1 Preliminaries

We start with some defintions:

Our state vector will be initially defined as $\mathbf{x}(t) = (\mathbf{r}(t), \dot{\mathbf{r}}(t), z(t))$, with $\mathbf{r}(t)$ being the position of the craft with the reference frame given as the surface of the planet. Additionally, $z(t) = \ln(m(t))$, so $\dot{z}(t) = \dot{m}(t)/m(t) = -\alpha\|\mathbf{T}_c(t)\|/m(t)$

Our ODE equation will be defined as

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{v} + C\mathbf{g}$$

With $\mathbf{v}$ as the control vector, and $\mathbf{g}$ as the constant vector for gravity. Here are the definitions of the matrices and control vector:

$$\sigma \equiv \Gamma/m, \quad \mathbf{u} \equiv \mathbf{T}_c/m, \quad z \equiv \ln m$$

$$A = \begin{bmatrix} \mathbf{0} & \mathbf{I} & 0 \\ -\mathbf{S}(\omega)^2 & -2\mathbf{S}(\omega) & 0 \\ \mathbf{0} & \mathbf{0} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \mathbf{0} & 0 \\ \mathbf{I} & 0 \\ \mathbf{0} & -\alpha \end{bmatrix}$$

$$C = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \\ 0_{1\times 3} \end{bmatrix}$$

$$\mathbf{S}(\omega) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \text{ where } \mathbf{S}(\omega)\mathbf{v} = \omega \times \mathbf{v}$$

$$\mathbf{v} = \begin{bmatrix} \mathbf{u}(t) \\ \sigma \end{bmatrix} = \sum_{j=0}^{N} \mathbf{p}_j \phi_j(t) = \Upsilon_k \eta$$

$$\phi_j = \begin{cases} 1, & t \in [t_j, t_{j+1}) \\ 0 & \text{otherwise} \end{cases}$$

$$\Upsilon_k = \begin{bmatrix} \mathbf{I}_4\phi_0 & \mathbf{I}_4\phi_1 & \dots & \mathbf{I}_4\phi_N \end{bmatrix}, \quad \eta = \begin{bmatrix} \mathbf{p}_0 \\ \vdots \\ \mathbf{p}_N \end{bmatrix}$$

$$E = [I_{3\times 3} \;\; 0_{3\times 4}], \quad F = [0_{1\times 6} \;\; 1], \quad E_{\mathrm{u}} = [I_{3\times 3} \;\; 0_{3\times 1}], \quad E_{\mathrm{v}} = [0_{3\times 3} \;\; I_{3\times 3} \;\; 0_{3\times 1}]$$

$$S \equiv \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{c} \equiv \frac{\mathbf{e}_1}{\tan\gamma} > 0$$

The solution of the ODE system (in discrete time with $t_k = k\Delta t$) is given by

$$\mathbf{x}_k(t) = \Phi_k \begin{bmatrix} \mathbf{r}_0 \\ \dot{\mathbf{r}}_0 \\ \ln m_0 \end{bmatrix} + \Psi_k \eta + \Lambda_k$$

$$\Phi_k = e^{Ak\Delta t}, \quad \Psi_k = \left(\int_0^{\Delta t} e^{A\tau} B \, d\tau \right) \Upsilon_k, \quad \Lambda_k = \int_{(k-1)\Delta t}^{k\Delta t} e^{A\tau} C\mathbf{g} \, d\tau$$

$$\mathbf{X} = \{\mathbf{r} \in \mathbb{R}^3 \big| \; \|S(\mathbf{r} - \mathbf{r}(t_N))\| \le \mathbf{c}^T(\mathbf{r} - \mathbf{r}(t_N))\}$$

$$z_0(t) = \ln(m_0 - \alpha\rho_2 t)$$

We now create our planetary soft landing problem.

2 Planetary Soft Landing Problem

$$\min_{N,\eta} \|E\mathbf{x}_N\|^2$$

Subject to

$$\|E_{\mathrm{u}}\Upsilon_k\eta\| \le \mathbf{e}_4^T\Upsilon_k\eta, \quad \hat{\mathbf{n}}_T E_{\mathrm{u}}\Upsilon_k\eta \ge \cos(\theta)\,\mathbf{e}_4^T\Upsilon_k\eta, \quad k = 0,\ldots,N$$

$$\rho_1 e^{-z_0(t_k)}\left[1 - (F\mathbf{x}_k - z_0(t_k)) + \frac{(F\mathbf{x}_k - z_0(t_k))^2}{2}\right] \le \mathbf{e}_4^T\Upsilon_k\eta \le \rho_2 e^{-z_0(t_k)}[1 - (F\mathbf{x}_k - z_0(t_k))]$$

$$E\mathbf{x}_k \in \mathbf{X}, \quad k = 1,\ldots,N$$

$$F\mathbf{x}_N \ge \ln m_N$$

$$\mathbf{x}_N^T\mathbf{e}_1 = 0, \quad E_{\mathrm{v}}\mathbf{x}_N^T = \mathbf{0}$$

$$\mathbf{x}_k(t) = \Phi_k \begin{bmatrix} \mathbf{r}_0 \\ \dot{\mathbf{r}}_0 \\ \ln m_0 \end{bmatrix} + \Psi_k\eta + \Lambda_k, \quad k = 1,\ldots,N$$