Definition 1.2.2 (Soundness of a Hilbert proof system). $\mathfrak{D}$ is called *sound* if, for each formula-set $\mathfrak{F}$ and each formula ϕ: $\mathfrak{F} \vdash_{\mathfrak{D}} \phi$ implies $\mathfrak{F} \Vdash \phi$ ■

Thus, soundness requires that all $\mathfrak{D}$-derivations are based on logical consequences. In particular, if $\mathfrak{D}$ is sound then all $\mathfrak{D}$-provable formulas are tautologies.

Lemma 1.2.3 (Criterion for soundness). *$\mathfrak{D}$ is sound if and only if all proof rules in $\mathfrak{D}$ are sound in the following sense:*

(i) all instances of the axioms in $\mathfrak{D}$ are valid and

(ii) whenever $(\phi_1, \ldots, \phi_n, \phi)$ is an instance of a proof rule in $\mathfrak{D}$ then $\{\phi_1, \ldots, \phi_n\} \Vdash \phi$.

The proof of Lemma 1.2.3 is left as an exercise.

Completeness of a Hilbert proof system means that the consequence relation $\Vdash$ is covered by the derivation relation $\vdash_{\mathfrak{D}}$ in the sense that whenever formula ϕ is a consequence of the formula-set $\mathfrak{F}$ then ϕ is $\mathfrak{D}$-provable from $\mathfrak{F}$. Weak completeness just requires that all tautologies are $\mathfrak{D}$-provable.

Definition 1.2.4 (Completeness and weak completeness of a Hilbert proof system). $\mathfrak{D}$ is called *complete* iff for each formula ϕ we have: $\mathfrak{F} \Vdash \phi$ implies $\mathfrak{F} \vdash_{\mathfrak{D}} \phi$. The notion *weak completeness* is used for proof systems satisfying the property that all valid formulas are $\mathfrak{D}$-derivable, i.e., $\Vdash \phi$ implies $\vdash_{\mathfrak{D}} \phi$. ■

Hence, if $\mathfrak{D}$ is sound and complete then the derivation-relation $\vdash_{\mathfrak{D}}$ yields a syntactic (operational) characterization of the declarative notion of logical consequences:

$$\mathfrak{F} \vdash_{\mathfrak{D}} \phi \quad \text{iff} \quad \mathfrak{F} \Vdash \phi$$

In particular, for sound and (weakly) complete deductive systems we have:

$$\vdash_{\mathfrak{D}} \phi \quad \text{iff} \quad \varnothing \Vdash \phi \quad \text{iff} \quad \phi \text{ is valid.}$$

Theorem 1.2.5 (Gödel's completeness theorem). *There exists a Hilbert proof system that is sound and complete for FOL.*

We will not provide a full proof for Gödel's completeness theorem, but present a Hilbert proof system for first-order logic and outline the proof of its soundness and completeness. First, we will assume the decidability of the sets $Pred_n$ and $Func_m$ in the underlying vocabulary and the variable-set Var. Later, in Remark 1.2.15 on page 29, we will explain how this assumption can be relaxed.

We start with a series of axiom schemata for the propositional logic fragment of FOL. The axiom schemata (A1)–(A8) shown in Fig. 5 together with modus ponens constitute a sound and complete Hilbert proof system, called $\mathfrak{D}_{PL}$, for propositional logic. The decidability of the underlying vocabulary and variable-sets ensures that the sets consisting of the instances of each axiom and the modus ponens are indeed decidable. The greek letters Φ, Ψ and Θ serve as formula symbols. Furthermore, the presented axioms rely on the assumption that the syntax uses disjunction $\vee$ and negation $\neg$ as basis operators Implication is defined as usual by $\phi \to \psi \stackrel{\text{def}}{=} \neg\phi \vee \psi$. Used as axioms for FOL, the axiom schemata in Fig. 5 are sound, i.e., all instances of these axioms are propositional tautologies.

