1.7.3 Uncountable quantification

We now provide an example for a logic that has the compactness property, but fails to fulfill the condition of the downward Löwenheim-Skolem theorem. For this purpose, we regard the logic

$$\mathrm{FOL}(\overset{2^{\mathbb{N}}}{\exists})$$

which extends FOL by formulas of the form $\overset{2^{\mathbb{N}}}{\exists} x.\phi$, stating the existence of *uncountably many* elements x such that ϕ holds.[4] Structures and interpretations and the satisfaction relation $\models$ for the FOL-fragment are defined as for standard FOL. The semantics of the new quantifier is given by:

$$\begin{array}{rll}(\mathcal{A},\mathcal{V}) \models \overset{2^{\mathbb{N}}}{\exists} x.\phi & \text{iff} & \text{there exist uncountably many } a \in A \text{ with } (\mathcal{A},\mathcal{V}[x := a]) \models \phi \\ & \text{iff} & \text{the set } \big\{a \in A : (\mathcal{A},\mathcal{V}[x := a]) \models \phi\big\} \text{ is uncountable}\end{array}$$

where A is the domain of $\mathcal{A}$. The dual operator is $\overset{2^{\mathbb{N}}}{\forall} x.\phi \overset{\text{def}}{=} \neg \overset{2^{\mathbb{N}}}{\exists} x.\neg\phi$ stating that there are only countably many elements x where ϕ does not hold. That is:

$$(\mathcal{A},\mathcal{V}) \models \overset{2^{\mathbb{N}}}{\forall} x.\,\phi \quad \text{iff} \quad \text{the set } \big\{a \in A : (\mathcal{A},\mathcal{V}[x := a]) \not\models \phi\big\} \text{ is countable}$$

For example:

$$\begin{array}{rcl}(\mathbb{R},\leqslant) & \models & \overset{2^{\mathbb{N}}}{\forall} y \overset{2^{\mathbb{N}}}{\exists} x.\, x < y \\ (\mathbb{R},\leqslant) & \not\models & \overset{2^{\mathbb{N}}}{\exists} x \overset{2^{\mathbb{N}}}{\forall} y.\, x < y\end{array}$$

The formula $\overset{2^{\mathbb{N}}}{\forall} y \overset{2^{\mathbb{N}}}{\exists} x.\, x < y$ holds for the ordered reals, since for each real number a, the set of real numbers b with $b < a$ is uncountable. Thus:

$$\big\{\, a \in \mathbb{R} : (\mathbb{R},\leqslant,[y := a]) \not\models \overset{2^{\mathbb{N}}}{\exists} x.\, x < y \,\big\} \;=\; \varnothing$$

The formula $\overset{2^{\mathbb{N}}}{\exists} x \overset{2^{\mathbb{N}}}{\forall} y.\, x < y$ does not hold for $(\mathbb{R},\leqslant)$ as the set of real numbers a such that $a < b$ for only countably many $b \in \mathbb{R}$ is empty, and hence not uncountable.

Obviously, for each structure $\mathcal{A}$ we have:

$$\begin{array}{lll}\mathcal{A} \models \overset{2^{\mathbb{N}}}{\exists} x.\, \mathit{true} & \text{iff} & \mathcal{A} \text{ is uncountable} \\ \mathcal{A} \models \overset{2^{\mathbb{N}}}{\forall} x.\, \mathit{false} & \text{iff} & \mathcal{A} \text{ is countable}\end{array}$$

Thus, $\mathrm{FOL}(\overset{2^{\mathbb{N}}}{\exists})$ can characterize both the class of countable structures and the class of uncountable structures. Hence, the conditions of the downward Löwenheim-Skolem Theorem (stating the existence of countable models for satisfiable sentences) and the upward Löwenheim-Skolem Theorem "from infinite to larger models" are violated for FOL with uncountable quantification. Note that the sentence

$$\overset{2^{\mathbb{N}}}{\exists} x.\, \mathit{true}$$

[4]The abstract syntax of $\mathrm{FOL}(\overset{2^{\mathbb{N}}}{\exists})$ is as for $\mathrm{FOL}(\overset{\infty}{\exists})$ except that we deal with formulas $\overset{2^{\mathbb{N}}}{\exists} x.\phi$ rather than $\overset{\infty}{\exists} x.\phi$.

has uncountable, but no countable models. Similarly, the upward Löwenheim-Skolem theorem "from infinite to larger models" (cf. Theorem 1.5.4 on page 73) does not hold, as

$$\neg \overset{2^{\mathbb{N}}}{\exists} x.\, \mathit{true} \quad \equiv \quad \overset{2^{\mathbb{N}}}{\forall} x.\, \mathit{false}$$

has countable, but no uncountable models. However, FOL($\overset{2^{\mathbb{N}}}{\exists}$) enjoys the *compactness property* (i.e., finitary satisfiability agrees with satisfiability), since it has sound and complete deductive calculi. E.g., we may extend a sound and complete Hilbert calculus for FOL by the axiom schemata (Q1)-(Q4) in Figure 16, which can shown to be sound and complete for FOL with uncountable quantification.

