Wrap-up Advanced Logics

Summer semester 2021

July 13, 2021

Technische Universität Dresden[1]

FOL: syntax

Vocabulary: $Voc = (Pred, Func)$

$$t ::= x \mid f(t_1, \ldots, t_n) \qquad \text{(terms)}$$

$$\varphi ::= \textbf{true} \mid P(t_1, \ldots, t_n) \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \forall x.\varphi \mid \underbrace{t_1 = t_2}_{\text{optional}} \qquad \text{(formulas)}$$

Syntactic sugar:

$$\begin{aligned} t_1 \neq t_2 &= \neg(t_1 = t_2) \\ \textbf{false} &= \neg\textbf{true} \\ \varphi_1 \vee \varphi_2 &= \neg(\neg\varphi_1 \wedge \neg\varphi_2) \\ \exists x.\varphi &= \neg\forall x.\neg\varphi \\ &\vdots \end{aligned}$$

FOL: semantics

Interpretation: $\mathcal{I} = (\mathcal{A}, \mathcal{V})$
$\mathcal{A}$ ... structure
$\mathcal{V}$ ... valuation

$$
\begin{aligned}
x^{\mathcal{I}} &= \mathcal{V}(x) \\
f(t_1, \ldots, t_m)^{\mathcal{I}} &= f^{\mathcal{A}}(t_1^{\mathcal{I}}, \ldots, t_m^{\mathcal{I}}) \\
\mathcal{I} &\models \mathsf{true} \\
&\vdots \\
\mathcal{I} &\models \forall x.\varphi && \text{iff} \quad \mathcal{I}[x := a] \models \varphi \text{ for all } a \in \mathcal{A} \\
\mathcal{I} &\models (t_1 = t_2) && \text{iff} \quad t_1^{\mathcal{I}} = t_2^{\mathcal{I}}
\end{aligned}
$$

FOL: sound and complete Hilbert proof system

Hilbert proof system: decidable set of proof rules (axioms are special instances of proof rules)

A rule:

$$\frac{\Phi_1, \ldots, \Phi_n}{\Phi} \quad \begin{array}{l} \longleftarrow \quad \text{premise} \\ \longleftarrow \quad \text{consequence} \end{array}$$

Derivation

Definition (Derivation)
If $\mathfrak{F}$ is a formula-set then a *derivation* in $\mathfrak{D}$ from $\mathfrak{F}$ denotes a **finite sequence** $\psi_1, \ldots, \psi_m$ of formulas such that for each $i \in \{1, \ldots, m\}$ at least one of the following conditions holds:

- ψ_i is an element of $\mathfrak{F}$
- ψ_i is an instance of an axiom
- there exists an instance $(\phi_1, \ldots, \phi_n, \phi)$ of some n-ary proof rule in $\mathfrak{D}$ such that $n \geqslant 1$ and $\psi_i = \phi$ and $\{\phi_1, \ldots, \phi_n\} \subseteq \{\psi_1, \ldots, \psi_{i-1}\}$.

Complete and sound Hilbert proof systems

A Hilbert proof system is

- *complete*, if for each formula φ and formula set $\mathfrak{F}$:

$$\mathfrak{F} \Vdash \varphi \implies \mathfrak{F} \vdash_{\mathfrak{D}} \varphi$$

- *weak complete*, if for each formula φ:

$$\Vdash \varphi \implies \vdash_{\mathfrak{D}} \varphi$$

- *sound*, if for each formula φ and formula set $\mathfrak{F}$:

$$\mathfrak{F} \vdash_{\mathfrak{D}} \varphi \implies \mathfrak{F} \Vdash \varphi$$

Theorem (Gödel's completeness theorem)
There exists a Hilbert proof system that is sound and complete for FOL.

The complete and sound Hilbert proof system for FOL in the script is called $\mathfrak{D}_{FOL}$ (without equality page 21, with equality page 26).

Check soundness of a Hilbert proof system

Lemma (Criterion for soundness)
$\mathfrak{D}$ is sound if and only if all proof rules in $\mathfrak{D}$ are sound in the following sense:

- *all instances of the axioms in $\mathfrak{D}$ are valid and*
- *whenever $(\phi_1, \ldots, \phi_n, \phi)$ is an instance of a proof rule in $\mathfrak{D}$ then $\{\phi_1, \ldots, \phi_n\} \Vdash \phi$.*

Semi-decidability of FOL-tautologies

Lemma
The set of all valid FOL-formulas (over some fixed recursively enumerable vocabulary Voc and recursively enumerable variable-set Var) is recursively enumerable.

- Set of all FOL-formulas is recursively enumerable
- Fix a complete and sound Hilbert proof system $\mathfrak{D}$
- Generate algorithmically every finite sequence $\varphi_0, \dots, \varphi_m$
    - If $\varphi_0, \dots, \varphi_m$ is a $\mathfrak{D}$-derivation , output φ_m
    - Otherwise, just continue

Lemma
The set of all valid FOL-formulas is undecidable.

And therefore the satisfiability problem, the equivalence problem and the consequence problem are undecidable.

Proof sketch

by the reduction from Post's correspondence problem (PCP). Take an instance $K = (u_1, v_1), \ldots, (u_n, v_n)$ with $u_i, v_i \in \{0, 1\}^*$.

