Finite variable logic

Let us conclude this section with a few remarks on the fragment of FOL($\bigwedge$) that uses only finitely many variables. More precisely, if $k \in \mathbb{N}$ then FOL$^{\leqslant k}$($\bigwedge$)-formulas are FOL($\bigwedge$)-formulas ϕ where the total number of variables that appear (free or bounded) in ϕ is less or equal k. These logics FOL$^{\leqslant k}$($\bigwedge$) are often called *finite variable logics*. Several FOL($\bigwedge$)-formulas with infinitely many (bounded) variables can be rewritten into equivalent FOL$^{\leqslant k}$($\bigwedge$)-formulas, by reusing names of bounded variables (and giving up the requirement that no variable has both free and bounded occurrences). For example, reachability has been defined in FOL($\bigwedge$) by

$$(x = y) \vee \bigvee_{n \geqslant 1} \exists z_0 \ldots \exists z_n. \big(\, z_0 = x \wedge z_n = y \wedge \bigwedge_{0 \leqslant i < n} E(z_i, z_{i+1}) \,\big),$$

which is a formula that uses infinitely many variables. However, the use of variables can be more sparse. Indeed, there is an alternative (equivalent) characterization of reachability in FOL$^{\leqslant k}$($\bigwedge$) where $k = 3$. The idea is to use formulas $\phi_n(x,y)$ that assert the existence of a path from x to y of length n and to characterize reachability by the formula:

$$\phi(x,y) \;=\; \bigvee_{n \geqslant 0} \phi_n(x,y)$$

The ϕ_n's are defined inductively. For $n = 0$ and $n = 1$ we may deal with the atomic formulas $x = y$ and $E(x,y)$, respectively. For $n \geqslant 1$, formula $\phi_{n+1}(x,y)$ should be equivalent to $\exists z_n.(\, E(x,z_n) \wedge \phi_n[x/z_n]\,)$. As long as the variables $z_1, z_2, z_3, \ldots$ are pairwise distinct, the formula $\phi(x,y)$ that results by the infinite disjunction of the formulas $x = y$ and $E(x,y)$ and $\phi_{n+1}(x,y) = \exists z_n.(\, E(x,z_n) \wedge \phi_n[x/z_n]\,)$ for $n \geqslant 1$ correctly characterizes reachability, but it uses infinitely many variables. For a FOL$^{\leqslant 3}$($\bigwedge$)-characterization of reachability, we have to reuse variable names. In fact, variable x does not appear in the subformula $\phi_n[x/z_n]$ of $\exists z_n.(\, E(x,z_n) \wedge \phi_n[x/z_n]\,)$. Hence, instead of introducing a fresh variable z_n we can reuse x. This is based on the general observation stating that if variable x can be replaced with variable z in the formula $\psi = \psi(x, \ldots)$ then:

$$\psi[x/z] \;\;\equiv\;\; \exists x. \big(\, x = z \wedge \psi \,\big)$$

This observation applied to $\psi = \phi_n(x,y)$ and leads to the following definition of formulas ϕ_n that use at most three variables (namely x, y and z):

$$\begin{aligned}
\phi_0(x,y) \quad &\stackrel{\text{def}}{=} \quad x = y \\
\phi_1(x,y) \quad &\stackrel{\text{def}}{=} \quad E(x,y) \\
\phi_{n+1}(x,y) \quad &\stackrel{\text{def}}{=} \quad \exists z. \big(\, E(x,z) \wedge \exists x.(\, z = x \wedge \phi_n(x,y)\,)\,\big)
\end{aligned}$$

for $n \geqslant 1$. Then,

$$\phi(x,y) \;\stackrel{\text{def}}{=}\; \bigvee_{n \geqslant 0} \phi_n(x,y)$$

is a FOL$^{\leqslant 3}$($\bigwedge$)-formula that characterizes reachability.

Although FOL$^{\leqslant k}$($\bigwedge$) is more expressive than FOL$^{\leqslant k}$ – as, e.g., reachability is definable in FOL$^{\leqslant 3}$($\bigwedge$), but not FO-definable – the distinguishing power of FOL$^{\leqslant k}$($\bigwedge$) and FOL$^{\leqslant k}$ for *finite structures* is the same. Here, FOL$^{\leqslant k}$ denotes the set of all FOL-formulas with at most k variables.

