2.2 Second-order prenex and Skolem forms

Each FOL-formula ϕ has an equivalent FOL-formula in prenex form and it can be transformed into a Skolem formula that is satisfiability-equivalent to ϕ. (Satisfiability-equivalence of two formulas ϕ and ϕ' means that either both ϕ and ϕ' are satisfiable or both are not satisfiable.)

We now adapt these concepts to second-order logic. For SOL, prenex forms can be obtained as in the first-order case. Furthermore, SOL even provides *equivalent* Skolem formulas for all FOL-formulas.

Definition 2.2.1 ((Strong) prenex form). A SOL-formula is said to be in *prenex form* (and called a SOL-prenex formula) if it consists of a quantifier prefix for first- and second-order variables, followed by a quantifier-free formula, called the *matrix*. Any SOL-formula that starts with a sequence of second-order quantifiers, followed by a sequence of first-order quantifiers and a quantifier-free formula is said to be in *strong prenex form*. ■

Thus, SOL-formulas in strong prenex form have the form

$$\underbrace{\mathbf{Q}_1 X_1\ \mathbf{Q}_2 X_1 \ldots \mathbf{Q}_n X_n}_{\text{second-order quantifiers}}\ \underbrace{Q_1' y_1\ Q_2' y_2 \ldots Q_m' y_m}_{\text{first-order quantifiers}}.\psi$$

where ψ is quantifier-free, $\mathbf{Q}_i X_i$ is a second-order (predicate or function) quantifier and $Q_j' y_j$ a first-order quantifier. The cases $n = 0$ or $m = 0$ are allowed. E.g., if ψ is quantifier-free, X, Y Z are predicate variables, F a function variable and x, y, z first-order variables then

$$\begin{aligned}
\phi_1 &= \forall X \exists F \exists y \forall Y \exists Z \forall x.\psi \\
\phi_2 &= \exists x \forall X \exists Y \exists Z \forall y \exists F.\psi \\
\phi_3 &= \exists F \forall X \exists Y \exists Z \forall y \exists x.\psi
\end{aligned}$$

are prenex SOL-formulas, but only ϕ_3 is in strong prenex SOL-form.

Using the same technique as it is known for FOL (see page 12), any SOL-formula can be transformed into an equivalent prenex SOL-formula. The so obtained prenex SOL-formula might start with an arbitrary mixture of first- and second-order quantifiers. An equivalent strong prenex form can be obtained by applying the following equivalence rules where quantification over first-order variables is mimicked by quantificaton over singleton sets:

$$\begin{aligned}
\exists x \overline{\mathbf{Q}}.\psi &\equiv \exists X \overline{\mathbf{Q}}.\big(\exists! x. X(x) \wedge \forall x.(X(x) \rightarrow \psi)\big) \\
\forall x \overline{\mathbf{Q}}.\psi &\equiv \forall X \overline{\mathbf{Q}}.\big(\exists! x. X(x) \rightarrow \forall x.(X(x) \rightarrow \psi)\big)
\end{aligned}$$

where $\overline{\mathbf{Q}}$ is an arbitrary sequence of first- or second-order quantifiers and X is a fresh predicate variable of arity 1 (where "fresh" means that X does not have free or bounded occurrences in ψ). Intuitively, in the first equivalence, set variable X represents a singleton set consisting of an element x that serves as a witness for $\exists x \overline{\mathbf{Q}}.\psi$. Similarly, in the second equivalence, instead of ranging over all individual elements x of the domain we range over all singleton sets $X = \{x\}$.