(A1)	*true*	(A5)	$(\Phi \vee \Psi) \rightarrow (\Psi \vee \Phi)$
(A2)	$\Phi \rightarrow \Phi$	(A6)	$(\Phi \rightarrow \Psi) \rightarrow \big((\Theta \rightarrow \Phi) \rightarrow (\Theta \rightarrow \Psi)\big)$
(A3)	$\Phi \vee \Phi \rightarrow \Phi$	(A7)	$(\Phi \rightarrow (\Theta \rightarrow \Psi)) \rightarrow (\Theta \rightarrow (\Phi \rightarrow \Psi))$
(A4)	$\Phi \rightarrow (\Phi \vee \Psi)$	(A8)	$(\Phi \rightarrow \Psi) \rightarrow \big((\Phi \vee \Theta) \rightarrow (\Psi \vee \Theta)\big)$

Figure 5: Axiom schemata of the Hilbert proof system $\mathfrak{D}_{\mathrm{PL}}$ for propositional logic

$$
\begin{aligned}
\psi_1 &= \phi_1 \rightarrow (\phi_1 \vee \phi_2) \\
\psi_2 &= (\phi_1 \vee \phi_2) \rightarrow ((\phi_1 \vee \phi_2) \vee \phi_3) \\
\psi_3 &= \big((\phi_1 \vee \phi_2) \rightarrow ((\phi_1 \vee \phi_2) \vee \phi_3)\big) \rightarrow \\
&\qquad \big((\phi_1 \rightarrow (\phi_1 \vee \phi_2)) \rightarrow (\phi_1 \rightarrow ((\phi_1 \vee \phi_2) \vee \phi_3))\big) \\
\psi_4 &= (\phi_1 \rightarrow (\phi_1 \vee \phi_2)) \rightarrow (\phi_1 \rightarrow ((\phi_1 \vee \phi_2) \vee \phi_3)) \\
\psi_5 &= \phi_1 \rightarrow ((\phi_1 \vee \phi_2) \vee \phi_3)
\end{aligned}
$$

Figure 6: Example for a $\mathfrak{D}_{\mathrm{PL}}$-proof

Example 1.2.6 ($\mathfrak{D}_{\mathrm{PL}}$-derivation). The formula sequence shown in Fig. 6 is an example for a $\mathfrak{D}_{\mathrm{PL}}$-proof. Here, ϕ_1, ϕ_2, ϕ_3 are arbitrary formulas of propositional or first-order logic. Note that ψ_1 are ψ_2 are instances of axiom (A4) while ψ_3 is an instance of axiom (A6). Furthermore, $\psi_3 = \psi_2 \rightarrow \psi_4$. Thus, formula ψ_4 arises from the modus ponens and formulas ψ_2 and ψ_3. Similarly, as $\psi_4 = \psi_1 \rightarrow \psi_5$, the last formula ψ_5 is obtained by the modus ponens applied to ψ_1 and ψ_4. ■

Example 1.2.7 (Derived proof rule: chain rule). In $\mathfrak{D}_{\mathrm{PL}}$ or any other Hilbert proof system $\mathfrak{D}$ that uses the modus ponens as proof rule and where the instances of axiom schema (A6) are $\mathfrak{D}$-provable, the following chain rule is a *derived proof rule*:

$$
\frac{\Phi \rightarrow \Psi, \quad \Psi \rightarrow \Theta}{\Phi \rightarrow \Theta} \qquad \text{(chain rule)}
$$

The meaning of this derived rule is that whenever $\mathfrak{F}$ is a formula-set and ϕ, ψ are formulas such that $\mathfrak{F} \vdash_{\mathfrak{D}} \phi \rightarrow \psi$ and $\mathfrak{F} \vdash_{\mathfrak{D}} \psi \rightarrow \theta$ then $\mathfrak{F} \vdash_{\mathfrak{D}} \phi \rightarrow \theta$. The correctness of the chain rule is established by showing that

$$
\big\{\phi \rightarrow \psi, \psi \rightarrow \theta\big\} \vdash_{\mathfrak{D}} \phi \rightarrow \theta
$$

where ϕ, ψ and θ are arbitrary formulas. We provide here a $\mathfrak{D}$-derivation of the formula $\phi \rightarrow \theta$

from the formula-set $\mathfrak{F} = \{\phi \to \psi, \psi \to \theta\}$:

$$\begin{aligned}
\psi_1 &= \psi \to \theta \\
\psi_2 &= \phi \to \psi \\
\psi_3 &= (\psi \to \theta) \to \big((\phi \to \psi) \to (\phi \to \theta)\big) \\
\psi_4 &= (\phi \to \psi) \to (\phi \to \theta) \\
\psi_5 &= \phi \to \theta
\end{aligned}$$