Theorem 1.7.9 (FOL$^{\leqslant k}$- vs FOL$^{\leqslant k}(\bigwedge)$-equivalence). *Let $\mathcal{A}$ and $\mathcal{B}$ finite structures for some vocabulary Voc. Then, the following two statements are equivalent:*

(a) $\mathcal{A}$ and $\mathcal{B}$ satisfy the same FOL$^{\leqslant k}$*-sentences.*

(b) $\mathcal{A}$ and $\mathcal{B}$ satisfy the same FOL$^{\leqslant k}(\bigwedge)$*-sentences.*

Proof. The implication (b) $\Longrightarrow$ (a) is obvious, since FOL$^{\leqslant k}$ is a sublogic of FOL$^{\leqslant k}(\bigwedge)$. To prove the implication (a) $\Longrightarrow$ (b), we fix two finite structures $\mathcal{A}$ and $\mathcal{B}$ over the same vocabulary with domains A and B, respectively, and suppose that $\mathcal{A}$ and $\mathcal{B}$ agree on all FOL$^{\leqslant k}$-sentences. Furthermore, we fix k pairwise distinct variables $x_1, \ldots, x_k$ such that all formulas under consideration use at most these variables. We provide a transformation

$$\phi \mapsto \widetilde{\phi} = \widetilde{\phi}_{\mathcal{A},\mathcal{B}}$$

which assigns to each FOL$^{\leqslant k}(\bigwedge)$-formula ϕ a FOL$^{\leqslant k}$-formula $\widetilde{\phi}$ such that

(1) $\mathit{Free}(\widetilde{\phi}) \subseteq \mathit{Free}(\phi)$

(2) for all $\overline{a} \in A^k$: $\quad (\mathcal{A}, \overline{a}) \models \widetilde{\phi}$ iff $(\mathcal{A}, \overline{a}) \models \phi$

(3) for all $\overline{b} \in B^k$: $\quad (\mathcal{B}, \overline{b}) \models \widetilde{\phi}$ iff $(\mathcal{B}, \overline{b}) \models \phi$

As mentioned above, the variables that appear free or bounded in ϕ and $\widetilde{\phi}$ are contained in $\{x_1, \ldots, x_k\}$. The notation $(\mathcal{A}, \overline{a})$ serves as an abbreviation for the interpretation with structure $\mathcal{A}$ and the variable valuation that interpretes the i-th variable x_i by the i-th element of $\overline{a}$. Note that with conditions (2) and (3), we just assert the equivalence of ϕ and $\widetilde{\phi}$ *with respect to* the given structures $\mathcal{A}$ and $\mathcal{B}$, but not general equivalence. In fact, the definition of the formulas $\widetilde{\phi} = \widetilde{\phi}_{\mathcal{A},\mathcal{B}}$ will depend on the given structures $\mathcal{A}$ and $\mathcal{B}$.

The definition of the FOL-formulas $\widetilde{\phi}$ is by structural induction. We put $\widetilde{\phi} \stackrel{\text{def}}{=} \phi$ if $\phi = \mathit{true}$ or ϕ is atomic. In this case, conditions (1)-(3) are obvious. For negation we define:

$$\widetilde{\neg\phi} \stackrel{\text{def}}{=} \neg\widetilde{\phi}$$

Note that the induction hypothesis immediately yields that conditions (1)-(3) are fulfilled. For *universal quantification*, i.e., a given FOL$^{\leqslant k}(\bigwedge)$-formula $\forall y.\phi$ with $y \in \{x_1, \ldots, x_k\}$, we define:

$$\widetilde{\forall y.\phi} \stackrel{\text{def}}{=} \forall y.\widetilde{\phi}$$

Condition (1) is clear from the induction hypothesis applied to ϕ. Let us check condition (2). The argument for condition (3) is analogous. For simplicity we assume that $y = x_1$. By induction hypothesis we have

$$(\mathcal{A}, a_1, a_2, \ldots, a_k) \models \widetilde{\phi} \quad \text{iff} \quad (\mathcal{A}, a_1, a_2, \ldots, a_k) \models \phi$$

for all $a_1, a_2, \ldots, a_k \in A$ where a_i stands for the interpretation of x_i. Since

$$y = x_1 \notin \mathit{Free}(\forall y.\phi) \ \text{ and } \ y \notin \mathit{Free}(\forall y.\widetilde{\phi}),$$

the interpretation of variable y is irrelevant for $\forall y.\phi$ and $\widetilde{\forall y.\phi} = \forall y.\widetilde{\phi}$. Thus:

$$
\begin{array}{rlcl}
 & (\mathcal{A}, a_1, a_2, \ldots, a_k) & \models & \widetilde{\forall y.\phi} \\
\text{iff} & (\mathcal{A}, a_1, a_2, \ldots, a_k) & \models & \forall y.\widetilde{\phi} \\
\text{iff} & (\mathcal{A}, a', a_2, \ldots, a_k) & \models & \widetilde{\phi} \quad \text{for all } a' \in A \\
\text{iff} & (\mathcal{A}, a', a_2, \ldots, a_k) & \models & \phi \quad \text{for all } a' \in A \\
\text{iff} & (\mathcal{A}, a_1, a_2, \ldots, a_k) & \models & \forall y.\phi
\end{array}
$$

It remains to consider the case where ϕ arises by a countable conjunction. By duality it suffices to consider the case of an *countable disjunction*, say

$$\phi \;=\; \bigvee_{i \in I} \phi_i$$

where ϕ (and the ϕ_i's) are $\text{FOL}^{\leqslant k}(\bigwedge)$-formulas where all variables that have free or bounded occurrences in ϕ are contained in $\{x_1, \ldots, x_k\}$. The goal is now to identify *finitely many* subformulas ϕ_i such that the truth value of ϕ under each $\mathcal{A}$- and $\mathcal{B}$-interpretation just depends on the truth values of these ϕ_i's.

- For each $\overline{a} \in A^k$ where $(\mathcal{A}, \overline{a}) \models \phi$ there exists an index $i(\overline{a}) \in I$ such that:

  $$(\mathcal{A}, \overline{a}) \models \phi_{i(\overline{a})}$$

  Let $I_{\mathcal{A}} \stackrel{\text{def}}{=} \big\{ i(\overline{a}) : \overline{a} \in A^k \text{ s.t. } (\mathcal{A}, \overline{a}) \models \phi \big\}$.

- Similarly, whenever $\overline{b} \in B^k$ and $(\mathcal{B}, \overline{b}) \models \phi$ then there is an index $j(\overline{b}) \in I$ such that:

  $$(\mathcal{B}, \overline{b}) \models \phi_{j(\overline{b})}$$

  Let $I_{\mathcal{B}} \stackrel{\text{def}}{=} \big\{ j(\overline{b}) : \overline{b} \in B^k \text{ s.t. } (\mathcal{B}, \overline{b}) \models \phi \big\}$.

As $\mathcal{A}$ and $\mathcal{B}$ are finite, the index-sets $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ are finite subsets of I. We define

$$\widetilde{\phi} \;\stackrel{\text{def}}{=}\; \bigvee_{i \in I_{\mathcal{A}} \cup I_{\mathcal{B}}} \widetilde{\phi}_i$$

Since $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ are finite and the $\widetilde{\phi}_i$'s are $\text{FOL}^{\leqslant k}$-formulas, $\widetilde{\phi}$ is a $\text{FOL}^{\leqslant k}$-formula. Let us check that conditions (1), (2) and (3) hold for $\widetilde{\phi}$. Condition (1) is obvious. We check condition (2). The proof for (3) is analogous. Let $\overline{a} \in A^k$.

- Suppose first that $(\mathcal{A}, \overline{a}) \models \phi$. Then, there is an index $i \in I_{\mathcal{A}}$ with $i = i(\overline{a})$ and

  $$(\mathcal{A}, \overline{a}) \models \phi_{i(\overline{a})}.$$

  The induction hypothesis yields that $(\mathcal{A}, \overline{a}) \models \widetilde{\phi}_{i(\overline{a})}$. But then

  $$(\mathcal{A}, \overline{a}) \models \widetilde{\phi} \;=\; \ldots \vee \widetilde{\phi}_{i(\overline{a})} \vee \ldots$$

- Suppose that $(\mathcal{A}, \overline{a}) \models \widetilde{\phi}$. Then, $(\mathcal{A}, \overline{a}) \models \widetilde{\phi}_i$ for some index $i \in I_{\mathcal{A}} \cup I_{\mathcal{B}}$. The induction hypothesis yields:
$$(\mathcal{A}, \overline{a}) \models \phi_i$$
But then $(\mathcal{A}, \overline{a}) \models \phi$.