The correctness of the above equivalences relies on the fact that

$$\exists X\, \overline{\mathbf{Q}}.\big(\exists! x. X(x) \wedge \forall x.(X(x) \rightarrow \psi)\big) \equiv \exists X.\big(\exists! x. X(x) \wedge \forall x.(X(x) \rightarrow \overline{\mathbf{Q}}.\psi)\big)$$

as X does not appear in the FO- or SO-variables in the quantifier sequence $\overline{\mathbf{Q}}$. Similarly:

$$\forall X \overline{\mathbf{Q}}. \big(\exists! x. X(x) \;\rightarrow\; \forall x. (X(x) \rightarrow \psi) \big) \;\equiv\; \forall X \big(\exists! x. X(x) \;\rightarrow\; \forall x. (X(x) \rightarrow \overline{\mathbf{Q}}. \psi) \big)$$

Given a prenex SOL-formula ϕ, we may use the above equivalence rules as rewrite rules to generate an equivalent SOL-formula that starts with a prefix of second-order quantifiers followed by a FOL-formula (when the predicate and function variables are viewed as predicate and function symbols, respectively). The inner FOL-formula can then be replaced with an equivalent prenex formula which yields a SOL-formula ϕ' which is in strong prenex form and equivalent to ϕ. Thus, we have:

Lemma 2.2.2. *Each SOL-formula ϕ can be transformed into an equivalent SOL-formula ϕ' in strong prenex form such that $\|\phi'\| = \mathcal{O}(\,\|\phi\|\,)$.*

We now consider Skolem SOL-formulas and will show that any FOL-formula is equivalent to such a second-order Skolem form.

Definition 2.2.3 (Skolem SOL-formula). A SOL-formula ϕ is said to be in *Skolem form* (and called a Skolem SOL-formula) if it starts with a possibly empty prefix of existential first-order quantifiers and second-order function quantifiers, followed by a FOL-formula in Skolem form, i.e., it has the form:

$$\underbrace{\exists\,\exists\,\exists\,\ldots\,\exists}_{\substack{\text{existential FO or}\\ \text{SO-function quantifiers}}}\;\underbrace{\forall x_1 \ldots \forall x_n}_{\substack{\text{universal}\\ \text{FO quantifiers}}}.\,\psi$$

where ψ is quantifier-free, called the *matrix*. ■

Since the above definition of Skolem SOL-formulas permits an arbitrary mixture of existential FO- and SO-quantifiers in front of the universal FO-quantifiers, Skolem SOL-formulas need not to be in strong prenex form. However, as the order of consecutive existential SO- or FO-quantifiers is irrelevant, any Skolem SOL-formula ϕ can be transformed into an equivalent Skolem SOL-formula ϕ' in strong prenex form: just move all existential SO-quantifiers to the left of the existential FO-quantifiers.

Example 2.2.4 (From FOL to Skolem SOL-forms). The FOL-formula $\forall x \exists y. P(x, y)$ is equivalent to the Skolem SOL-formula

$$\exists F \forall x. P(x, F(x)).$$

The fact that $\forall x \exists y.\, P(x, y)$ is a logical consequence of $\exists F \forall x.\, P(x, F(x))$ is obvious. Let us check the other direction, i.e., we show that

$$\forall x \exists y. P(x, y) \;\Vdash\; \exists F \forall x. P(x, F(x)).$$

Let $\mathcal{A}$ be a model for $\forall x \exists y. P(x, y)$ with domain A. Then, for each $a \in A$ there exists $a' \in A$ such that $(a, a') \in P^{\mathcal{A}}$. Using the axiom of choice we obtain a function

$$F^{\mathcal{A}} : A \rightarrow A \text{ such that } (a, F^{\mathcal{A}}(a)) \in P^{\mathcal{A}} \text{ for all } a \in A.$$