(Q1)	$\overset{2^{\mathbb{N}}}{\exists} x.\Phi \to \overset{2^{\mathbb{N}}}{\exists} y.\, \Phi[x/y]$ if x can be replaced in Φ with y (see page 8)
(Q2)	$\neg \overset{2^{\mathbb{N}}}{\exists} x.\, (\, x = y \vee x = z\,)$ where y and z are variables such that $x \notin \{y, z\}$
(Q3)	$\forall x.(\Phi \to \Psi) \;\to\; \big(\overset{2^{\mathbb{N}}}{\exists} x.\Phi \;\to \overset{2^{\mathbb{N}}}{\exists} x.\Psi\big)$
(Q4)	$\big(\neg \overset{2^{\mathbb{N}}}{\exists} x \exists y.\, \Phi \wedge \overset{2^{\mathbb{N}}}{\exists} y \exists x.\Phi\,\big) \;\to\; \exists x \overset{2^{\mathbb{N}}}{\exists} y.\, \Phi$

Figure 16: Axiom schemata for $\overset{2^{\mathbb{N}}}{\exists} x$

We skip here a proof for the completeness of these axioms for FOL($\overset{2^{\mathbb{N}}}{\exists}$) (in combination with the axioms and rules of some sound and complete Hilbert proof system for FOL), but explain the soundness. The meaning of (Q1) is simply renaming of bounded variables. The soundness of (Q2) follows by the fact that singleton sets and sets with two elements are countable.

Soundness of (Q3). The third axiom (Q3) relies on the observation that supersets of uncountable sets are uncountable too. Formally, suppose that $\mathcal{I} = (\mathcal{A}, \mathcal{V})$ is a model for

$$\forall x.\, (\phi \to \psi) \quad \text{and} \quad \overset{2^{\mathbb{N}}}{\exists} x.\, \phi$$

with domain A. Since $\mathcal{I} \models \forall x.(\phi \to \psi)$ we have:

$$A_\phi \overset{\text{def}}{=} \big\{a \in A : \mathcal{I}[x := a] \models \phi\big\} \subseteq \big\{a \in A : \mathcal{I}[x := a] \models \psi\big\} \overset{\text{def}}{=} A_\psi$$

As $\mathcal{I} \models \overset{2^{\mathbb{N}}}{\exists} x.\phi$, the set A_ϕ is uncountable. But then A_ψ (as a superset of A_ϕ) is uncountable too. Thus:

$$\mathcal{I} \models \overset{2^{\mathbb{N}}}{\exists} x.\, \psi \qquad \blacksquare$$

Soundness of (Q4). The last axiom (Q4) states that if the union of countably many sets is uncountable then at least one of these sets is uncountable. Let us check this. Suppose $\mathcal{I} = (\mathcal{A}, \mathcal{V})$ is an interpretation such that

$$\mathcal{I} \models \neg \overset{2^{\mathbb{N}}}{\exists} x \exists y.\phi \quad \text{and} \quad \mathcal{I} \models \overset{2^{\mathbb{N}}}{\exists} y \exists x.\phi$$

with domain A. For $a \in A$ let

$$A_a \overset{\text{def}}{=} \{ b \in A : \mathcal{I}[x := a, y := b] \models \phi \}$$

Then, $\mathcal{I}[x := a] \models \exists y.\phi$ if and only if $A_a \neq \varnothing$. As $\mathcal{I} \models \neg \overset{2^{\mathbb{N}}}{\exists} x \exists y.\phi$ the set

$$\begin{aligned} C &\overset{\text{def}}{=} \{ a \in A : A_a \neq \varnothing \} \\ &= \{ a \in A : \mathcal{I}[x := a] \models \exists y.\phi \} \end{aligned}$$

is countable. For each element $b \in A$ we have:

$$\begin{aligned} \mathcal{I}[y := b] \models \exists x.\phi \quad & \text{iff} \quad \text{there exists } a \in A \text{ such that } \mathcal{I}[x := a, y := b] \models \phi \\ & \text{iff} \quad b \in A_a \text{ for some } a \in C \\ & \text{iff} \quad b \in \bigcup_{a \in C} A_a \end{aligned}$$

