The winning objective for the duplicator (who aims to establish the equivalence of $\mathcal{A}$ and $\mathcal{B}$) is to establish a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$. As the meaning of the constant symbols $c_1,\ldots,c_\ell$ under both structures is supposed to be public for both opponents, the winning criterion adds the tuples $\overline{c}^{\mathcal{A}}$ and $\overline{c}^{\mathcal{B}}$ to the outcome $\langle \overline{a}, \overline{b} \rangle$.

Definition 1.6.13 (Winning outcomes). Recall that *Const* $= \{c_1,\ldots,c_\ell\}$ is the set of constant symbols in the underlying finite, relational vocabulary. An outcome

$$\langle \overline{a}, \overline{b} \rangle \; = \; \langle (a_1,\ldots,a_k), (b_1,\ldots,b_k) \rangle \in A^k \times B^k$$

of a k-round play is said to be *winning (for the duplicator)* iff the pair

$$\langle\, (\overline{a}, \overline{c}^{\mathcal{A}}),\, (\overline{b}, \overline{c}^{\mathcal{B}})\,\rangle \; = \; \langle\, (a_1,\ldots,a_k, c_1^{\mathcal{A}},\ldots,c_\ell^{\mathcal{A}}),\, (b_1,\ldots,b_k, c_1^{\mathcal{B}},\ldots,c_\ell^{\mathcal{B}})\,\rangle \in A^{k+\ell} \times B^{k+\ell}$$

defines a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$. Otherwise, i.e., if $\langle\, (\overline{a}, \overline{c}^{\mathcal{A}}),\, (\overline{b}, \overline{c}^{\mathcal{B}})\,\rangle$ does not define a partial isomorphism, the spoiler wins. ■

For the special case $k = 0$ we get that the 0-round outcome, i.e., the initial game configuration $\langle \varepsilon, \varepsilon \rangle$ where no move has been performed so far, is winning if and only if the pair $\langle \overline{c}^{\mathcal{A}}, \overline{c}^{\mathcal{B}} \rangle$ defines a partial isomorphism. Here, ε denotes the empty tuple. Note that

- if the pair $\langle \overline{c}^{\mathcal{A}}, \overline{c}^{\mathcal{B}} \rangle$ does not define a partial isomorphism then there is no winning outcome. For example, if there are constant symbols c and d such that $c^{\mathcal{A}} = d^{\mathcal{A}}$, while $c^{\mathcal{B}} \neq d^{\mathcal{B}}$, then there is no winning outcome at all.
- if a k-round outcome $\langle (a_1,\ldots,a_k), (b_1,\ldots,b_k) \rangle$ is winning then the outcomes

  $$\langle (a_1,\ldots,a_i), (b_1,\ldots,b_i) \rangle$$

  of rounds $i = 0, 1, \ldots, k{-}1$, must be winning too. In fact, we even have the following stronger condition. If a k-round outcome $\langle (a_1,\ldots,a_k), (b_1,\ldots,b_k) \rangle$ is winning then so are all pairs $\langle (a_{j_1},\ldots,a_{j_m}), (b_{j_1},\ldots,b_{j_m}) \rangle$ where $j_1,\ldots,j_m$ is an arbitrary sequence of indices in $\{1,2,\ldots,k\}$.

Next we define an equivalence $\cong_k$ between structures that identifies exactly those structures $\mathcal{A}$ and $\mathcal{B}$ where the duplicator can play in such a way that the outcome $\langle \overline{a}, \overline{b} \rangle$ after the k-th round is winning, no matter how the spoiler behaves. We formalize this by means of winning strategies for the duplicator. Informally, a *strategy* for the duplicator is a function that provides the duplicator's decisions. That is, a k-round strategy takes as input the outcome $\langle (a_1,\ldots,a_i), (b_1,\ldots,b_i) \rangle$ of the i-th round (where $0 \leqslant i < k$) and the move of the spoiler in the $(i{+}1)$-st round, i.e., an element a_{i+1} of structure $\mathcal{A}$ or an element b_{i+1} of structure $\mathcal{B}$. The strategy then provides a response by the duplicator, i.e., an element b_{i+1} of structure $\mathcal{B}$, if the spoiler has chosen an element of structure $\mathcal{A}$, or an element a_{i+1} of structure $\mathcal{A}$ if the spoiler has chosen an element of structure $\mathcal{B}$. A *k-round strategy* can be formalized as a function