Hence, $\mathcal{A}$ is a model for $\exists F \forall x.\, P(x, F(x))$. ■

While FOL fails to provide equivalent Skolem FOL-formulas for all formulas (e.g., $\forall x \exists y.\, P(x,y)$ has no equivalent Skolem FOL-formula), any FOL-formula can be transformed into an *equivalent* SOL-formula in Skolem form. For this, we use roughly the same transformation as for constructing satifiability-equivalent Skolem FOL-formulas from given FOL-formulas. The major difference is that now we may use existential SO-quantification to bound the auxiliary fresh function variables. The algorithm works as follows. We first transform the given FOL-formula into prenex form. As long as the given formula has the form

$$\phi \;=\; \underbrace{\exists\,\exists\,\ldots\,\exists}_{\substack{\text{existential}\\ \text{SO/FO-quantifiers}}}\;\underbrace{\forall x_1\,\forall x_2\,\ldots\,\forall x_k}_{\substack{\text{universal}\\ \text{FO-quantifiers}}}\;\underbrace{\exists y}_{\substack{\text{existential}\\ \text{FO-quantifier}}}\,.\psi$$

where $k \geqslant 1$, we eliminate the FO-quantifier $\exists y$ by introducing an SO-quantifier $\exists F$ with a fresh k-ary function variable F and replacing the above formula with

$$\exists F\,\exists\,\exists\,\ldots\,\exists\,\forall x_1\,\forall x_2\ldots\forall x_k.\; \psi[y/F(x_1,\ldots,x_k)]$$

The resulting SOL-formula ϕ_{Skolem} is in Skolem form and equivalent to the original FOL-formula ϕ. Formulas ϕ and ϕ_{Skolem} have the same length (since the transformation just relies on replacing FO-quantifiers with SO-quantifiers and variables with terms, but does not increase the number of operators). The word-length of ϕ_{Skolem}, however, can be quadratic in the word-length of ϕ. We obtain:

Theorem 2.2.5 (Existence of equivalent Skolem SOL-formulas). *For each FOL-formula ϕ there exists an equivalent SOL-Skolem formula of word-length $\mathcal{O}(\|\phi\|^2)$.*

Example 2.2.6 (Skolem SOL-formulas). For the FOL-formula

$$\phi \;=\; \forall u\,\exists v\,\forall w\,\exists x\,\forall y\,\exists z.\, P(u,v,w,x,y,z),$$

we first remove the existential FO-quantification $\exists v$ by introducing a new unary function variable H and replacing ϕ with

$$\phi' \;=\; \exists H\,\forall u\,\forall w\,\exists x\,\forall y\,\exists z.\, P\big(u,H(u),w,x,y,z\big).$$

We then have to remove the quantifier $\exists x$ by introducing a new binary function variable G and replacing ϕ' with

$$\phi'' \;=\; \exists G\,\exists H\,\forall u\,\forall w\,\forall y\,\exists z.\, P\big(u,H(u),w,G(u,w),y,z\big).$$

It remains to remove the quantifier $\exists z$. For this we add a fresh ternary function variable F and obtain a Skolem SOL-formula that is equivalent to ϕ:

$$\phi_{\text{Skolem}} \;=\; \exists F\,\exists G\,\exists H\,\forall u\,\forall w\,\forall y.\, P\big(u,H(u),w,G(u,w),y,F(u,w,y)\big).$$

For another example, the FOL-sentence $\exists v \forall w \exists x \forall y \exists z.\, P(v,w,x,y,z)$ is equivalent to

$$\exists F\,\exists v\,\forall w\,\forall y\,\exists z.\, P\big(v,w,F(v,w),y,z\big).$$

This formula is equivalent to $\exists G\,\exists F\,\exists v\,\forall w\,\forall y.\, P\big(v,w,F(v,w),y,G(v,w,y)\big)$. ∎

2.3 Properties of SOL

Each of the extensions of FOL studied in the end of Chapter 1 (the logics FOL($\overset{\infty}{\exists}$), FOL($\overset{2^{\mathbb{N}}}{\exists}$) and FOL($\bigwedge$)) violates exactly one of the conditions imposed by the compactness theorem or the downward Löwenheim-Skolem theorem. We now show that for SOL none of these two conditions holds. The crucial point is that SOL provides characterizations of both finite sets and countable sets.

