1.5 Compactness property and Löwenheim-Skolem Theorems

The fact that FOL has sound and complete deductive calculi has several important consequences. We start with the *compactness property*.

Let us first state the compactness property for deductive systems. Let $\mathfrak{D}$ be a Hilbert proof system. Then, for each formula-set $\mathfrak{F}$ and formula ϕ we have:

$$\mathfrak{F} \vdash_{\mathfrak{D}} \phi \quad \text{iff} \quad \begin{cases} \text{there exists a finite subset } \mathfrak{F}_0 \\ \text{of } \mathfrak{F} \text{ such that } \mathfrak{F}_0 \vdash_{\mathfrak{D}} \phi \end{cases}$$

This statement is a simple consequence of the fact that $\mathfrak{D}$-proofs from $\mathfrak{F}$ are *finite* formula-sequences. Hence, only finitely many formulas of $\mathfrak{F}$ can appear in a given $\mathfrak{D}$-proof $\psi_1, \psi_2, \ldots, \psi_m$ from $\mathfrak{F}$, and we get $\mathfrak{F}_0 \vdash_{\mathfrak{D}} \phi$ where $\mathfrak{F}_0 = \mathfrak{F} \cap \{\psi_1, \ldots, \psi_m\}$. If $\mathfrak{D}$ is sound and complete then the above yields:

$$\mathfrak{F} \Vdash \phi \quad \text{iff} \quad \begin{cases} \text{there exists a finite subset } \mathfrak{F}_0 \\ \text{of } \mathfrak{F} \text{ such that } \mathfrak{F}_0 \Vdash \phi \end{cases}$$

This is the compactness property for the consequence relation, which holds for any logic that has sound and complete proof systems. In particular, it holds for FOL. As a consequence we obtain that the notions "satisfiability" and "finitary satisfiability" agree for sets of FOL-formulas. The latter refers to the property that each finite subset has a model. More precisely, a formula-set $\mathfrak{F}$ is said to be *finitely satisfiable* iff each finite subset of $\mathfrak{F}$ is satisfiable.

Theorem 1.5.1 (Compactness Theorem). *Let $\mathfrak{F}$ be a set of FOL-formulas. Then, $\mathfrak{F}$ is satisfiable iff every finite subset of $\mathfrak{F}$ is satisfiable.*

Proof. The implication "$\Longrightarrow$" is obvious, as any model for $\mathfrak{F}$ is also a model for all its subsets. Let us now turn to the proof of the implication "$\Longleftarrow$". Assume by contradiction that $\mathfrak{F}$ is not satisfiable, while any finite subset of $\mathfrak{F}$ has a model. As $\mathfrak{F}$ is supposed to be unsatisfiable, we have $\mathfrak{F} \Vdash$ *false*. The compactness property for the consequence relation (see above) yields the existence of some finite subset $\mathfrak{F}_0$ of $\mathfrak{F}$ with $\mathfrak{F}_0 \Vdash$ *false*. Hence, $\mathfrak{F}_0$ is not satisfiable. This contradicts the assumption. $\square$

The above proof of Theorem 1.5.1 uses the compactness property for the consequence relation to show the satisfiability of finitely satisfiable formula-sets. Vice versa, the compactness property for the consequence relation can be derived from Theorem 1.5.1 as follows:

$$\begin{array}{ll} & \mathfrak{F} \Vdash \phi \\ \text{iff} & \mathfrak{F} \cup \{\neg\phi\} \text{ is not satisfiable} \\ \text{iff} & \mathfrak{F} \cup \{\neg\phi\} \text{ is not finitely satisfiable} \\ \text{iff} & \text{there exists a finite subset } \mathfrak{F}_0 \text{ of } \mathfrak{F} \text{ such that } \mathfrak{F}_0 \cup \{\neg\phi\} \text{ is not satisfiable} \\ \text{iff} & \text{there exists a finite subset } \mathfrak{F}_0 \text{ of } \mathfrak{F} \text{ such that } \mathfrak{F}_0 \Vdash \phi \end{array}$$

Thus, the statement of Theorem 1.5.1 is equivalent to the compactness property for the consequence relation.

Another characteristic feature of FOL are the Löwenheim-Skolem Theorems that state the existence of models of a certain cardinality for satisfiable sets of formulas. Recall that the given vocabulary and set of variables are supposed to be recursively enumerable. This ensures that the set of all FOL-formulas and any subset thereof is countable.