As $\mathcal{I} \models \overset{2^{\mathbb{N}}}{\exists} y \exists x.\phi$, the set

$$B \overset{\text{def}}{=} \{ b \in A : \mathcal{I}[y := b] \models \exists x.\phi \} = \bigcup_{a \in C} A_a$$

is uncountable. But then there must be an element $a_0 \in C$ such that A_{a_0} is uncountable, as otherwise B would be a countable union of countable sets, and therefore countable. Hence:

$$\mathcal{I}[x := a_0] \models \overset{2^{\mathbb{N}}}{\exists} y.\phi$$

This yields $\mathcal{I} \models \exists x \overset{2^{\mathbb{N}}}{\exists} y.\phi$. ■

Applying the same arguments as in Theorem 1.5.1 on page 71 we obtain:

Corollary 1.7.13 (Compactness property for FOL($\overset{2^{\mathbb{N}}}{\exists}$)). *Each finitely satisfiable set of* FOL($\overset{2^{\mathbb{N}}}{\exists}$)*-formulas is satisfiable.*

As FOL($\overset{2^{\mathbb{N}}}{\exists}$) enjoys the compactness property also the statement of the upward Löwenheim-Skolem theorem "from finite to infinite models" (cf. Theorem 1.5.2 on page 72) holds. In fact, a careful inspection of the proof of Theorem 1.5.2 yields that the same argument applies for FOL extended by uncountable quantification. In summary, concerning the compactness property and Löwenheim-Skolem theorems, FOL with infinite quantification behaves dually to FOL with uncountable quantification.

FOL($\overset{2^{\mathbb{N}}}{\exists}$) is powerful enough to characterize the rational numbers with the natural linear order $\leqslant$ ("less or equal") on the rational numbers. Let us see why. Since also $(\mathbb{R}, \leqslant)$ is a dense, linear order without endpoints, the FOL-formulas in Definition 1.4.13 (page 53) have models that are not isomorphic to $(\mathbb{Q}, \leqslant)$. By the upward Löwenheim-Skolem theorem, the class of structures that are isomorphic to $(\mathbb{Q}, \leqslant)$ is not FO-definable. However, Lemma 1.4.15 on page 54, shows that all *countable* models for these formulas are isomorphic to $(\mathbb{Q}, \leqslant)$. Hence, using the FOL-formulas for linear, dense orders without endpoints in Definition 1.4.13 together with the formula

$$\neg \overset{2^{\mathbb{N}}}{\exists} x.\, \mathit{true}$$

(which characterizes countable structures), we obtain a $\mathrm{FOL}(\overset{2^{\mathbb{N}}}{\exists})$-characterization of the ordered rationals. That is:

Lemma 1.7.14. *The class of structures that are isomorphic to* $(\mathbb{Q}, \leqslant)$ *is* $\mathrm{FOL}(\overset{2^{\mathbb{N}}}{\exists})$*-definable.*

Clearly, the natural numbers viewed as structure over the empty vocabulary have a $\mathrm{FOL}(\overset{2^{\mathbb{N}}}{\exists})$-characterization by means of the formula-set

$$\big\{ \overset{2^{\mathbb{N}}}{\forall} x.\, \mathit{false} \big\} \cup \big\{ \psi_n : n \geqslant 2 \big\}$$

where the ψ_n's are FOL-sentences that define sets with n or more elements. However, the uncountable quantifier does not provide the possibility to characterize the natural numbers with the successor function and constant 0.

Theorem 1.7.15. *There is no set* $\mathfrak{F}$ *of* $\mathrm{FOL}(\overset{2^{\mathbb{N}}}{\exists})$*-sentences over* Voc_{nat} *such that* $(\mathbb{N}, \mathit{succ}, 0) \models \mathfrak{F}$ *and all countable models for* $\mathfrak{F}$ *are isomorphic to* $(\mathbb{N}, \mathit{succ}, 0)$.

Proof. The argument is roughly the same as in the proof of Theorem 1.6.6 on page 86.