Assume *Voc* consists the unary predicate symbol P, two unary function symbols f_0 and f_1, and a constant c. We define

$$f_{l_1 l_2 \ldots l_n}(t) = f_{l_n}(\ldots f_2(f_1(t)))$$

Define

$$\varphi_K = \psi_1 \wedge \psi_2 \to \theta$$

$$\psi_1 = \bigwedge_{1 \le i \le n} P(f_{u_i}(c), f_{v_i}(c))$$

$$\psi_2 = \forall x \forall y. \left(P(x, y) \to \bigwedge_{1 \le i \le n} P(f_{u_i}(x), f_{v_i}(y)) \right)$$

$$\theta = \exists x. P(x, x)$$

Lemma
FOL-SAT-FIN is recursively enumerable.

Lemma
FOL-SAT-FIN is undecidable.

Theorem (Trakhtenbrot's theorem)
FOL-VALID-FIN is not recursively enumerable.

Monadic FOL

- purely relational
- only unary predicate symbols

Lemma (Finite model property)
If φ is a satisfiable MFO-formula (without equality) with k predicate symbols then φ has a model where the domain is a subset of $\{0, 1\}^k$.

Schönfinkel-Bernays formulas

Formulas of the form

$$\exists x_1 \ldots \exists x_n \forall y_1 \ldots \forall y_m . \varphi$$

where $n, m \geqslant 0$ and φ quantifier-free.

Lemma (Small model property)
Let ψ be a satisfiable Schönfinkel-Bernays sentence with n existential quantifiers and ℓ constant symbols. Then, ψ has a model with at most $n + \ell$ elements.

Proof.

- Let $\mathcal{A}$ be a model of ψ
- $a_1, \ldots, a_n$ the values of $x_1, \ldots, x_n$ under $\mathcal{A}$
- Structure with $\{a_1, \ldots, a_n\} \cup \left\{c_1^{\mathcal{A}}, \ldots, c_\ell^{\mathcal{A}}\right\}$ and the same behavior as $\mathcal{A}$ is a model for ψ

Definition (FO-theory)
Let $\mathfrak{T}$ be a set of FOL-formulas. $\mathfrak{T}$ is called *closed under logical consequences* if for all formulas ϕ: $\mathfrak{T} \vdash \phi$ implies $\phi \in \mathfrak{T}$. $\mathfrak{T}$ is called an *FO-theory* if $\mathfrak{T}$ is true in some structure and closed under logical consequences.

Definition (complete FO-theory)
A theory $\mathfrak{T}$ is called *complete* if it is maximal satisfiable over some structure, i.e., for all sentences ϕ: either $\phi \in \mathfrak{T}$ or $\neg\phi \in \mathfrak{T}$.

Definition (FO-theory of a structure)
If $\mathcal{A}$ is a structure for some vocabulary *Voc* then the first-order theory of $\mathcal{A}$, denoted $Th(\mathcal{A})$, defined by

$$Th(\mathcal{A}) = \{\varphi : \mathcal{A} \models \varphi\}$$

consists of all FOL-formulas ϕ over *Voc* that hold over $\mathcal{A}$.

Theorem (Characterization of complete theories)
Let $\mathfrak{T}$ be a FO-theory. Then, the following statements are equivalent:

(a) $\mathfrak{T}$ *is complete (i.e., for each sentence* ϕ *either* $\phi \in \mathfrak{T}$ *or* $\neg\phi \in \mathfrak{T}$*).*

(b) $\mathfrak{T} = Th(\mathcal{A})$ *for some structure* $\mathcal{A}$*.*

(c) $\mathfrak{T} = Th(\mathcal{A})$ *for all structures* $\mathcal{A}$ *such that* $\mathcal{A} \models \mathfrak{T}$*.*

Theorem (Semi-decidablity and decidability of complete theories)
Let $\mathfrak{T}$ be a complete theory. Then, $\mathfrak{T}$ is recursively enumerable if and only if $\mathfrak{T}$ is decidable.

Definition (Axiomatizable theories)
A theory $\mathfrak{T}$ is said to be *axiomatizable* if there exists a decidable set $\mathfrak{F}$ of formulas such that $\mathfrak{T}$ agrees with the logical closure of $\mathfrak{F}$, i.e.,

$$\mathfrak{T} \;=\; \{\phi \,:\, \mathfrak{F} \Vdash \phi\} = Cl(\mathfrak{F}).$$

In this case, $\mathfrak{F}$ is called an *axiomatization* for $\mathfrak{T}$.

Theorem (Axiomatizability and sound and complete deductive calculi)
Let $\mathfrak{T}$ be a FO-theory. Then, $\mathfrak{T}$ is axiomatizable iff $\mathfrak{T}$ has a sound and complete deductive calculus.

A proof system $\mathfrak{D}$ is sound and complete for $\mathfrak{T}$ if $\vdash_{\mathfrak{D}} \varphi$ iff $\varphi \in \mathfrak{T}$ for all sentences φ

Theorem (Axiomatizability and sound and complete deductive calculi)
Let $\mathfrak{T}$ be a FO-theory. Then, $\mathfrak{T}$ is axiomatizable iff $\mathfrak{T}$ has a sound and complete deductive calculus.

Proof.

- For direction “$\Rightarrow$” take an arbitrary and complete Hilbert proof system $\mathfrak{D}$ and an axiomatization $\mathcal{F}$ for $\mathfrak{T}$. Then $\mathfrak{D} + \mathcal{F}$ is a sound and complete deductive calculus.

