REMINDERS : HW 3a & 3b ARE UP!

→ always leave time to troubleshoot
→ PLAY w/ your code

Questions?

Last wk : ways to integrate

HW: Bessel Function as an integral
1st "REAL WORLD" APPLICATION
└ integrand known, not easy to integrate analytically.

TODAY: ERROR ANALYSIS ; let's expand f about the two endpoints of a small differential element

WRITE $f_i = f(x_i)$

$f'_i = f'(x_i)$

etc.

expand about x_{i-1} & x_i

$$f(x) = f_{i-1} + (x - x_{i-1}) f'_{i-1} + \tfrac{1}{2}(x-x_{i-1})^2 f''_{i-1} + \cdots$$
$$= f_i + (x - x_i) f'_i + \tfrac{1}{2}(x - x_i)^2 f''_{i} + \cdots$$

$$\int_{x_{i-1}}^{x_i} f(x)\,dx = \int_{u=0}^{u=\Delta x} du \left[f_{i-1} + u f'_{i-1} + \tfrac{1}{2}u^2 f''_{i-1} + \cdots \right]$$
$$= \int_{v=-\Delta x}^{v=0} dv \left[f_i + v f'_{i} + \tfrac{1}{2}v^2 f''_{i} + \cdots \right]$$

take the average of these

$$= \frac{1}{2}\left[\Delta x (f_{i-1} + f_i) + \tfrac{1}{2}\Delta x^2 (f'_{i-1} - f'_i) + \tfrac{1}{6}\Delta x^3 (f''_{i-1} + f''_i) + \cdots \right]$$

NOW LETS EXPRESS A FINITE INTEGRAL

$$\int_a^b f(x)\,dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x)\,dx$$

$$= \sum_{i=1}^{N} \Delta x \cdot \tfrac{1}{2}(f_{i-1}+f_i) \quad \leftarrow \text{trapezoidal!}$$

$$(\star) \quad + \sum_{i=1}^{N} \Delta x^2 \cdot \tfrac{1}{4}(f'_{i-1}-f'_i) \quad \leftarrow = -\tfrac{\Delta x^2}{4}(f'(b)-f'(a))$$

$$\underbrace{+ \sum_{i=1}^{N} \Delta x^3 \cdot \tfrac{1}{12}(f''_{i-1}+f''_i)} + \cdots \quad \Big\} \text{ error from the trap. rule}$$

OBSERVE: THIS IS JUST TRAP. RULE APPROX OF $\frac{1}{6}\int_a^b f''(x)\,dx \cdot \Delta x^2$

$$= \tfrac{1}{6}\left[f'(x)\right]_a^b$$

$$\Rightarrow \int_a^b f(x)\,dx = \sum_{i=1}^{N} \Delta x \cdot \tfrac{1}{2}(f_{i-1}+f_i)$$

$$\boxed{+ \tfrac{1}{12}\Delta x^2 (f'(b) - f'(a))} \text{ — error is } 2^{\text{nd}}\ \Theta$$

$$+ \Theta(\Delta x^4)$$

$\frac{1}{12}$ from cancellation w/ $\star$

ROUNDING ERROR (in base 10), $\boxed{C \sim \boxed{10^{-16}}}$

error const. (Newman §4.2)

HOW TO USE THIS:

error: $\sigma = C\,x$

"STD. DEV."

number on which you want error

$x_{OBS} = x_{true} + \varepsilon$

error (NOT KNOWN)

if we knew ε, then we could directly find x_{true}

$x_1 + x_2$ HAS σ given by $\sigma^2 = \sigma_1^2 + \sigma_2^2$

↳ also: $x_1 x_2$ OR $\frac{x_1}{x_2}$ HAS

$$\frac{\sigma^2}{x^2} = \frac{\sigma_1^2}{x_1^2} + \frac{\sigma_2^2}{x_2^2}$$

IN AN INTEGRAL: WE WANT $\int_a^b f(x)\,dx$

↳ ROUNDING ERROR HAS $\sigma \sim C \int_a^b f(x)\,dx$

WHAT IS THE Δx for WHICH $\sigma_{APPROX} \sim \sigma_{ROUND}$?