Formulas ψ_1, ψ_2 are elements of $\mathfrak{F}$. Formula ψ_3 is an instance of axiom schema (A6). Formula ψ_4 arises by the modus ponens applied to ψ_3 and $\psi_1 = \psi \to \theta$. Finally, formula ψ_5 is obtained by the modus ponens applied to ψ_4 and $\psi_2 = \phi \to \psi$. ■

Soundness of the Hilbert proof system $\mathfrak{D}_{\mathrm{PL}}$ is an easy verification. We just have to inspect all axioms and verify that their instances are valid. Completeness of $\mathfrak{D}_{\mathrm{PL}}$ for propositional logic is stated here without proof. In particular, $\mathfrak{D}_{\mathrm{PL}}$ is sound for FOL, but not complete since it lacks of any proof rule for the quantifiers and e.g. $\forall x.P(x) \nvdash_{\mathfrak{D}} P(c)$, although $P(c)$ is a consequence of $\forall x.P(x)$.

For FOL, we could deal with the axiom schemata of Fig. 5 in combination with several other proof rules for quantification and the equality symbol. However, since the validity problem for propositional logic is decidable, the set of all propositional tautologies over some fixed vocabulary and variable-set is decidable too, and so is the set of all generalizations of propositional tautologies. Hence, for FOL we can deal with the set of all generalizations of propositional tautologies as one axiom, called axiom (PT). In addition, we deal with axioms that are characteristic for the semantics of universal FO-quantification. Later, we will add further axioms for the equality symbol.

We start with FOL without equality. Let $\mathfrak{D}_{\mathrm{FOL}}$ denote the Hilbert proof system consisting of the modus ponens and the following four axioms:

Axiom (PT): the set of all *generalizations* of propositional tautologies

Axioms (Q1), (Q2), (Q3) for FO-quantification:
the set of all *generalizations* of the three formula schemata shown in Fig. 7 on page 22

(Q1)	$\forall x.\Phi \to \Phi[x/t]$	if x can be replaced with t in Φ
(Q2)	$\Phi \to \forall x.\Phi$	if $x \notin \mathit{Free}(\Phi)$
(Q3)	$\forall x.(\Phi \to \Psi) \to \big(\forall x.\Phi \to \forall x.\Psi\big)$	

Figure 7: Axiom schemata in $\mathfrak{D}_{\mathrm{FOL}}$ for FO-quantification

Remark 1.2.8 (Generalizations). Recall that if ϕ is a FOL-formula then all formulas of the form $\forall x_1 \ldots \forall x_n.\phi$ are called generalizations of ϕ. The reason why we have to deal with generalizations of the propositional tautologies (axiom (PT)) and of the axioms in Figure 7 is that otherwise FOL-tautologies like $\forall x.(P(x) \to P(x))$ or $\forall y(\forall x.Q(x,y) \to Q(y,y))$ could not be derived. ■

The first axiom (Q1) in Fig. 7 requires that t is a term which can replace x in Φ without invoking undesired bindings (see the explanations on page 8). Formally, the instances of this axiom are all formulas $\forall x.\phi \to \phi[x/t]$ such that for each variable y that appears in t, formula ϕ does not contain a subformula $\forall y.\psi$ or $\exists y.\psi$ where $x \in \mathit{Free}(\psi)$. Under this assumption we have

$$\forall x.\phi \Vdash \phi[x/t],$$

which is a consequence of the substitution lemma (see page 8). Let us briefly recall the argument. Suppose that $\mathcal{I} = (\mathcal{A}, \mathcal{V})$ is an interpretation with domain A such that $\mathcal{I} \models \forall x.\phi$. Then, $\mathcal{I}[x := a] \models \phi$ for all $a \in A$. With $a = t^{\mathcal{I}}$ we get $\mathcal{I}[x := t^{\mathcal{I}}] \models \phi$. The substitution lemma yields that $\mathcal{I} \models \phi[x/t]$. Dropping the requirement that x can be replaced with t in ϕ for axiom (Q1) would yield an incorrect axiom. E.g., for

$$\phi \ = \exists y.\, P(x,y)$$

the term $t = y$ does not yield a legal substitute for x (in the above sense), since the replacement of x with y would cause an undesired binding. Note that $\phi \to \phi[x/y]$ is not valid:

$$\forall x \underbrace{\exists y.\, P(x,y)}_{=\phi} \to \underbrace{\exists y.\, P(y,y)}_{=\phi[x/y]}$$