Example 2.3.1 (SOL-characterizations of infinite structures). The following SOL-sentence over the empty vocabulary characterizes the structures with an infinite domain:

$$\psi_\infty \stackrel{\text{def}}{=} \exists F.\Big(\underbrace{\forall x \forall y.(F(x) = F(y) \rightarrow x = y)}_{\text{injective}} \wedge \underbrace{\exists z \forall x.(F(x) \neq z)}_{\text{not surjective}}\Big)$$

where F is an unary function variable. Formula ψ_∞ asserts the existence of a function $F : A \rightarrow A$ that is injective (left subformula), but not surjective (right subformula). This holds for any infinite set A, but is impossible for finite sets. The latter is a well-known fact.

> For the sake of completeness, let us recall the argument. If $F : A \rightarrow A$ is injective and not surjective then an infinite subset of A can be generated as follows. Since F is not surjective, there is some $a_0 \in A \setminus F(A)$. We then define inductively $a_{n+1} = F(a_n)$ for all $n \in \mathbb{N}$. If we suppose that $a_n = a_m$ where $1 \leqslant n < m$ then by the injectivity of F we get $a_{n-1} = a_{m-1}$. We repeat this argument several times and obtain $a_0 = a_{m-n}$. But this is impossible as $a_0 \notin F(A)$, while $a_{m-n} = F(a_{m-n-1}) \in F(A)$. Hence, $\{a_n : n \in \mathbb{N}\}$ is an infinite subset of A.
>
> Vice versa, if A is infinite then we pick pairwise distinct elements $a_0, a_1, a_2, \ldots \in A$ and define function $F : A \rightarrow A$ by $F(a) = a$ if $a \notin \{a_n : n \in \mathbb{N}\}$ and $F(a_n) = a_{n+1}$ for all $n \in \mathbb{N}$. Then, F is injective, but not surjective.

Thus, the models for ψ_∞ are exactly the structures with an infinite domain. Instead of an unary function variable, one might also use a binary predicate variable X to characterize the infinite structures:

$$\psi'_\infty \stackrel{\text{def}}{=} \exists X.\big(\phi_1 \wedge \phi_2 \wedge \phi_3\big)$$

where

$$\begin{aligned}
\phi_1 &= \forall x \forall y \forall z.\big(X(x,y) \wedge X(y,z) \rightarrow X(x,z)\big)\\
\phi_2 &= \forall y \exists z. X(y,z)\\
\phi_3 &= \forall y. \neg X(y,y)
\end{aligned}$$

Formula $\phi_1 \wedge \phi_2$ states the transitivity and totality of X, while ϕ_3 asserts the irreflexivity. We show that for each structure A over the empty vocabulary, i.e., each nonempty set A:

$$A \text{ is infinite} \quad \text{iff} \quad A \models \psi'_\infty$$

Suppose first that A is a model for ψ'_∞. That is, $(A, [X := R]) \models \phi_1 \wedge \phi_2 \wedge \phi_3$ for some subset R of $A \times A$. An infinite sequence of pairwise distinct elements in A is obtained as follows.

Choose an arbitrary element $a_0 \in A$.
Since $(A, [X := R]) \models \phi_2$, we may pick an element $a_1 \in A$ such that $(a_0, a_1) \in R$.
Again by ϕ_2, there is an element $a_2 \in A$ where $(a_1, a_2) \in R$.
Again by ϕ_2, there is an element $a_3 \in A$ where $(a_2, a_3) \in R$.
$\vdots$

We continue in this way and obtain an infinite sequence $a_0\, a_1\, a_2\, a_3 \ldots$ of elements in A with $(a_i, a_{i+1}) \in R$ for all $i \in \mathbb{N}$.

- Since $(A, [X := R]) \models \phi_1$, we have $(a_i, a_j) \in R$ for all $0 \leqslant i < j$.
- Since $(A, [X := R]) \models \phi_3$, we get $a_i \neq a_j$ for all $i, j \in \mathbb{N}$.

