For an example how Ehrenfeucht-Fraïssé games can be used to show that certain properties are not FO-definable we consider games on finite linear orders. Recall that a linear order is a structure $(A, \sqsubseteq)$ such that $\sqsubseteq$ is a binary relation on A which is transitive, reflexive and antisymmetric and such that every two elements of A are comparable via $\sqsubseteq$, i.e., either $a \sqsubseteq b$ or $b \sqsubseteq a$ (or both, if $a = b$). See Definition 1.4.13 on page 53.

Notation 1.6.38 (Finite linear orders). We now consider games on *finite linear orders*, i.e., linear orders $\mathcal{A} = (A, \sqsubseteq)$ where A is finite. In this case, the elements of A can be enumerated in an increasing way, i.e., there is an enumeration $a_1, \ldots, a_m$ of the elements in A such that $a_1 \sqsubset a_2 \sqsubset \ldots \sqsubset a_m$. Typical examples for finite linear orders are finite subsets of $\mathbb{N}$ with the natural order $\leqslant$.

Element a_{i+1} is called the *(direct) successor* of a_i (for $1 \leqslant i < m$). We refer to a_1 as the *minimal element* and to a_m as the *maximal element*, and sometimes write $\min^{\mathcal{A}}$ and $\max^{\mathcal{A}}$ for the minimal and maximal element of $\mathcal{A}$, respectively. Furthermore, $m = |A|$ is called the *length* of $\mathcal{A}$. ∎

The vocabulary of linear orders agrees with the vocabulary Voc_{graph} for formalizing graph properties. However, we use here typical symbols like $\sqsubseteq$ for partial orders rather than the symbol E for the edge relation of a graph. Atomic formulas will be written in infix notation $x \sqsubseteq y$ rather than $\sqsubseteq(x, y)$.

Lemma 1.6.39. *Let $\mathcal{A} = (A, \sqsubseteq_A)$ and $\mathcal{B} = (B, \sqsubseteq_B)$ be finite linear orders and let $k \in \mathbb{N}$, $k \geqslant 2$: Then:*

$$\mathcal{A} \cong_k \mathcal{B} \quad \textit{iff} \quad (\mathcal{A}, \min^{\mathcal{A}}) \cong_k (\mathcal{B}, \min^{\mathcal{B}}) \quad \textit{iff} \quad (\mathcal{A}, \max^{\mathcal{A}}) \cong_k (\mathcal{B}, \max^{\mathcal{B}})$$

Proof. We only provide here the argument for the minimal elements. The treatment of the maximal elements is analogous.

It is obvious that $(\mathcal{A}, \min^{\mathcal{A}}) \cong_k (\mathcal{B}, \min^{\mathcal{B}})$ implies $\mathcal{A} \cong_k \mathcal{B}$. We now suppose that $\mathcal{A} \cong_k \mathcal{B}$ and show that there is a k-round winning strategy for $\mathcal{A}$ and $\mathcal{B}$ that responds with $\min^{\mathcal{A}}$ (resp. $\min^{\mathcal{B}}$) whenever the spoiler chooses $\min^{\mathcal{B}}$ (resp. $\min^{\mathcal{A}}$). Suppose, e.g., that $\mathcal{S}$ is a k-round winning strategy for $\mathcal{A} \cong_k \mathcal{B}$ such that

$$\mathcal{S}(\,(a_1, \ldots, a_i, \min^{\mathcal{A}}), (b_1, \ldots, b_i)\,) \;=\; b \text{ for some } b \in B \text{ with } \min^{\mathcal{B}} \sqsubset_B b$$

Then, $i = k-1$, as otherwise $\mathcal{S}$ would not be a winning strategy.

> Note that for $i < k-1$ the spoiler can select $\min^{\mathcal{B}}$ in round $i+2$ which yields a k-round outcome of the form $\langle (a_1, \ldots, a_k), (b_1, \ldots, b_k) \rangle$ with
>
> $$a_{i+1} = \min^{\mathcal{A}}, \;\; b_{i+1} = b \;\text{ and }\; b_{i+2} = \min^{\mathcal{B}}.$$
>
> But then $b_{i+2} \sqsubset_B b_{i+1}$ while $a_{i+2} \not\sqsubset_A a_{i+1}$, and hence, $\langle (a_1, \ldots, a_k), (b_1, \ldots, b_k) \rangle$ does not define a partial isomorphism. Thus, the spoiler wins this $\mathcal{S}$-play, which is impossible if $\mathcal{S}$ is a winning strategy.