Example 1.2.9 ($\mathfrak{D}_{\mathbf{FOL}}$-proof for bounded renaming). Using the Hilbert proof system $\mathfrak{D}_{\mathrm{FOL}}$ we provide a proof for bounded renaming, i.e., we show that

$$\forall x.\phi \ \vdash_{\mathfrak{D}_{\mathrm{FOL}}} \ \forall y.\phi[x/y],$$

provided that (i) $y \notin \mathit{Free}(\phi)$ and (ii) variable x can be replaced with y in ϕ, i.e., there is no subformula $\forall y.\psi$ (or $\exists y.\psi$) of ϕ such that $x \in \mathit{Free}(\psi)$. Under assumptions (i) and (ii), the formula-sequence shown in Fig. 8 on page 24 constitutes a $\mathfrak{D}_{\mathrm{FOL}}$-proof of the formula $\forall y.\phi[x/y]$ from $\forall x.\phi$.

Formula ψ_1 is the premise (an element of the underlying formula-set). Formula ψ_2 is an instance of axiom (Q2), since $y \notin \mathit{Free}(\forall x.\phi)$ by condition (i). The modus ponens yields formula ψ_3. Since x can be replaced in ϕ with y (condition (ii)), formula ψ_4 is a generalization of an instance of axiom (Q1). Axiom (Q3) yields formula ψ_5. Applying the modus ponens yields formulas ψ_6 and ψ_7. ■

Theorem 1.2.10 (Soundness and completeness of $\mathfrak{D}_{\mathbf{FOL}}$). *If $\mathfrak{F}$ is a set of FOL-formulas and ϕ a FOL-formula without equality then:*

$$\mathfrak{F} \Vdash \phi \quad \textit{iff} \quad \mathfrak{F} \vdash_{\mathfrak{D}_{FOL}} \phi$$

Proving the soundness of $\mathfrak{D}_{\mathrm{FOL}}$ requires checking that all instances of the axioms are valid formulas. This is obvious for axiom (PT). The validity of the generalizations of formulas of the form $\forall x.(\phi \to \psi) \to (\forall x.\phi \to \forall x.\psi)$ is an easy verification. The validity of $\phi \to \forall x.\phi$ if $x \notin \mathit{Free}(\phi)$ is clear, as ϕ and $\forall x.\phi$ are even equivalent. The validity of $\forall x.\phi \to \phi[x/t]$ under

$$
\begin{array}{lcll}
\psi_1 & = & \forall x.\phi & \text{(assumption)} \\
\psi_2 & = & \forall x.\phi \to \forall y \forall x.\phi & \text{(axiom (Q2))} \\
\psi_3 & = & \forall y \forall x.\phi & \text{(modus ponens)} \\
\psi_4 & = & \forall y.(\forall x.\phi \to \phi[x/y]) & \text{(axiom (Q1))} \\
\psi_5 & = & \forall y.(\forall x.\phi \to \phi[x/y]) \to (\forall y \forall x.\phi \to \forall y.\phi[x/y])) & \text{(axiom (Q3))} \\
\psi_6 & = & \forall y \forall x.\phi \to \forall y.\phi[x/y] & \text{(modus ponens)} \\
\psi_7 & = & \forall y.\phi[x/y] & \text{(modus ponens)}
\end{array}
$$

Figure 8: Example for a $\mathfrak{D}_{\mathrm{FOL}}$-proof

the assumption that x can be replaced with t in ϕ has been shown before. Soundness of the modus ponens is clear.

We now provide a proof sketch for the completeness of $\mathfrak{D}_{\mathrm{FOL}}$ as a proof system for FOL without equality. A simple key property of $\mathfrak{D}_{\mathrm{FOL}}$ and most other Hilbert-style proof systems for FOL that is useful at various places in the completeness proof, but also for our purposes, is the following deduction property:

Lemma 1.2.11 (Deduction property). *Let $\mathfrak{D}$ be a Hilbert proof system that is complete for propositional logic, i.e., $\vdash_{\mathfrak{D}} \phi$ for all formulas that are propositional tautologies, and uses the modus ponens, but no other proof rule of arity one or more. Then, for all sets $\mathfrak{F}$ of formulas and all formulas ϕ:*

$$\mathfrak{F} \vdash_{\mathfrak{D}} \phi \to \psi \quad \text{iff} \quad \mathfrak{F} \cup \{\phi\} \vdash_{\mathfrak{D}} \psi$$

