The following theorem shows that each satisfiable set of FOL-formulas has a countable model. For FOL without equality, we can even guarantee infinite countable models. To establish this result it is important that we require the variable-set *Var* and the vocabulary *Voc* to be *countable*. Our default-assumption that the underlying vocabulary *Voc* and the variable set *Var* are recursively enumerable is irrelevant. We just need here that *Var* and *Voc* are countable. Therefore, the set of all terms over (*Voc*, *Var*) and the set of all FOL-formulas over (*Voc*, *Var*) are countable too.

Theorem 1.5.5 (Downward Löwenheim-Skolem Theorem). *Each satisfiable set of FOL-formulas (over some countable vocabulary and countable variable-set) has a countable model. More precisely:*

- *Each satisfiable set of FOL-formulas without equality has an infinite countable model.*
- *Each satisfiable set of FOL-formulas with equality has a (finite or infinite) countable model.*

Proof. For the special case of a singleton formula-set consisting of a closed FOL-formula in Skolem form, i.e., sentences of the form $\phi = \forall x_1 \ldots\ldots \forall x_n.\psi$ where ψ is quantifier-free (see page 12), we may apply the Herbrand-theory, which yields that ϕ is satisfiable if and only if ϕ has a Herbrand-model. Since the Herbrand-universe is countable this yields the claim. For the general case, an analogous argument is applicable. The rough idea is to construct a *term model*, i.e., a model where the terms serve as elements of the domain. As stated above, the set of terms is countable since the vocabulary *Voc* and the variable-set *Var* are supposed to be countable. It is no restriction to assume that the variable set *Var* is infinite. Otherwise we can extend *Var* accordingly. This assumption ensures that the set of terms in infinite.

The idea is to extend the given formula-set $\mathfrak{F}$ to a *maximal satisfiable* formula-set $\mathfrak{F}^+$. More precisely, we construct a formula-set $\mathfrak{F}^+$ which enjoys the following properties:

(1) $\mathfrak{F} \subseteq \mathfrak{F}^+$ and $\mathfrak{F}^+$ is satisfiable.

(2) For each formula ϕ: either $\phi \in \mathfrak{F}^+$ or $\neg\phi \in \mathfrak{F}^+$.

(3) For each formula ϕ and variable x there is a constant symbol c such that:

$$\neg\forall x.\phi \ \rightarrow \neg\phi[x/c] \in \mathfrak{F}^+$$

Maximal satisfiability of $\mathfrak{F}^+$ means satisfiability (second part of (1)) and condition (2). The additional constraint (3) will be crucial to show that the term model that we are going to construct is indeed a model for $\mathfrak{F}^+$. Since $\mathfrak{F}$ is contained in $\mathfrak{F}^+$, the consructed term model is then a term model for $\mathfrak{F}$ too.

Our proof starts with the construction of a term model when $\mathfrak{F}^+$ is given, first for FOL without equality and then for FOL with equality. The second step is then to show the existence of a formula-set $\mathfrak{F}^+$ satisfying conditions (1), (2) and (3).

Construction of a term model for $\mathfrak{F}^+$ for FOL without equality. Let us suppose that a formula-set $\mathfrak{F}^+$ satisfying conditions (1), (2) and (3) has been constructed. We first explain the construction of a term model $\mathcal{I} = (\mathcal{A}, \mathcal{V})$ for $\mathfrak{F}^+$ for the case where $\mathfrak{F}$ is a set of FOL-formulas without equality. With the construction of $\mathfrak{F}^+$ presented below, also $\mathfrak{F}^+$ will not use the equality symbol. The vocabulary used in the formulas of $\mathfrak{F}^+$ will be a countable extension of the original vocabulary *Voc*. Assuming that the original variable-set *Var* is infinite, $\mathfrak{F}^+$ can use the same variable-set *Var*.

The domain A of $\mathcal{A}$ is the set of terms that can be built by the variables and constant and function symbols that appear in $\mathfrak{F}^+$. To ensure that the set of terms is infinite, we suppose that the vocabulary has at least one function symbol of arity $\geqslant 1$ (see above). Hence, A is countable and infinite.

