1 Nonconvex Minimum Landing Error Problem

Initially, we started with the following nonconvex problem. Define $\mathbf{x}(t) = (\mathbf{r}(t), \dot{\mathbf{r}}(t))$, and we use the coordinate system (z, x, y):

$$\min_{t_f, \mathbf{T}_c} \|S\mathbf{r}(t_f)\|$$

Where

$$\dot{\mathbf{x}} = A(\omega)\mathbf{x}(t) + B\left(\mathbf{g} + \frac{\mathbf{T}_c(t)}{m(t)}\right)$$

$$\dot{m}(t) = -\alpha\|\mathbf{T}_c(t)\|$$

$$\mathbf{x}(t) \in \mathbf{X} \equiv \{\mathbf{r} \in \mathbb{R}^3 \big| \ \|S(\mathbf{r} - \mathbf{r}(t_N))\| \leq \mathbf{c}^T(\mathbf{r} - \mathbf{r}(t_N))\}$$

$$0 < \rho_1 \leq \|\mathbf{T}_c(t)\| \leq \rho_2, \quad \hat{\mathbf{n}}^T\mathbf{T}_c(t) \geq \|\mathbf{T}_c(t)\|\cos\theta$$

$$m(0) = m_0, \quad m(t_f) \geq m_0 = m_f > 0$$

$$\mathbf{r}(0) = \mathbf{r}_0, \quad \dot{\mathbf{r}}(0) = \dot{\mathbf{r}}_0$$

$$\mathbf{e}_1^T\mathbf{r}(t_f) = 0, \quad \dot{\mathbf{r}}(t_f) = \mathbf{0}$$

$$S \equiv \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{c} \equiv \frac{\mathbf{e}_1}{\tan\gamma} > 0$$

$$A = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{S}(\omega)^2 & -2\mathbf{S}(\omega) \end{bmatrix}$$

$$B = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}$$

$$\mathbf{S}(\omega) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \text{ where } \mathbf{S}(\omega)\mathbf{v} = \omega \times \mathbf{v}$$

ω is the angular velocity of the planet with respect to the origin, which is the intended landing point. $\mathbf{n}^T$ is the pointing vector for the craft. ρ_1 and ρ_2 are the lower and upper thrust bounds, and α is a proportionality constant for fuel use. $\mathbf{g}$ is the constant vector for gravity.

Similarly, the corresponding minimum fuel use problem takes the same form as the previous question, but with a new objective and constraint:

$$\min_{t_f, \mathbf{T}_c} \int_0^{t_f} \|\mathbf{T}_c(t)\| dt$$

Including the constraint

$$\|S\mathbf{r}(t)\| \leq \|d^*_{P1}\|$$

Where d^*_{P1} is the solution to the minimum landing landing error problem.

There are several problems with this problem statement; the double inequality and the mass equation are nonconvex, and the ODE is not easily solvable by any measure, at least not in continuous time. So, the Blackmore papers introduce a slack variable Γ, which is related to the upper thrust bound. When they introduce Γ, they have to answer the following two questions:

1. How do we know that by solving the problem with the slack variable, we will in turn have a solution for the original nonconvex problem?
2. How do we know that the new problem is convex, and not just nonconvex like the previous problem?

The slack variable Γ is defined as having the following relations:

1. $\|\mathbf{T}_c(t)\| \leq \Gamma(t)$
2. $\hat{\mathbf{n}}^T\mathbf{T}_c(t) \geq \Gamma(t)\cos\theta$
3. $0 < \rho_1 \leq \Gamma(t) \leq \rho_2$

Since the inequalities above are convex, we introduce the following proposed convex problems to solve, as opposed to the previous.