Theorem (Axiomatizability and sound and complete deductive calculi)
Let $\mathfrak{T}$ be a FO-theory. Then, $\mathfrak{T}$ is axiomatizable iff $\mathfrak{T}$ has a sound and complete deductive calculus.

Proof.

- For direction "$\Leftarrow$" enumerate $\mathfrak{T}$: $\psi_0, \psi_1, \ldots$
- Set $\varphi_n = \psi_n \land \underbrace{\mathsf{true} \land \ldots \land \mathsf{true}}_{n \text{ times}}$ and $\mathfrak{F} = \{\varphi_n \,:\, n \in \mathbb{N}\}$
- For a formula φ with $|\varphi| = m$ to be in $\mathfrak{F}$, $\varphi_n = \varphi$ has to hold for a $n \leqslant m$.
- Enumerate $\varphi_0, \ldots, \varphi_m$ and check whether $\varphi = \varphi_i$ for a $i \in \{0, \ldots, m\}$

Corollary (Equivalence of axiomatizability and semi-decidability)
Let $\mathfrak{T}$ be a FO-theory. Then, $\mathfrak{T}$ has an axiomatization iff $\mathfrak{T}$ is recursively enumerable.

- $\mathit{Th}(\mathbb{N}, +, *, =)$ is neither decidable, recursively enumerable, axiomatizable, nor have a sound and complete deductive calculi (*First Gödel's incompleteness theorem*)
- $\mathit{Th}(\mathbb{N}, +, =)$ is decidable and axiomatizable (Presburger arithmetic)
- $\mathit{Th}(\mathbb{Q}, +, *, =)$ is not decidable, etc.
- $\mathit{Th}(\mathbb{R}, +, *, =)$ is decidable and axiomatizable (Tarski algebra)

Quantified Boolean formulas

- Syntax: $\varphi ::= \textbf{true} \mid q \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \forall p.\varphi$
- Syntactic sugar and semantics as expected

Theorem (Theorem by Stockmeyer and Meyer)
$\mathrm{QBF-TRUTH}$ *is PSPACE-complete.*

$\mathrm{QBF-TRUTH}$ *denotes the problem to decide whether a QBF-sentence is equivalent to* **true**.

Quantified Boolean formulas

- Syntax: $\varphi ::= \mathbf{true} \mid q \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \forall p.\varphi$
- Syntactic sugar and semantics as expected

Theorem (Theorem by Stockmeyer and Meyer)
$\mathrm{QBF-TRUTH}$ *is PSPACE-complete.*

- Membership by algorithm using a decision tree
- Hardness by reducing a polynomial space-bounded deterministic Turing machine to a QBF formula

Compactness theorem

Theorem (Compactness Theorem)
Let $\mathfrak{F}$ be a set of FOL-formulas. Then, $\mathfrak{F}$ is satisfiable iff every finite subset of $\mathfrak{F}$ is satisfiable.

Proof. "$\Rightarrow$" obvious.

Theorem (Compactness Theorem)
Let $\mathfrak{F}$ be a set of FOL-formulas. Then, $\mathfrak{F}$ is satisfiable iff every finite subset of $\mathfrak{F}$ is satisfiable.

Proof. "$\Leftarrow$"

- Proof by contradiction. Assume $\mathfrak{F}$ is unsatisfiable, but every finite subset is satisfiable.
- Hence, $\mathfrak{F} \Vdash$ **false**
- For every sound and complete Hilbert proof system:
  $\mathfrak{F} \vdash_{\mathfrak{D}} \varphi$ iff there exists a finite subset $\mathfrak{F}_0$ of $\mathfrak{F}$ s.t. $\mathfrak{F}_0 \vdash_{\mathfrak{D}} \varphi$
- Hence,
  $\mathfrak{F} \Vdash \varphi$ iff there exists a finite subset $\mathfrak{F}_0$ of $\mathfrak{F}$ s.t. $\mathfrak{F}_0 \Vdash \varphi$
- There exists a finite subset $\mathfrak{F}_0$ of $\mathfrak{F}$ such that $\mathfrak{F}_0 \Vdash$ **false**. Contradiction to $\mathfrak{F}_0$ being satisfiable

Upward Löwenheim-Skolem theorem

Theorem (Upward Löwenheim-Skolem Theorem)
Let $\mathfrak{F}$ be a set of formulas such that for each $n \in \mathbb{N}$ there exists a finite model $(\mathcal{A}_n, \mathcal{V}_n)$ for $\mathfrak{F}$ with at least n elements. Then, $\mathfrak{F}$ has an infinite model.

- Take $\psi_n = \exists x_1 \ldots \exists x_n. \bigwedge_{1 \leqslant i < j \leqslant n}(x_i \neq x_j)$ requiring at least n elements
- Every finite subset of $\mathfrak{F} \cup \{\psi_n \ : \ n \geq 2\}$ is satisfiable
- By compactness, $\mathfrak{F} \cup \{\psi_n \ : \ n \geq 2\}$ is satisfiable
- $\mathfrak{F} \cup \{\psi_n \ : \ n \geq 2\}$ has an infinite model, thus $\mathfrak{F}$

Theorem (Upward Löwenheim-Skolem Theorem for FOL without equality)
Let $\mathfrak{F}$ be a satisfiable set of FOL-formulas without equality and let C be a set. Then, $\mathfrak{F}$ is satisfiable over some structure where the domain is a superset of C.