We now define the variable valuation $\mathcal{V}$ and the meanings of the predicate and functions symbols in the term structure $\mathcal{A}$. Terms are interpreted by itself. That is:

- $\mathcal{V}(x) \stackrel{\text{def}}{=} x$ for each variable x,
- $c^{\mathcal{A}} \stackrel{\text{def}}{=} c$ for each constant symbol c,
- the function $f^{\mathcal{A}}$ for an m-ary function symbol f is given by $f^{\mathcal{A}}(t_1, \ldots, t_m) \stackrel{\text{def}}{=} f(t_1, \ldots, t_m)$.

The predicates $P^{\mathcal{A}}$ for the n-ary predicate symbols are given by:

$$P^{\mathcal{A}} \stackrel{\text{def}}{=} \{ (t_1, \ldots, t_n) \in A^n : P(t_1, \ldots, t_n) \in \mathfrak{F}^+ \}$$

For the interpretation $\mathcal{I} = (\mathcal{A}, \mathcal{V})$ we have $t^{\mathcal{I}} = t$ for all terms t. We now show:

$$\phi \in \mathfrak{F}^+ \quad \text{iff} \quad \mathcal{I} \models \phi \tag{*}$$

Obviously, (*) yields that $\mathcal{I}$ is a model for $\mathfrak{F}^+$, and hence, also for $\mathfrak{F}$ (see condition (1)).

Before proving (*), we observe that the maximal satisfiability of $\mathfrak{F}^+$ yields that $\mathfrak{F}^+$ is closed under consequences, i.e.:

$$\text{if } \mathfrak{F}^+ \Vdash \psi \text{ then } \psi \in \mathfrak{F}^+.$$

Note that if $\mathcal{J}$ is an interpretation with $\mathcal{J} \models \mathfrak{F}^+$ and $\mathfrak{F}^+ \Vdash \psi$ then $\mathcal{J} \models \psi$, and therefore, $\mathcal{J} \not\models \neg\psi$. But then $\neg\psi \notin \mathfrak{F}^+$. The maximality condition (2) yields $\psi \in \mathfrak{F}^+$.

In particular, $\mathfrak{F}^+$ is closed under equivalence $\equiv$. That is:

$$\text{if } \phi \in \mathfrak{F}^+ \text{ and } \phi \equiv \psi \text{ then } \psi \in \mathfrak{F}^+.$$

To see this, suppose $\phi \in \mathfrak{F}^+$ and $\phi \equiv \psi$. Then, $\phi \Vdash \psi$, and therefore $\mathfrak{F}^+ \Vdash \psi$. As $\mathfrak{F}^+$ is closed under consequences, we get $\psi \in \mathfrak{F}^+$.

Furthermore, for all formulas ψ, ψ_1 and ψ_2 we have:

$$\begin{array}{rcl} \neg\psi \in \mathfrak{F}^+ & \text{iff} & \psi \notin \mathfrak{F}^+ \\ \psi_1 \wedge \psi_2 \in \mathfrak{F}^+ & \text{iff} & \psi_1 \in \mathfrak{F}^+ \text{ and } \psi_2 \in \mathfrak{F}^+ \end{array}$$

The first statement is immediate from the maximal satisfiability of $\mathfrak{F}^+$. The second statement follows by the facts that $\mathfrak{F}^+$ is closed under consequences and $\psi_1 \wedge \psi_2 \Vdash \psi_i$, $i = 1, 2$, and $\{\psi_1, \psi_2\} \Vdash \psi_1 \wedge \psi_2$.

Proof of (*). We now prove (*) by induction on the length k of formulas.