2 Convex Problems

$$\min_{t_f,\mathbf{T}_c} \|E\mathbf{r}(t_f)\|$$

Where

$$\dot{\mathbf{x}} = A(\omega)\mathbf{x}(t) + B\left(\mathbf{g} + \frac{\mathbf{T}_c(t)}{m(t)}\right)$$
$$\dot{m}(t) = -\alpha\Gamma(t), \quad \|\mathbf{T}_c(t)\| \leq \Gamma(t)$$
$$\mathbf{x}(t) \in \mathbf{X}$$
$$0 < \rho_1 \leq \Gamma(t) \leq \rho_2, \quad \hat{\mathbf{n}}^T\mathbf{T}_c(t) \geq \Gamma(t)\cos\theta$$
$$m(0) = m_0, \quad m(t_f) \geq m_0 = m_f > 0$$
$$\mathbf{r}(0) = \mathbf{r}_0, \quad \dot{\mathbf{r}}(0) = \dot{\mathbf{r}}_0$$
$$\mathbf{e}_1^T\mathbf{r}(t_f) = 0, \quad \dot{\mathbf{r}}(t_f) = \mathbf{0}$$

Similarly, the corresponding minimum fuel use problem takes the same form as the previous question, but with a new objective and constraint:

$$\min_{t_f,\mathbf{T}_c,\Gamma} \int_0^{t_f} \Gamma(t)dt$$

Including the constraint

$$\|E\mathbf{r}(t)\| \leq \|d^*_{P2}\|$$

3 Analysis

To answer the first question, we define $\mathcal{F}_e$ and $\mathcal{F}_f$ to be the solution sets for the nonconvex landing error problem and the corresponding minimal fuel use problem, and the optimal solution sets are $\mathcal{F}_e^*$ and $\mathcal{F}_f^*$. Similarly, we define $\mathcal{F}_{re}$ and $\mathcal{F}_{rf}$ as the solutions to the convexified minimum landing error and fuel use problems. We consider the following statements:

1. $\mathcal{F}_f \subseteq \mathcal{F}_e$, $\mathcal{F}_f^* \subseteq \mathcal{F}_e^*$, $\mathcal{F}_{rf} \subseteq \mathcal{F}_{re}$, $\mathcal{F}_{rf}^* \subseteq \mathcal{F}_{re}^*$
2. $\mathcal{F}_f \subseteq \mathcal{F}_{rf}$ in the sense that it is equivalent to the subset of solutions from $\mathcal{F}_{rf}$ when $\Gamma = \|\mathbf{T}_c(t)\|$
3. $\mathcal{F}_f^* \subseteq \mathcal{F}_{rf}^*$ in the sense that it is equivalent to the subset of solutions from $\mathcal{F}_{rf}^*$ when $\Gamma = \|\mathbf{T}_c(t)\|$

To prove the first set of statements, remember that the fuel use problem depends on the previous problem, so if you solve the fuel use problem, you also solve the landing error problem. So $\mathcal{F}_f \subseteq \mathcal{F}_e$. The same argument goes for all of the other statements. For the second and third statements, consider that if $\Gamma = \|\mathbf{T}_c(t)\|$, then any solution to the convex problems are equivalent to a solution from the nonconvex problems.

4 Re-definitions

We start with some definitions: Our state vector will be now be defined as $\mathbf{x}(t) = (\mathbf{r}(t), \dot{\mathbf{r}}(t), z(t))$, with $z(t) = \ln(m(t))$, so $\dot{z}(t) = \dot{m}(t)/m(t) = -\alpha\|\mathbf{T}_c(t)\|/m(t)$.

Our ODE equation will be defined as

$$\dot{\mathbf{x}} = A\mathbf{x} + B(\mathbf{v} + C\mathbf{g})$$

With $\mathbf{v}$ as the control vector. Here are the definitions of the matrices and control vector:

$$\sigma \equiv \Gamma/m, \quad \mathbf{u} \equiv \mathbf{T}_c/m, \quad z \equiv \ln m$$

$$A = \begin{bmatrix} \mathbf{0} & \mathbf{I} & 0 \\ -\mathbf{S}(\omega)^2 & -2\mathbf{S}(\omega) & 0 \\ \mathbf{0} & \mathbf{0} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \mathbf{0} & 0 \\ \mathbf{I} & 0 \\ \mathbf{0} & -\alpha \end{bmatrix}$$