By assumption (a), $\mathcal{A}$ and $\mathcal{B}$ satisfy the same $\mathrm{FOL}^{\leqslant k}$-sentences. Let ϕ be an arbitrary $\mathrm{FOL}^{\leqslant k}(\bigwedge)$-sentence. We regard the $\mathrm{FOL}^{\leqslant k}$-formula $\widetilde{\phi}$ such that (1), (2) and (3) hold. Then, $\widetilde{\phi}$ is a $\mathrm{FOL}^{\leqslant k}$-sentence (by condition (1)). Since $\mathcal{A}$ and $\mathcal{B}$ yield the same truth value for $\widetilde{\phi}$, conditions (2) and (3) yield:

$$\mathcal{A} \models \phi \quad \text{iff} \quad \mathcal{A} \models \widetilde{\phi} \quad \text{iff} \quad \mathcal{B} \models \widetilde{\phi} \quad \text{iff} \quad \mathcal{B} \models \phi$$

Thus, $\mathcal{A}$ and $\mathcal{B}$ fulfill the same $\mathrm{FOL}^{\leqslant k}(\bigwedge)$-sentences. □

1.7.2 Infinite quantification

We now consider two extensions of FOL with existential quantifiers stating the existence of infinitely or uncountably many elements with a certain property.

We first consider the logic $\mathrm{FOL}(\overset{\infty}{\exists})$ which extends FOL by formulas of the form

$$\overset{\infty}{\exists} x.\phi$$

stating that *there exist infinitely many* x such that ϕ holds. Thus, the abstract syntax of $\mathrm{FOL}(\overset{\infty}{\exists})$ is given by:

$$\phi \ ::= \ \mathit{true} \ \big| \ P(t_1, \ldots, t_n) \ \big| \ t_1 = t_2 \ \big| \ \neg\phi \ \big| \ \phi_1 \wedge \phi_2 \ \big| \ \forall x.\phi \ \big| \ \overset{\infty}{\exists} x.\phi$$

Structures and interpretations and the satisfaction relation $\models$ for the FOL-fragment of $\mathrm{FOL}(\overset{\infty}{\exists})$ are defined as for standard FOL. The semantics of the new operator is given by:

$$\begin{aligned} (\mathcal{A}, \mathcal{V}) \models \overset{\infty}{\exists} x.\phi \quad & \text{iff} \quad \text{there exists infinitely many } a \in A \text{ with } (\mathcal{A}, \mathcal{V}[x := a]) \models \phi \\ & \text{iff} \quad \text{the set } \big\{ a \in A : (\mathcal{A}, \mathcal{V}[x := a]) \models \phi \big\} \text{ is infinite} \end{aligned}$$

where A is the domain of $\mathcal{A}$. The dual quantifier $\overset{\infty}{\forall} x.\phi \overset{\text{def}}{=} \neg \overset{\infty}{\exists} x.\neg\phi$ states that ϕ holds *for almost all* x, i.e., for all but finitely many x:

$$\begin{aligned} (\mathcal{A}, \mathcal{V}) \models \overset{\infty}{\forall} x.\phi \quad & \text{iff} \quad (\mathcal{A}, \mathcal{V}) \not\models \overset{\infty}{\exists} x.\neg\phi \\ & \text{iff} \quad \text{the set } \big\{ a \in A : (\mathcal{A}, \mathcal{V}[x := a]) \models \neg\phi \big\} \text{ is finite} \\ & \text{iff} \quad \text{the set } \big\{ a \in A : (\mathcal{A}, \mathcal{V}[x := a]) \not\models \phi \big\} \text{ is finite} \end{aligned}$$

For example, we have:

$$\begin{aligned} (\mathbb{N}, \leqslant) & \models \forall x \overset{\infty}{\exists} y.\, x < y \\ (\mathbb{N}, \leqslant) & \models \forall x \overset{\infty}{\forall} y.\, x < y \end{aligned}$$

Note that $\forall x \ \overset{\infty}{\exists} y.\ x < y$ holds for the natural numbers, since for each $n \in \mathbb{N}$ the set $\{m \in \mathbb{N} : n < m\}$ is infinite. The second formula holds as for each natural number n, almost all natural numbers m are larger than n.