Basis of induction: $k = |\phi| = 0$. For $\phi = \mathit{true}$ we have $\mathit{true} \in \mathfrak{F}^+$ (since $\mathfrak{F}^+$ is maximal satisfiable) and $\mathfrak{I} \models \mathit{true}$. We now regard an atomic formula $\phi = P(t_1, \ldots, t_n)$. Then:

$$\begin{array}{lll} & P(t_1,\ldots,t_n) \in \mathfrak{F}^+ & \\ \text{iff} & \left(t_1^{\mathfrak{I}},\ldots,t_n^{\mathfrak{I}}\right) \in P^{\mathcal{A}} & \text{(by definition of } P^{\mathcal{A}}\text{)} \\ \text{iff} & (t_1,\ldots,t_n) \in P^{\mathcal{A}} & \text{(as } t^{\mathfrak{I}} = t \text{ for all terms } t\text{)} \\ \text{iff} & \mathfrak{I} \models P(t_1,\ldots,t_n) & \text{(by definition of } \models\text{)} \end{array}$$

As we deal with FOL without equality, we do not consider atomic formulas of the form $t_1 = t_2$.

Induction step. Let $k = |\phi| \geqslant 1$ and let us suppose that (*) holds for all formulas of length $\leqslant k-1$ (induction hypothesis). If $\phi = \neg\psi$ then

$$\begin{array}{lll} & \neg\psi \in \mathfrak{F}^+ & \\ \text{iff} & \psi \notin \mathfrak{F}^+ & \text{(as } \mathfrak{F}^+ \text{ is maximal satisfiable, see above)} \\ \text{iff} & \mathfrak{I} \not\models \psi & \text{(induction hypothesis)} \\ \text{iff} & \mathfrak{I} \models \neg\psi = \phi & \text{(semantics of } \neg\text{)} \end{array}$$

The treatment of $\phi = \psi_1 \wedge \psi_2 \in \mathfrak{F}^+$ is analogous:

$$\begin{array}{lll} & \psi_1 \wedge \psi_2 \in \mathfrak{F}^+ & \\ \text{iff} & \{\psi_1, \psi_2\} \subseteq \mathfrak{F}^+ & \text{(as } \mathfrak{F}^+ \text{ is maximal satisfiable, see above)} \\ \text{iff} & \mathfrak{I} \models \psi_1 \text{ and } \mathfrak{I} \models \psi_2 & \text{(induction hypothesis)} \\ \text{iff} & \mathfrak{I} \models \psi_1 \wedge \psi_2 \ = \ \phi & \text{(semantics of } \wedge\text{)} \end{array}$$

Let now $\phi = \forall y.\psi$.

"$\Longleftarrow$": Suppose $\mathfrak{I} \models \forall y.\psi$. Then, $\mathfrak{I}[y := s] \models \psi$ for all terms s. In particular, we have:

$$\mathfrak{I}[y := c] \models \psi$$

where c is a constant symbol as in condition (3) such that $\neg\forall y.\psi \rightarrow \neg\psi[y/c] \in \mathfrak{F}^+$. As $\psi[y/c]$ is a logical consequence of $\forall y.\psi$ and $\mathfrak{I}$ is a model for $\forall y.\psi$, we get:

$$\mathfrak{I} \models \psi[y/c]$$

Since $\big|\psi[y/c]\big| = \big|\psi\big| = \big|\phi\big| - 1 = k-1$ we may apply the induction hypothesis and get:

$$\psi[y/c] \in \mathfrak{F}^+$$

The formula $\neg\forall y.\psi \rightarrow \neg\psi[y/c]$ is equivalent to $\psi[y/c] \rightarrow \forall y.\psi$. Hence:

$$\psi[y/c] \rightarrow \forall y.\psi \in \mathfrak{F}^+$$

Furthermore, $\forall y.\psi$ is a logical consequence of the formulas $\psi[y/c]$ and $\psi[y/c] \rightarrow \forall y.\psi$. Since both formulas belong to $\mathfrak{F}^+$, this yields:

$$\mathfrak{F}^+ \Vdash \forall y.\psi$$

Since $\mathfrak{F}^+$ is closed under consequences (see above), we obtain $\forall y.\psi \in \mathfrak{F}^+$.