$$C = \begin{bmatrix} \mathbf{I} \\ 0_{1\times 3} \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} \mathbf{u}(t) \\ \sigma \end{bmatrix} = \sum_{j=0}^{N} \mathbf{p}_j \phi_j(t) = \Upsilon_k \eta$$

$$\phi_j = \begin{cases} 1, & t \in [t_j, t_{j+1}) \\ 0 & \text{otherwise} \end{cases}$$

$$\Upsilon(t_k) = \begin{bmatrix} \mathbf{I}_4\phi_0 & \mathbf{I}_4\phi_1 & \dots & \mathbf{I}_4\phi_N \end{bmatrix}, \quad \eta = \begin{bmatrix} \mathbf{p}_0 \\ \vdots \\ \mathbf{p}_N \end{bmatrix}$$

$$E = [I_{3\times 3} \;\; 0_{3\times 4}], \quad F = [0_{1\times 6} \;\; 1], \quad E_{\mathrm{u}} = [I_{3\times 3} \;\; 0_{3\times 1}], \quad E_{\mathrm{v}} = [0_{3\times 3} \;\; I_{3\times 3} \;\; 0_{3\times 1}]$$

4.1 Detour: Solving the ODE

To solve an ODE of the form

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

We define the fundamental solution matrix to be

$$\dot{U}(t) = A(t)U(t), \quad U(t_0) = I$$

so

$$U(t) = e^{\int_{t_0}^{t} A(\tau - t_0)d\tau}$$

the general solution for the ODE given above, by variation of parameters, is

$$\mathbf{x}(t) = U(t)\mathbf{x}_0 + U(t)\int_{t_0}^{t} U^{-1}(\tau)B(\tau)\mathbf{u}(\tau)d\tau$$

We define

$$\Phi(t,\tau) \equiv U(t)U^{-1}(\tau)$$

And so the solution is equivalent to

$$\mathbf{x}(t) = U(t)U^{-1}(t_0)\mathbf{x}_0 + \int_{t_0}^{t} U(t)U^{-1}(\tau)B(\tau)\mathbf{u}(\tau)d\tau = \Phi(t,t_0)\mathbf{x}_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)\mathbf{u}(\tau)d\tau$$

We can't really use a continuous definition for our system, so to start, remember that A and B are constant matrices, and that $t_0 = 0$:

$$\mathbf{x}(t) = \Phi(t,0)\mathbf{x}_0 + \int_{0}^{t} \Phi(t,\tau)B\mathbf{u}(\tau)d\tau$$

where

$$\Phi(t,\tau) = e^{A(t-\tau)}$$

In this case, $A(t-t_0) = A \cdot (t-t_0)$. In order to create a recursive formula for the ODE, we can compare the difference between two state vectors with no control ($\mathbf{u}(t) = \mathbf{0}$), which is given by the formula $\mathbf{x}(t) = \Phi(t,\tau)\mathbf{x}(\tau)$. Say that two state vectors differ in time by Δt, then for example

$$\Phi(t_k + \Delta t, t_k) = e^{A\Delta t}$$

Similarly, when we add a control part, we only account for the change in control over the brief period of time Δt:

$$\begin{aligned}\mathbf{x}(t_k + \Delta t) &= \Phi(t_k + \Delta t, t_k)\mathbf{x}_k + \int_{t_k}^{t_k+\Delta t} \Phi(t_k + \Delta t, \tau) B\mathbf{u}(\tau) d\tau \\ &= e^{A\Delta t}\mathbf{x}_k + \int_{t_k}^{t_k+\Delta t} e^{A(t_k+\Delta t-\tau)} B\mathbf{u}(\tau) d\tau\end{aligned}$$

If we introduce a limit on the top bound of the integral ($\mathbf{u}(t)$ is discontinuous if treated as a discrete function, with a hole at $t_k + \Delta t$), then we can remove the control vector:

$$\mathbf{x}(t_k + \Delta t) = e^{A\Delta t}\mathbf{x}_k + \lim_{s\to(t_k+\Delta t)} \left(\int_{t_k}^{s} e^{A(s-\tau)} B d\tau\right)\mathbf{u}(t_k)$$