Theorem 1.5.2 (Upward Löwenheim-Skolem Theorem "from finite to infinite models"). *Let $\mathfrak{F}$ be a set of formulas such that for each $n \in \mathbb{N}$ there exists a finite model $(\mathcal{A}_n, \mathcal{V}_n)$ for $\mathfrak{F}$ with at least n elements. Then, $\mathfrak{F}$ has an infinite model.*

Proof. For $n \geqslant 2$, let ψ_n be the following FOL-sentence over the empty vocabulary that holds for exactly those structures $\mathcal{A}$ of size $\geqslant n$:

$$\psi_n \stackrel{\text{def}}{=} \exists x_1 \ldots \exists x_n. \bigwedge_{1 \leqslant i < j \leqslant n} (x_i \neq x_j)$$

We now consider the formula-set:

$$\mathfrak{G} \stackrel{\text{def}}{=} \mathfrak{F} \cup \{ \psi_n : n \geqslant 2 \}$$

and show that $\mathfrak{G}$ is satisfiable. For this, we use the compactness theorem and show that $\mathfrak{G}$ is finitely satisfiable. Let $\mathfrak{G}_0$ be a finite subset of $\mathfrak{G}$. Then, there exists some $n \geqslant 2$ such that $\mathfrak{G}_0$ is a subset $\mathfrak{F} \cup \{\psi_2, \ldots, \psi_n\}$. But then $(\mathcal{A}_n, \mathcal{V}_n) \models \mathfrak{G}_0$.

Let $(\mathcal{A}, \mathcal{V})$ be a model for $\mathfrak{G}$ and A its domain. Since $\mathfrak{F} \subseteq \mathfrak{G}$, $(\mathcal{A}, \mathcal{V})$ is a model for $\mathfrak{F}$ too. Moreover, $\mathcal{A}$ is infinite as $\mathcal{A} \models \psi_n$ for each $n \geqslant 2$. □

In the above theorem it is important that we require the existence of finite models of arbitrary size. From the existence of a finite model we cannot derive the existence of an infinite model. For example, the FOL-sentence

$$\phi = \forall x \forall y. (x = y)$$

is satisfiable, and any structure with a singleton domain is a model for ϕ. But there is no infinite model for ϕ, even no model with two or more elements. However, for FOL without equality, satisfiable formula-sets have arbitrarily large models. More precisely, we show in the next theorem that for FOL without equality the assumption that there are finite models of arbitrary size is irrelevant and infinite models can be guaranteed for all satisfiable sets of FOL-formulas without equality.

Theorem 1.5.3 (Upward Löwenheim-Skolem Theorem for FOL without equality). *Let $\mathfrak{F}$ be a satisfiable set of FOL-formulas without equality and let C be a set. Then, $\mathfrak{F}$ is satisfiable over some structure where the domain is a superset of C.*

Proof. Let $(\mathcal{B}, \mathcal{W})$ be a model for $\mathfrak{F}$ and let B be the domain of $\mathcal{B}$. W.l.o.g., $B \cap C = \varnothing$. We define a structure $\mathcal{A}$ with domain

$$A \stackrel{\text{def}}{=} B \cup C$$

as follows. To provide the interpretations of the predicate and function symbols, we fix an element $b_0 \in B$ and define a surjective function $h : A \to B$ by: $h(b) = b$ for $b \in B$ and $h(c) = b_0$ if $c \in C$. For each n-ary predicate symbol P we put

$$P^{\mathcal{A}} \stackrel{\text{def}}{=} \{ (a_1, \ldots, a_n) \in B^n : (h(a_1), \ldots, h(a_n)) \in P^{\mathcal{B}} \}.$$

The function $f^{\mathcal{A}} : A^m \to A$ for each m-ary function symbol f is defined by:

$$f^{\mathcal{A}}(a_1,\ldots,a_m) \stackrel{\text{def}}{=} f^{\mathcal{B}}\big(h(a_1),\ldots,h(a_m)\big)$$

Then, $f^{\mathcal{A}}(\overline{a}) \in B$ and therefore:

$$h(f^{\mathcal{A}}(a_1,\ldots,a_m)) = f^{\mathcal{A}}(a_1,\ldots,a_m) = f^{\mathcal{B}}\big(h(a_1),\ldots,h(a_m)\big)$$

Furthermore, we define a variable valuation $\mathcal{V} : Var \to A$ by $\mathcal{V}(x) = \mathcal{W}(x)$ for all variables $x \in Var$. (Note that $\mathcal{W}(x) \in B \subseteq A$.) Hence, $h : A \to B$ is a surjective homomorphism from $\mathcal{A}$ to $\mathcal{B}$ and the given variable valuation $\mathcal{W} : Var \to B$ has the form $\mathcal{W} = h \circ \mathcal{V}$. Since we deal here with FOL without equality, we have:

$$(\mathcal{A},\mathcal{V}) \models \phi \text{ iff } (\mathcal{B},\mathcal{W}) \models \phi$$

for all formulas ϕ (see page 11). Since $(\mathcal{B},\mathcal{W})$ is a model for $\mathfrak{F}$, so is $(\mathcal{A},\mathcal{V})$. □

In particular, any satisfiable set of FOL-formulas without equality has infinite models. Note that this does not contradict the lack of the finite model property (Theorem 1.3.2 on page 34), which states that there are satisfiable formulas that do not have any finite model and implies that there are FOL-formulas that hold for all finite structures without being valid. The above theorem just asserts the existence of infinite models for satisfiable formulas.