$$\mathcal{S} : \bigcup_{0 \leqslant i < k} (\, A^{i+1} \times B^i \,\cup\, A^i \times B^{i+1}\,) \to A \cup B,$$

where $A^0 = B^0 = \{\varepsilon\}$ consists of the empty tuple, such that for all $0 \leqslant i < k$ and elements $a_1,\ldots,a_i,a_{i+1} \in A$ and $b_1,\ldots,b_i,b_{i+1} \in B$:

$$\begin{aligned} \mathcal{S}(\,(a_1,\ldots,a_i),\, (b_1,\ldots,b_i,b_{i+1})\,) \;&\in A \\ \mathcal{S}(\,(a_1,\ldots,a_i,a_{i+1}),\, (b_1,\ldots,b_i)\,) \;&\in B \end{aligned}$$

Thus, if the spoiler chooses $\mathcal{A}$ and the element $a_{i+1} \in A$ in the i-th round then strategy $\mathcal{S}$ tells the duplicator to pick the element $b_{i+1} = \mathcal{S}(\,(a_1,\ldots,a_i,a_{i+1}),\,(b_1,\ldots,b_i)\,) \in B$.

A $\mathcal{S}$-*outcome* is any pair $\langle(a_1,\ldots,a_k),(b_1,\ldots,b_k)\rangle \in A^k \times B^k$ such that for each index i with $0 \leqslant i < k$:

$$a_{i+1} \;=\; \mathcal{S}(\,(a_1,\ldots,a_i),\,(b_1,\ldots,b_i,b_{i+1})\,) \;\text{ or }\; b_{i+1} \;=\; \mathcal{S}(\,(a_1,\ldots,a_i,a_{i+1}),\,(b_1,\ldots,b_i)\,)$$

That is, the $\mathcal{S}$-outcomes are exactly the outcomes that can be obtained when the duplicator's decisions rely on strategy $\mathcal{S}$, while the spoiler can behave in an arbitrary way. Strategy $\mathcal{S}$ is called a k-*round winning strategy* (for the duplicator) iff each $\mathcal{S}$-outcome is winning, i.e., iff for each $\mathcal{S}$-outcome $\langle\overline{a},\overline{b}\rangle \in A^k \times B^k$ the pair $\langle\,(\overline{a},\overline{c}^{\mathcal{A}}),\,(\overline{b},\overline{c}^{\mathcal{B}})\,\rangle$ defines a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Definition 1.6.14 (k-round game equivalence $\cong_k$). Let $k \in \mathbb{N}$ and $\mathcal{A}$ and $\mathcal{B}$ structures for *Voc*. Then, k-round-game-equivalence of $\mathcal{A}$ and $\mathcal{B}$ is defined by:

$$\mathcal{A} \cong_k \mathcal{B} \quad \overset{\text{def}}{\Longleftrightarrow} \quad \begin{cases} \text{the duplicator has a } k\text{-round winning strategy} \\ \text{in the Ehrenfeucht-Fraïssé game for } \mathcal{A} \text{ and } \mathcal{B} \end{cases}$$

For the case $k = 0$ this means: $\mathcal{A} \cong_0 \mathcal{B}$ iff $\langle\overline{c}^{\mathcal{A}},\overline{c}^{\mathcal{B}}\rangle$ defines a partial isomorphism. ■

Above we observed that if an outcome $\langle(a_1,\ldots,a_k),(b_1,\ldots,b_k)$ after the k-th round is winning then the outcome $\langle(a_1,\ldots,a_i),(b_1,\ldots,b_i)$ of the i-th round is winning too. This yields the monotonicity of the game-equivalence relations in the number of rounds:

Lemma 1.6.15. *If $\mathcal{A} \cong_k \mathcal{B}$ and $k \geqslant 1$ then $\mathcal{A} \cong_i \mathcal{B}$ for all $i \in \mathbb{N}$, $i < k$.*

Proof. obvious, as each k-round winning strategy is also an i-round winning strategy, provided that $i < k$. □

Example 1.6.16 (Games on sets). For an example, we regard the empty vocabulary. Thus, structures are just nonempty sets. Assuming that A and B are sets with $|A|$, $|B| \geqslant k$ we have $A \cong_k B$. Let us see how a winning strategy for the duplicator works.