Proof.

- Take a model $(\mathcal{B}, \mathcal{V}_{\mathcal{B}})$
- Take an arbitrary element $b_0 \in \mathrm{Dom}(\mathcal{B})$
- Define surjective homomorphism $h : \mathrm{Dom}(\mathcal{B}) \cup C \to \mathrm{Dom}(\mathcal{B})$ by taking the identity for elements of B and b_0 for elements of C
- Define a new model in accordance to h

Upward Löwenheim-Skolem Theorem for FOL with equality

Theorem (Upward Löwenheim-Skolem Theorem for FOL with equality)
Let $\mathfrak{F}$ be a satisfiable set of FOL-formulas that is satisfiable over some infinite structure and let C be a set. Then, $\mathfrak{F}$ is satisfiable over some structure where the domain is a superset of C.

Proof.

- Extend vocabulary by fresh constant c for every $c \in C$
- By compactness theorem $\mathfrak{G} = \mathfrak{F} \cup \{c \neq d \,:\, c, d \in C,\, c \neq d\}$ is satisfiable
- Therefore $\mathfrak{G}$ (and $\mathfrak{F}$) have a model with a domain being a superset of C

Theorem (Downward Löwenheim-Skolem Theorem)
Each satisfiable set of FOL-formulas (over some countable vocabulary and countable variable-set) has a countable model. More precisely:

- *Each satisfiable set of FOL-formulas without equality has an infinite countable model.*
- *Each satisfiable set of FOL-formulas with equality has a (finite or infinite) countable model.*

Downward Löwenheim-Skolem Theorem – Proof idea

- Construct formula set $\mathfrak{F}^+$ with the following properties:
  (1) $\mathfrak{F} \subseteq \mathfrak{F}^+$ and $\mathfrak{F}^+$ is satisfiable.
  (2) For each formula ϕ: either $\phi \in \mathfrak{F}^+$ or $\neg\phi \in \mathfrak{F}^+$.
  (3) For each formula ϕ and variable x there is a constant symbol c such that:
  $$\neg\forall x.\phi \ \rightarrow \neg\phi[x/c] \in \mathfrak{F}^+$$
- Construct term model $\mathcal{I}$
  - for without equality: directly
  - for with equality: quotient-based

Definition (FO-definability of a class of structures)
Given a vocabulary *Voc*, a class $\mathcal{C}$ of structures for *Voc* is said to be *FO-definable* if there exists a FOL-sentence ϕ over the vocabulary *Voc*, such that exactly the structures $\mathcal{A} \in \mathcal{C}$ are models for ϕ, i.e.,

$$\mathcal{C} = \{\mathcal{A} : \mathcal{A} \text{ is a structure over } Voc \text{ such that } \mathcal{A} \models \phi\}$$

Simple examples:

- groups,
- linear orders, ...

Theorem (The class of finite structures is not FO-definable)
There is no vocabulary Voc and set $\mathfrak{F}$ consisting of FOL-formulas over Voc such that for all structures $\mathcal{A}$ for Voc:

$$\mathcal{A} \text{ is a model for } \mathfrak{F} \text{ iff } \mathcal{A} \text{ is finite.}$$

Theorem (The class of finite structures is not FO-definable)
There is no vocabulary Voc and set $\mathfrak{F}$ consisting of FOL-formulas over Voc such that for all structures $\mathcal{A}$ for Voc:

$$\mathcal{A} \textit{ is a model for } \mathfrak{F} \textit{ iff } \mathcal{A} \textit{ is finite.}$$

Proof. By the upward Löwenheim-Skolem-Theorem

Theorem (The natural numbers cannot be characterized in FOL)
There is no satisfiable set $\mathfrak{F}$ of FOL-sentences over Voc_{nat} such that $(\mathbb{N}, succ, 0) \models \mathfrak{F}$ and all countable models for $\mathfrak{F}$ are isomorphic to $(\mathbb{N}, succ, 0)$.

Theorem (The natural numbers cannot be characterized in FOL)
There is no satisfiable set $\mathfrak{F}$ of FOL-sentences over Voc_{nat} such that $(\mathbb{N}, succ, 0) \models \mathfrak{F}$ and all countable models for $\mathfrak{F}$ are isomorphic to $(\mathbb{N}, succ, 0)$.

Proof.

- Assume by contradiction, there exists such an $\mathfrak{F}$
- Consider the set $\mathfrak{G} = \mathfrak{F} \cup \{x \neq succ^n(0) \;:\; n \in \mathbb{N}\}$
- By compactness and the Downward-Löwenheim-Skolem theorem, there is a countable model $(\mathcal{B}', n')$ with $(\mathcal{B}', n') \models \mathfrak{G}$
- By $(\mathcal{B}', n') \models \{x \neq succ^n(0) \;:\; n \in \mathbb{N}\}$ $\mathcal{B}'$ is non-isomorphic to $(\mathbb{N}, succ, 0)$, but still $\mathcal{B}' \models \mathfrak{F}$

Complexity of checking FO-definable graph properties

Theorem (FO-definable graph properties are in PTIME)
For each formula ϕ over Voc_{graph} the problem ϕ-GRAPH is solvable in time polynomial in the size of the input-graph. (Formula ϕ is viewed to be fixed and the size of a finite graph is the total number of nodes and edges.)

By recursive analysis.