There is a property of integrals that

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

$$\int_a^b f(x-a)dx = \int_a^b f(b-x)dx$$

So,

$$\int_{t_0}^{s} e^{A(s-\tau)} B d\tau = \int_{t_0}^{s} e^{A(\tau-t_0)} B d\tau$$

We can make the simple variable change $t = \tau - t_k$:

$$= \int_0^{\Delta t} e^{At} B dt$$

And so we have obtained our formulas for the matrices for the discrete-time approximate solution to the ODE. With $t_k = k\Delta t$, we have

$$\mathbf{x}_k = \Phi\mathbf{x}_{k-1} + \Psi(\Upsilon_{k-1}\eta + C\mathbf{g})$$

$$\Phi = e^{A\Delta t}, \quad \Psi = \int_0^{\Delta t} e^{A\tau} B\, d\tau$$

$$z_0(t) = \ln(m_0 - \alpha\rho_2 t)$$

The only thing that is approximate about this is the way we have defined Δt; if we had took the limit so that it approached dt, we would have recovered the continuous time solution, just in a really strange form.

We now create our planetary soft landing problem.

5 Minimum Landing Error Problem

$$\min_{N,\eta} \|E\mathbf{x}_N\|^2$$

Subject to

$$\|E_{\mathrm{u}}\Upsilon_k\eta\| \le \mathbf{e}_4^T\Upsilon_k\eta, \quad \hat{\mathbf{n}}^T E_{\mathrm{u}}\Upsilon_k\eta \ge \cos(\theta)\,\mathbf{e}_4^T\Upsilon_k\eta, \quad k = 0,\ldots,N$$

$$\rho_1 e^{-z_0(t_k)}\left[1 - (F\mathbf{x}_k - z_0(t_k)) + \frac{(F\mathbf{x}_k - z_0(t_k))^2}{2}\right] \le \mathbf{e}_4^T\Upsilon_k\eta \le \rho_2 e^{-z_0(t_k)}[1 - (F\mathbf{x}_k - z_0(t_k))]$$

$$E\mathbf{x}_k \in \mathbf{X}, \quad k = 1, \dots, N$$

$$F\mathbf{x}_N \geq \ln m_N$$

$$\mathbf{x}_N^T \mathbf{e}_1 = 0, \quad E_\mathrm{v}\mathbf{x}_N^T = \mathbf{0}$$

$$\mathbf{x}_k = \Phi\mathbf{x}_{k-1} + \Psi(\Upsilon_{k-1}\eta + C\mathbf{g}), \quad k = 1, \dots, N$$

Next is the minimum fuel use problem.

$$\min_{N,\eta} \int_0^{t_N} \sigma(t)dt$$

Subject to

$$\|E_\mathrm{u}\Upsilon_k\eta\| \leq \mathbf{e}_4^T\Upsilon_k\eta, \quad \hat{\mathbf{n}}^T E_\mathrm{u}\Upsilon_k\eta \geq \cos(\theta)\,\mathbf{e}_4^T\Upsilon_k\eta, \quad k = 0, \dots, N$$

$$\rho_1 e^{-z_0(t_k)}\left[1 - (F\mathbf{x}_k - z_0(t_k)) + \frac{(F\mathbf{x}_k - z_0(t_k))^2}{2}\right] \leq \mathbf{e}_4^T\Upsilon_k\eta \leq \rho_2 e^{-z_0(t_k)}[1 - (F\mathbf{x}_k - z_0(t_k))]$$

$$E\mathbf{x}_k \in \mathbf{X}, \quad k = 1, \dots, N$$

$$F\mathbf{x}_N \geq \ln m_N$$

$$\mathbf{x}_N^T \mathbf{e}_1 = 0, \quad E_\mathrm{v}\mathbf{x}_N^T = \mathbf{0}$$

$$\mathbf{x}_k = \Phi\mathbf{x}_{k-1} + \Psi(\Upsilon_{k-1}\eta + C\mathbf{g}), \quad k = 1, \dots, N$$

Additionally,

$$\|E\mathbf{x}_N\| \leq \|d_{P3}^*\|$$

Where d_P^* is the solution for the previous problem.