Proof. We first show the implication $\Longrightarrow$ that holds for each Hilbert proof system where the modus ponens is a primary or derivable proof rule. Suppose $\mathfrak{F} \vdash_{\mathfrak{D}} \phi \to \psi$. Then, $\mathfrak{F} \cup \{\phi\} \vdash_{\mathfrak{D}} \phi \to \psi$. As $\mathfrak{F} \cup \{\phi\} \vdash_{\mathfrak{D}} \phi$, the modus ponens is applicable and yields $\mathfrak{F} \cup \{\phi\} \vdash_{\mathfrak{D}} \psi$.

The proof for the implication "$\Longleftarrow$" can be established by induction on the length m of a shortest $\mathfrak{D}$-proof for ψ from $\mathfrak{F} \cup \{\phi\}$.

Basis of induction. If $m = 1$ then ψ is either an element of $\mathfrak{F} \cup \{\phi\}$ or an instance of an axiom. In case $\psi = \phi$ we immediately get $\mathfrak{F} \vdash_{\mathfrak{D}} \phi \to \psi$ since then $\phi \to \psi$ is a propositional tautology. Suppose now that $\psi \in \mathfrak{F}$ or ψ is an instance of an axiom. Then, $\mathfrak{F} \vdash_{\mathfrak{D}} \psi$. The formula $\psi \to (\phi \to \psi)$ is a propositional tautology. (Note that $\psi \to (\phi \to \psi)$ is defined as $\neg\psi \vee (\neg\phi \vee \psi)$, which is equivalent to *true*.) Hence, we have $\vdash_{\mathfrak{D}} \psi \to (\phi \to \psi)$ and therefore

$$\mathfrak{F} \vdash_{\mathfrak{D}} \psi \to (\phi \to \psi)$$

Then, we apply the modus ponens to obtain $\mathfrak{F} \vdash_{\mathfrak{D}} \phi \to \psi$.

Step of induction. We suppose that a shortest $\mathfrak{D}$-proof for ψ from $\mathfrak{F} \cup \{\phi\}$ has length $m \geqslant 2$. Suppose ψ is obtained by applying the modus ponens to formulas θ and $\theta \to \psi$ in a $\mathfrak{D}$-proof for ψ from $\mathfrak{F} \cup \{\phi\}$ of length m. As θ and $\theta \to \psi$ have $\mathfrak{D}$-proofs from $\mathfrak{F} \cup \{\phi\}$ of length at most $m-1$, we may apply the induction hypothesis and obtain:

$$
\begin{array}{lcl}
\mathfrak{F} & \vdash_{\mathfrak{D}} & \phi \to \theta \\
\mathfrak{F} & \vdash_{\mathfrak{D}} & \phi \to (\theta \to \psi)
\end{array}
$$

The formula

$$\sigma \stackrel{\text{def}}{=} (\phi \to \theta) \to \big((\phi \to (\theta \to \psi)) \to (\phi \to \psi) \big)$$

is a propositional tautology. To see this, consider the underlying propositional formula

$$\varphi \stackrel{\text{def}}{=} (q \to r) \to \big((q \to (r \to p)) \to (q \to p) \big)$$

where p, q and r are boolean variables and show that φ evaluates to true under each assignment $\mathfrak{I}$ for p, q and r. If $\mathfrak{I} \not\models q$ then $\mathfrak{I}$ is a model for each formula of the form $q \to \ldots$. Hence, $\mathfrak{I} \models \varphi$. If $\mathfrak{I} \models q$ and $\mathfrak{I} \not\models r$ then $\mathfrak{I} \not\models q \to r$, and therefore, $\mathfrak{I}$ is a model for each formula of the form $(q \to r) \to \ldots$. Finally, if $\mathfrak{I} \models r$ then $\mathfrak{I} \models q \to (r \to p)$ iff $\mathfrak{I} \models q \to p$, and hence $\mathfrak{I} \models \varphi$.