"$\Longrightarrow$": Suppose now that $\forall y.\psi \in \mathfrak{F}^+$. We have to show that $\mathcal{I}[y := s] \models \psi$ for all terms $s \in A$. In the sequel, let s be a fixed term. By renaming bounded variables of ψ, we switch from ψ to an equivalent formula ψ' of the same length such that ψ' contains no quantifiers for the variable that appear in s. This ensures that y can be replaced with s in ψ'.

As $\forall y.\psi \equiv \forall y.\psi'$ and as we suppose that $\forall y.\psi \in \mathfrak{F}^+$ we also have $\forall y.\psi' \in \mathfrak{F}^+$. Remind that $\mathfrak{F}^+$ is closed under equivalence $\equiv$. Since variable y can be replaced in ψ' with the term s we have:

$$\forall y.\psi' \Vdash \psi'[y/s]$$

As $\mathfrak{F}^+$ is closed under logical consequences, we obtain:

$$\psi'[y/s] \in \mathfrak{F}^+$$

Since the length of $\psi'[y/s]$ agrees with the length of ψ' and ψ and is smaller than the length of $\phi = \forall y.\psi$ (more precisely, we have $\left|\psi'[y/s]\right| = \left|\psi'\right| = \left|\psi\right| = k-1$), we may apply the induction hypothesis, which yields:

$$\mathcal{I} \models \psi'[y/s]$$

As variable y can be replaced with term s in ψ' we get by the substitution lemma (see page 8):

$$\begin{array}{lll} & \mathcal{I} \models \psi'[y/s] & \\ \text{iff} & \mathcal{I}[y := s] \models \psi' & \text{(substitution lemma)} \\ \text{iff} & \mathcal{I}[y := s] \models \psi & \text{(as } \psi' \equiv \psi\text{)} \end{array}$$

This shows $\mathcal{I}[y := s] \models \psi$ for all terms s, and therefore $\mathcal{I} \models \forall y.\psi$.

Construction of a term model for $\mathfrak{F}^+$ for FOL with equality. To treat sets of FOL-formulas with equality, a countable (possibly finite) model for $\mathfrak{F}^+$ is obtained by the quotient structure $\mathcal{A}/\!\sim$ where $\mathcal{A}$ is the term structure as above and $\sim$ denotes the equivalence that identifies exactly those terms t_1 and t_2 such that $\mathfrak{F}^+ \Vdash (t_1 = t_2)$. We then have:

$$t_1 \sim t_2 \quad \text{iff} \quad \mathfrak{F}^+ \Vdash (t_1 = t_2) \quad \text{iff} \quad t_1 = t_2 \in \mathfrak{F}^+$$

The quotient structure $\mathcal{A}/\!\sim$ is obtained as follows. The domain of $\mathcal{A}/\!\sim$ is the quotient space of A with respect to $\sim$, i.e., $Dom^{\mathcal{A}/\sim}$ equals

$$A/\!\sim \;=\; \big\{ [t] : t \in A \big\}$$

where

$$[t] \stackrel{\text{def}}{=} \big\{ t' : t \sim t' \big\}$$

denotes the equivalence class of t. For each n-ary predicate symbol P we put

$$P^{\mathcal{A}/\sim} \stackrel{\text{def}}{=} \big\{ ([t_1], \ldots, [t_n]) \,:\, P(t_1, \ldots, t_n) \in \mathfrak{F}^+ \big\}.$$

Note that, since $\mathfrak{F}^+$ is maximal satisfiable, we have for all terms $t_1, \ldots, t_n, s_1, \ldots, s_n$:

$$\text{if } t_i \sim s_i \text{ for } i = 1, \ldots, n \text{ then:} \quad P(t_1, \ldots, t_n) \in \mathfrak{F}^+ \;\text{ iff }\; P(s_1, \ldots, s_n) \in \mathfrak{F}^+$$

Thus, if $T_1,\dots,T_n \in A/\!\sim$ then

$$(T_1,\dots,T_n) \in P^{\mathcal{A}/\sim} \quad \text{iff} \quad P(t_1,\dots,t_n) \in \mathfrak{F}^+ \text{ for all terms } t_i \in T_i,\ i=1,\dots,n.$$