Let $\langle(a_1,\ldots,a_i),(b_1,\ldots,b_i)\rangle$ be the current game configuration (i.e., outcome) after the i-th round, where $0 \leqslant i < k$. W.l.o.g., the spoiler chooses structure $\mathcal{A}$ in the first step.

- If the spoiler picks an element $a_{i+1} \in A \setminus \{a_1,\ldots,a_i\}$ then the duplicator responds with an element $b_{i+1} \in B \setminus \{b_1,\ldots,b_i\}$.
- If the spoiler picks an element a_{i+1} in $\{a_1,\ldots,a_i\}$, say $a_{i+1} = a_j$, then the duplicator chooses the element $b_{i+1} = b_j$.

It is now easy to see that the outcome $\langle(a_1,\ldots,a_m),(b_1,\ldots,b_m)\rangle$ of each round $m \leqslant k$ defines a partial isomorphism, as $\{a_1,\ldots,a_m\}$ and $\{b_1,\ldots,b_m\}$ viewed as multisets are isomorphic. Thus, the duplicator wins after k rounds. ■

Example 1.6.17 (Games on graphs). We consider the two graphs shown in Figure 14 viewed as structures over *Voc*$_{\text{graph}}$. We have $\mathcal{A} \cong_0 \mathcal{B}$ as there are no constant symbols in *Voc*$_{\text{graph}}$, but

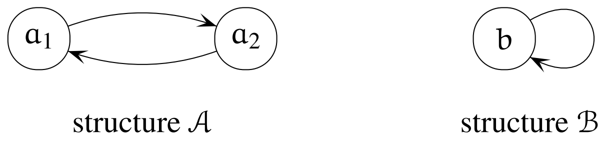

Figure 14: Game on graphs: $\mathcal{A} \cong_0 \mathcal{B}$, but $\mathcal{A} \not\cong_1 \mathcal{B}$

$\mathcal{A} \not\cong_1 \mathcal{B}$. The latter follows by the fact that none of the possible 1-round outcomes $\langle a_1, b\rangle$ or $\langle a_2, b\rangle$ defines a partial isomorphism, since (b, b) is an edge in $\mathcal{B}$, while (a, a) is not an edge in $\mathcal{A}$ for both nodes $a = a_1$ and $a = a_2$ in $\mathcal{A}$.

Let us now regard the graphs shown in Figure 15 on page 95. Graph $\mathcal{A}$ consisting of a simple cycle of length 3. Graph $\mathcal{B}$ has the infinite node-set $\mathbb{Z}$ and an edge from each element n to its successor $n+1$. Then, $\mathcal{A} \cong_1 \mathcal{B}$ since no node in $\mathcal{A}$ or $\mathcal{B}$ has a self-loop and there are no constant symbols. Hence, all pairs $\langle a, b\rangle$ consisting of a node a in $\mathcal{A}$ and a node b in $\mathcal{B}$ define a partial isomorphism.

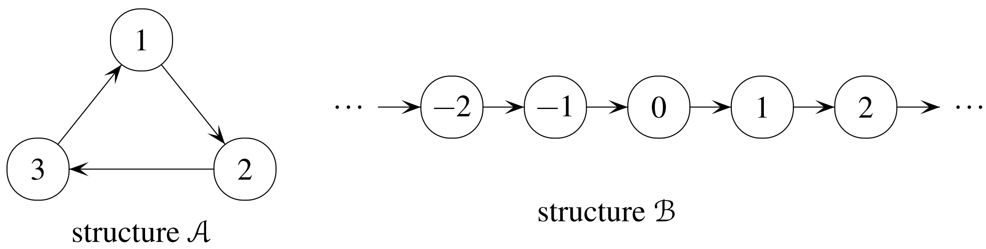

Figure 15: Game on graphs: $\mathcal{A} \cong_1 \mathcal{B}$, but $\mathcal{A} \not\cong_2 \mathcal{B}$