As $\mathfrak{D}$ is supposed to be complete for propositional logic, we get $\vdash_{\mathfrak{D}} \sigma$. The modus ponens applied to σ and $\phi \to \theta$ yields:

$$\mathfrak{F} \vdash_{\mathfrak{D}} \big(\phi \to (\theta \to \psi)\big) \to (\phi \to \psi)$$

Applying the modus ponens again yields the desired statement: $\mathfrak{F} \vdash_{\mathfrak{D}} \phi \to \psi$. □

In Lemma 1.2.11, we required that there is no other proper proof rule of arity one or more than the modus ponens. This requirement can be relaxed and the deduction property can be established for each Hilbert proof system that is complete for propositional tautologies and for which the induction step in the proof of Lemma 1.2.11 can be provided for each of its proof rules.

Remark 1.2.12 (Deduction property vs modus ponens). Whenever $\mathfrak{D}$ is a Hilbert proof system that enjoys the deduction property, then the modus ponens is a derivable proof rule. To see this suppose that $\mathfrak{F} \vdash_{\mathfrak{D}} \phi$ and $\mathfrak{F} \vdash_{\mathfrak{D}} \phi \to \psi$. The goal is to show that $\mathfrak{F} \vdash_{\mathfrak{D}} \psi$.

- As $\mathfrak{F} \vdash_{\mathfrak{D}} \phi$ there exists a $\mathfrak{D}$-proof $\phi_1, \phi_2, \ldots, \phi_n$ for ϕ from $\mathfrak{F}$.
- As $\mathfrak{F} \vdash_{\mathfrak{D}} \phi \to \psi$ we have $\mathfrak{F} \cup \{\phi\} \vdash_{\mathfrak{D}} \psi$ by the deduction property. Hence, there is a $\mathfrak{D}$-proof $\psi_1, \psi_2, \ldots, \psi_m$ for ψ from $\mathfrak{F} \cup \{\phi\}$.

But then the formula-sequence $\phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi_m$ is a $\mathfrak{D}$-proof for ψ from $\mathfrak{F}$. ■

The major part in completeness proofs for deductive calculi is to establish a connection between the semantic notion of satisfiability and the syntactic notion of *consistency*. There are many equivalent definitions of the *(in)consistency* of a formula-set $\mathfrak{F}$ with respect to $\mathfrak{D}$. One possibility is to define $\mathfrak{D}$-inconsistency for a formula-set $\mathfrak{F}$ by the requirement that *false* is $\mathfrak{D}$-provable from $\mathfrak{F}$. An alternative condition for the $\mathfrak{D}$-inconsistency of a formula-set $\mathfrak{F}$ is to require that all formulas are $\mathfrak{D}$-provable from $\mathfrak{F}$. Another possibility is to require that $(\mathfrak{F} \vdash_{\mathfrak{D}} \phi)$ and $(\mathfrak{F} \vdash_{\mathfrak{D}} \neg\phi)$ for some formula ϕ. For all standard Hilbert-style calculi, these notions of inconsistency are equivalent. In the next lemma, we prove the equivalence of these conditions for the $\mathfrak{D}$-inconsistency under the assumption that $\mathfrak{D}$ is complete for propositional logic, i.e., whenever ϕ is a propositional tautology then $\vdash_{\mathfrak{D}} \phi$, and uses the modus ponens.

Lemma 1.2.13 (Characterizations of $\mathfrak{D}$-inconsistency). *Let $\mathfrak{D}$ be a Hilbert proof system for FOL that is complete for propositional logic and enjoys the deduction property. Then, for all formula-sets $\mathfrak{F}$ the following statements are equivalent:*

(a) $\mathfrak{F} \vdash_{\mathfrak{D}} \mathit{false}$.

(b) There exists a formula ψ *such that* $\mathfrak{F} \vdash_{\mathfrak{D}} \psi$ *and* $\mathfrak{F} \vdash_{\mathfrak{D}} \neg\psi$.

(c) $\mathfrak{F} \vdash_{\mathfrak{D}} \phi$ *for all formulas* ϕ.