Assume there is a satisfiable set $\mathfrak{F}$ of $\mathrm{FOL}(\overset{2^{\mathbb{N}}}{\exists})$-sentences such that $(\mathbb{N}, \mathit{succ}, 0) \models \mathfrak{F}$ and all countable models are isomorphic to $(\mathbb{N}, \mathit{succ}, 0)$. Let x be a variable and let

$$\mathfrak{G} \;\stackrel{\text{def}}{=}\; \mathfrak{F} \cup \big\{ x \neq \mathit{succ}^n(0) : n \geqslant 1 \big\} \cup \big\{ \overset{2^{\mathbb{N}}}{\forall} y.\, \mathit{false} \big\}$$

Every finite subset $\mathfrak{G}'$ of $\mathfrak{G}$ is satisfiable, since there is some formula "$x \neq \mathit{succ}^n(0)$" which is not contained in $\mathfrak{G}'$. Then, $(\mathbb{N}, \mathit{succ}, 0, [x := n]) \models \mathfrak{G}'$. Hence, $\mathfrak{G}$ is finitely satisfiable. By the compactness theorem (Corollary 1.7.13), $\mathfrak{G}$ is satisfiable. Let $(\mathcal{A}, \mathcal{V})$ be a model for $\mathfrak{G}$ with domain A. Then, A must be countable, since

$$\mathcal{A} \models \overset{2^{\mathbb{N}}}{\forall} y.\, \mathit{false},$$

and $\mathcal{A} \models \mathfrak{F}$. However, $\mathcal{A}$ is not isomorphic to $(\mathbb{N}, \mathit{succ}, 0)$ since $a \stackrel{\text{def}}{=} \mathcal{V}(x)$ is an element in A that cannot be obtained by finite applications of $\mathit{succ}^{\mathcal{A}}$ to $0^{\mathcal{A}}$. Therefore, the structure of any model for $\mathfrak{G}$ induces a model for $\mathfrak{F}$ that is not isomorphic to $(\mathbb{N}, \mathit{succ}, 0)$. Contradiction. $\square$

2 Second-order logic

Second-order logic (SOL) extends FOL by quantification over predicates and functions. This yields a very powerful and elegant logic that can express many interesting properties, including various properties where FOL fails. The price we have to pay is the lack of sound and complete proof systems, the compactness property and the Löwenheim-Skolem theorems. Given the undecidability results for FOL, no algorithms for checking satisfiability or validity of SOL formulas can be expected. For the monadic fragment of SOL, however, there is an interesting connection to regular languages, which yields the decidability of the satisfiability and related problems.

2.1 Syntax and semantics of SOL

In the sequel, we assume a fixed vocabulary $Voc = (Pred, Func)$ as for FOL and deal with three kinds of variables:

- first-order variables $x, y, z, \ldots \in Var$ that represent elements of the domain of a structure,
- predicate variables $X, Y, Z, \ldots \in PVar$ that stand for relations over the domain,
- function variables $F, G, H, \ldots \in FVar$ that stand for functions over the domain.

Predicate and function variables are often called second-order variables. As for predicate and function symbols, they are assumed to be classified according to their arity. I.e., we assume *PVar* to be partitioned into pairwise disjoint, recursively enumerable sets $PVar_1, PVar_2, \ldots$ where $PVar_n$ is the set of n-ary predicate variables. An analogous partition is required for the function variables. That is, we require *FVar* to be the union of pairwise disjoint, recursively enumerable sets $FVar_1, FVar_2, \ldots$ where $FVar_m$ is the set of m-ary function variables. We skip here 0-ary function variables ("constant variables") because they would play the same role as first-order variables. Furthermore, we require that the sets *Var*, *PVar* and *FVar* are pairwise disjoint and that there is no overlap with the symbols in the vocabulary.

Syntax of SOL. The abstract syntax of terms and formulas of SOL over *Voc*, *Var*, *PVar*, *FVar* is given by:

$$t \ ::= \ \underbrace{x \mid f(t_1, \ldots, t_m)}_{\text{as in FOL}} \mid F(t_1, \ldots, t_m)$$

$$\phi \ ::= \ \underbrace{true \mid P(t_1, \ldots, t_n) \mid t_1 = t_2 \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid \forall x.\phi}_{\text{as in FOL}}$$

$$X(t_1, \ldots, t_n) \mid \underbrace{\forall X.\phi \mid \forall F.\phi}_{\text{second-order quantification}}$$

where F is a function variable and X an n-ary predicate variable. The symbols x, f, P are as in the abstract syntax for FOL (cf. page 2). As in the first-order case, the above abstract syntax uses negation $\neg$, conjunction $\wedge$ and universal quantification $\forall$ as basis, which allows to derive other boolean connectives like disjunction $\vee$, implication $\rightarrow$, etc., and existential quantification in the standard way. For instance, existential first- and second-order quantification is obtained by:

$$\exists x.\phi \stackrel{\text{def}}{=} \neg\forall x.\neg\phi, \quad \exists X.\phi \stackrel{\text{def}}{=} \neg\forall X.\neg\phi \quad \text{and} \quad \exists F.\phi \stackrel{\text{def}}{=} \neg\forall F.\neg\phi.$$