The above shows that we may replace $\mathcal{S}$ with another k-round winning strategy $\mathcal{S}'$ which agrees with $\mathcal{S}$ for the i-round outcomes for $0 \leqslant i < k-1$ and for the $(k-1)$-round outcomes where the spoiler did not choose a minimal element. If in the last round (round k) the spoiler picks the minimum of one of the structures $\mathcal{A}$ or $\mathcal{B}$, then $\mathcal{S}'$ answers with the minimum of the other structure. This strategy $\mathcal{S}'$ is, in fact, a k-round winning strategy for the extended structures $(\mathcal{A}, \min^{\mathcal{A}})$ and $(\mathcal{B}, \min^{\mathcal{B}})$. □

Note that the above lemma is wrong if k=1. Consider a singleton finite linear order $\mathcal{A}$ and a finite linear order $\mathcal{B}$ with two or more elements. Then,

$$\mathcal{A} \cong_1 \mathcal{B}, \text{ but } (\mathcal{A}, \min^{\mathcal{A}}) \not\cong_1 (\mathcal{B}, \min^{\mathcal{B}}),$$

as the duplicator has no appropriate answer in $(\mathcal{A}, \min^{\mathcal{A}})$ when the spoiler chooses an element of $(\mathcal{B}, \min^{\mathcal{B}})$ that is different from $\min^{\mathcal{B}}$. For $k \geqslant 3$, we even have

$$\mathcal{A} \cong_k \mathcal{B} \text{ iff } (\mathcal{A}, \min^{\mathcal{A}}, \max^{\mathcal{A}}) \cong_k (\mathcal{B}, \min^{\mathcal{B}}, \max^{\mathcal{B}}).$$

We have to exclude here the case k=2, since $\mathcal{A} \cong_2 \mathcal{B}$ holds for all linear orders with three or more elements, while $(\mathcal{A}, \min^{\mathcal{A}}, \max^{\mathcal{A}}) \not\cong_2 (\mathcal{B}, \min^{\mathcal{B}}, \max^{\mathcal{B}})$, if $\mathcal{A}$ has exactly three elements (namely $\min^{\mathcal{A}}$ and $\max^{\mathcal{A}}$ and a single element between $\min^{\mathcal{A}}$ and $\max^{\mathcal{A}}$) and $\mathcal{B}$ has four elements, say

$$\min^{\mathcal{B}} = b_1 \sqsubset b_2 \sqsubset b_3 \sqsubset b_4 = \max^{\mathcal{B}}.$$

Then, the duplicator looses any play where the spoiler selects elements b_2 and b_3 in the first and second round.

Notation 1.6.40 (Down-/upward closure of linear orders). If $\mathcal{A} = (A, \sqsubseteq)$ is a linear order and $a \in A$ then $\mathcal{A} \downarrow a$ denotes the substructure that results from $\mathcal{A}$ by deleting all elements a' that are strictly larger than a, i.e., $\mathcal{A} \downarrow a = (A \downarrow a, \sqsubseteq')$ where $A \downarrow a$ denotes the *downward closure* of a and $\sqsubseteq'$ for $\sqsubseteq$ restricted to the elements in $A \downarrow a$. That is,

$$A \downarrow a \stackrel{\text{def}}{=} \{a' \in A : a' \sqsubseteq a\} \quad \text{and} \quad \sqsubseteq' \stackrel{\text{def}}{=} \sqsubseteq \cap (A \downarrow a \times A \downarrow a).$$

$\mathcal{A} \uparrow a$ has an analogous meaning and stands for the substructure that is obtained by considering all elements in the *upward closure* $A \uparrow a$ of a. Here, $A \uparrow a$ consists of the elements $a' \in A$ with $a \sqsubseteq a'$. ■

Lemma 1.6.41 (Concatenation of linear orders). *Let* $\mathcal{A} = (A, \sqsubseteq_A)$ *and* $\mathcal{B} = (B, \sqsubseteq_B)$ *be two finite linear orders and* $a \in A$, $b \in B$. *Then, for all* $k \geqslant 2$*:*

$$\mathcal{A} \downarrow a \cong_k \mathcal{B} \downarrow b \textit{ and } \mathcal{A} \uparrow a \cong_k \mathcal{B} \uparrow b \quad \textit{iff} \quad (\mathcal{A}, a) \cong_k (\mathcal{B}, b)$$

Proof. "$\Longleftarrow$": obvious since each k-round winning strategy for $(\mathcal{A}, a) \cong_k (\mathcal{B}, b)$ can serve as a k-round winning strategy for $\mathcal{A} \downarrow a \cong_k \mathcal{B} \downarrow b$ and for $\mathcal{A} \uparrow a \cong_k \mathcal{B} \uparrow b$.