Moreover, we have $\mathcal{A} \not\cong_2 \mathcal{B}$. To see why, we observe that the spoiler wins each play where she selects structure $\mathcal{B}$ and the element $b_1 = 0$ in the first round and the element $b_2 = 2$ (or any other element different from 0, -1 and 1) in the second round. The reason is that the duplicator can choose an arbitrary node $a_1 \in \{1,2,3\}$ in the first round, but has to pick either a_1 again or a predecessor or successor of a_1 in the second round. But since $b_1 = 0$ and $b_2 = 2$ are not connected via an edge in $\mathcal{B}$ and $b_1 \neq b_2$, the duplicator has no chance to make a move such that the resulting outcome of the second round defines a partial isomorphism. ■

Notation 1.6.18 (Equivalence $\cong_k$ for pointed structures). Let $\mathcal{A}$ and $\mathcal{B}$ be structures with domains A and B, respectively. If $a \in A$ and $b \in B$ then the notation

$$(\mathcal{A}, a) \cong_k (\mathcal{B}, b)$$

will be used to denote that the duplicator has a k-round winning strategy when a and b are treated as the meaning of a new constant symbol. In particular, for $k \geqslant 1$, this requires the existence of a k-round strategy which responds with b (resp. a) whenever the spoiler chooses a (resp. b). More precisely: $(\mathcal{A}, a) \cong_k (\mathcal{B}, b)$ iff there exists a k-round strategy $\mathcal{S}$ for the duplicator such that for each $\mathcal{S}$-outcome $\langle \overline{a}, \overline{b}\rangle \in A^k \times B^k$ the pair

$$\langle\, (\overline{a}, \overline{c}^{\mathcal{A}}, a), (\overline{b}, \overline{c}^{\mathcal{B}}, b)\,\rangle$$

defines a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$. Each such strategy $\mathcal{S}$ will be called a winning strategy for $(\mathcal{A}, a) \cong_k (\mathcal{B}, b)$. ■

Lemma 1.6.19 (Recursive characterization of $\cong_k$). *Let $\mathcal{A}$, $\mathcal{B}$ structures for the same relational vocabulary Voc. Then, we have:*

(a) $\mathcal{A} \cong_0 \mathcal{B}$ *iff* $\mathcal{A}$ *and* $\mathcal{B}$ *fulfill the same atomic sentences.*

(b) *For* $k \geqslant 1$*:* $\mathcal{A} \cong_k \mathcal{B}$ *iff*

(1) *for each* $a \in A$ *there exists* $b \in B$ *such that* $(\mathcal{A}, a) \cong_{k-1} (\mathcal{B}, b)$,

(2) *for each* $b \in B$ *there exists* $a \in A$ *such that* $(\mathcal{A}, a) \cong_{k-1} (\mathcal{B}, b)$.

Proof. Recall that we assume that $c_1, \ldots, c_\ell$ are the constant symbols of *Voc* and that we write, e.g., $\overline{c}^{\mathcal{A}}$ for the tuple $(c_1^{\mathcal{A}}, \ldots, c_\ell^{\mathcal{A}})$.

ad (a). The outcome after a 0-round play is the tuple $\langle \varepsilon, \varepsilon \rangle$ where ε denotes the empty sequence. Thus:

$\mathcal{A} \cong_0 \mathcal{B}$

iff the 0-round outcome $\langle \varepsilon, \varepsilon \rangle$ is winning

iff $\langle \overline{c}^{\mathcal{A}}, \overline{c}^{\mathcal{B}} \rangle$ defines a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$

iff the following two conditions hold:

- for all $1 \leqslant i < j \leqslant \ell$: $c_i^{\mathcal{A}} = c_j^{\mathcal{A}}$ iff $c_i^{\mathcal{B}} = c_j^{\mathcal{B}}$
- for all n-ary predicate symbols P and $i_1, \ldots, i_n \in \{1, \ldots, \ell\}$:
$$(c_{i_1}^{\mathcal{A}}, \ldots, c_{i_n}^{\mathcal{A}}) \in P^{\mathcal{A}} \text{ iff } (c_{i_1}^{\mathcal{B}}, \ldots, c_{i_n}^{\mathcal{B}}) \in P^{\mathcal{B}}$$

iff $\mathcal{A}$ and $\mathcal{B}$ fulfill the same atomic sentences

Note that the atomic sentences over *Voc* are the formulas of the form "$c_{i_1} = c_{i_2}$" or $P(c_{i_1}, \ldots, c_{i_n})$ where P is an n-ary predicate symbol and $i_1, \ldots, i_n \in \{1, \ldots, \ell\}$.

