Implicit integration

in static case:

$$F_i(u) = F_{ext}$$

At cycle j $F_i(u)^{j+1}$ can be expanded based on taylor expansion:

$$F_i(u)^{j+1} = F_i(u)^j + \left.\frac{\partial F_i}{\partial u}\right|_{u^j} (u^{j+1} - u^j) + \dots$$

therefore; Truncating to the first 2 terms:

$$F_i(u)^{j+1} = F_i(u)^j + K^j \Delta u = F_{ext}$$

$$K^j \Delta u = F_{ext} - F_{int} = \text{Residual}$$

K^j is the tangent stiffness matrix

dynamic case with Newmark:

$$(F_i)_{t+1} = F_{ext\,t+1}$$

$$\downarrow$$

$$\underbrace{m\ddot{u}_{t+1} + C\dot{u}_{t+1} + (F_i)_{t+1}}_{\downarrow} = F_{ext\,t+1}$$

$$(\hat{F}_i)_{t+1} = F_{ext,\,t+1}$$

$$(\hat{F}_i)^{j+1}_{t+1} = \hat{F}^j_{i\,t+1} + \frac{\partial \hat{F}^j_{i,t}}{\partial u_{t+1}} \Delta u^j = F_{ext\,t+1}$$

$$\Delta u^j = u^{j+1}_{t+1} - u^j_{t+1}$$

$$\frac{\partial}{\partial u^i_{t+1}} (\hat{F}_i)^j_t = M \frac{\partial \ddot{u}}{\partial u_{t+1}} + C \frac{\partial \dot{u}}{\partial u_{t+1}} + \frac{\partial F_i}{\partial u_{t+1}}$$

$$\ddot{u}_{t+1} = \frac{1}{\beta \Delta t^2} \Delta u - \frac{1}{\beta \Delta t} \dot{u}_t + \left(\frac{1}{2\beta} - 1\right) \ddot{u}_t$$

$$\frac{\partial \ddot{u}}{\partial \ddot{u}_{t+1}} = \frac{1}{\beta \Delta t^2}$$

$$\dot{u}_{t+1} = \frac{\gamma}{\beta \Delta t} \Delta u + \left(1 - \frac{\gamma}{\beta}\right) \dot{u}_t + \left(1 - \frac{\gamma}{2\beta}\right) \Delta t\, \ddot{u}_t$$

$$\frac{\partial \dot{u}}{\partial \ddot{u}_{t+1}} = \frac{\gamma}{\beta \Delta t}$$

$$\frac{\partial}{\partial u^j_{t+1}} (\hat{F}_i)^j_t = M \frac{1}{\beta \Delta t^2} + C \frac{\gamma}{\beta \Delta t} + \frac{\partial F_i}{\partial u_{t+1}}$$

$$= (K_T)^j_{t+1}$$

Because

$$(\hat{F}_i)^j_{t+1} + \frac{\partial \hat{F}_i}{\partial u_{t+1}} \Delta u^j = F_{ext\,t+1}$$

$$\frac{\partial \hat{F}_i}{\partial u_{t+1}} = (K)^j_{t+1}$$

$$(K)^j_{t+1}\, \Delta u^j = F_{ext,\,t+1} - (\hat{F}_i)^j_{t+1} = \text{Residual} = R^j_{t+1}$$

$$R^j_{t+1} = F^i_{ext,\,t+1} - (K_T)^j_{t+1} \Delta u^i - (\hat{F}_i)^j_{t+1} = M\ddot{u}_{t+1} + C\dot{u}_{t+1} + (F_i)_{t+1}\, u_{t+1}$$

$$(F_i)_{t+1} = (K_T)_{t+1}$$

$$R^i_{t+1} = F^i_{ext,\,t+1} - (K_T)^i_{t+1} \Delta u^i - M\left[\frac{1}{\beta \Delta t^2} \Delta u - \frac{1}{\beta \Delta t} \dot{u}_t - \left(\frac{1}{2\beta} - 1\right) \ddot{u}_t\right]$$

$$- C\left[\frac{\gamma}{\beta \Delta t} \Delta u + \left(1 - \frac{\gamma}{\beta}\right) \dot{u}_t + \left(1 - \frac{\gamma}{2\beta}\right) \Delta t\, \ddot{u}_t\right]$$

$$- K\, \Delta u$$

$$\hat{R}^{i}_{t+1} = F^{j}_{ex,t+1} - F^{j}_{int,t+1} - M\left[\frac{1}{\beta\Delta t^2} u_{t+1} - \frac{1}{\beta\Delta t^2} u_t - \frac{1}{\beta\Delta t}\dot{u}_t - \left(\frac{1}{2\beta}-1\right)\ddot{u}_t\right]$$

$$- C\left[\frac{\gamma}{\beta\Delta t} u_{t+1} - \frac{\gamma}{\beta\Delta t} u_t + \left(1-\frac{\gamma}{\beta}\right)\dot{u}_t + \left(1-\frac{\gamma}{2\beta}\right)\Delta t\,\ddot{u}_t\right]$$

$$- K\,\Delta u_{\cancel{t+1}}$$

$$\underset{= 0}{\underline{\text{Residual}}} = F^{j}_{ext,t+1} - F^{j}_{int,t+1} - \Delta u \underbrace{\left[\frac{1}{\beta\Delta t^2} M + \frac{\gamma}{\beta\Delta t} C + K\right]}_{\hat{K}} + \dot{u}_t\left[-\frac{1}{\beta\Delta t} M - \left(1-\frac{\gamma}{\beta}\right)C\right] + \ddot{u}_t\left[-\left(\frac{1}{2\beta}-1\right)M + \left(1-\frac{\gamma}{2\beta}\right)\Delta t\, C\right]$$

$$\hat{K}\,\Delta u = F^{j}_{ext,t+1} - F^{j}_{int,t+1} + \dot{u}_t\left[-\frac{1}{\beta\Delta t} M - \left(1-\frac{\gamma}{\beta}\right)C\right] + \ddot{u}_t\left[-\left(\frac{1}{2\beta}-1\right)M + \left(-1-\frac{\gamma}{2\beta}\right)\Delta t\, C\right]$$