Proof. (a) $\Longrightarrow$ (b): We put $\psi = \mathit{true}$. Since true is a propositional tautology we have $\vdash_{\mathfrak{D}} \mathit{true}$, and therefore $\mathfrak{F} \vdash_{\mathfrak{D}} \psi$. By definition, $\neg\psi = \neg\mathit{true} = \mathit{false}$. Hence, $\mathfrak{F} \vdash_{\mathfrak{D}} \neg\psi$ by assumption (a).
(b) $\Longrightarrow$ (c): Let $\mathfrak{F} \vdash_{\mathfrak{D}} \psi$ and $\mathfrak{F} \vdash_{\mathfrak{D}} \neg\psi$ and ϕ an arbitrary formula. Since

$$\psi \rightarrow (\neg\psi \rightarrow \phi) \equiv \neg\psi \vee \neg\neg\psi \vee \phi \equiv \mathit{true}$$

we get a $\mathfrak{D}$-proof for ϕ as follows:

(i)	$\mathfrak{F} \vdash_{\mathfrak{D}} \psi$	(assumption)
(ii)	$\mathfrak{F} \vdash_{\mathfrak{D}} \psi \rightarrow (\neg\psi \rightarrow \phi)$	(propositional tautology)
(iii)	$\mathfrak{F} \vdash_{\mathfrak{D}} \neg\psi \rightarrow \phi$	(modus ponens applied to (i) and (ii))
(iv)	$\mathfrak{F} \vdash_{\mathfrak{D}} \neg\psi$	(assumption)
(v)	$\mathfrak{F} \vdash_{\mathfrak{D}} \phi$	(modus ponens applied to (iii) and (iv))

(c) $\Longrightarrow$ (a): obvious $\square$

In the sequel, $\mathfrak{D}$ is supposed to be a Hilbert proof system for FOL that is complete for propositional logic and enjoys the deduction property, see Lemma 1.2.11 on page 24). The notion $\mathfrak{D}$-*inconsistency* of a formula-set $\mathfrak{F}$ means that the conditions of Lemma 1.2.13 are fulfilled. Formula-set $\mathfrak{F}$ is called $\mathfrak{D}$-*consistent* if $\mathfrak{F}$ is not $\mathfrak{D}$-inconsistent. The following theorem shows that – under mild assumptions – to establish completeness of a Hilbert proof system $\mathfrak{D}$ it suffices to provide models for $\mathfrak{D}$-consistent sets.

Theorem 1.2.14 (Completeness vs satisfiability/consistency). *Given a Hilbert proof system* $\mathfrak{D}$ *for FOL that is complete for propositional logic and enjoys the deduction property, then the following statements (1) and (2) are equivalent:*

(1) $\mathfrak{D}$ *is complete, i.e., for each formula-set* $\mathfrak{F}$ *and formula* ϕ*:* $\mathfrak{F} \Vdash \phi$ *implies* $\mathfrak{F} \vdash_{\mathfrak{D}} \phi$

(2) $\mathfrak{D}$*-consistency implies satisfiability, i.e., each* $\mathfrak{D}$*-consistent formula-set is satisfiable.*

Proof. To see that (1) implies (2) we may use the following arguments.

	$\mathfrak{F}$ is not satisfiable	
$\Longrightarrow$	$\mathfrak{F} \Vdash \mathit{false}$	
$\Longrightarrow$	$\mathfrak{F} \vdash_{\mathfrak{D}} \mathit{false}$	(completeness, assumption (1))
$\Longrightarrow$	$\mathfrak{F}$ is $\mathfrak{D}$-inconsistent	(definition of $\mathfrak{D}$-inconsistency)

To see that (2) implies (1) we make use of the deduction property stating that $\mathfrak{F} \cup \{\phi\} \vdash_{\mathfrak{D}} \psi$ if and only if $\mathfrak{F} \vdash_{\mathfrak{D}} (\phi \rightarrow \psi)$. With this property we may argument as follows:

$$
\begin{array}{lll}
& \mathfrak{F} \Vdash \phi & \\
\Longrightarrow & \mathfrak{F} \cup \{\neg\phi\} \text{ is not satisfiable} & \\
\Longrightarrow & \mathfrak{F} \cup \{\neg\phi\} \text{ is } \mathfrak{D}\text{-inconsistent} & \text{(assumption (2))} \\
\Longrightarrow & \mathfrak{F} \cup \{\neg\phi\} \vdash_{\mathfrak{D}} \textit{false} & \text{(def. of } \mathfrak{D}\text{-inconsistency)} \\
\Longrightarrow & \mathfrak{F} \vdash_{\mathfrak{D}} (\neg\phi \to \textit{false}) & \text{(deduction property)} \\
\Longrightarrow & \mathfrak{F} \vdash_{\mathfrak{D}} \phi &
\end{array}
$$