ad (b). Let now $k \geqslant 1$. Let us first assume that $\mathcal{A} \cong_k \mathcal{B}$. We now show statement (1). Let $a \in A$ and let $\mathcal{S}$ be a k-round winning strategy for the duplicator. Strategy $\mathcal{S}$ tells the duplicator to answer by an element
$$b = \mathcal{S}(\langle a, \varepsilon \rangle) \in B$$
in structure $\mathcal{B}$, provided that the spoiler has chosen structure $\mathcal{A}$ and the element $a \in A$ in the first round. Using strategy $\mathcal{S}$ also for the next $k-1$ rounds, the duplicator will win from the game position $\langle a, b \rangle$ within $k-1$ rounds. But this yields a $(k-1)$-round winning strategy for $(\mathcal{A}, a) \cong_{k-1} (\mathcal{B}, b)$. This shows statement (1). Statement (2) follows by a symmetric argument.

Vice versa, (1) and (2) induce a strategy for the duplicator to react on the spoiler's first move such that she will win the play after $k-1$ further rounds when following the winning strategies for $(\mathcal{A}, a) \cong_{k-1} (\mathcal{B}, b)$ from the second round on. Hence, there is a winning strategy for k rounds, which yields $\mathcal{A} \cong_k \mathcal{B}$.

For the sake of completeness let us formalize the last argument and provide the precise definition of a winning strategy $\mathcal{S}$ for $\mathcal{A} \cong_k \mathcal{B}$. By assumptions (1) and (2) and the axiom of choice, there exist functions $f : A \to B$ and $g : B \to A$ such that
$$(\mathcal{A}, a) \cong_{k-1} (\mathcal{B}, f(a)) \quad \text{and} \quad (\mathcal{A}, g(b)) \cong_{k-1} (\mathcal{B}, b)$$

for all $a \in A$ and $b \in B$. For each pair (a,b) with $(\mathcal{A},a) \cong_{k-1} (\mathcal{B},b)$ we pick a winning strategy $\mathcal{S}_{a,b}$ for $(\mathcal{A},a) \cong_{k-1} (\mathcal{B},b)$. The definition of $\mathcal{S}$ is then as follows:

$$\begin{aligned}
\mathcal{S}(a,\varepsilon) &\stackrel{\text{def}}{=} f(a) \\
\mathcal{S}(\varepsilon,b) &\stackrel{\text{def}}{=} g(b) \\
\mathcal{S}(\,(a,a_1,\ldots,a_i),\ (b,b_1,\ldots,b_i,b_{i+1})\,) &\stackrel{\text{def}}{=} \mathcal{S}_{a,b}(\,(a_1,\ldots,a_i),\ (b_1,\ldots,b_i,b_{i+1})\,) \\
\mathcal{S}(\,(a,a_1,\ldots,a_i,a_{i+1}),\ (b,b_1,\ldots,b_i)\,) &\stackrel{\text{def}}{=} \mathcal{S}_{a,b}(\,(a_1,\ldots,a_i,a_{i+1}),\ (b_1,\ldots,b_i)\,)
\end{aligned}$$

for all $a, a_1, \ldots, a_{i+1} \in A$, $b, b_1, \ldots, b_{i+1} \in B$ and $0 \leqslant i < k-1$. Since we are only interested in the $\mathcal{S}$-outcomes, the definition of $\mathcal{S}_{a,b}$ is irrelevant if $(\mathcal{A},a) \not\cong_{k-1} (\mathcal{B},b)$. Each $\mathcal{S}$-outcome has the form

$$\langle (a,a_1,\ldots,a_{k-1}),(b,b_1,\ldots,b_{k-1})\rangle$$

where $(\mathcal{A},a) \cong_{k-1} (\mathcal{B},b)$ and $\langle (a_1,\ldots,a_{k-1}),(b_1,\ldots,b_{k-1})\rangle$ is a $\mathcal{S}_{a,b}$-outcome. As $\mathcal{S}_{a,b}$ is a $(k-1)$-round winning strategy for $(\mathcal{A},a) \cong_{k-1} (\mathcal{B},b)$, the pair

$$\langle\,(a_1,\ldots,a_{k-1},\overline{c}^{\mathcal{A}},a),\,(b_1,\ldots,b_{k-1},\overline{c}^{\mathcal{B}},b)\,\rangle$$

defines a partial isomorphism. But then the pair

$$\langle (a,a_1,\ldots,a_{k-1},\overline{c}^{\mathcal{A}}),(b,b_1,\ldots,b_{k-1},\overline{c}^{\mathcal{B}})\rangle$$

induced by the given $\mathcal{S}$-outcome defines a partial isomorphism too. Hence, the duplicator wins all $\mathcal{S}$-plays. □