As the formula $\forall x \forall y.(x = y)$ shows (see above), the statement of Theorem 1.5.3 does not hold for FOL with equality. However, the statement of Theorem 1.5.3 also holds for FOL with equality and satisfiable sets with at least one infinite model.

Theorem 1.5.4 (Upward Löwenheim-Skolem Theorem "from infinite to larger models"). *Let $\mathfrak{F}$ be a formula-set that is satisfiable over some infinite structure and let C be a set. Then, $\mathfrak{F}$ is satisfiable over some structure where the domain is a superset of C.*

Proof. In this proof, we drop the default assumption that vocabularies and variable-sets are recursively enumerable und make use of the fact that FOL over arbitrary vocabularies and variable-sets has sound and complete proof systems. See Remark 1.2.15 on page 29.

Let *Voc* be the underlying vocabulary of formula-set $\mathfrak{F}$. We now regard the vocabulary Voc_C which extends *Voc* by fresh constant symbols $c \in C$, where we assume w.l.o.g. that none of the elements in C appears in *Voc*. Let

$$\mathfrak{G} \stackrel{\text{def}}{=} \mathfrak{F} \cup \big\{\text{"}c \neq d\text{"} : c,d \in C, c \neq d\big\}$$

where "$c \neq d$" is viewed as a formulas over Voc_C.

Claim. $\mathfrak{G}$ is satisfiable.

Proof of the claim. We show that $\mathfrak{G}$ is finitely satisfiable. For this, we take a finite subset $\mathfrak{G}_0$ of $\mathfrak{G}$ and show that $\mathfrak{G}_0$ is satisfiable. There exists pairwise distinct elements $c_1,\ldots,c_n \in C$ such that

$$\mathfrak{G}_0 \subseteq \mathfrak{F} \cup \big\{\text{"}c_i \neq c_j\text{"} : 1 \leqslant i,j \leqslant n, i \neq j\big\}.$$

Note that each finite subset of $\mathfrak{G}$ contains only finitely many of the literals "$c \neq d$".

Let $(\mathcal{B},\mathcal{W})$ be a model for $\mathfrak{F}$ with an infinite domain B. We pick pairwise distinct elements $b_1,\ldots,b_n \in B$. Structure $\mathcal{B}_0$ extends $\mathcal{B}$ by meanings for the constant symbols $c_1,\ldots,c_n$:

$$c_i^{\mathcal{B}_0} \stackrel{\text{def}}{=} b_i \text{ for } 1 \leqslant i \leqslant n.$$

Obviously, $(\mathcal{B}_0, \mathcal{W})$ is a model for $\mathfrak{F}$ (as $(\mathcal{B}, \mathcal{W})$ is a model for $\mathfrak{F}$) and a model for the formulas "$c_i \neq c_j$" for $1 \leqslant i, j \leqslant n$, $i \neq j$. Hence:

$$(\mathcal{B}_0, \mathcal{W}) \models \mathfrak{G}_0$$

The compactness property yields that $\mathfrak{G}$ is satisfiable. This completes the proof of the claim. $\rfloor$

We now take a model $(\mathcal{A}, \mathcal{V})$ for $\mathfrak{G}$. Let A be the domain of $\mathcal{A}$. Since $\mathcal{A}$ is a model for the formulas "$c \neq d$" where c, $d \in C$, $c \neq d$, the elements $c^{\mathcal{A}} \in A$ for $c \in C$ are pairwise distinct. Thus, we may assume that $C \subseteq A$. $\square$

Hence, any set $\mathfrak{F}$ of FOL-formulas that has some infinite model $\mathcal{A}$ then for each cardinal κ larger than the cardinality of (the domain of) $\mathcal{A}$, there is a model $\mathcal{B}$ for $\mathfrak{F}$ of cardinality at least κ. To see this, we take $C = 2^A$ in Theorem 1.5.4 where A is the domain of $\mathcal{A}$ and 2^A denotes the powerset of A. For instance, any formula set that has an infinite countable model also has a model where the domain is a superset of $\mathbb{R}$ or a superset of the powerset of the reals or a superset of the powerset of the powerset of the reals, and so on. We state here without proof that, given an infinite model $\mathcal{A}$ for $\mathfrak{F}$, then for each cardinal κ larger than the cardinality of $\mathcal{A}$ there is a model $\mathcal{B}$ for $\mathfrak{F}$ of cardinality (exactly) κ. This yields, e.g., that any formula-set that has a model where the domain is $\mathbb{N}$ also has a model with domain $\mathbb{R}$ and a model with domain $2^{\mathbb{R}}$, and so on.