6 Simplifying the problems

We need to simplify this further a little, hopefully solve this problem a little faster. We let the control vector and the state vector be part of the same matrix, so we have the final state vector

$$\mathbf{y}_k = \begin{bmatrix} \mathbf{x}_k \\ \mathbf{p}_k \end{bmatrix}$$

So we can define the matrix $\mathbf{y} = \begin{bmatrix} \mathbf{y}_0 & \dots & \mathbf{y}_N \end{bmatrix}^T$, and so we redefine the ODE system to be

$$D\mathbf{y}_k = \Omega(\mathbf{y}_{k-1} + Z\mathbf{g})$$

Where

$$\Omega = \begin{bmatrix} \Phi_{7\times7} & \Psi_{7\times4} \end{bmatrix}, \quad Z = \begin{bmatrix} \mathbf{0}_{7\times3} \\ \mathbf{I}_{3\times3} \\ \mathbf{0}_{1\times3} \end{bmatrix}, \quad D = \begin{bmatrix} I_{7\times7} & 0_{7\times4} \end{bmatrix}$$

We can combine two constraints together to get

$$P\mathbf{y}_N = \mathbf{0} \tag{1}$$

Where

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0_{1\times5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0_{1\times5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0_{1\times5} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0_{1\times5} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0_{1\times5} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0_{1\times5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0_{1\times5} \end{bmatrix}$$

We can rewrite our ODE as the set of equations

$$\begin{bmatrix} I_{11\times 11} & & & & \\ -\Omega_{7\times 11} & D_{7\times 11} & & & \\ & \ddots & \ddots & & \\ & & & -\Omega_{7\times 11} & D_{7\times 11} \\ & & & & P \end{bmatrix} \begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_N \end{bmatrix} = \begin{bmatrix} \mathbf{y}_0 \\ \Omega Z\mathbf{g} \\ \vdots \\ \Omega Z\mathbf{g} \\ 0_{7\times 1} \end{bmatrix} \tag{2}$$

We can also rewrite the glideslope constraint as

$$\mathbf{X} = \{\mathbf{y}_k \in \mathbb{R}^{(11N)\times 1} \big| \ \|S(\mathbf{y}_k - \mathbf{y}_N)\| \leq \mathbf{c}^T(\mathbf{y}_k - \mathbf{y}_N)\} \tag{3}$$

Where

$$S = \begin{bmatrix} 0 & 1 & 0 & 0_{1\times 8} \\ 0 & 0 & 1 & 0_{1\times 8} \end{bmatrix}$$

The same can be done for the control constraints as well:

$$\|E_u \mathbf{y}_k\| \leq \mathbf{e}_{11}^T \mathbf{y}_k \tag{4}$$

$$\hat{\mathbf{n}}^T E_u \mathbf{y}_k \geq \cos{(\theta)} \mathbf{e}_{11}^T \mathbf{y}_k \tag{5}$$

$$\rho_1 e^{-z_0(t_k)} \left[1 - (\mathbf{e}_7^T \mathbf{y}_k - z_0(t_k)) + \frac{(\mathbf{e}_7^T \mathbf{y}_k - z_0(t_k))^2}{2}\right] \leq \mathbf{e}_{11}^T \mathbf{y}_k \leq \rho_2 e^{-z_0(t_k)}[1 - (\mathbf{e}_7^T \mathbf{y}_k - z_0(t_k))]$$

$$\left[\left(1 + z_0(t_k) + \frac{z_0^2(t_k)}{2}\right) - \left(\mathbf{e}_7^T + \frac{\mathbf{e}_{11}^T}{\rho_1 e^{-z_0(t_k)}} + z_0(t_k)\mathbf{e}_7^T\right)\mathbf{y}_k + \frac{(\mathbf{e}_7^T \mathbf{y}_k)^2}{2}\right] \leq 0$$