Rechability

Theorem (Reachability is not FO-definable)
There is no FOL-formula $\phi(x, y)$ over Voc_{graph} such that for all graphs $\mathcal{G}$ and nodes a, b in $\mathcal{G}$ we have:

$$(\mathcal{G}, a, b) \models \phi(x, y) \text{ iff } b \text{ is reachable from } a \text{ in } \mathcal{G}$$

- Find formulas:
  - φ for strongly connectedness,
  - ψ for having outdegree exactly 1
- $\varphi \wedge \psi$ characterizes exactly simple cycles, i.e., cycles where every node is visited exactly once
- By the upward Löwenheim-Skolem theorem, $\varphi \wedge \psi$ has an infinite model. But there are no infinite simple cycles.

Ehrenfeucht-Fraïssé games

- Two-player game on two structures $\mathcal{A}$ and $\mathcal{B}$: Spoiler and duplicator
- In every round spoiler chooses an element from $\mathcal{A}$ or $\mathcal{B}$
- Duplicator responds with an element from the other structure
- k-round outcome: $\langle (a_1, \ldots, a_k), (b_1, \ldots, b_k) \rangle$

Partial isomorphism

Definition (Partial isomorphism)
Let $\mathcal{A}$ and $\mathcal{B}$ be structures for the same finite and relational vocabulary *Voc* with domains $A = \mathrm{Dom}^{\mathcal{A}}$ and $B = \mathrm{Dom}^{\mathcal{B}}$, and let $\overline{a} = (a_1, \dots, a_k) \in A^k$, $\overline{b} = (b_1, \dots, b_k) \in B^k$. The pair $\langle \overline{a}, \overline{b} \rangle$ is said to define a *partial isomorphism* between $\mathcal{A}$ and $\mathcal{B}$ iff the following conditions hold:

(1) $a_i = a_j \quad \text{iff} \quad b_i = b_j \quad \text{for all } 1 \leq i < j \leq k$

(2) for all constant symbols $c \in \mathit{Const}$ and all $i \in \{1, \dots, k\}$:
$a_i = c^{\mathcal{A}} \quad \text{iff} \quad b_i = c^{\mathcal{B}}$

(3) for each n-ary predicate symbol P and each index-tuple $(i_1, \dots, i_n) \in \{1, \dots, k\}^n$:

$$(a_{i_1}, \dots, a_{i_n}) \in P^{\mathcal{A}} \;\text{ iff }\; (b_{i_1}, \dots, b_{i_n}) \in P^{\mathcal{B}}$$

Partial isomorphism

$$\overline{a} = (a_1, \ldots, a_i, \ldots, a_j, \ldots, a_k) \in A^k$$

$$\overline{b} = (b_1, \ldots, b_i, \ldots, b_j, \ldots, b_k) \in B^k$$

Partial isomorphism

$$\overline{a} = (a_1, \ldots, a_i, \ldots, a_j, \ldots, a_k) \in A^k$$

$$(a_i, a_j, a_1) \in P^{\mathcal{A}}$$

$$\Longleftrightarrow$$

$$(b_i, b_j, b_1) \in P^{\mathcal{B}}$$

$$\overline{b} = (b_1, \ldots, b_i, \ldots, b_j, \ldots, b_k) \in B^k$$

Definition (Winning outcomes)
Recall that $\mathit{Const} = \{c_1, \ldots, c_\ell\}$ is the set of constant symbols in the underlying finite, relational vocabulary. An outcome

$$\langle \overline{a}, \overline{b} \rangle \;=\; \langle (a_1, \ldots, a_k), (b_1, \ldots, b_k) \rangle \in A^k \times B^k$$

of a k-round play is said to be *winning (for the duplicator)* iff the pair

$$\langle\, (\overline{a}, \overline{c}^{\mathcal{A}}),\, (\overline{b}, \overline{c}^{\mathcal{B}})\,\rangle \;=\; \langle\, (a_1, \ldots, a_k, c_1^{\mathcal{A}}, \ldots, c_\ell^{\mathcal{A}}),\, (b_1, \ldots, b_k, c_1^{\mathcal{B}}, \ldots, c_\ell^{\mathcal{B}})\,\rangle$$

defines a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$. Otherwise, i.e., if $\langle\, (\overline{a}, \overline{c}^{\mathcal{A}}),\, (\overline{b}, \overline{c}^{\mathcal{B}})\,\rangle$ does not define a partial isomorphism, the spoiler wins.

Strategies

... are functions

$$\mathcal{S} \;:\; \bigcup_{0 \leq i < k} (\, A^{i+1} \times B^{i} \;\cup\; A^{i} \times B^{i+1} \,) \;\to\; A \cup B,$$

where $A^0 = B^0 = \{\varepsilon\}$ consists of the empty tuple, such that for all $0 \leq i < k$ and elements $a_1, \ldots, a_i, a_{i+1} \in A$ and $b_1, \ldots, b_i, b_{i+1} \in B$:

$$\begin{aligned} \mathcal{S}(\,(a_1, \ldots, a_i), (b_1, \ldots, b_i, b_{i+1})\,) \quad &\in\; A \\ \mathcal{S}(\,(a_1, \ldots, a_i, a_{i+1}), (b_1, \ldots, b_i)\,) \quad &\in\; B \end{aligned}$$

Definition (k-round game equivalence $\cong_k$)
Let $k \in \mathbb{N}$ and $\mathcal{A}$ and $\mathcal{B}$ structures for *Voc*. Then, k-round-game-equivalence of $\mathcal{A}$ and $\mathcal{B}$ is defined by:

$$\mathcal{A} \cong_k \mathcal{B} \quad \overset{\text{def}}{\Longleftrightarrow} \quad \left\{ \begin{array}{l} \text{the duplicator has a } k\text{-round winning strategy} \\ \text{in the Ehrenfeucht-Fraïssé game for } \mathcal{A} \text{ and } \mathcal{B} \end{array} \right.$$

For the case $k = 0$ this means: $\mathcal{A} \cong_0 \mathcal{B}$ iff $\langle \overline{c}^{\mathcal{A}}, \overline{c}^{\mathcal{B}} \rangle$ defines a partial isomorphism.