The goal is now to show that the game-equivalence $\cong_k$ of structures $\mathcal{A}$ and $\mathcal{B}$ implies that $\mathcal{A}$ and $\mathcal{B}$ cannot be distinguished by FOL-sentences where the nesting of quantifiers is bounded above by k. Even the converse holds, i.e., structures that agree on all such FOL-sentences are identified by $\cong_k$.

Notation 1.6.20 (Quantifier rank, FOL[k]-formulas). The *quantifier rank* of a FOL-formula (over an arbitrary vocabulary) is defined as the maximal depth of quantifier nesting. Formally,

$$\begin{aligned}
qr(\phi) &\stackrel{\text{def}}{=} 0 \quad \text{if } \phi = \mathit{true} \text{ or } \phi \text{ is an atomic formula} \\
qr(\phi_1 \wedge \phi_2) &\stackrel{\text{def}}{=} \max\{qr(\phi_1), qr(\phi_2)\} \\
qr(\neg\phi) &\stackrel{\text{def}}{=} qr(\phi) \\
qr(\forall x.\phi) &\stackrel{\text{def}}{=} qr(\phi) + 1
\end{aligned}$$

We write FOL[k] for the set of FOL-formulas of quantifier rank at most k where the vocabulary is assumed to be fixed. If $x_1, \ldots, x_n$ are pairwise distinct variables then

$$\mathrm{FOL}[k](x_1,\ldots,x_n)$$

denotes the set of all formulas ϕ over the given vocabulary with $qr(\phi) \leqslant k$ and $\mathit{Free}(\phi) \subseteq \{x_1,\ldots,x_n\}$. Occasionally, we write $\mathrm{FOL}[k](\varnothing)$ to denote the set of all FOL[k]-*sentences*, i.e., formulas $\phi \in \mathrm{FOL}[k]$ with $\mathit{Free}(\phi) = \varnothing$. ■

E.g., the quantifier rank of an atomic formula or any boolean combination of atomic formulas is 0. Formulas $\exists x.P(x,y)$ and $\exists x.P(x,y) \wedge \forall y.R(y)$ have quantifier rank 1. An example for a formula with quantifier rank 2 is

$$\exists z.R(z) \wedge \forall x \exists y.P(x,y),$$

while the quantifier rank of $\exists z.\big(R(z) \wedge \forall x \exists y.(P(x,y) \wedge P(y,z)) \big)$ is 3.

Before studying the relation between $\cong_k$ and equivalence of structures for FOL[k]-formulas, we will establish some important properties of FOL[k]. The first observation is that, up to logical equivalence $\equiv$, there are only finitely many FOL[k]-sentences. Stated differently, for each $k \in \mathbb{N}$ there exist FOL[k]-sentences $\phi_1, \ldots, \phi_r$ such that for each FOL[k]-sentence ψ there is some index $i \in \{1, \ldots, r\}$ with $\psi \equiv \phi_i$. This, of course, requires a finite relational vocabulary.

Lemma 1.6.21 (Finiteness of FOL[k]($x_1, \ldots, x_n$) up to equivalence). *Given a finite relational vocabulary and n variables $x_1, \ldots, x_n$, then, for each $k \in \mathbb{N}$, there are only finitely many equivalence classes of formulas $\phi \in$ FOL[k]$(x_1, \ldots, x_n)$.*

Proof. As the vocabulary is supposed to be relational, i.e., there are no function symbols of arity $\geqslant 1$. Hence, all terms are either constant symbols or variables. Furthermore, the set of constant symbols is finite, say *Const* $= \{c_1, \ldots, c_\ell\}$. The argument is now by induction on k:

Basis of induction. FOL[0]$(x_1, \ldots, x_n)$ consists of all boolean combinations of *true* and the atomic formulas $t_1 = t_2$ and $P(t_1, \ldots, t_m)$ where the t_i's are either constant symbols or in $\{x_1, \ldots, x_n\}$. Thus, the t_i's belong to the finite set $\{c_1, \ldots, c_\ell, x_1, \ldots, x_n\}$. However, up to equivalence $\equiv$, there are only finitely many such boolean combinations.