Free and bounded occurrences of predicate and function variables are defined as for first-order variables. As for FOL, bounded first- and second-order variables can be renamed to ensure that there are no (first-order, predicate or function) variables that have both free and bounded occurrences in a given formula. Furthermore, the concept of bounded renaming permits to assume that all quantifiers of a given formula bind different variables. We write $\mathit{Free}(\phi)$ for the set of first-order variables that appear free in ϕ. $\mathit{PFree}(\phi)$ denotes the set of predicate variables that have free occurrences in ϕ. Similarly, we write $\mathit{FFree}(\phi)$ for the set of function variables that have free occurrences in ϕ.

Semantics of SOL. SOL-interpretations extend FOL-interpretations by providing meanings for the predicate and function variables. More precisely, an interpretation for the tuple (Voc, Var, $\mathit{PVar}, \mathit{FVar}$) is a pair $\mathcal{I} = (\mathcal{A}, \mathcal{V})$ consisting of a structure $\mathcal{A}$ for Voc (defined as for FOL, see page 7) and a variable valuation $\mathcal{V}$ that assigns

- to each first-order variable x an element $\mathcal{V}(x) = x^{\mathcal{I}} \in A$
- to each n-ary predicate variable X a relation $\mathcal{V}(X) = X^{\mathcal{I}} \subseteq A^n$,
- to each m-ary function variable F a function $\mathcal{V}(F) = F^{\mathcal{I}} : A^m \to A$

where $A = \mathit{Dom}^{\mathcal{A}}$. Given an $\mathcal{A}$-interpretation $\mathcal{I}$, the elements $t^{\mathcal{I}} \in \mathit{Dom}^{\mathcal{A}}$ for the terms are defined in the obvious way, i.e., we extend the corresponding definition for FOL by

$$F(t_1,\ldots,t_m)^{\mathcal{I}} \stackrel{\text{def}}{=} F^{\mathcal{I}}(t_1^{\mathcal{I}},\ldots,t_m^{\mathcal{I}})$$

where F is an m-ary function variable. Similarly, satisfaction of the new atomic formulas $X(t_1,\ldots,t_n)$ where X is an n-ary predicate variable is given by:

$$\mathcal{I} \models X(t_1,\ldots,t_n) \quad \text{iff} \quad (t_1^{\mathcal{I}},\ldots,t_n^{\mathcal{I}}) \in X^{\mathcal{I}}$$

The satisfaction relation $\models$ for the FOL-fragment of SOL is defined as for FOL. The only new features are:

$$\begin{array}{lll} (\mathcal{A},\mathcal{V}) \models \forall X.\phi & \text{iff} & (\mathcal{A},\mathcal{V}[X := R]) \models \phi \ \text{ for all relations } R \subseteq A^n \\ (\mathcal{A},\mathcal{V}) \models \forall F.\phi & \text{iff} & (\mathcal{A},\mathcal{V}[F := f]) \models \phi \ \text{ for all functions } f : A^m \to A \end{array}$$

where $A = \mathit{Dom}^{\mathcal{A}}$. Notions like model, satisfiability, validity, logical implication, equivalence, positive normal form, etc., are defined as for FOL. In the context of SOL, a *generalization* of a formula ϕ denotes a formula of the form $\forall\ldots\forall.\phi$ with an arbitrary prefix of universal FO- and SO-quantifiers.

Example 2.1.1 (Leibniz axiom for equality). If t_1 and t_2 are terms then the atomic formula $t_1 = t_2$ is equivalent to the formula

$$\forall X.\,(\,X(t_1) \leftrightarrow X(t_2)\,),$$

where X is an unary predicate variable, stating that two things are equal if and only if there is no property that distinguishes them. This is the so-called *Leibniz axiom* for equality.