Similarly, if f is an m-ary function symbol then we define:

$$f^{\mathcal{A}/\sim}([t_1],\dots,[t_m]) \ \stackrel{\text{def}}{=}\ [\, f(t_1,\dots,t_m)\,]$$

This function $f^{\mathcal{A}/\sim}$ is well-defined since for all terms $t_1,\dots,t_m$, $s_1,\dots,s_m$ such that $t_i \sim s_i$ for $1 \leqslant i \leqslant m$ we have $f(t_1,\dots,t_m) \sim f(s_1,\dots,s_m)$. The variable valuation $\mathcal{V}' : \mathit{Var} \to A/\!\sim$ is given by $\mathcal{V}'(x) \stackrel{\text{def}}{=} [x]$. It is now easy to see that

$$(\mathcal{A}/\!\sim,\mathcal{V}') \models \phi \quad \text{iff} \quad \phi \in \mathfrak{F}^+$$

In particular, $(\mathcal{A}/\!\sim,\mathcal{V}')$ is a countable (possibly finite) model for $\mathfrak{F}^+$, and hence also for $\mathfrak{F}$.

Construction of $\mathfrak{F}^+$. To realize condition (3) on page 75 we extend the given vocabulary *Voc* by pairwise distinct new constant symbols $c_1,c_2,c_3,\dots$. Let *Voc** be the resulting vocabulary. As *Voc* is countable, so is the extended vocabulary *Voc**. We now consider an arbitrary enumeration

$$(\phi_1,x_1),\ (\phi_2,x_2),\ (\phi_3,x_3),\ \dots$$

of all pairs (ϕ,x) where ϕ is a formula over the extended vocabulary *Voc** and x a variable in *Var*. Let

$$\psi_n \ \stackrel{\text{def}}{=}\ \neg\forall x_n.\phi_n \to \neg\phi_n[x_n/c_{i_n}]$$

where c_{i_n} is the "first" constant symbol in $c_1,c_2,c_3,\dots$ such that c_{i_n} does not occur in $\phi_1,\dots,\phi_n$, $\psi_1,\dots,\psi_{n-1}$. Here, "first" means that i_n is the smallest index i such that c_i does not appear in $\phi_1,\dots,\phi_n$ and $i \notin \{i_1,\dots,i_{n-1}\}$. Such an index i exists, as there are only finitely many constant symbols in $\phi_1,\dots,\phi_n$. Furthermore, we have $i_1 < i_2 < i_3 < \dots$. Let

$$\mathfrak{G} \ \stackrel{\text{def}}{=}\ \{\psi_1,\psi_2,\psi_3,\dots\}$$

Claim. $\mathfrak{F} \cup \mathfrak{G}$ is satisfiable.

Proof. We will make use of the compactness property and show that each of the formula-sets

$$\mathfrak{H}_0 \stackrel{\text{def}}{=} \mathfrak{F} \quad \text{and} \quad \mathfrak{H}_n \stackrel{\text{def}}{=} \mathfrak{F} \cup \{\psi_1,\dots,\psi_n\} \ \text{ for } n \geqslant 1$$

is satisfiable. More precisely, we will construct inductively models for $\mathfrak{H}_n$.

We pick a model $(\mathcal{A},\mathcal{V})$ for $\mathfrak{F}$. (Recall that the given formula-set $\mathfrak{F}$ is supposed to be satisfiable.) Let A be the domain of $\mathcal{A}$. The idea is to extend $\mathcal{A}$ by adding interpretations for the new constant symbols $c_1,c_2,c_3,\dots$. We do this in an inductive way and define structures $\mathcal{B}_n$ that extend $\mathcal{A}$ by interpretations for the new constant symbols that appear in $\mathfrak{H}_n$ such that $(\mathcal{B}_n,\mathcal{V})$ is a model for $\mathfrak{H}_n$. More precisely, besides the symbols in the original vocabulary *Voc*, structure $\mathcal{B}_n$ provides meanings for the constant symbols $c_{i_1},\dots,c_{i_n}$ and all new constant symbols c_i where $i \notin \{i_1,i_2,i_3,\dots\}$. Structures $\mathcal{B}_n$ and $\mathcal{A}$ have the same domain A and assign the same meaning to all predicate and function symbols that occur in $\mathfrak{F}$.