Thus, $\{a_i : i \in \mathbb{N}\}$ is an infinite subset of A. This shows that any model for ψ'_∞ is infinite. Vice versa, any infinite structure is a model for ψ'_∞. To see this, we consider an infinite set A and pick pairwise distinct elements $a_0, a_1, a_2, \ldots \in A$. Let

$$R \stackrel{\text{def}}{=} \left\{ (a_i, a_j) : 0 \leqslant i < j \right\} \cup \left\{ (a, a_j) : a \in A \setminus \{a_0, a_1, a_2, \ldots\}, j \geqslant 0 \right\}$$

Then, R is total, transitive and irreflexive. Thus, $(A, [X := R])$ is a model for $\phi_1 \wedge \phi_2 \wedge \phi_3$. Therefore, $A \models \psi'_\infty$. ■

The SO-definability of the class of finite structures is obtained by considering the negation of the SOL-sentences ψ_∞ or ψ'_∞ in Example 2.3.1. That is, for each nonempty set A:

$$A \text{ is finite} \quad \text{iff} \quad A \models \neg\psi_\infty \quad \text{iff} \quad A \models \neg\psi'_\infty$$

As in Section 1.7.1, the fact that the class of finite structures is SO-definable can be used to show that the condition of the compactness theorem does not hold for SOL:

Theorem 2.3.2 (Lack of compactness property for SOL). *There exists a set of SOL-sentences that is finitely satisfiable, but not satisfiable.*

Proof. We consider the set $\mathfrak{F} \stackrel{\text{def}}{=} \left\{ \psi_{\text{fin}} \right\} \cup \left\{ \psi_2, \psi_3, \psi_4, \ldots \right\}$ where ψ_{fin} is a SOL-sentence over the empty vocabulary that defines the class of finite structures (see Example 2.3.1 above) and ψ_n a FOL-formula representing the structures with at least n elements, e.g.,

$$\psi_n \;=\; \exists x_1 \ldots \exists x_n . \bigwedge_{1 \leqslant i < j \leqslant n} x_i \neq x_j$$

(see the proof of Theorem 1.5.2 on page 72). Then, each finite subset of $\mathfrak{F}$ is satisfiable, but there is no model for $\mathfrak{F}$. □

Example 2.3.3 (SOL-characterization of countable sets). Second-order logic over the empty vocabulary is even powerful enough to distinguish countable from uncountable sets. For this, let us first observe that for each set A:

$$A \text{ is countable} \quad \text{iff} \quad \left\{ \begin{array}{l} \text{there exists a linear order } \sqsubseteq \text{ on } A \text{ such that for} \\ \text{each } a \in A \text{ the downward closure } A \downarrow a \text{ is finite} \end{array} \right.$$

Recall that $A \downarrow a \ = \ \{b \in A : b \sqsubseteq a\}$.

"$\Longrightarrow$": Suppose A is countable. Let $a_0\, a_1\, a_2 \ldots$ be an enumeration of A such that $a_i \neq a_j$ for $i \neq j$. (This enumeration is finite, if A is finite, and it might be non-recursive if A is infinite.) Let $\sqsubseteq$ be the unique linear order on A such that $a_0 \sqsubset a_1 \sqsubset a_2 \sqsubset \ldots$. Then, for each element $a = a_n \in A$, the downward closure $A \downarrow a \ = \ \{a_0, a_1, \ldots, a_n\}$ is finite.

"$\Longleftarrow$": Suppose that $\sqsubseteq$ is a linear order on A such that for each $a \in A$ the downward closure $A \downarrow a$ is finite. For $a \in A$, let

$$\mathit{rank}(a) \ \stackrel{\text{def}}{=} \ |A \downarrow a|$$

be the size of the downward closure of a. That is, if $i = \mathit{rank}(a)$ then a is the i-th element of $\mathcal{A}$. Since $\sqsubseteq$ is a linear order, given $a, b \in A$ with $a \neq b$, then either $a \sqsubset b$ or $b \sqsubset a$, i.e., either $A \downarrow a \subsetneqq A \downarrow b$ or $A \downarrow b \subsetneqq A \downarrow a$. Hence:

$$\begin{array}{lclcl}
\mathit{rank}(a) < \mathit{rank}(b) & \text{iff} & A \downarrow a \ \subsetneqq \ A \downarrow b & \text{iff} & a \sqsubset b \\
\mathit{rank}(a) = \mathit{rank}(b) & \text{iff} & A \downarrow a \ = \ A \downarrow b & \text{iff} & a = b
\end{array}$$