"$\Longrightarrow$": Suppose the duplicator has winning strategies for $\mathcal{A} \downarrow a \cong_k \mathcal{B} \downarrow b$ and $\mathcal{A} \uparrow a \cong_k \mathcal{B} \uparrow b$. We may assume w.l.o.g. (see Lemma 1.6.39) that the winning strategy for $\mathcal{A} \downarrow a \cong_k \mathcal{B} \downarrow b$

responds with the maximal element a resp. b if and only if the spoiler chooses b resp. a. Similarly, for the winning strategy $\mathcal{A} \uparrow a \cong_k \mathcal{B} \uparrow b$ we assume that the duplicator selects the minimal element a resp. b if and only if the spoiler chooses b resp. a.

To establish $(\mathcal{A}, a) \cong_k (\mathcal{B}, b)$ we have to provide a k-round winning strategy for the duplicator. Suppose that the decisions of the duplicator (by means of this strategy) for the first i rounds are already defined such that each outcome $\langle(a_1, \ldots, a_i), (b_1, \ldots, b_i)\rangle$ of round i defines a partial isomorphism when a and b are treated as the interpretation of a constant symbol. Thus, $a_j \sqsubseteq_{\mathcal{A}} a$ iff $b_j \sqsubseteq_{\mathcal{B}} b$ and $a \sqsubseteq_{\mathcal{A}} a_j$ iff $b \sqsubseteq_{\mathcal{B}} b_j$ for all $1 \leqslant j \leqslant i$. In particular:

$$\{j \in \{1, \ldots, i\} : a_j \sqsubseteq_A a\} \quad = \quad \{j \in \{1, \ldots, i\} : b_j \sqsubseteq_B b\} \quad = \quad \{j_1, \ldots, j_\ell\}$$

for integers $j_1, \ldots, j_\ell$ with $1 \leqslant j_1 < \ldots < j_\ell \leqslant i$. Removing all elements $a_j \sqsupset_A a$ and $b_j \sqsupset_B b$ from $\langle(a_1, \ldots, a_i), (b_1, \ldots, b_i)\rangle$ yields the ℓ-round outcome $\langle(a_{j_1}, \ldots, a_{j_\ell}), (b_{j_1}, \ldots, b_{j_\ell})\rangle$, which can serve as input for the given winning strategy for $\mathcal{A} \downarrow a \cong_k B \downarrow b$. Similarly, removing all elements $a_j \sqsubset_A a$ and $b_j \sqsubset_B b$ from $\langle(a_1, \ldots, a_i), (b_1, \ldots, b_i)\rangle$ yields an outcome that can serve as input for the given winning strategy for $\mathcal{A} \uparrow a \cong_k B \uparrow b$. W.l.o.g., the spoiler chooses $\mathcal{A}$ in the first step of round $i{+}1$.

- If the spoiler picks an element $a_{i+1} \sqsubseteq_A a$ then the duplicator applies the winning strategy for $\mathcal{A} \downarrow a \cong_k \mathcal{B} \downarrow b$ and the game configuration that results from $\langle(a_1, \ldots, a_i), (b_1, \ldots, b_i)\rangle$ by removing all elements $a_j \sqsupset_A a$ and $b_j \sqsupset_B b$.

- Similarly, if the spoiler chooses an element $a_{i+1} \sqsupset_A a$ then the duplicator applies the winning strategy for
$$\mathcal{A} \uparrow a \cong_k \mathcal{B} \uparrow b$$
and the game configuration that results from $\langle(a_1, \ldots, a_i), (b_1, \ldots, b_i)\rangle$ by removing all elements $a_j \sqsubset_A a$ and $b_j \sqsubset_B b$.

This strategy is, in fact, winning and yields $(\mathcal{A}, a) \cong_k (\mathcal{B}, b)$. □

Lemma 1.6.42 (Games on finite linear orders). *If $\mathcal{A} = (A, \sqsubseteq_A)$ and $\mathcal{B} = (B, \sqsubseteq_B)$ are finite linear orders such that $|A|, |B| \geqslant 2^k$ for some $k \in \mathbb{N}$ then $\mathcal{A} \cong_k \mathcal{B}$.*

Proof. The proof is by induction on k. Since the underlying vocabulary has no constant symbols and $a \sqsubseteq a$ for all elements a of a linear order we have $\mathcal{A} \cong_1 \mathcal{B}$ for all nonempty finite linear orders $\mathcal{A}$ and $\mathcal{B}$. Thus, the cases $k{=}0$ or $k{=}1$ hold without any assumptions on the lengths of $\mathcal{A}$ and $\mathcal{B}$. For $k{=}2$ we need the assumption that $|A|, |B| \geqslant 2^2 = 4$. In particular, the maximal element is strictly larger than the minimal element and there are at least two elements different from the minimal and maximal element. This ensures that $\mathcal{A} \cong_2 \mathcal{B}$.