The compactness property does not hold for $\mathrm{FOL}(\overset{\infty}{\exists})$ since

$$\phi_{\text{fin}} \stackrel{\text{def}}{=} \neg \overset{\infty}{\exists} x.\, \mathit{true} \ \equiv\ \overset{\infty}{\forall} x.\, \mathit{false}$$

characterizes the finite structures. Thus, the set $\{\,\phi_{\text{fin}}\,\} \cup \{\,\psi_n : n \geqslant 2\,\}$ is finitely satisfiable, but not satisfiable. Here, the ψ_n's are FOL-formulas that characterize structures with n or more elements (see e.g. the proof of Theorem 1.7.6 on page 120).

The existence of a $\mathrm{FOL}(\overset{\infty}{\exists})$-sentence ϕ_{fin} that holds exactly for the finite structures also yields that the upward Löwenheim-Skolem theorem "from finite to infinite models" (cf. Theorem 1.5.2 on page 72) does not hold for $\mathrm{FOL}(\overset{\infty}{\exists})$. Note that for each $n \in \mathbb{N}$, $n \geqslant 1$, ϕ_{fin} has a finite model $\mathcal{A}_n$ with n elements, but there is no infinite model for ϕ_{fin}.

The other part of the upward Löwenheim-Skolem theorem ("from infinite to larger models", cf. Theorem 1.5.4 on page 73) is violated by $\mathrm{FOL}(\overset{\infty}{\exists})$ too. To see this, we show that the natural numbers with the natural order have a characterization in $\mathrm{FOL}(\overset{\infty}{\exists})$:

Example 1.7.10 ($\mathrm{FOL}(\overset{\infty}{\exists})$-characterization of the ordered natural numbers). We regard the structure $(\mathbb{N}, \leqslant)$ and provide a $\mathrm{FOL}(\overset{\infty}{\exists})$-sentence ϕ_{nat} over the vocabulary consisting of a binary predicate symbol $\sqsubseteq$ such that ϕ_{nat} holds exactly for the structures that are isomorphic to $(\mathbb{N}, \leqslant)$. As usual we write $x \sqsubseteq y$ for $x \neq z \wedge x \sqsubseteq y$. Let

$$\phi_{\text{nat}} \stackrel{\text{def}}{=} \psi_{\text{LO}} \wedge \psi_1 \wedge \psi_2$$

where ψ_{LO} is a FOL-sentence that formalizes the conditions of linear orders (see Definition 1.4.13 on page 53) and where ψ_1 and ψ_2 are the following sentences:

$$\begin{aligned} \psi_1 &\stackrel{\text{def}}{=} \forall x\, \overset{\infty}{\forall} y.\ x \sqsubset y \\ \psi_2 &\stackrel{\text{def}}{=} \forall x\, \exists y.\ x \sqsubset y \end{aligned}$$

Sentence ψ_1 asserts that for each element x almost all elements y are larger than x, i.e., the number of elements y with $y \leqslant x$ is finite. Note that:

$$\begin{aligned} & (\mathbb{N}, \leqslant, [x := n]) \models \overset{\infty}{\forall} y.\ x \sqsubset y \\ \text{iff} \quad & (\mathbb{N}, \leqslant, [x := n]) \not\models \overset{\infty}{\exists} y.\ \neg(x \sqsubset y) \\ \text{iff} \quad & (\mathbb{N}, \leqslant, [x := n]) \not\models \overset{\infty}{\exists} y.\ y \sqsubseteq x \\ \text{iff} \quad & \{m \in \mathbb{N} : m \leqslant n\} \text{ is finite} \end{aligned}$$

The latter certainly holds for all natural numbers n. Thus, $(\mathbb{N}, \leqslant) \models \psi_1$. Sentence ψ_2 formalizes the condition that there are no maximal elements. Clearly, we have $(\mathbb{N}, \leqslant) \models \psi_2$ and therefore:

$$(\mathbb{N}, \leqslant) \models \phi_{\text{nat}}$$

Suppose now that $\mathcal{A} = (A, \sqsubseteq)$ is a model for ϕ_{nat}. The goal is to show that $\mathcal{A}$ is isomorphic to $(\mathbb{N}, \leqslant)$. As $\mathcal{A} \models \psi_{\text{LO}} \wedge \psi_2$, $\mathcal{A}$ is a linear order without maximal elements. In particular, A

must be infinite. As $\mathcal{A} \models \psi_1$, for each element $a \in A$ the downward-closure of a, i.e., the set $A \downarrow a \ = \{b \in A : b \sqsubseteq a\}$, is finite. Let

$$\mathit{rank}(a) \stackrel{\text{def}}{=} \left| A \downarrow a \right|$$

If $a, b \in A$ and $a \sqsubset b$ then $A \downarrow a$ is a proper subset of $A \downarrow b$, and therefore $\mathit{rank}(a) < \mathit{rank}(b)$. As every two elements in A are comparable via $\sqsubseteq$ we get for all $a, b \in A$:

$$\begin{array}{lll} a \sqsubset b & \text{iff} & \mathit{rank}(a) < \mathit{rank}(b) \\ a = b & \text{iff} & \mathit{rank}(a) = \mathit{rank}(b) \end{array}$$

Thus, the rank function $\mathit{rank} : A \to \mathbb{N}$ is injective and a homomorphism. It is also surjective.