But this yields that the function $A \to \mathbb{N}$, $a \mapsto \mathit{rank}(a)$ is an injective function from A to $\mathbb{N}$. Therefore, A is countable. $\rfloor$

This observation leads to the following SOL-sentence ϕ_{ctbl} over the empty vocabulary that defines the class of countable sets. Let

$$\phi_{\text{ctbl}} \ \stackrel{\text{def}}{=} \ \exists \sqsubseteq . \Big(\phi_{\text{lin\_order}}(\sqsubseteq) \ \wedge \ \forall x \exists X. \big(\psi_{\text{fin}}(X) \wedge \forall y. (\, x \sqsubseteq y \leftrightarrow X(y)\,)\,\big)\Big)$$

where $\sqsubseteq$ is used here as a binary predicate variable. X is a monadic predicate variable and

- $\phi_{\text{lin\_order}}(\sqsubseteq)$ is a FOL-formula stating that $\sqsubseteq$ is a linear order (see Definition 1.4.13 on page 53),
- $\psi_{\text{fin}}(X)$ is a SOL-formula stating that X is a finite set. Here, we may use the characterization of finite sets as in Example 2.3.1 (page 145) and define $\psi_{\text{fin}}(X)$ as follows:

  $$\psi_{\text{fin}}(X) \ \stackrel{\text{def}}{=} \ \forall F. \big(\psi_{\text{inj}}(X,F) \wedge \forall x. (X(x) \to X(F(x))) \ \to \ \psi_{\text{sur}}(X,F) \big)$$

  where X is s set variable (i.e., an unary predicate variable) and F an unary function variable. Subformula $\forall x. (X(x) \to X(F(x)))$ asserts that the image of X under F is contained in X. Thus, F can be viewed as function from X to X. Formula $\psi_{\text{inj}}(X,F)$ asserts the injectivity and $\psi_{\text{sur}}(X,F)$ the surjectivity of F.

  $$\begin{array}{lcl}
  \psi_{\text{inj}}(X,F) & = & \forall x \forall y. \big((\, X(x) \wedge X(y) \wedge F(x) = F(y)\,) \to x = y \,\big) \\
  \psi_{\text{sur}}(X,F) & = & \forall z. \big(X(z) \to \exists x. (\, X(x) \wedge F(x) = z \big)
  \end{array}$$

  Thus, formula $\psi_{\text{fin}}(X)$ states that each injective function $F : X \to X$ is surjective.

Then, ϕ_{ctbl} defines the class of countable sets, while $\neg\phi_{\text{ctbl}}$ holds exactly for the uncountable sets. ■

We are now ready to prove that neither the downward Löwenheim-Skolem theorem and nor the upward Löwenheim-Skolem theorems hold for second-order logic.

Theorem 2.3.4 (The Löwenheim-Skolem Theorems do not hold for SOL).

(a) There exists a SOL-sentence that has finite models of arbitrary size $n \geqslant 1$*, but has no infinite model.*

(b) There exists a SOL-sentence that has an infinite, but no uncountable model.

(c) There exists a satisfiable SOL-sentence that has no countable model.