In the last step we used the completeness of $\mathfrak{D}$ for propositional tautologies and the fact that the modus ponens is derivable from the deduction property (see Remark 1.2.12). Note that $(\neg\phi \to \textit{false}) \to \phi$ is a propositional tautology. Since $\mathfrak{D}$ is supposed to be complete for propositional logic, we have $\mathfrak{F} \vdash_{\mathfrak{D}} (\neg\phi \to \textit{false}) \to \phi$. But then we can apply the modus ponens to obtain $\mathfrak{F} \vdash_{\mathfrak{D}} \phi$. □

The argument used in Theorem 1.2.14 does not use any specific properties of FOL. Thus, (1) and (2) are equivalent for each logic and deductive calculus that meets the deduction property and is complete for propositional logic. In an analogous way, one can show that $\mathfrak{D}$ is sound if and only if each satisfiable formula-set is $\mathfrak{D}$-consistent. Thus, for sound and complete proof systems, the notions $\mathfrak{D}$-consistency and satisfiability agree. More precisely:

$$
\mathfrak{D} \text{ is sound and complete} \quad \text{iff} \quad \begin{cases} \text{for all formula-sets } \mathfrak{F}\text{:} \\ \mathfrak{F} \text{ is } \mathfrak{D}\text{-consistent} \iff \mathfrak{F} \text{ is satisfiable} \end{cases}
$$

On the basis of Theorem 1.2.14, completeness of $\mathfrak{D}_{\text{FOL}}$ can be established by providing a model for each $\mathfrak{D}_{\text{FOL}}$-consistent formula-set $\mathfrak{F}$. This can be done by extending $\mathfrak{F}$ to a maximal $\mathfrak{D}_{\text{FOL}}$-consistent formula-set $\mathfrak{F}^+$, which means a set of formulas that subsumes $\mathfrak{F}$, is $\mathfrak{D}_{\text{FOL}}$-consistent and satisfies the following two properties:

- for each formula ϕ either $\phi \in \mathfrak{F}^+$ or $\neg\phi \in \mathfrak{F}^+$
- if $\neg\forall x.\phi \in \mathfrak{F}^+$ then there exists a constant symbol c such that $\neg\phi[x/c] \in \mathfrak{F}^+$.

For this formula-set $\mathfrak{F}^+$, a *term model* $(\mathcal{A}, \mathcal{V})$ can be defined (i.e., a model where the domain consists of the terms of the underlying vocabulary) which interpretes any term by itself and yields $(\mathcal{A}, \mathcal{V}) \models \phi$ if and only if $\phi \in \mathfrak{F}^+$. But then, $(\mathcal{A}, \mathcal{V})$ is also a model of $\mathfrak{F}$. We skip here the details of the construction of $\mathfrak{F}^+$ and its term model, as we will deal with similar constructions at several other places.

Additional axioms for equality. To deal with FOL with equality, some additional axioms are needed. First we need an axiom stating the reflexivity of equality. For this we deal with an axiom consisting of all generalizations of the formulas $t = t$ where t is a term. Second, axioms are required that assert that equality is preserved by function application and that the predicates do not distinguish between equal terms. That is, if f is an m-ary function symbol and P an n-ary predicate symbol then we deal with the axioms consisting of all generalizations of the formulas:

$$
\bigwedge_{1 \leqslant i \leqslant m} (t_i = s_i) \;\to\; f(t_1, \ldots, t_m) = f(s_1, \ldots, s_m)
$$

$$
\bigwedge_{1 \leqslant i \leqslant n} (t_i = s_i) \;\to\; \big(P(t_1, \ldots, t_n) \to P(s_1, \ldots, s_n)\big)
$$

where the t_i's and s_i's are terms and f is an m-ary function symbol and P either an n-ary predicate symbol of the underlying vocabulary or $n = 2$ and P stands for the equality symbol. The latter means that we also deal with the axiom consisting of the generalizations of

$$(t_1 = s_1 \wedge t_2 = s_2) \rightarrow (t_1 = t_2 \rightarrow s_1 = s_2)$$

In fact, the Hilbert proof system that extends $\mathfrak{D}_{\mathrm{FOL}}$ by these axioms for equality can shown to be sound and complete for FOL with equality.