In the induction step $k{-}1 \Longrightarrow k$ ($k \geqslant 3$), we suppose that $|A|, |B| \geqslant 2^k$. We show that the conditions (1) and (2) of the recursive characterization of $\cong_k$ in Lemma 1.6.19 (page 96) hold. By symmetry it suffices to establish condition (1). So, we take an element $a \in A$. The goal is to find an element $b \in B$ such that $(\mathcal{A}, a) \cong_{k-1} (\mathcal{B}, b)$. For this, we apply Lemma 1.6.41 and look for an element $b \in B$ such that the downward closures $\mathcal{A} \downarrow a$ and $\mathcal{B} \downarrow b$ are $\cong_{k-1}$-equivalent and the upward closures $\mathcal{A} \uparrow a$ and $\mathcal{B} \uparrow b$ are $\cong_{k-1}$-equivalent.

- If $\mathcal{A} \downarrow a$ contains fewer than 2^{k-1} elements and a is the j-smallest element of $\mathcal{A}$ then we define b as the j-smallest element of $\mathcal{B}$. Then, $j < 2^{k-1}$ and $\mathcal{A} \downarrow a$ and $\mathcal{B} \downarrow b$ are isomorphic. In particular,
$$\mathcal{A} \downarrow a \cong_{k-1} \mathcal{B} \downarrow b.$$
The length of $\mathcal{A} \uparrow a$ and $\mathcal{B} \uparrow b$ is at least
$$2^k - j + 1 \; > \; 2^k - 2^{k-1} \; = \; 2^{k-1}.$$
The induction hypothesis yields $\mathcal{A} \uparrow a \cong_{k-1} \mathcal{B} \uparrow b$. Hence, $(\mathcal{A}, a) \cong_{k-1} (\mathcal{B}, b)$ by Lemma 1.6.41.

- An analogous argument applies if $\mathcal{A} \uparrow a$ contains fewer than 2^{k-1} elements.

- Let us now assume that $\mathcal{A} \uparrow a$ and $\mathcal{A} \downarrow a$ have at least 2^{k-1} elements. Since $\mathcal{B}$ has 2^k or more elements there exists $b \in B$ such that $\mathcal{B} \downarrow b$ and $\mathcal{B} \uparrow b$ have at least 2^{k-1} elements, e.g., we may take the 2^{k-1}-smallest element of $\mathcal{B}$. By the induction hypothesis:
$$\mathcal{A} \uparrow a \cong_{k-1} \mathcal{B} \uparrow b \quad \text{and} \quad \mathcal{A} \downarrow a \cong_{k-1} \mathcal{B} \downarrow b.$$
Again, Lemma 1.6.41 yields $(\mathcal{A}, a) \cong_{k-1} (\mathcal{B}, b)$.

□

We are now ready to prove that the class of finite linear orders of even length cannot be characterized in FOL. This result is stronger than Theorem 1.6.5 on page 85 stating that the class of finite structures of even cardinality is not FO-definable, since there we used the empty vocabulary, while we suppose here the vocabulary of linear orders which consists of a binary predicate symbol $\sqsubseteq$. However, dealing with richer vocabularies, the class of finite structures of even length might have characterizations in FOL.

Theorem 1.6.43 (Finite linear orders of even length are not FO-definable). *There is no FOL-sentence ϕ over the vocabulary consisting of a binary predicate symbol $\sqsubseteq$ such that for each finite linear order $\mathcal{A} = (A, \sqsubseteq)$:*
$$\mathcal{A} \models \phi \quad \textit{iff} \quad |A| \textit{ is even}$$

Proof. The idea is to apply Lemma 1.6.42 for showing that the criterion for non-FO-definability stated in Corollary 1.6.37 is satisfied. This amounts proving that for each $k \geqslant 1$ there are finite linear orders $\mathcal{A} = \mathcal{A}_k$ and $\mathcal{B} = \mathcal{B}_k$ such that $\mathcal{A} \cong_k \mathcal{B}$ and the length of $\mathcal{A}$ is even, while $\mathcal{B}$ has odd length. We consider the finite linear orders $\mathcal{A} = (A, \leqslant)$ and $\mathcal{B} = (B, \leqslant)$ with the domains
$$A = \{1, \ldots, 2^k\} \qquad \text{and} \qquad B = \{1, \ldots, 2^k+1\}$$
and the natural linear order $\leqslant$. Then, $|A|$ is even and $|B|$ is odd (since $k \geqslant 1$) and $\mathcal{A} \cong_k \mathcal{B}$ by Lemma 1.6.42. □

In Theorem 1.6.10 on page 89 we used the Upward Löwenheim-Skolem Theorem to show that reachability is not FO-definable. But there we ranged over all graph structures, including infinite ones. We are now in the position to show that FOL is even not powerful enough to characterize reachability in finite graphs:

Theorem 1.6.44 (Reachability on finite graphs is not FO-definable). *There is no FOL-formula $\phi(x,y)$ over the vocabulary Voc_{graph} such that for all finite graphs $\mathcal{G}$ and nodes a, b in $\mathcal{G}$:*

$$(\mathcal{G}, a, b) \models \phi(x,y) \quad \textit{iff} \quad b \textit{ is reachable from } a \textit{ in } \mathcal{G}$$

Proof. Suppose by contradiction that such a FOL-formula $\phi(x,y)$ over $\mathit{Voc}_{\text{graph}}$ (with a single binary predicate symbol E for the edge relation) exists. Then, $\forall x \forall y.\phi(x,y)$ is a FOL-sentence that characterizes strong connectivity of finite graphs. I.e., for all finite graphs $\mathcal{G}$:

$$\mathcal{G} \models \forall x \forall y.\phi(x,y) \quad \text{iff} \quad \mathcal{G} \text{ is strongly connected}$$

We show that under this assumption finite linear orders of even length would be FO-definable (which yields a contradiction to Theorem 1.6.43). For this, we will define a FOL-formula $\psi(z_1,z_2)$ over the vocabulary of linear orders (with a single binary predicate symbol $\sqsubseteq$) such that:

$$|A| \text{ is odd} \quad \text{iff} \quad \mathcal{G}_{\mathcal{A}} \text{ is strongly connected}$$

where $\mathcal{G}_{\mathcal{A}} = (A, E^{\mathcal{A}})$ is the directed graph with the following edge relation:

$$E^{\mathcal{A}} \stackrel{\text{def}}{=} \big\{ (a,a') \in A \times A : (\mathcal{A}, a, a') \models \psi(z_1,z_2) \big\}$$

The remaining arguments are then as follows. Let $\theta \stackrel{\text{def}}{=} \neg \forall x \forall y.\phi(x,y)$ and

$$\theta' \stackrel{\text{def}}{=} \theta\big[E(z_1,z_2)/\psi(z_1,z_2) : z_1,z_2 \in \mathit{Var}\big]$$

be the FOL-sentence over the vocabulary of linear orders that results from θ by replacing any atom $E(z_1,z_2)$ with the formula $\psi(z_1,z_2)$. Then, for each finite linear order $\mathcal{A} = (A, \sqsubseteq)$:

$$\mathcal{A} \models \theta' \quad \text{iff} \quad \mathcal{G}_{\mathcal{A}} \models \theta$$

This can be shown by considering the transformation

$$\chi \mapsto \chi' \stackrel{\text{def}}{=} \chi\big[E(z_1,z_2)/\psi(z_1,z_2) : z_1,z_2 \in \mathit{Var}\big]$$

that maps an arbitrary FOL-formula χ over the vocabulary of graphs to a formula χ' over the vocabulary of linear orders. By structural induction it can then be shown that for all formulas χ and all finite linear orders $\mathcal{A}$ and variable valuations $\mathcal{V}$:

$$(\mathcal{G}_{\mathcal{A}}, \mathcal{V}) \models \chi \quad \text{iff} \quad (\mathcal{A}, \mathcal{V}) \models \chi'$$

In the basis of induction we have to consider the cases where χ is *true*, an atomic formula of the form $z_1 = z_2$ or $E(z_1,z_2)$ where z_1,z_2 are variables. (Recall that the vocabulary of graphs has no function symbols. Thus, terms are just variables.) The first two cases are obvious. The argument for atoms of the form $\chi = E(z_1,z_2)$ uses the definition of the graph $\mathcal{G}_{\mathcal{A}}$. Let $a_1 = \mathcal{V}(z_1)$ and $a_2 = \mathcal{V}(z_2)$. Then:

$$(\mathcal{G}_{\mathcal{A}}, \mathcal{V}) \models \underbrace{E(z_1,z_2)}_{=\chi} \ \text{ iff } \ (a_1,a_2) \in E^{\mathcal{A}} \ \text{ iff } \ (\mathcal{A}, a_1, a_2) \models \underbrace{\psi(z_1,z_2)}_{=\chi'}$$

The step of induction is straightforward and omitted here. With $\chi = \theta$ and $\chi' = \theta'$ we get:

$$\begin{array}{lll} \mathcal{A} \models \theta' & \text{iff} & \mathcal{G}_{\mathcal{A}} \models \theta = \neg\forall x \forall y.\phi(x,y) \\ & \text{iff} & \mathcal{G}_{\mathcal{A}} \not\models \forall x \forall y.\phi(x,y) \\ & \text{iff} & \mathcal{G}_{\mathcal{A}} \text{ is not strongly connected} \\ & \text{iff} & |A| \text{ is even} \end{array}$$

But such a formula θ' does not exist by Theorem 1.6.43. Contradiction.