Rank-k-types

Quantifier rank of an FOL-formula φ is its maximal nesting depth.

As before, let *Voc* be a finite relational vocabulary. If $\mathcal{A}$ is a structure for *Voc* then

$$\mathit{FOL}[k](\mathcal{A}) = \{\,\phi \,:\, \phi \text{ is a } \mathit{FOL}[k]\text{-sentence with } \mathcal{A} \models \phi\,\}$$

is called the *rank-k-type* over $\mathcal{A}$. A formula-set $\mathfrak{R}$ is said to be a rank-k type if $\mathfrak{R} = \mathit{FOL}[k](\mathcal{A})$ for some structure $\mathcal{A}$ for *Voc*.

Lemma (FOL[k]-definability of the rank-k-types)
Let Voc be a finite relational vocabulary and let $\mathfrak{R}_1, \ldots, \mathfrak{R}_s$ be the rank-k-types of Voc. Then, there are FOL[k]-sentences $\theta_1, \ldots, \theta_s$ such that for all $i \in \{1, \ldots, s\}$:

- $\mathfrak{R}_i \;=\; \{\, \psi \,:\, \psi$ *is a FOL[k]-sentence with* $\theta_i \Vdash \psi \,\}$
- *for each structure $\mathcal{A}$ we have:* $\quad \mathcal{A} \models \theta_i \quad$ *iff* $\quad FOL[k](\mathcal{A}) = \mathfrak{R}_i$.

Furthermore, each FOL[k]-sentence is equivalent to the disjunction of some of the θ_i's.

Theorem (Ehrenfeucht-Fraïssé-Theorem)
Let Voc be a finite relational vocabulary and $\mathcal{A}$ and $\mathcal{B}$ two structures for Voc. Then, the following two statements are equivalent:

(a) $\mathcal{A} \cong_k \mathcal{B}$

(b) *$\mathcal{A}$ and $\mathcal{B}$ agree on all FOL[k]-sentences, i.e., $\mathcal{A} \models \psi$ iff $\mathcal{B} \models \psi$ for all FOL[k]-sentence ψ*

Proof by induction on k

Corollary (FOL-equivalence of structures)
Let $\mathcal{A}$ and $\mathcal{B}$ structures over the same finite relational vocabulary Voc. Then, the following statements are equivalent:

(a) $\mathcal{A} \cong \mathcal{B}$

(b) $\mathcal{A}$ *and* $\mathcal{B}$ *fulfill the same FOL-sentences over Voc.*

(c) $Th(\mathcal{A}) = Th(\mathcal{B})$

$$\mathcal{A} \cong \mathcal{B} \text{ iff } \mathcal{A} \cong_k \mathcal{B} \text{ for all } k \in \mathbb{N}$$

FOL-equivalence of structures

Corollary (FOL-equivalence of structures)
Let $\mathcal{A}$ and $\mathcal{B}$ structures over the same finite relational vocabulary Voc. Then, the following statements are equivalent:

(a) $\mathcal{A} \cong \mathcal{B}$

(b) *$\mathcal{A}$ and $\mathcal{B}$ fulfill the same FOL-sentences over Voc.*

(c) $Th(\mathcal{A}) = Th(\mathcal{B})$

$$\mathcal{A} \cong \mathcal{B} \text{ iff } \mathcal{A} \cong_k \mathcal{B} \text{ for all } k \in \mathbb{N}$$

Example: Every two dense linear orders without endpoints satisfy the same FOL-sentences, e.g., $Th(\mathbb{R}, \leqslant) = Th(\mathbb{Q}, \leqslant)$

Theorem (FO-definability and Ehrenfeucht-Fraïssé games)
Let Voc be a finite relational vocabulary and $\mathcal{C}$ a class of structures for Voc. Then, the following statements are equivalent:

(a) *$\mathcal{C}$ is FO-definable.*

(b) *There exists $k \in \mathbb{N}$ such that for all structures $\mathcal{A}$ and $\mathcal{B}$ over Voc:*

$$\text{if } \mathcal{A} \in \mathcal{C} \text{ and } \mathcal{A} \cong_k \mathcal{B} \text{ then } \mathcal{B} \in \mathcal{C}.$$

Theorem (FO-definability and Ehrenfeucht-Fraïssé games)
Let Voc be a finite relational vocabulary and $\mathcal{C}$ a class of structures for Voc. Then, the following statements are equivalent:

(a) *$\mathcal{C}$ is FO-definable.*

(b) *There exists $k \in \mathbb{N}$ such that for all structures $\mathcal{A}$ and $\mathcal{B}$ over Voc:*

$$\text{if } \mathcal{A} \in \mathcal{C} \text{ and } \mathcal{A} \cong_k \mathcal{B} \text{ then } \mathcal{B} \in \mathcal{C}.$$

Example: Connectivity of finite undirected graphs is not FO-definable.