@ this point, no need to try smaller Δx.

$$\frac{1}{12}\Delta x^2 \left(f'(a) - f'(b) \right) \simeq C \int_a^b f(x)\,dx$$

$$\Delta x = \sqrt{\frac{12\, C \int_a^b f(x)\,dx}{f'(a) - f'(b)}}$$

$\frac{b-a}{N}$

$$N \approx (b-a)\underbrace{\sqrt{\frac{f'(a)-f'(b)}{12\int_a^b f(x)dx}}}_{\text{so if this is } \Theta(1)} C^{-1/2}$$

$$C^{-1/2} = 10^8$$

HUGE # for N.
COULD DO BETTER
W/ SIMPSON'S
RULE W/ SMALLER N

analog for simpson's rule (exercise)

$$N = (b-a)\sqrt[4]{\frac{f'''(a)-f'''(b)}{90\int_a^b f(x)dx}}\ \boxed{C^{-1/4}}$$

$$C^{-1/4} \sim 10^4$$

SO DON'T GO
BEYOND A FEW THOUSAND

HOW MUCH DATA ARE WE ACTUALLY USING

$$\underbrace{f_0 = f(a)\ ,\quad f_1 = f(x_1)\ ,\quad f_2\ ,\ \dots}$$

N values of f ... BUT ALSO CHOSE N POINTS TO SAMPLE

$$\underbrace{x_0 = a\ ,\ x_1, \dots, x_N = b}_{\text{assumed evenly spaced}}$$

$\boxed{\text{2N pieces of data}}$

RECALL: for POLYN OF DEG = 2 $\quad f(y) = \underbrace{Ay^2 + By + C}_{\text{3 PIECES of DATA}}$

in general, deg = N, takes ~~[illegible]~~ (N+1) data.

So 2N PIECES OF DATA $\longrightarrow$ POLY OF DEG $\boxed{2N-1}$

so parabola w/ 1 (x, y) PAIR??

how? eg. (x_0, y_0) @ $f'(x_0) = 0$

so: there is data in picking <u>where</u> we sample

... just tricky how to <u>use</u> this information.

<u>INTERPOLATING FUNCTION</u> : define nice, smooth func.

$$\phi_k(x) = \prod_{\substack{m=1 \\ m \neq k}}^{N} \frac{(x - x_m)}{(x_k - x_m)}$$

k^{th}

denom is nonzero constant

poly of DEG. (N-1)

eg. $\{x_1, x_2, x_3\}$

$$\phi_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

<u>PROPERTY</u>: if $x_m \in \{x_1 \ldots\}$, then

$$\phi_k(x_m) = \begin{cases} 1 & \text{if } m=k \\ 0 & \text{otherwise} \end{cases} \equiv \delta_{mk}$$

for discrete points x_m

for the mathematically inclined: δ_{mk} looks like a metric... or something useful for <u>projection</u> ...

<u>HOW TO USE THIS</u>:

if we knew $\{x_k\}$ & $\{f_k\}$, claim that

$$f(x) \approx F(x) \equiv \sum_{k=1}^{N} f(x_k)\,\phi_k(x)$$

PROPERTIES: $F(x_k) = f_k$ ✓

POLYN. OF DEG $(N-1)$ ← way better than simpson

only (unique) func. w/ these prop.

$$\int_a^b f(x)\,dx \approx \int_a^b F(x)\,dx$$

$$= \sum_{k=1}^{N} f(x_k) \underbrace{\int_a^b \phi_k(x)\,dx}_{w_k}$$

<u>observe</u> :

$\sum_k f(x_k) w_k$ is the same form as SIMPSON'S RULE, TRAP. RULE, ...

$\int_a^b \phi_k(x)\,dx$ only dep on $\{x_k\}$, <u>not</u> the function values!

<u>BUT HOW TO CALCULATE w_k?</u>

$$w_k = \underbrace{\int_a^b \phi_k(x)\,dx}$$

just another integral.

... use Simpson's rule.

(WHAAAAT?!)

<u>INDEP</u> of f!

SO JUST DO IT ONCE & FOR ALL.

... still dep on a & b

<u>Better</u> : CHANGE VARIABLES :

$$x = \tfrac{1}{2}(b-a)\,y + \tfrac{1}{2}(b+a)$$

s.t. $y = -1 \longrightarrow x = a$

$y = 1 \longrightarrow x = b$

then calculate a fiducial weight

$$\hat{w}_k = \int_{-1}^{1} \phi_k(y)\, dy$$

$\{y_k\}$ — ASSUME WE HAVE THESE

$y = y(x_k)$

given $\{y_k\}$ can calculate these once & for all.

