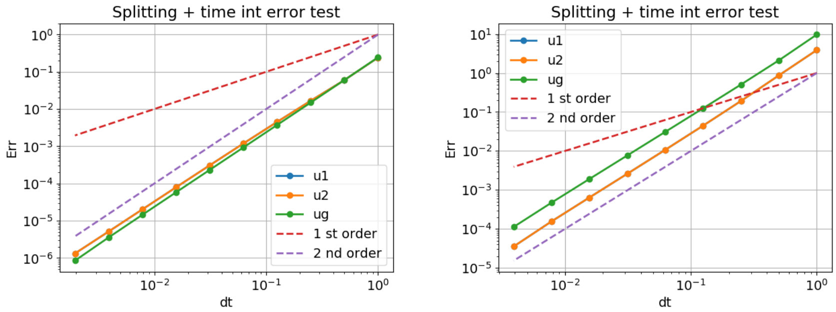


Figure 50: Test 2.2.3, NNWR, SDIRK2. Left: 1D, Right: 2D

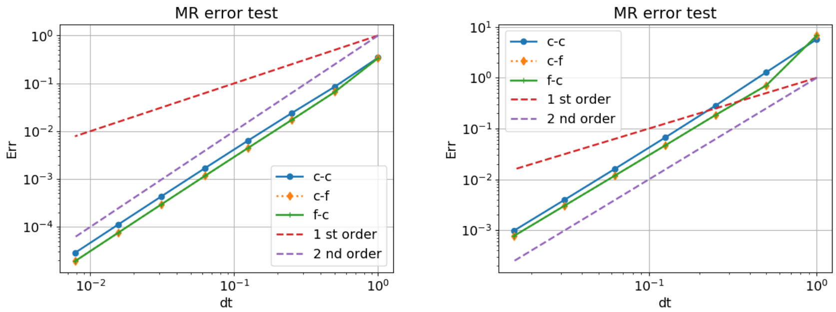


Figure 51: Test 2.2.5, NNWR, SDIRK2. Left: 1D, Right: 2D

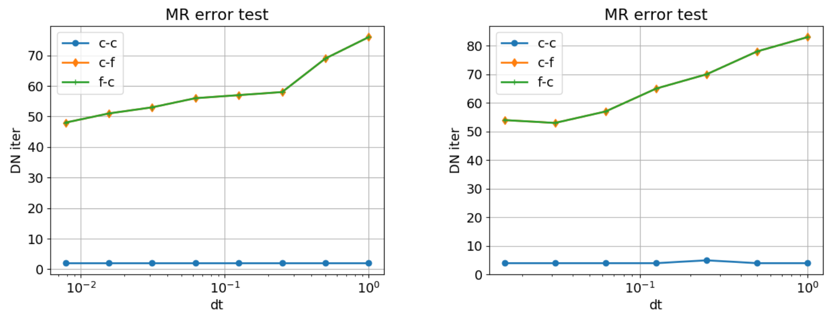


Figure 52: Test 2.2.5 iterations, NNWR, SDIRK2. Left: 1D, Right: 2D

4.2.1 Air-Steel

Summary: X

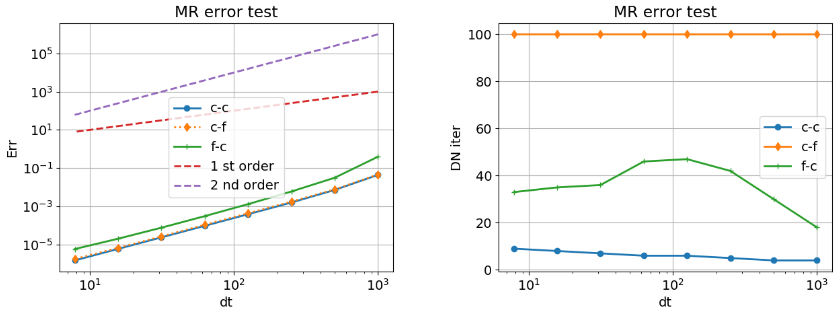


Figure 53: Test 2.2.5 for Air-Steel coupling, NNWR, SDIRK2. 1D

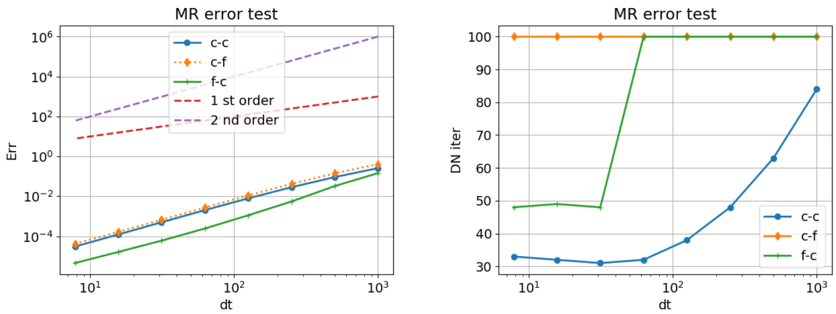


Figure 54: Test 2.2.5 for Air-Steel coupling, NNWR, SDIRK2. 2D

4.2.2 Air-Water

Summary:

X

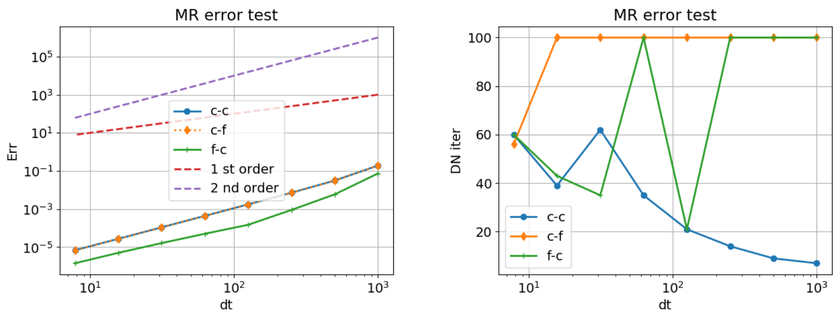


Figure 55: Test 2.2.5 for Air-Water coupling, NNWR, SDIRK2. 1D

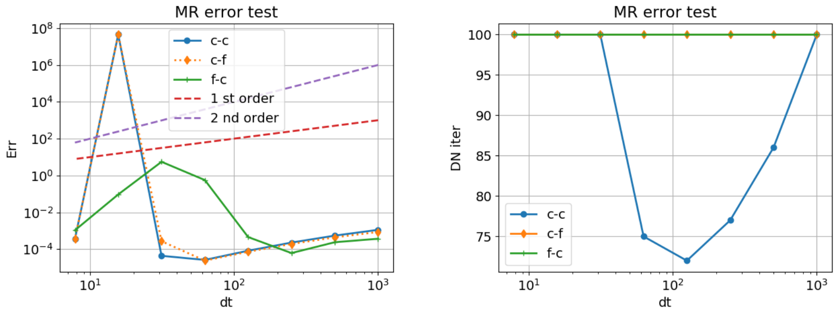


Figure 56: Test 2.2.5 for Air-Water coupling, NNWR, SDIRK2. 2D

4.2.3 Water-Steel

Summary: wait for 2D result

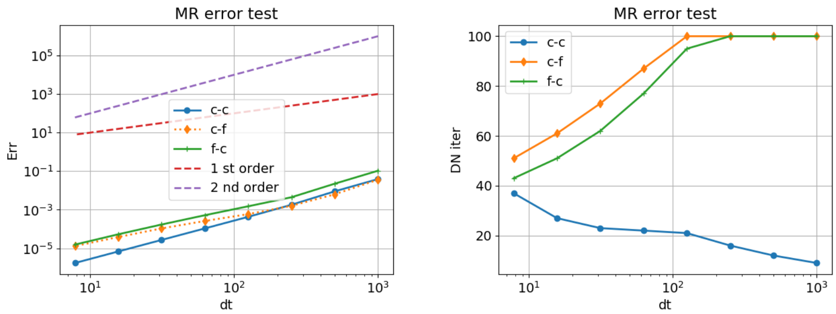


Figure 57: Test 2.2.5 for Water-Steel coupling, NNWR, SDIRK2. 1D

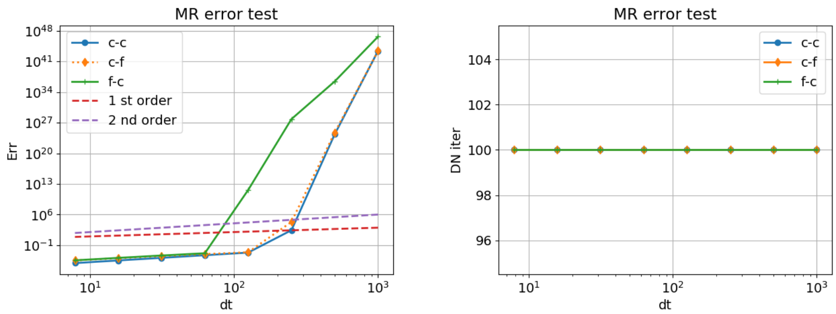


Figure 58: Test 2.2.5 for Water-Steel coupling, NNWR, SDIRK2. 2D

4.3 SDIRK2 time adaptive, single adaptive

Double adaptive: Adaptivity only for the Dirichlet stage

Figures: 59. Figures 60, 61 and 62 for different material parameters.