It remains to provide the definition of the formula $\psi(z_1,z_2)$ over the vocabulary of linear orders such that the graph $\mathcal{G}_{\mathcal{A}} = (A, E^{\mathcal{A}})$ is strongly connected if and only if the length of $\mathcal{A}$ is odd. Let

$$z_1 \lhd z_2 \quad \stackrel{\text{def}}{=} \quad z_1 \sqsubset z_2 \wedge \forall z.\big((z \sqsubseteq z_1) \vee (z_2 \sqsubseteq z) \big)$$

That is, $z_1 \lhd z_2$ denotes that z_2 is the (direct) successor of z_1 in the given linear order. Furthermore, we define formulas that state that z is the first (minimal) element and the last (maximal) element of the given finite linear order:

$$\mathit{first}(z) \ \stackrel{\text{def}}{=}\ \forall z'.z \sqsubseteq z' \qquad \text{and} \qquad \mathit{last}(z) \ \stackrel{\text{def}}{=}\ \forall z'.z' \sqsubseteq z$$

We then put:

$$\psi \ \stackrel{\text{def}}{=}\ \psi_1 \vee \psi_2 \vee \psi_3$$

where ψ_1, ψ_2 and ψ_3 are as follows. Formula ψ_1 formalizes that z_2 is the successor of the successor of z_1:

$$\psi_1(z_1,z_2) \ \stackrel{\text{def}}{=}\ \exists z.\big((z_1 \lhd z) \wedge (z \lhd z_2) \big)$$

Formula ψ_2 requires z_1 to be the predecessor of the maximal element and z_2 to be the minimal element:

$$\psi_2(z_1,z_2) \ \stackrel{\text{def}}{=}\ \mathit{first}(z_2) \wedge \exists z.((z_1 \lhd z) \wedge \mathit{last}(z))$$

Formula ψ_3 is the dual of ψ_2 and states that z_1 is the maximal element, while z_2 is the second element (i.e., the successor of the minimal element):

$$\psi_3(z_1,z_2) \ \stackrel{\text{def}}{=}\ \mathit{last}(z_1) \wedge \exists z.((z \lhd z_2) \wedge \mathit{first}(z))$$

We then have:

$$|A| \text{ is odd} \quad \text{iff} \quad \mathcal{G}_{\mathcal{A}} \text{ is strongly connected}$$

Let us check this. Suppose first that $|A|$ is odd, say $A = \{a_1, \ldots, a_{2k+1}\}$ where $a_1 \lhd a_2 \lhd \ldots \lhd a_{2k+1}$. Then:

$$\begin{array}{rcll} E^{\mathcal{A}} & = & \{ (a_{2i-1}, a_{2i+1}) : 1 \leqslant i \leqslant k \} & \cup \\ & & \{ (a_{2i}, a_{2i+2}) : 1 \leqslant i < k \} & \cup \\ & & \{ (a_{2k}, a_1),\ (a_{2k+1}, a_2) \} & \end{array}$$

But then $a_1\, a_3 \ldots a_{2k+1}\, a_2\, a_4 \ldots a_{2k}\, a_1$ is a path in $\mathcal{G}_{\mathcal{A}}$ that runs through all elements of A. Thus, $\mathcal{G}_{\mathcal{A}}$ is strongly connected.

Vice versa, if $|A|$ is even, say $A = \{a_1, \ldots, a_{2k}\}$ where $a_1 \lhd a_2 \lhd \ldots \lhd a_{2k}$, then:

$$\begin{array}{rcll} E^{\mathcal{A}} & = & \{ (a_{2i-1}, a_{2i+1}) : 1 \leqslant i < k \} & \cup \\ & & \{ (a_{2i}, a_{2i+2}) : 1 \leqslant i < k \} & \cup \\ & & \{ (a_{2k-1}, a_1), (a_{2k}, a_2) \} & \end{array}$$

The elements $a_1, a_3, \ldots, a_{2k-1}$ with odd index constitute a strongly connected component, but there is no edge to one of the elements $a_2, a_4, \ldots, a_{2k}$ with even index. More precisely, $\mathcal{G}_\mathcal{A}$ consists of two disjoint simple cycles, namely $a_1\, a_3 \ldots a_{2k-1}\, a_1$ and $a_2\, a_4 \ldots a_{2k}\, a_2$. Thus, $\mathcal{G}_\mathcal{A}$ is not strongly connected. $\square$

We finally show that connectivity of finite undirected graphs is not FO-definable. In our approach, we define an undirected graph as a pair $\mathcal{G} = (A, E)$ where A is a set and E a symmetric binary relation on A. Thus, we view undirected graphs as special instances of directed graphs and use FOL-formulas with equality over the vocabulary Voc_{graph} to specify properties for them. Connectivity of an undirected graph means that for all nodes a, b with $a \neq b$ there is a path $a_0\, a_1 \ldots a_n$ from $a = a_0$ to $a_n = b$ such that $n \geqslant 1$ and $a_0, a_1, \ldots, a_n$ are pairwise distinct.