Countable Conjunctions: Syntax and semantics

- allows additionally $\bigwedge_{i \in I} \varphi_i$ for a countable set I,
- Syntactic sugar: $\bigvee_{i \in I} \varphi_i = \neg \bigwedge_{i \in I} \neg \varphi_i$,
- Semantics: $(\mathcal{A}, \mathcal{V}) \models \bigwedge_{i \in I} \varphi_i$ iff $(\mathcal{A}, \mathcal{V}) \models \varphi_i$ for all $i \in I$,
- $\bigwedge_{i \in \varnothing} \varphi_i \equiv \mathsf{true}$ and $\bigvee_{i \in \varnothing} \varphi_i \equiv \mathsf{false}$

Countable conjunction and finite structures

ψ_{fin} holds exactly for finite structures:

$$\psi_n = \exists x_1 \ldots \exists x_n . \bigwedge_{1 \leqslant i < j \leqslant n} (x_i \neq x_j)$$

$$\psi_{\mathrm{fin}} = \bigvee_{n \in \mathbb{N}} \neg \psi_n$$

Countable conjunction and finite structures

ψ_{fin} holds exactly for finite structures:

$$\psi_n = \exists x_1 \ldots \exists x_n . \bigwedge_{1 \leqslant i < j \leqslant n} (x_i \neq x_j)$$

$$\psi_{\text{fin}} = \bigvee_{n \in \mathbb{N}} \neg \psi_n$$

ψ_{even} holds exactly for finite, even structures:

$$\psi_n = \exists x_1 \ldots \exists x_n . \left(\bigwedge_{1 \leqslant i < j \leqslant n} (x_i \neq x_j) \wedge \forall y . \bigvee_{1 \leqslant i \leqslant n} y = x_i \right)$$

$$\psi_{\text{fin}} = \bigvee_{n \in \mathbb{N}} \psi_{2n}$$

Countable conjunction and natural numbers

Assume a constant symbol 0 and a unary function *succ*

$$
\begin{aligned}
&\forall x. succ(x) \neq 0 \\
\wedge &\forall x \forall y. (succ(x) = succ(y) \rightarrow x = y) \\
\wedge &\forall x. \bigvee_{n \geqslant 0} x = succ^n(0)
\end{aligned}
$$

characterizes the natural numbers, i.e., every model is isomorphic to $(\mathbb{N}, succ, 0)$.

Other $FOL(\bigwedge)$-definable examples

- Reachability in graphs,
- Periodic groups

Theorem (Lack of compactness property for *FOL*($\bigwedge$))
There exists an unsatisfiable countable set of FOL($\bigwedge$)-sentences that is finitely satisfiable.

$$\mathfrak{F} = \{\psi_{\mathrm{fin}}\} \cup \{\psi_n \,:\, n \geqslant 2\}$$

is finitely satisfiable, but not satisfiable

ψ_n characterizes all structures having at least n elements

Upward Löwenheim-Skolem theorem and *FOL*($\bigwedge$)

As consequence of the definability of finite structures, the upward Löwenheim-Skolem theorem ("from finite to infinite") does not hold.

ψ_{fin} has for every $n \in \mathbb{N}$ a model of cardinality n, but no infinite model.

As consequence of the definability of finite structures, the upward Löwenheim-Skolem theorem (“from finite to infinite”) does not hold.

ψ_{fin} has for every $n \in \mathbb{N}$ a model of cardinality n, but no infinite model.

Also, as $(\mathbb{N}, succ, 0)$ can be characterized in $FOL(\bigwedge)$, the upward Löwenheim-Skolem theorem (“from infinite to larger models”) does not hold.

Calculus for $FOL(\bigwedge)$

Theorem
There is no sound and complete calculus for $FOL(\bigwedge)$.

- Existence of a sound and complete calculus $\mathfrak{D}$ would yield

$$\mathfrak{F} \models \varphi \text{ implies } \mathfrak{G} \models \varphi \text{ for a finite } \mathfrak{G} \subseteq \mathfrak{F}$$

- But then

$$\begin{aligned} \mathfrak{F} \text{ unsatisfiable iff } & \mathfrak{F} \models \textbf{false} \\ \text{iff } & \mathfrak{G} \models \textbf{false} \text{ for a finite } \mathfrak{G} \subseteq \mathfrak{F} \\ \text{iff } & \text{there exists an unsatisfiable, finite } \mathfrak{G} \subseteq \mathfrak{F} \end{aligned}$$

- Contradiction to the non-existing compactness property for $FOL(\bigwedge)$

Downward Löwenheim-Skolem theorem for $FOL(\bigwedge)$

Theorem
Each satisfiable $FOL(\bigwedge)$-sentence has a countable (possibly finite) model.