For the sake of completeness, let us formally prove the equivalence of the atom $t_1 = t_2$ and the formula $\forall X. (X(t_1) \leftrightarrow X(t_2))$. Obviously, we have:

$$t_1 = t_2 \;\Vdash\; \forall X. (X(t_1) \leftrightarrow X(t_2))$$

We now show that $t_1 = t_2$ is a consequence of $\forall X. (X(t_1) \leftrightarrow X(t_2))$. Let $\mathcal{I} = (\mathcal{A}, \mathcal{V})$ be an interpretation such that

$$\mathcal{I} \models \forall X. (X(t_1) \leftrightarrow X(t_2))$$

and let A be the domain of structure $\mathcal{A}$. Then, $\mathcal{I}[X := B] \models X(t_1) \leftrightarrow X(t_2)$ for each subset B of A. Let

$$b \stackrel{\text{def}}{=} t_1^{\mathcal{I}} \quad \text{and} \quad B \stackrel{\text{def}}{=} \{b\}.$$

Then, $\mathcal{I}[X := B] \models X(t_1)$, and therefore $\mathcal{I}[X := B] \models X(t_2)$. But then $t_2^{\mathcal{I}} \in B = \{b\}$, i.e.,

$$t_1^{\mathcal{I}} \;=\; b \;=\; t_2^{\mathcal{I}}.$$

This yields $\mathcal{I} \models t_1 = t_2$. ■

Thus, the atomic formulas $t_1 = t_2$ could have been skipped from the syntax of SOL. There are, however, non-standard semantics for SOL where the equivalence of $t_1 = t_2$ and $\forall X. (X(t_1) \leftrightarrow X(t_2))$ does not hold and where the equality symbol cannot be derived from the other operators. Also the concept of function variables and second-order quantification over function variables is redundant and could have been skipped from the basis syntax.

Example 2.1.2 (Comprehensive formulas). If X is an n-ary predicate variable and ϕ a formula with $X \notin \mathit{PFree}(\phi)$ then the SOL-formula

$$\exists X \forall \overline{y}. \big(X(\overline{y}) \leftrightarrow \phi\big)$$

is valid, where $\overline{y} = (y_1, \ldots, y_n)$ is a tuple of pairwise distinct first-order variables. Assuming that $\mathit{Free}(\phi) = \{y_1, \ldots, y_n\}$ then this formula stands for the fact that there is a relation that characterizes the n-tuples where ϕ holds. However, the above formula is valid, no matter whether the y_i's are free in ϕ or ϕ has other free first- or second-order variables. To see this, consider an interpretation $\mathcal{I} = (\mathcal{A}, \mathcal{V})$ with domain A and put

$$R \;\stackrel{\text{def}}{=}\; \big\{\, \overline{a} \in A^n : \mathcal{I}[\overline{y} := \overline{a}] \models \phi \,\big\}.$$

Then, $\mathcal{I}[X := R] \models \forall \overline{y}\big(X(\overline{y}) \leftrightarrow \phi\big)$. Hence, $\mathcal{I}$ is a model for the above formula. Formulas of this type are called *comprehensive*. Analogous (valid) comprehensive formulas exist for function variables. They have the form

$$\forall \overline{y}\, \exists! z. \phi \;\rightarrow\; \exists F \forall \overline{y} \forall z. \big(F(\overline{y}) = z \leftrightarrow \phi\big)$$

where $F \notin \mathit{FFree}(\phi)$. As for FOL, the notation $\exists!$ is used as a short form notation for "there is a unique element ...", see page 14. ■

Example 2.1.3 (SOL-characterization of the natural numbers). The standard *induction principle* states that any set $N \subseteq \mathbb{N}$ which contains 0 and is closed under the successor function $succ : \mathbb{N} \to \mathbb{N}$, $succ(n) = n+1$, is, in fact, the set of natural numbers. In SOL, this can be formulated as

$$\phi_{\text{ind}} \stackrel{\text{def}}{=} \forall X. \Big(\big(X(0) \wedge \forall y.(X(y) \to X(succ(y))) \big) \to \forall y.X(y) \Big)$$

where X is an unary predicate variable. The underlying vocabulary is Voc_{nat} consisting of an unary function symbol *succ* and a constant symbol 0. In fact, ϕ_{ind} together with the FOL-formulas

$$\phi_1 \stackrel{\text{def}}{=} \forall x.\, succ(x) \neq 0$$
$$\phi_2 \stackrel{\text{def}}{=} \forall x \forall y.\, \big(succ(x) = succ(y) \to x = y\big)$$

characterizes the natural numbers, i.e., the structure $(\mathbb{N}, succ, 0)$. That is, $(\mathbb{N}, succ, 0) \models \mathfrak{F}$ and any model for $\mathfrak{F} = \{\phi_1, \phi_2, \phi_{\text{ind}}\}$ is isomorphic to the structure $(\mathbb{N}, succ, 0)$.