Theorem 1.6.45 (Connectivity of finite undirected graphs is not FO-definable). *There is no FOL-sentence ϕ over Voc_{graph} such that for all finite undirected graphs $\mathcal{G} = (A, E)$:*

$$\mathcal{G} \textit{ is connected} \quad \textit{iff} \quad \mathcal{G} \models \phi$$

Proof. We apply Corollary 1.6.37 (see page 108) and show that for each $k \in \mathbb{N}$ there exist two undirected graphs $\mathcal{A} = \mathcal{A}_k$ and $\mathcal{B} = \mathcal{B}_k$ such that $\mathcal{A}$ is connected, $\mathcal{B}$ is not connected and $\mathcal{A} \cong_k \mathcal{B}$. Graph $\mathcal{A}$ consists of a simple cycle with 2^{k+1} nodes. Say the node-set A of $\mathcal{A}$ and the edge relation $E^\mathcal{A}$ are given by:

$$A \;=\; \big\{ v_j : j \in \mathbb{N}, 0 \leqslant j < 2^{k+1} \big\}, \qquad E^\mathcal{A} \stackrel{\text{def}}{=} \big\{ (v_j, v_{j+1}), (v_{j+1}, v_j) : 0 \leqslant j < 2^{k+1} \big\}$$

where $v_{2^{k+1}}$ is identified with v_0. Graph $\mathcal{B}$ consists of two disjoint copies of this simple cycle, say the node set of $\mathcal{B}$ is

$$B \;=\; \big\{ w_j, u_j : j \in \mathbb{N}, 0 \leqslant j < 2^{k+1} \big\}$$

and the edge relation in $\mathcal{B}$ is

$$E^\mathcal{B} \;=\; \big\{ (w_j, w_{j+1}), (w_{j+1}, w_j), (u_j, u_{j+1}), (u_{j+1}, u_j) : 0 \leqslant j < 2^{k+1} \big\}$$

where $w_{2^{k+1}} = w_0$ and $u_{2^{k+1}} = u_0$. Obviously $\mathcal{A}$ is connected, while $\mathcal{B}$ is not. To prove that $\mathcal{A} \cong_k \mathcal{B}$ we define a distance function d for the nodes in $\mathcal{A}$ and $\mathcal{B}$ (similar to the one used in Example 1.6.34 on page 104). Given two nodes v, w in either $\mathcal{A}$ or $\mathcal{B}$ then $d(v, w)$ denotes the length of a shortest path from v to w, if such a path exists. Otherwise $d(v, w) = \infty$. Since the edge relations are symmetric, so is the distance function, i.e., $d(v, w) = d(w, v)$. Furthermore, we have:

$$d(v_h, v_j) \;=\; d(w_h, w_j) \;=\; d(u_h, u_j) \;=\; |h - j|$$

and $d(w_h, u_j) \;=\; d(u_j, w_h) \;=\; \infty$ for all h and $j \in \big\{0, 1, \ldots, 2^{k+1} - 1\big\}$. Note that

$$d(v, w) = 0 \;\text{ iff }\; v = w \qquad \text{and} \qquad d(v, w) = 1 \;\text{ iff }\; (v, w) \text{ is an edge.}$$

The goal is now to design a k-round winning strategy $\mathcal{S}$ such that for each possible game configuration $\langle (a_1, \ldots, a_i), (b_1, \ldots, b_i) \rangle \in A^i \times B^i$ after the i-th round we have for all $h, j \in \{1, \ldots, i\}$:

(1) If $d(a_h, a_j) < 2^{k-i+1}$ then $d(a_h, a_j) \;=\; d(b_h, b_j)$.

(2) If $d(a_h, a_j) \geqslant 2^{k-i+1}$ then $d(b_h, b_j) \geqslant 2^{k-i+1}$.

For the last round (i.e., $i = k$) we get that for all $h, j \in \{1, \ldots, k\}$:

- if $a_h = a_j$ then $d(a_h, a_j) = 0 = d(b_h, b_j)$ (by (1)), and therefore $b_h = b_j$.
- if (a_h, a_j) is an edge in $\mathcal{A}$ then $d(a_h, a_j) = 1 < 2 = 2^1$ and $d(b_h, b_j) = 1$ (by (1)). Thus, (b_h, b_j) is an edge in $\mathcal{B}$.

By the symmetry of conditions (1) and (2), we can also conclude that $b_h = b_j$ yields $a_h = a_j$ and that $(b_h, b_j) \in E^{\mathcal{B}}$ implies $(a_h, a_j) \in E^{\mathcal{A}}$. Hence, all outcomes after the k-th round that satisfy conditions (1) and (2) are winning.