- For $n = 0$ we define $\mathcal{B}_0$ as the structure that agrees with $\mathcal{A}$ for all predicate and function symbols in the original vocabulary *Voc*. Furthermore, we pick an arbitrary element $a_0 \in A$ and define
$$c_i^{\mathcal{B}_0} \stackrel{\text{def}}{=} a_0$$
for all constant symbols c_i such that $i \notin \{i_1, i_2, \ldots\}$. Clearly, $(\mathcal{B}_0, \mathcal{V})$ is a model for $\mathfrak{H}_0 = \mathfrak{F}$. Recall that none of the constant symbols c_i appears in $\mathfrak{F}$. Thus, the interpretations $(\mathcal{A}, \mathcal{V})$ and $(\mathcal{B}_0, \mathcal{V})$ yield the same truth value for all formulas of $\mathfrak{F}$.

- For $n \geqslant 1$, structure $\mathcal{B}_n$ agrees with $\mathcal{B}_{n-1}$ on all predicate and function symbols that appear in $\mathfrak{F}$ and the constant symbols c_i where either $i \in \{i_1, \ldots, i_{n-1}\}$ or $i \notin \{i_1, i_2, \ldots\}$. In particular, $\mathcal{B}_n$ and $\mathcal{B}_{n-1}$ agree on all symbols that appear in $\mathfrak{H}_{n-1}$. The task is now to define an appropriate interpretation for c_{i_n}. For this, we evaluate $\forall x_n.\phi_n$ over $(\mathcal{B}_{n-1}, \mathcal{V})$. Note that for the constant symbols c_i that appear in ϕ_n we have: either $i \notin \{i_1, i_2, \ldots\}$ or $i = i_k$ for some $k < n$. Thus, $\mathcal{B}_{n-1}$ provides meanings for all symbols in ϕ_n.

  - If $(\mathcal{B}_{n-1}, \mathcal{V}) \not\models \forall x_n.\phi_n$ then there exists an element $a_n \in A$ such that:
  $$(\mathcal{B}_{n-1}, \mathcal{V}[x_n := a_n]) \not\models \phi_n$$
  In this case, we define $c_{i_n}^{\mathcal{B}_n} \stackrel{\text{def}}{=} a_n$.
  - If $(\mathcal{B}_{n-1}, \mathcal{V}) \models \forall x_n.\phi_n$ then we define $c_{i_n}^{\mathcal{B}_n} \stackrel{\text{def}}{=} a_0$ where a_0 is as in the case $n = 0$ (or any other element in structure $\mathcal{A}$).

  Since $\mathcal{B}_n$ and $\mathcal{B}_{n-1}$ agree on all symbols that appear in $\mathfrak{H}_{n-1}$, $(\mathcal{B}_n, \mathcal{V})$ is a model for $\mathfrak{H}_{n-1}$. Furthermore, the choice of $c_{i_n}^{\mathcal{B}_n}$ and the fact that c_{i_n} does not appear in ϕ_n yields that
  $$\begin{array}{lcll} (\mathcal{B}_n, \mathcal{V}) & \not\models & \phi_n[x_n/c_{i_n}] & \text{if } (\mathcal{B}_{n-1}, \mathcal{V}) \not\models \forall x_n.\phi_n \\ (\mathcal{B}_n, \mathcal{V}) & \models & \forall x_n.\phi_n & \text{if } (\mathcal{B}_{n-1}, \mathcal{V}) \models \forall x_n.\phi_n \end{array}$$
  In both cases we get:
  $$(\mathcal{B}_n, \mathcal{V}) \models \psi_n = \neg\forall x_n.\phi_n \to \neg\phi_n[x_n/c_{i_n}]$$
  Hence, $(\mathcal{B}_n, \mathcal{V}) \models \mathfrak{H}_n$.