Let us see why. It is obvious that $(\mathbb{N}, succ, 0)$ is a model for ϕ_1, ϕ_2 and ϕ_{ind}, and hence, for $\mathfrak{F}$. Suppose now that $\mathcal{A} = (A, succ^{\mathcal{A}}, 0^{\mathcal{A}})$ is a model for $\mathfrak{F}$. We define a function $h : \mathbb{N} \to A$ inductively by:

$$h(0) \stackrel{\text{def}}{=} 0^{\mathcal{A}} \quad \text{and} \quad h(n+1) \stackrel{\text{def}}{=} succ^{\mathcal{A}}(h(n)) \text{ for } n \in \mathbb{N}.$$

Since $\mathcal{A} \models \phi_1 \wedge \phi_2$, the elements $h(n)$ are pairwise distinct and we have $succ^{\mathcal{A}}\big(h(n)\big) = h(n+1)$ for all $n \geqslant 0$. Let us now consider the set

$$B \stackrel{\text{def}}{=} h(\mathbb{N}) = \{h(n) : n \in \mathbb{N}\}.$$

The goal is to show that $A = B$, which yields the surjectivity of h. Clearly, we have $0^{\mathcal{A}} = h(0) \in B$ and with $b = h(n) \in B$ we also have $succ^{\mathcal{A}}(b) = h(n+1) \in B$. Thus, the subformula $X(0) \wedge \forall y.(X(y) \to X(succ(y)))$ of ϕ_{ind} holds in $\mathcal{A}$ when B serves as meaning for X. That is:

$$(\mathcal{A}, [X := B]) \models X(0) \wedge \forall y.(X(y) \to X(succ(y)))$$

Since $\mathcal{A} \models \phi_{\text{ind}}$, we obtain $(\mathcal{A}, [X := B]) \models \forall y.X(y)$. This yields that $B = A$. Hence, h is an isomorphism from $(\mathbb{N}, succ, 0)$ to $\mathcal{A}$. ■

The above example shows that SOL can characterize the natural numbers with the successor function and the constant 0, while FOL cannot (cf. Theorem 1.6.6 on page 86). This result can be extended to the natural numbers with addition and multiplication (i.e., the structure $(\mathbb{N}, +, *)$) where formulas ϕ_1, ϕ_2 have to be replaced with appropriate FOL-sentences that specify the essential properties of addition and multiplication.

Example 2.1.4 (Well-ordering, (least) upper bounds). A *well-ordering* is a partial order $\sqsubseteq$ on a set A (cf. Definition 1.4.13 on page 53) such that any nonempty subset B of A has a *least element*, i.e., an element $b \in B$ such that $b \sqsubseteq b'$ for all $b' \in B$. In SOL, this condition can be formalized as follows:

$$\phi_{\text{wo}} \stackrel{\text{def}}{=} \forall X. \Big(\underbrace{\exists z.X(z)}_{X \neq \varnothing} \to \exists y. \underbrace{\big(X(y) \wedge \forall z.(X(z) \to y \sqsubseteq z)\big)}_{y \text{ is least element of } X} \Big)$$

Given a partial order $(A,\sqsubseteq)$ and a subset B of A, an element $a \in A$ is called an *upper bound* for B if $b \sqsubseteq a$ for all elements $b \in B$. a is called a *least upper bound* of B if in addition $a \sqsubseteq a'$ for each upper bound a' of B. In the ordered field of reals any bounded nonempty set has a least upper bound. This property is given by the SOL-sentence:

$$\phi_{\mathrm{lub}} \stackrel{\mathrm{def}}{=} \forall X. \Big(\big(\underbrace{\exists z. X(z)}_{X \neq \varnothing} \wedge \underbrace{\exists y \forall z. (X(z) \rightarrow z \sqsubseteq y)}_{X \text{ is bounded}} \big) \rightarrow \exists y. \psi_{\mathrm{lub}}(y,X) \Big)$$

where subformula $\psi_{\mathrm{lub}}(y,X)$ states that y is least upper bound of X:

$$\psi_{\mathrm{lub}}(y,X) \stackrel{\mathrm{def}}{=} \forall y'. \big(\forall z. (X(z) \rightarrow z \sqsubseteq y') \leftrightarrow y \sqsubseteq y'\big)$$

Note that $\psi_{\mathrm{lub}}(y,X) \Vdash \forall z. (X(z) \rightarrow z \sqsubseteq y)$ Hence, $\psi_{\mathrm{lub}}(y,X)$ asserts that (1) y is an upper bound of X and (2) $y \sqsubseteq y'$ for each other upper bound y' of X.