Step of induction . Let us now assume that $k \geqslant 1$. Then, FOL[k]$(x_1, \ldots, x_n)$ is the set of all formulas that can be written as boolean combinations of formulas $\psi \in$ FOL[k$-$1]$(x_1, \ldots, x_n)$ and formulas $\forall y.\psi$ where $\psi \in$ FOL[k$-$1]$(x_1, \ldots, x_n, y)$. Note that, due to the possibility of bounded renaming, one variable $y = y_k$ is sufficient to cover all equivalence classes. By induction hypothesis, there are only finitely many equivalence classes of formulas in FOL[k$-$1]$(x_1, \ldots, x_n)$ and in FOL[k$-$1]$(x_1, \ldots, x_n, y)$. This yields the claim. □

Corollary 1.6.22 (Only finitely many FOL[k]-sentences up to equivalence). *For each $k \in \mathbb{N}$ and finite relational vocabulary, the set of* FOL[k]*-sentences is finite up to equivalence $\equiv$.*

Notation 1.6.23 (Rank-k-types). As before, let *Voc* be a finite relational vocabulary. If $\mathcal{A}$ is a structure for *Voc* then

$$\begin{aligned} \text{FOL[k]}(\mathcal{A}) &\overset{\text{def}}{=} \big\{ \phi \in \text{FOL[k]} \,:\, \mathit{Free}(\phi) = \varnothing \text{ and } \mathcal{A} \models \phi \big\} \\ &= \big\{ \phi \,:\, \phi \text{ is a FOL[k]-sentence with } \mathcal{A} \models \phi \big\} \end{aligned}$$

is called the *rank-*k*-type* over $\mathcal{A}$. A formula-set $\mathfrak{R}$ is said to be a rank-k type if $\mathfrak{R} =$ FOL[k]$(\mathcal{A})$ for some structure $\mathcal{A}$ for *Voc*. ■

The rank-k-type of a structure $\mathcal{A}$ can be viewed as the FO-theory of $\mathcal{A}$ restricted to the FOL[k]-sentences. The following is immediate from the definition of rank-k-types:

Lemma 1.6.24. FOL[k]$(\mathcal{A})$ = FOL[k]$(\mathcal{B})$ *iff* $\mathcal{A}$ *and* $\mathcal{B}$ *agree on all* FOL[k]*-sentences.*

Rank-k-types are sets of FOL[k]-sentences that are closed under equivalence, i.e., if $\mathfrak{R}$ is a rank-k-type and $\phi \in \mathfrak{R}$ then all FOL[k]-sentences ψ with $\phi \equiv \psi$ are contained in $\mathfrak{R}$. Note that if $\phi \in \mathfrak{R} = \text{FOL}[k](\mathcal{A})$ then $\mathcal{A} \models \phi$, and therefore $\mathcal{A} \models \psi$ for all formulas ψ such that $\phi \equiv \psi$. Hence, by Corollary 1.6.22:

Corollary 1.6.25 (Finiteness of the set of rank-k-types). *For finite relational vocabulary Voc, the set*

$$\{ \text{FOL}[k](\mathcal{A}) \,:\, \mathcal{A} \text{ is a structure for } Voc \}$$

consisting of all rank-k-types over Voc is finite.

Even a stronger result can be established that asserts that each rank-k-type $\mathfrak{R}$ is *defined* by some FOL[k]-formula θ in the sense that $\mathfrak{R}$ agrees with the set of FOL[k]-sentences that are consequences of θ. More precisely:

Lemma 1.6.26 (FOL[k]-definability of the rank-k-types). *Let Voc be a finite relational vocabulary and let $\mathfrak{R}_1, \ldots, \mathfrak{R}_s$ be the rank-k-types of Voc. Then, there are FOL[k]-sentences $\theta_1, \ldots, \theta_s$ such that for all $i \in \{1, \ldots, s\}$:*

- $\mathfrak{R}_i = \{ \psi : \psi$ *is a* FOL[k]*-sentence with* $\theta_i \Vdash \psi \}$
- *for each structure $\mathcal{A}$ we have:* $\mathcal{A} \models \theta_i$ *iff* $\text{FOL}[k](\mathcal{A}) = \mathfrak{R}_i$.