Summary: It works some

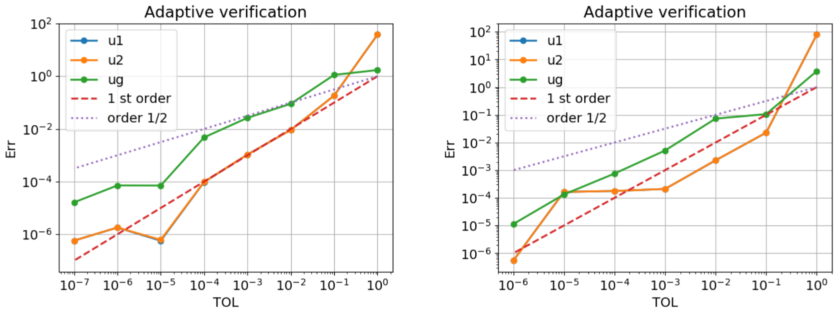


Figure 59: Test 2.2.3, NNWR, Time adaptive SDIRK2, single adaptive. Left: 1D, Right: 2D

4.3.1 Air-Steel

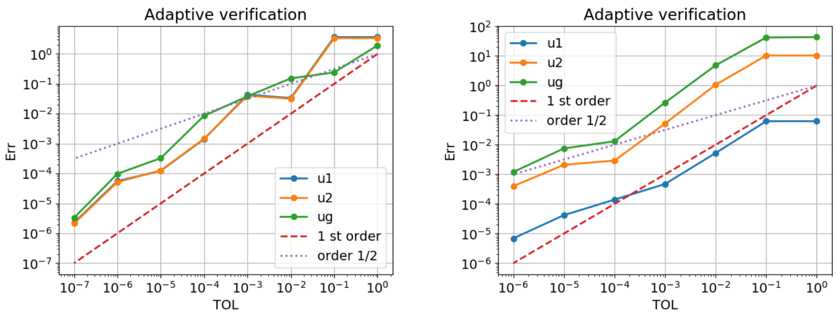


Figure 60: Test 2.2.5 for Air-Steel coupling, NNWR, SDIRK2 TA single. Left: 1D. Right: 2D

4.3.2 Air-Water

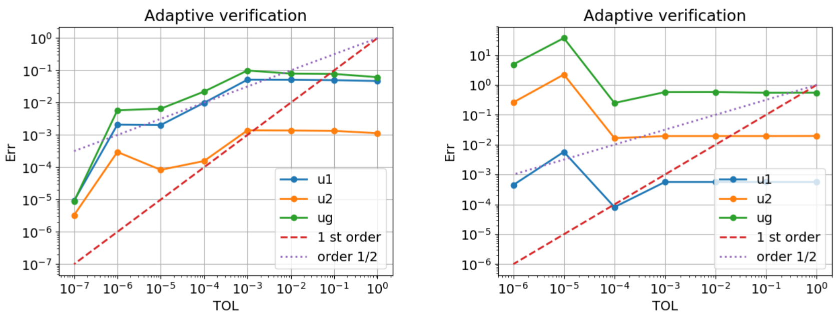
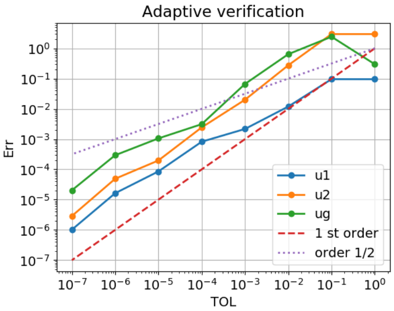


Figure 61: Test 2.2.5 for Air-Water coupling, NNWR, SDIRK2 TA single. Left: 1D. Right: 2D

4.3.3 Water-Steel



NNWR/TA/water_steel/water_steelverify_adaptive_dim_2_n_32_

Figure 62: Test 2.2.5 for Water-Steel coupling, NNWR, SDIRK2 TA single. Left: 1D. Right: 2D

4.4 SDIRK2 time adaptive, double adaptive

Single adaptive: Adaptivity for both the Dirichlet and Neumann stages, independently. Same initial timesteps in Dirichlet and Neumann stages

Does this actually make sense?

Figures: 63. Figures 64, 65 and 66 for different material parameters.

Summary:

x

NNWR/NNWR/TA/verify_adaptive_dim_1_n_32_order_2.png-2.png

Figure 63: Test 2.2.3, NNWR, Time adaptive SDIRK2, double adaptive. Left: 1D, Right: 2D

4.4.1 Air-Steel

NNWR/NNWR/TA/air_steel/air_steelverify_adaptive_dim_1_n_32_order_2.png-2.png

Figure 64: Test 2.2.5 for Air-Steel coupling, NNWR, SDIRK2 TA double. Left: 1D. Right: 2D

4.4.2 Air-Water

NNWR/NNWR/TA/air_water/air_waterverify_adaptive_dim_1_n_32_order_2.png-2.png

Figure 65: Test 2.2.5 for Air-Water coupling, NNWR, SDIRK2 TA double. Left: 1D. Right: 2D

4.4.3 Water-Steel

[illegible]

Figure 66: Test 2.2.5 for Water-Steel coupling, NNWR, SDIRK2 TA double.
Left: 1D. Right: 2D