It is well-known that each ordered field that satisfies ϕ_{lub} is isomorphic to the ordered field of reals. Recall that an ordered field is a field $(\mathbb{K},+,*,0,1)$ with a linear order $\sqsubseteq$ such that for all a, b, $c \in \mathbb{K}$ the following two conditions (i) and (ii) hold:

(i) $a \sqsubseteq b$ implies $a+c \sqsubseteq b+c$

(ii) if $0 \sqsubseteq a$ and $0 \sqsubseteq b$ then $0 \sqsubseteq a * b$.

Thus, using a vocabulary $\mathit{Voc}_{\mathrm{ord\_field}}$ with binary function symbols $+$ and $*$, constant symbols 0,1 and a binary predicate symbol $\sqsubseteq$, the FOL-sentences for the axioms of an ordered field together with ϕ_{lub} constitute a SOL-sentence that characterizes the structure $(\mathbb{R},+,*,0,1,\leqslant)$ up to isomorphism. ■

Example 2.1.5 (Class of periodic groups is SO-definable). Recall that a group $\mathcal{A} = (A,\circ,e)$ is said to be periodic iff for each element $a \in A$ there exists a positive integer $n \geqslant 1$ such that $a^n = e$. We saw in Theorem 1.6.7 on page 87 that the class of periodic groups has no characterization in first-order logic. In SOL, the class of periodic groups is defined by the sentence $\phi_{\mathrm{group}} \wedge \psi$ where ϕ_{group} is a FOL-sentence for the group axioms and

$$\psi \stackrel{\mathrm{def}}{=} \forall X. \big((\forall x. X(x,x) \wedge \forall x \forall y. (X(x,y) \rightarrow X(x, y \circ x))) \rightarrow \forall x. X(x,e) \big)$$

Then, for each structure $\mathcal{A} = (A,\circ,e)$ over the vocabulary of groups we have:

$$\mathcal{A} \models \phi_{\mathrm{group}} \wedge \psi \quad \text{iff} \quad \mathcal{A} \text{ is a periodic group}$$

"$\Longrightarrow$": Let us first check that each group $\mathcal{A}$ with $\mathcal{A} \models \psi$ is periodic. We regard the set

$$R \stackrel{\mathrm{def}}{=} \big\{ (a,a^n) \in A^2 : n \geqslant 1 \big\}.$$

Obviously, we have $(a,a) \in R$. And, whenever a, $b \in A$ and $(a,b) \in R$ then $b = a^n$ for some $n \geqslant 1$. Hence $(a, b \circ a) = (a, a^{n+1}) \in R$. This yields:

$$\begin{aligned} (\mathcal{A},[X := R]) &\models \forall x. X(x,x) \\ (\mathcal{A},[X := R]) &\models \forall x \forall y. (X(x,y) \rightarrow X(x, y \circ x)) \end{aligned}$$

As $\mathcal{A} \models \psi$ we get:

$$(\mathcal{A}, [X := R]) \models \forall x.X(x, e)$$

Thus, $(a, e) \in R$ for each $a \in A$. But this yields that for each element $a \in A$ there is some $n \geqslant 1$ with $a^n = e$.

"$\Longleftarrow$": Let us now suppose that $\mathcal{A}$ is a periodic group. We have to show that $\mathcal{A} \models \psi$. Let $R \subseteq A^2$ be a binary relation on A such that

$$(\mathcal{A}, [X := R]) \models \forall x.X(x, x) \wedge \forall x \forall y.\,(\,X(x, y) \to X(x, y \circ x)\,)$$

The task is to show that $(\mathcal{A}, [X := R]) \models \forall x.X(x, e)$, i.e., $(a, e) \in R$ for all $a \in A$.

- As $(\mathcal{A}, [X := R]) \models \forall x.X(x, x)$, we have: $\{(a, a) : a \in A\} \subseteq R$.
- As $(\mathcal{A}, [X := R]) \models \forall x \forall y.\,(\,X(x, y) \to X(x, y \circ x)\,)$: if $(a, b) \in R$ then $(a, b \circ a) \in R$.

By induction on n we obtain $(a, a^n) \in R$ for all $a \in R$ and $n \geqslant 1$. By assumption, $\mathcal{A}$ is periodic. Hence, for each $a \in A$ there is some $n \geqslant 1$ with $a^n = e$. But then $(a, e) \in R$ for each element $a \in A$. ■