Proof. Statement (a) follows from the fact that there is a SOL-sentence that holds exactly for the finite structures. Statements (b) and (c) follow from the fact that SOL provides a characterization of the (un)countable structures. More precisely, the upward Löwenheim-Skolem Theorem ("from infinite to larger models") does not hold for SOL (statement (b)), as ϕ_{ctbl} is a satisfiable SOL-formula (e.g., $\mathbb{N}$ or $\mathbb{Q}$ is a model) without any uncountable model. Similarly, the downward Löwenheim-Skolem Theorem does not hold for SOL (statement (c)), as $\neg\phi_{\text{ctbl}}$ is a satisfiable SOL-sentence (e.g., $\mathbb{R}$ is a model) without any countable model. □

Clearly, as FOL is a sublogic of SOL the validity problem for SOL is undecidable. The same holds for the satisfiability, equivalence and consequence problem. Since SOL does not fulfill the compactness property, there is even *no* deductive calculus $\mathfrak{D}$ for SOL such that, for all SOL-formula sets $\mathfrak{F}$ and SOL-formulas ϕ: $\mathfrak{F} \Vdash \phi$ iff $\mathfrak{F} \vdash_{\mathfrak{D}} \phi$. But there could still be a sound and *weakly complete* deductive calculus $\mathfrak{D}$ for the valid SOL-formulas, i.e., a mechanic way to derive the SOL-tautologies:

$$\Vdash \phi \quad \text{iff} \quad \vdash_{\mathfrak{D}} \phi$$

We now show that there is even no sound and weakly complete proof system for second-order logic. This result goes back to Gödel and is known as *Gödel's second incompleteness theorem*. Our argument uses the fact that the set of formulas that can be derived from the axioms and rules of a proof system is recursively enumerable. Thus, sound and weakly complete proof systems could only exists for SOL if the set of valid SOL-formulas would be recursively enumerable. This, however, is wrong, provided that the underlying vocabulary is rich enough to encode an unbounded two-dimensional grid (and the computations of Turing machines). When establishing the undecidability of FOL we used the vocabulary Voc_ω that contains infinitely many predicate and function symbols of any arity and mentioned that one binary predicate symbol is sufficient for this purpose.

Theorem 2.3.5 (SOL-VALID is not recursively enumerable). *The set of valid SOL-sentences over some vocabulary with at least one binary predicate symbol is not recursively enumerable.*

Proof. The proof of Theorem 2.3.5 relies on the following facts:

(1) There exists a SOL-sentence ϕ_{fin} over the empty vocabulary that defines the class of finite structures. This has been shown in Example 2.3.1 on page 145. E.g., we might deal with $\phi_{\text{fin}} = \neg\psi_\infty$ or $\phi_{\text{fin}} = \neg\psi'_\infty$ where ψ_∞ and ψ'_∞ are as in Example 2.3.1.

(2) The set FOL-VALID-FIN of FOL-sentences that hold for all finite structures is not recursively enumerable. This has been shown in Theorem 1.3.7 on page 42 (Trakhtenbrot's Theorem).

Using (1) and (2), the proof for Theorem 2.3.5 is as follows. For each FOL-sentence ψ:

$$\begin{array}{ll} & \psi \in \text{FOL-VALID-FIN} \\ \text{iff} & \mathcal{A} \models \psi \text{ for all finite structures } \mathcal{A} \\ \text{iff} & \text{the SOL-sentence } \phi_{\text{fin}} \to \psi \text{ is valid} \\ \text{iff} & \phi_{\text{fin}} \to \psi \in \text{SOL-VALID} \end{array}$$

Hence, assuming that there is a recursive enumeration for all valid SOL-sentences, then we could use this infinite procedure to generate all FOL-sentences that are valid over all finite structures as follows: we apply the fictive recursive enumeration procedure to generate all valid SOL-sentences and check for each of them whether it has the form $\phi_{\text{fin}} \to \psi$ for some FOL-sentence ψ. If so, then we output ψ. This would yield a recursive enumeration of all formulas $\psi \in$ FOL-VALID-FIN, and contradicts (2). □

Corollary 2.3.6 (Gödel's incompleteness theorem for SOL). *There is no sound and weakly complete deductive calculus for the set of valid SOL-sentences.*