The construction of the $\mathcal{B}_n$'s shows that each of the sets $\mathfrak{H}_n = \mathfrak{F} \cup \{\psi_1, \ldots, \psi_n\}$ is satisfiable. But then $\mathfrak{F} \cup \mathfrak{G} = \mathfrak{F} \cup \{\psi_1, \psi_2, \psi_3, \ldots\}$ is finitely satisfiable, and therefore satisfiable (by the compactness theorem). This completes the proof of the claim. $\rfloor$

We extend $\mathfrak{F} \cup \mathfrak{G}$ to a maximal satisfiable set. This can be done by picking an interpretation $\mathcal{I}$ that is a model for $\mathfrak{F} \cup \mathfrak{G}$. We then define $\mathfrak{F}^+$ as the set of formulas ϕ that hold under $\mathcal{I}$:
$$\mathfrak{F}^+ \stackrel{\text{def}}{=} \{\, \phi : \mathcal{I} \models \phi \,\}$$

We then have:

(1) $\mathfrak{F} \subseteq \mathfrak{F}^+$ (as $\mathcal{I}$ is a model for $\mathfrak{F} \cup \mathfrak{G}$) and $\mathcal{I}$ is a model for $\mathfrak{F}^+$ (by definition of $\mathfrak{F}^+$)

(2) for each formula ϕ: either $\phi \in \mathfrak{F}^+$ or $\neg\phi \in \mathfrak{F}^+$ (as $\mathcal{I} \models \phi$ iff $\mathcal{I} \not\models \neg\phi$)

(3) for each pair (ϕ, x) consisting of a formula ϕ and a variable x we have $(\phi, x) = (\phi_n, x_n)$ for some $n \geqslant 1$ and

$$\psi_n \;=\; \neg\forall x.\phi \rightarrow \neg\phi[x/c_{i_n}] \in \mathfrak{G} \subseteq \mathfrak{F}^+$$

Thus, conditions (1)-(3) hold. □

The downward Löwenheim-Skolem-Theorem ensures the existence of countable models for satisfiable formula-sets. However, all models might be infinite, even for singleton formula-sets, as FOL does not have the finite model property (see Theorem 1.3.2 on page 34).

The combination of the Löwenheim-Skolem theorems yields:

Corollary 1.5.6. *Each set of FOL-formulas (over some countable vocabulary and variable-set) that has finite models of arbitrary size also has an infinite countable and an uncountable model.*

For the special case of FOL without equality the condition "that has finite models of arbitrary size" can be replaced with the condition "that has some finite model", as we saw in the upward Löwenheim-Skolem Theorem for FOL without equality (see Theorem 1.5.3 on page 72).

The Löwenheim-Skolem theorems yield the existence of *nonstandard models* for "standard" FO-theories. E.g., by the upward Löwenheim-Skolem theorem, the FO-theory of the reals $Th(\mathbb{R}, +, *, =)$ has a model $\mathcal{A}$ where the cardinality of the domain is a superset of $2^{\mathbb{R}}$, and hence, $\mathcal{A}$ and $(\mathbb{R}, +, *)$ are not isomorphic. Similarly, the theory $Th(\mathbb{N}, +, *, =)$ of arithmetic (cf. Example 1.4.5 on page 47) has an uncountable model $\mathcal{A}$, and therefore, $\mathcal{A}$ and $(\mathbb{N}, +, *)$ are not isomorphic. But even among the *countable* models for the FO-theory $Th(\mathbb{N}, +, *, =)$ of arithmetic, there are structures that are not isomorphic to $(\mathbb{N}, +, *)$.

Theorem 1.5.7 (Nonstandard countable model for the theory of arithmetic). *Let Voc be the vocabulary consisting of two binary function symbols + and ∗. There exists a countable structure $\mathcal{A} = (A, +^{\mathcal{A}}, *^{\mathcal{A}})$ such that the following conditions (1) and (2) hold:*

(1) $Th(\mathcal{A}) = Th(\mathbb{N}, +, *, =)$, *i.e.,* $\mathcal{A}$ *and* $(\mathbb{N}, +, *)$ *are models for the same FOL-formulas over* Voc.