Let

$$A = \frac{\mathbf{e}_7^T}{\sqrt{2}}, \quad b^T = -\left(\mathbf{e}_7^T + \frac{\mathbf{e}_{11}^T}{\rho_1 e^{-z_0(t_k)}} + z_0(t_k)\mathbf{e}_7^T\right)$$

$$c = \left(1 + z_0(t_k) + \frac{z_0^2(t_k)}{2}\right)$$

Thus the equation can be written as a quadratic equation

$$\mathbf{y}_k^T A^T A \mathbf{y}_k + b^T \mathbf{y}_k + c \leq 0$$

In this, $A^T A$ would be a diagonal matrix of all zeroes except for a 1 at the 7th column. There is a known conversion to a SOCP for this, which is

$$\left\| \begin{matrix} (1 + b^T \mathbf{y}_k + c)/2 \\ A\mathbf{y}_k \end{matrix} \right\|_2 \leq \pm(1 - b^T \mathbf{y}_k - c)/2$$

Equivalently, we change the left side to

$$\left\| \begin{bmatrix} b^T/2 \\ A \end{bmatrix} \mathbf{y}_k + \begin{bmatrix} (1+c)/2 \\ 0 \end{bmatrix} \right\|_2 \leq \pm(1 - b^T \mathbf{y}_k - c)/2$$

Additionally we have

$$(\mathbf{e}_{11}^T + \rho_2 e^{-z_0(t_k)} \mathbf{e}_7^T)\mathbf{y}_k \leq \rho_2 e^{-z_0(t_k)}(1 + z_0(t_k))$$

The main problem is that we can only really go with one of these, and the question is for what range of values is $(1 - b^T \mathbf{y}_k - c)/2$ positive or negative? Since α, ρ_1, ρ_2, and m_0 are all positive, we can find that, not perfectly, but for the range of total allowable flight time determined by the time that $z_0 = -\infty$, $(1 - b^T \mathbf{y}_k - c)/2$ is almost always positive for realistic values of σ and Γ (Γ, σ both positive), and so we will remove the $\pm$:

$$\left\| \begin{bmatrix} b^T/2 \\ A \end{bmatrix} \mathbf{y}_k + \begin{bmatrix} (1+c)/2 \\ 0 \end{bmatrix} \right\|_2 \leq (1 - b^T \mathbf{y}_k - c)/2 \tag{6}$$

$$(\mathbf{e}_{11}^T + \rho_2 e^{-z_0(t_k)}\mathbf{e}_7^T)\mathbf{y}_k \leq \rho_2 e^{-z_0(t_k)}(1 + z_0(t_k)) \tag{7}$$

For the fuel constraint, we have two possible options; one is to simply go with the constraint

$$\mathbf{e}_7^T \mathbf{y}_k \geq \ln m_N \tag{8}$$

Sometimes the program produces impossible solutions where the mass increases (usually because the constraints aren't as tight as they could be), so an alternative is to add the matrix constraint

$$\begin{bmatrix} -\mathbf{e}_7^T & \mathbf{e}_7^T & & \\ & \ddots & \ddots & \\ & & -\mathbf{e}_7^T & \mathbf{e}_7^T \end{bmatrix} \begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_N \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{9}$$

7 Stupid Syntax

ECOS requires the SOC constraints to be in a strange format, which takes the following form for an equation.

$$Gx + s = h, \quad s = \begin{bmatrix} s_0 \\ \mathbf{s}_1 \end{bmatrix}$$

$$s = h - Gx$$

$$s \in \{s = (s_0, \mathbf{s}_1) | \|\mathbf{s}_1\| \leq s_0\}$$

If we have the form

$$\|Ax + b\| = c^T x + d$$

Then

$$-G = \begin{bmatrix} c^T \\ A \end{bmatrix}$$

$$h = \begin{bmatrix} d \\ b \end{bmatrix}$$

So that

$$h - Gx = \begin{bmatrix} c^T x + d \\ Ax + b \end{bmatrix}$$

8 Final Problem Statement

$$\min_{N,\eta} \int_0^{t_N} \sigma(t) dt$$

The Equality constraints are

$$\begin{bmatrix} I_{11\times11} & & & \\ -\Omega_{7\times11} & D_{7\times11} & & \\ & \ddots & \ddots & \\ & & -\Omega_{7\times11} & D_{7\times11} \\ & & & P_{7\times11} \end{bmatrix} \begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_N \end{bmatrix} = \begin{bmatrix} \mathbf{y}_0 \\ \Omega Z \mathbf{g} \\ \vdots \\ \Omega Z \mathbf{g} \\ 0_{7\times1} \end{bmatrix}$$