Furthermore, each FOL[k]*-sentence is equivalent to the disjunction of some of the* θ_i*'s.*

Proof. By Lemma 1.6.21, there are finitely many FOL[k]-sentences $\phi_1, \ldots, \phi_r$ such that each sentence $\phi \in \text{FOL}[k]$ is equivalent to some of the ϕ_i's. W.l.o.g., the ϕ_i's are pairwise not equivalent. Each subset I of $\{1, \ldots, r\}$ specifies a rank-k-type $\mathfrak{R}_I$. More precisely, for $I \subseteq \{1, \ldots, r\}$, let θ_I be the following FOL[k]-sentence:

$$\theta_I \stackrel{\text{def}}{=} \bigwedge_{i \in I} \phi_i \wedge \bigwedge_{\substack{1 \leqslant i \leqslant r \\ i \notin I}} \neg \phi_i$$

Furthermore, we define:

$$\mathfrak{R}_I \stackrel{\text{def}}{=} \{ \psi : \psi \text{ is a FOL[k]-sentence with } \theta_I \Vdash \psi \}$$

We now show that:

(1) $\text{FOL}[k](\mathcal{A}) = \mathfrak{R}_I$ for each model $\mathcal{A}$ for θ_I

(2) for each structure $\mathcal{A}$ there exists $I \subseteq \{1, \ldots, r\}$ such that $\mathcal{A} \models \theta_I$ and $\text{FOL}[k](\mathcal{A}) = \mathfrak{R}_I$.

Note that by (1) and (2) we get that $\mathfrak{R}_I$ is a rank-k-type, provided that θ_I is satisfiable, and vice versa, each rank-k-type agrees with one of the formula-sets $\mathfrak{R}_I$. Thus, there are at most 2^r rank-k types, and each of them is defined by a FOL[k]-sentence as stated in the lemma.

ad (1). Let $\mathcal{A}$ be a structure such that $\mathcal{A} \models \theta_I$. We have to show that $\text{FOL}[k](\mathcal{A}) = \mathfrak{R}_I$.

"$\subseteq$": Let $\psi \in \mathrm{FOL}[k](\mathcal{A})$. Then, ψ is a FOL[k]-sentence such that $\mathcal{A} \models \psi$. There is some index $i \in \{1,\dots,r\}$ such that $\psi \equiv \phi_i$. Hence, $\mathcal{A} \models \phi_i$. As $\mathcal{A} \models \theta_I$ and θ_I is a conjunction of the formulas ϕ_j for $j \in I$ and the formulas $\neg\phi_j$ for $j \notin I$, we get $i \in I$ and

$$\theta_I = \dots \wedge \phi_i \wedge \dots \Vdash \phi_i \equiv \psi,$$

and therefore $\theta_I \Vdash \psi$. But then $\psi \in \mathfrak{R}_I$.

"$\supseteq$": Let $\psi \in \mathfrak{R}_I$. Then, ψ is a FOL[k]-sentence such that $\theta_I \Vdash \psi$. As $\mathcal{A}$ is a model for θ_I we have $\mathcal{A} \models \psi$. But then $\psi \in \mathrm{FOL}[k](\mathcal{A})$.

ad (2). Let $\mathcal{A}$ be a structure and $I \stackrel{\text{def}}{=} \{ i \in \{1,\dots,r\} : \mathcal{A} \models \phi_i \}$. Obviously, we then have $\mathcal{A} \models \theta_I$, and therefore $\mathrm{FOL}[k](\mathcal{A}) = \mathfrak{R}_I$ (by (1)).

It remains to explain the last statement of the lemma asserting that each FOL[k]-sentence is equivalent to a disjunction of some of the θ_I's. Let ψ be a FOL[k]-sentence. Then, $\psi \equiv \phi_i$ for some $i \in \{1,\dots,r\}$. Let $\mathfrak{J}$ be the set of all index-sets $I \subseteq \{1,\dots,r\}$ where θ_I is satisfiable and $i \in I$. Then, ψ is equivalent to $\bigvee\limits_{I \in \mathfrak{J}} \theta_I$. $\square$