(2) $\mathcal{A}$ *and* $(\mathbb{N}, +, *)$ *are not isomorphic.*

Proof. If x is a variable and $n \in \mathbb{N}$ then we define the following formulas:

$$\begin{aligned}
\phi_0(x) &\stackrel{\text{def}}{=} \textit{zero}(x) \\
\phi_1(x) &\stackrel{\text{def}}{=} \textit{one}(x) \\
\phi_{n+1}(x) &\stackrel{\text{def}}{=} \exists y \exists z. \big(\phi_n(y) \wedge \textit{one}(z) \wedge y + z = x\big)
\end{aligned}$$

where

$$\textit{zero}(x) \stackrel{\text{def}}{=} \forall y.(x + y = y) \qquad \textit{one}(x) \stackrel{\text{def}}{=} \forall y.(x * y = y)$$

Let $\mathcal{N} \stackrel{\text{def}}{=} (\mathbb{N}, +, *)$. Then, we write $(\mathcal{N}, m)$ to denote any interpretation with structure $\mathcal{N}$ and a variable valuation $\mathcal{V}$ such that $\mathcal{V}(x) = m$. Obviously, for all n, $m \in \mathbb{N}$:

$$\begin{aligned}
(\mathcal{N}, m) \models \textit{zero}(x) &\quad \text{iff} \quad m = 0 \\
(\mathcal{N}, m) \models \textit{one}(x) &\quad \text{iff} \quad m = 1 \\
(\mathcal{N}, m) \models \phi_n(x) &\quad \text{iff} \quad m = n
\end{aligned}$$

We now consider the following formula-set $\mathfrak{F}$ that extends the theory of arithmetic by the formulas $\neg\phi_n(x)$:

$$\mathfrak{F} \stackrel{\text{def}}{=} Th(\mathcal{N}) \cup \left\{\neg\phi_n(x) : n \in \mathbb{N}\right\}$$

Claim: $\mathfrak{F}$ is satisfiable.

Proof of the claim. For each finite subset $\mathfrak{G}$ of $\mathfrak{F}$, there is some $n \in \mathbb{N}$ such that $\neg\phi_n(x) \notin \mathfrak{G}$. But then $(\mathcal{N}, n) \models \mathfrak{G}$. Thus, $\mathfrak{F}$ is finitely satisfiable. By the compactness theorem (see page 71), $\mathfrak{F}$ is satisfiable. $\rfloor$

The downward Löwenheim-Skolem theorem states the existence of a countable model $(\mathcal{A}, \mathcal{V})$ for $\mathfrak{F}$. We now show that $\mathcal{A}$ fulfills conditions (1) and (2).

ad (1). Remind from Theorem 1.4.6 on page 48 that for each complete theory $\mathfrak{T}$ and each structure $\mathcal{A}$ we have $\mathcal{A} \models \mathfrak{T}$ if and only if $\mathfrak{T} = Th(\mathcal{A})$. With $\mathfrak{T} = Th(\mathcal{N})$ we get $Th(\mathcal{A}) = Th(\mathcal{N})$ as $\mathcal{N}$ is a model for $Th(\mathcal{N})$.

ad (2). Let $a \stackrel{\text{def}}{=} \mathcal{V}(x)$. Structures $\mathcal{A}$ and $\mathcal{N}$ are not isomorphic as $(\mathcal{A}, a) \models \neg\phi_n(x)$ for all $n \in \mathbb{N}$, while $(\mathcal{N}, n) \models \phi_n(x)$ for each $n \in \mathbb{N}$. $\square$

We finally remark that the existence of countable models for satisfiable formula-sets and the compactness property are characteristic for FOL in the sense that no proper extension of FOL enjoys these two properties. We will not provide the proof for this result (see e.g. [EFT84]), which is known as the *Theorem by Lindström*. Instead, in Section 1.7 and 2 we will consider some special extensions of FOL and see that at least one of these properties is violated.

Before doing so, we will discuss the limitations of FOL and show that several "natural" mathematical properties cannot be expressed in FOL.