The SOC constraints are

$$\|E_u \mathbf{y}_k\| \leq \mathbf{e}_{11}^T \mathbf{y}_k$$

$$\mathbf{y} \in \mathbf{X} = \{\mathbf{y} \in \mathbb{R}^{(11N)\times1} | \ \|S(\mathbf{y}_k - \mathbf{y}_N)\| \leq \mathbf{c}^T(\mathbf{y}_k - \mathbf{y}_N)\}$$

$$\left\| \begin{bmatrix} b^T/2 \\ A \end{bmatrix} \mathbf{y}_k + \begin{bmatrix} (1+c)/2 \\ 0 \end{bmatrix} \right\|_2 \leq (1 - b^T \mathbf{y}_k - c)/2$$

$$\|E\mathbf{x}_N - \mathbf{q}\| \le \|d_P^* - \mathbf{q}\|$$

The Linear Constraints are

$$\mathbf{e}_7^T\mathbf{y}_k \ge \ln m_N$$
$$0 \le \rho_2 e^{-z_0(t_k)}[1 - (\mathbf{e}_7^T\mathbf{y}_k - z_0(t_k))] - \mathbf{e}_{11}^T\mathbf{y}_k$$
$$0 \le \hat{\mathbf{n}}^T E_u\mathbf{y}_k - \cos(\theta)\mathbf{e}_{11}^T\mathbf{y}_k$$

The Blackmore papers do not mention this, but minimal cost functions have to be of the form $c^T x$ for an SOCP. The objective to minimize distance involves minimizing a norm, which cannot be represented as $c^T x$. So, we introduce one more variable, d_P. We introduce the following minimal landing error problem.

9 Minimum Landing Error Problem (SOCP form)

$$\min d_P$$

Where

$$\|S\mathbf{y}_N\|_2 \le d_P$$

$$\begin{bmatrix} I_{11\times 11} & & & \\ -\Omega_{7\times 11} & D_{7\times 11} & & \\ & \ddots & \ddots & \\ & & -\Omega_{7\times 11} & D_{7\times 11} \\ & & & P_{7\times 11} \end{bmatrix} \begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_N \end{bmatrix} = \begin{bmatrix} \mathbf{y}_0 \\ \Omega Z\mathbf{g} \\ \vdots \\ \Omega Z\mathbf{g} \\ 0_{7\times 1} \end{bmatrix}$$

The SOC constraints are

$$\|E_u\mathbf{y}_k\| \le \mathbf{e}_{11}^T\mathbf{y}_k$$
$$\mathbf{y} \in \mathbf{X} = \{\mathbf{y} \in \mathbb{R}^{(11N)\times 1} \big| \ \|S(\mathbf{y}_k - \mathbf{y}_N)\| \le \mathbf{c}^T(\mathbf{y}_k - \mathbf{y}_N)\}$$
$$\left\| \begin{bmatrix} b^T/2 \\ A \end{bmatrix} \mathbf{y}_k + \begin{bmatrix} (1+c)/2 \\ 0 \end{bmatrix} \right\|_2 \le (1 - b^T\mathbf{y}_k - c)/2$$
$$\|S\mathbf{y}_N\| \le \|d_P^*\|$$

The Linear Constraints are

$$\mathbf{e}_7^T\mathbf{y}_k \ge \ln m_N$$
$$0 \le \rho_2 e^{-z_0(t_k)}[1 - (\mathbf{e}_7^T\mathbf{y}_k - z_0(t_k))] - \mathbf{e}_{11}^T\mathbf{y}_k$$
$$0 \le \hat{\mathbf{n}}^T E_u\mathbf{y}_k - \cos(\theta)\mathbf{e}_{11}^T\mathbf{y}_k$$