> Let us see why. Since A is infinite, for each $n \in \mathbb{N}$ there is an element $a \in A$ with rank $\geqslant n$. Given an element $a \in A$ of rank n then, for each $m \in \mathbb{N}$ where $m < n$, there is an element of rank m. To see this, we regard the strictly increasing sequence $a_1, a_2, \ldots, a_n$ of the elements in $A \downarrow a$ and then take the m-th element a_m of this sequence. Hence, for each $n \in \mathbb{N}$ there exists a unique element $a_n \in A$ with $\mathit{rank}(a_n) = n$.

Hence, the rank function is an isomorphism from $\mathcal{A}$ to the ordered natural numbers. This example shows that there is a FOL($\overset{\infty}{\exists}$)-sentence ϕ that is satisfiable over some infinite structure, but all models for ϕ are countable. Thus, the upward Löwenheim-Skolem theorem "from infinite to larger models" does not hold. ■

The above results show that FOL($\overset{\infty}{\exists}$) violates the compactness property and the upward Löwenheim-Skolem theorems (both versions "from finite to infinite models" and "from infinite to larger models").

However, the downward Löwenheim-Skolem theorem holds for FOL($\overset{\infty}{\exists}$), since it lies between FOL and FOL($\bigwedge$), and the latter provides countable models for satisfiable sentences (see Theorem 1.7.8).

Theorem 1.7.11 (Embedding of FOL($\overset{\infty}{\exists}$) into FOL($\bigwedge$)). *For each* FOL($\overset{\infty}{\exists}$)*-formula* ϕ *there exists a* FOL($\bigwedge$)*-formula* $\overline{\phi}$ *over the same vocabulary such that* ϕ *and* $\overline{\phi}$ *have the same models.*

Proof. We provide a transformation $\phi \mapsto \overline{\phi}$ which assigns to each FOL($\overset{\infty}{\exists}$)-formula ϕ a FOL($\bigwedge$)-formula $\overline{\phi}$ with the same predicate and function symbols and the same free variables and such that ϕ and $\overline{\phi}$ have the same models. The definition of $\overline{\phi}$ is by structural induction:

$$\begin{array}{lll} \overline{\phi} & \stackrel{\text{def}}{=} & \phi \qquad\qquad \text{if } \phi = \mathit{true} \text{ or } \phi \text{ is an atomic formula} \\ \overline{\neg\phi} & \stackrel{\text{def}}{=} & \neg\overline{\phi} \\ \overline{\phi_1 \wedge \phi_2} & \stackrel{\text{def}}{=} & \overline{\phi_1} \wedge \overline{\phi_2} \\ \overline{\forall x.\phi} & \stackrel{\text{def}}{=} & \forall x.\overline{\phi} \\ \overline{\overset{\infty}{\exists} x.\phi} & \stackrel{\text{def}}{=} & \bigwedge\limits_{n \geqslant 1} \exists x_1 \ldots \exists x_n. \left(\overline{\phi}[x/x_1] \wedge \ldots \overline{\phi}[x/x_n] \wedge \bigwedge\limits_{1 \leqslant i < j \leqslant n} x_i \neq x_j \right) \\ & & \text{where } x_1, x_2, \ldots \text{ is a sequence of pairwise distinct fresh variables.} \end{array}$$

It is now easy to see that $\mathfrak{I} \models \phi$ iff $\mathfrak{I} \models \overline{\phi}$ for all interpretations $\mathfrak{I}$. $\square$

The combination of Theorem 1.7.11 and Theorem 1.7.8 (page 122) yields:

Theorem 1.7.12 (Downward Löwenheim-Skolem Theorem for FOL($\overset{\infty}{\exists}$)). *Each satisfiable* FOL($\overset{\infty}{\exists}$)*-sentence has a countable (possibly finite) model.*