It remains to explain how the duplicator can preserve the distance conditions (1) and (2), which are obvious for the initial game configuration $\langle \varepsilon, \varepsilon \rangle$. For the first round the duplicator may pick an arbitrary node in the structure not chosen by the spoiler. Assume now that $1 \leqslant i < k$ and that the current game configuration after the i-th round is $\langle (a_1, \ldots, a_i), (b_1, \ldots, b_i) \rangle$ such that (1) and (2) hold. Let us suppose that the spoiler picks $a = a_{i+1} \in A$.

Case 1. There exists an element $a_h \in \{a_1, \ldots, a_i\}$ such that $d(a_h, a) < 2^{k-i}$.

Then, for all elements $a_j \in \{a_1, \ldots, a_i\}$ where $d(a_j, a) < 2^{k-i}$ we have:

$$d(a_h, a_j) \leqslant d(a_h, a) + d(a, a_j) < 2^{k-i} + 2^{k-i} = 2^{k-i+1}$$

That is, all elements a_j in the $(2^{k-i}-1)$-neighbourhood of $a = a_{i+1}$ are in the $(2^{k-i+1}-1)$-neighbourhood of a_h. But then we have $d(b_h, b_j) = d(a_h, a_j)$ for all these indices j. The duplicator chooses a "corresponding" element $b = b_{i+1} \in B$ in the $(2^{k-i}-1)$-neighbourhood of b_h. That is, b is a node in $\mathcal{B}_k$ such that

$$d(b, b_j) = d(a, a_j) < 2^{k-i+1}$$

for all indices $j \in \{1, \ldots, i\}$ where $d(a_h, a_j) = d(b_h, b_j) < 2^{k-i+1}$. In particular, this choice of b ensures that:

$$\text{whenever } j \in \{1, \ldots, i\} \text{ and } d(a, a_j) < 2^{k-i} \text{ then } d(b, b_j) = d(a, a_j).$$

For the indices $j \in \{1, \ldots, i\}$ where a_j is outside the $(2^{k-i+1}-1)$-neighbourhood of a_h, we have:

$$2^{k-i+1} \leqslant d(a_j, a_h) \leqslant d(a_j, a) + d(a, a_h)$$

and therefore

$$d(a_j, a) \geqslant 2^{k-i+1} - \underbrace{d(a, a_h)}_{<2^{k-i}} > 2^{k-i}$$

Since $d(a_j, a_h) \geqslant 2^{k-i+1}$ implies $d(b_j, b_h) \geqslant 2^{k-i+1}$ (by assumption) we also have

$$d(b_j, b) \geqslant \underbrace{d(b_j, b_h)}_{\geqslant 2^{k-i+1}} - \underbrace{d(b, b_h)}_{<2^{k-i}} > 2^{k-i}$$

Thus, for all indices $j \in \{1, \ldots, i\}$:

$$\text{either } d(a, a_j) = d(b, b_j) < 2^{k-i} \text{ or } d(a, a_j), d(b, b_j) \geqslant 2^{k-i}$$

Case 2. The spoiler chooses an element $a = a_{i+1} \in A$ such that $d(a_j, a) \geqslant 2^{k-i}$ for all $j \in \{1, \ldots, i\}$. Then, the duplicator can choose an arbitrary element $b = b_{i+1} \in B$ with $d(b_j, b) \geqslant 2^{k-i}$ for all $j \in \{1, \ldots, i\}$. Note that such an element exists since the number of elements b' with $d(b_j, b') \geqslant 2^{k-i}$ for $1 \leqslant j \leqslant i$ is at least

$$2^{k+1} - i \cdot (2^{k-i+1} - 1) \;>\; 2^{k+1} - i \cdot 2^{k-i+1} \;=\; 2^{k+1} - \underbrace{\frac{i}{2^i}}_{\leqslant 1} \cdot 2^{k+1} \;\geqslant\; 0.$$

The above calculation uses 2^{k+1} as a lower bound for the number of nodes in $\mathcal{B}$ (graph $\mathcal{B}$ even has 2^{k+2} nodes) and makes use of the fact that the number of elements b'' with $d(b', b'') < 2^{k-i}$ is

$$2 \cdot (2^{k-i} - 1) + 1 \;=\; 2^{k-i+1} - 1.$$

Thus, the total number of nodes which are in the $(2^{k-i} - 1)$-neighbourhood of some node b_j for $1 \leqslant j \leqslant i$ is at most $i \cdot (2^{k-i+1} - 1)$.

If the spoiler selects structure $\mathcal{B}$ and an element $b = b_{i+1}$ in $\mathcal{B}$ then the strategy for the duplicator that ensures conditions (1) and (2) is defined in an analogous way. Note that in the second case we calculated with 2^{k+1} as a lower bound for the number of nodes in $\mathcal{B}$. So, the above calculation also applies to the case where the duplicator has to choose a node in $\mathcal{A}$. $\square$