$$= \int_{-1}^{1} \phi_k(x(y))\, d\, y(x)$$

$$= \int_a^b \phi_k(x) \frac{2}{b-a}\, dx$$

$$= \frac{+2}{b-a} \underbrace{\int_a^b \phi_k(x)\, dx}_{= w_k}$$

$$\Rightarrow \boxed{w_k = \tfrac{1}{2}(b-a)\, \hat{w}_k}$$

(weights for $x \in [a,b]$ are easy to derive from weights for $y \in [-1,1]$

But how to pick $\{y_k\}$?

$$\{y_k\} \updownarrow \{x_k\}$$

INSPIRATION: Legendre polynomials

$\exists$ special functions $P_N(x)$, POLY OF DEG. N

SUPPOSE f is POLYN of deg $(2N-1)$

$$f(x) = \underset{\substack{\uparrow \\ \text{degree } (N-1) \text{ or less}}}{g(x)}\, P_N(x) + \underset{\substack{\uparrow \\ \text{remainder: deg } (N-1) \text{ OR LESS}}}{r(x)}$$

$$\int_{-1}^{1} f(x)\,dx = \underbrace{\int_{-1}^{1} g(x) P_N(x)\,dx}_{=\,0 \text{ by orthogonality}} + \underbrace{\int_{-1}^{1} r(x)\,dx}_{\deg \le (N-1)}$$

$g(x)$ has deg $(N-1)$ so no overlap w/ P_N!

$$= \int_{-1}^{1} r(x)\,dx$$

APPROX w/ N interpol. functions. (EXACT)

$$= \sum_{k=1}^{N} \underbrace{\left(\int_{-1}^{1} \phi_k(x)\,dx\right)}_{\hat{w}_k} r(x_k)$$

So: given $f \rightarrow r(x)$. JUST DO DIVISION.
BUT THIS IS HARD! WHERE TO SAMPLE?

FACT: P_N HAS N ZEROS BETWEEN -1 & 1

↳ KNOWN.

SO CHOOSE to sample @ these zeros

$\{\tilde{x}_k\}$ s.t. $P_N(\tilde{x}_k)=0$

↑ N points

DEF. WEIGHTS: $\tilde{\phi}_k$ w/ SAMPLE POINTS $\{\tilde{x}_k\}$

$$\tilde{w}_k = \int_{-1}^{1} \tilde{\phi}_k(x)\,dx$$

↑ CAN CALCULATE ALL OF THESE ONCE & FOR ALL FOR A GIVEN N

SO: methodology: known list for given N

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{k}^{N} \tilde{w}_k f(\tilde{x}_k)$$

$$\approx \sum_{k}^{N} \tilde{w}_k \left[\cancel{q(\tilde{x}_k) P_N(\tilde{x}_k)} + r(\tilde{x}_k) \right]$$

0 from $P_N(\tilde{x}_k)=0$

$$\approx \sum_{k}^{N} \tilde{w}_k\, r(\tilde{x}_k)$$ ☺

on the other hand:

if f is DEG. $\leq 2N-1$

$$\int_{-1}^{1} f(x)\,dx = \underbrace{\int_{-1}^{1} q(x) P_N(x)\,dx}_{=0} + \int_{-1}^{1} r(x)\,dx$$

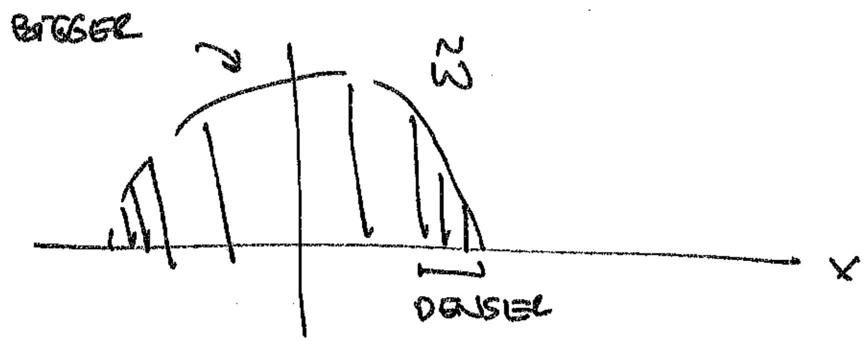
$$= \boxed{\sum_{k=1}^{N} \tilde{w}_k\, r(\tilde{x}_k)}$$

matches ☺!!

so ☺ is exact in this case!

Result

$$\tilde{\omega}_k = \left[\frac{2}{1-x^2} \left(\frac{dP_n}{dx} \right)^{-2} \right]_{x = x_k}$$



↳ result: <u>Gaussian Quadrature</u>