- Assume an uncountable model $\mathcal{B}$ with domain B
- Define a sequence of countable subsets of B $B_0, B_1, \ldots$ s.t.
  (i) if $\psi = \psi(x_1, \ldots, x_n, x) \in \mathrm{subf}^+(\phi)$ and $b_1, \ldots, b_n \in B_i$ such that
  $$(\mathcal{B}, [x_1 := b_1, \ldots, x_n := b_n]) \models \exists x.\psi$$
  then there exists some $a \in B_{i+1}$ with $(\mathcal{B}, [x_1 := b_1, \ldots, x_n := b_n, x := a]) \models \psi$.
  (ii) for each n-ary function symbol f of Voc and all elements $b_1, \ldots, b_n \in B_i$ we have:
  $$f^{\mathcal{B}}(b_1, \ldots, b_n) \in B_{i+1}$$
- Define $A = \bigcup_{i \geqslant 0} B_i$ and a model A accordingly

Infinite quantification: Syntax and Semantics

Syntax:

- FOL with $\overset{\infty}{\exists}\, x.\varphi$
- syntactic sugar: $\overset{\infty}{\forall}\, x.\varphi = \neg \overset{\infty}{\exists}\, x.\neg\varphi$

Semantics:

$\overset{\infty}{\exists}\, x.\varphi$ iff there exists infinitely many $a \in \mathcal{A}$ with $(\mathcal{A}, \mathcal{V}[x := a]) \models \varphi$

$\overset{\infty}{\forall}\, x.\varphi$ iff for almost all $a \in \mathcal{A}$ holds: $(\mathcal{A}, \mathcal{V}[x := a]) \models \varphi$

$FOL(\overset{\infty}{\exists})$ and Finite Structures

- $\psi_{\text{fin}} = \neg \overset{\infty}{\exists} x.$**true** characterizes finite structures
- Analogously to $FOL(\bigwedge)$ the following theorems are violated to:
    - compactness theorem,
    - upward Löwenheim-Skolem theorem "from finite to infinite models"

Assume a relational vocabulary with $\sqsubseteq$ as binary predicate.
$\varphi_{\mathrm{nat}} = \psi_{\mathrm{LO}} \wedge \psi_1 \wedge \psi_2$ with

$$\psi_{\mathrm{LO}} \text{ describing } \sqsubseteq \text{ as linear order}$$

$$\psi_1 = \forall x. \overset{\infty}{\forall} y.x \sqsubset y$$

$$\psi = \forall x.\exists y.x \sqsubset y$$

holds true exactly for those structures isomorphic to $(\mathbb{N}, \leqslant)$.

$FOL(\overset{\infty}{\exists})$ and the Natural Numbers

Assume a relational vocabulary with $\sqsubseteq$ as binary predicate.
$\varphi_{\mathrm{nat}} = \psi_{\mathrm{LO}} \wedge \psi_1 \wedge \psi_2$ with

$$\psi_{\mathrm{LO}} \text{ describing } \sqsubseteq \text{ as linear order}$$
$$\psi_1 = \forall x. \overset{\infty}{\forall} y. x \sqsubset y$$
$$\psi = \forall x. \exists y. x \sqsubset y$$

holds true exactly for those structures isomorphic to $(\mathbb{N}, \leqslant)$.
Consequence: The upward Löwenheim-Skolem theorem (“from infinite to larger models”) does not hold.

Embedding of $FOL(\overset{\infty}{\exists})$ into $FOL(\bigwedge)$

Theorem (Embedding of $FOL(\overset{\infty}{\exists})$ into $FOL(\bigwedge)$)
For each $FOL(\overset{\infty}{\exists})$-formula ϕ there exists a $FOL(\bigwedge)$-formula $\overline{\phi}$ over the same vocabulary such that ϕ and $\overline{\phi}$ have the same models.

Trick:

$$\overline{\overset{\infty}{\exists}\, x.\varphi} = \bigwedge_{n\geqslant 1} \exists x_1 \ldots \exists x_n.(\overline{\varphi}[x/x_1] \wedge \ldots \wedge \overline{\varphi}[x/x_n] \wedge \bigwedge_{1\leq i<j\leq n} x_i \neq x_j$$

where $x_1, x_2, \ldots$ is a sequence of pairwise distinct fresh variables.

Embedding of $FOL(\overset{\infty}{\exists})$ into $FOL(\bigwedge)$

Theorem (Embedding of $FOL(\overset{\infty}{\exists})$ into $FOL(\bigwedge)$)
For each $FOL(\overset{\infty}{\exists})$-formula ϕ there exists a $FOL(\bigwedge)$-formula $\overline{\phi}$ over the same vocabulary such that ϕ and $\overline{\phi}$ have the same models.

Trick:

$$\overline{\overset{\infty}{\exists}\, x.\varphi} = \bigwedge_{n \geqslant 1} \exists x_1 \ldots \exists x_n.(\overline{\varphi}[x/x_1] \wedge \ldots \wedge \overline{\varphi}[x/x_n] \wedge \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j$$

where $x_1, x_2, \ldots$ is a sequence of pairwise distinct fresh variables.

Consequence: The downward Löwenheim-Skolem theorem holds for $FOL(\overset{\infty}{\exists})$.

Uncountable quantification: Syntax and Semantics

Syntax:

- FOL with $\overset{2^{\mathbb{N}}}{\exists} x.\varphi$
- syntactic sugar: $\overset{2^{\mathbb{N}}}{\forall} x.\varphi = \neg \overset{2^{\mathbb{N}}}{\exists} x.\neg\varphi$

Semantics:

$\overset{2^{\mathbb{N}}}{\exists} x.\varphi$ iff there exists uncountably many $a \in \mathcal{A}$ with

$$(\mathcal{A}, \mathcal{V}[x := a]) \models \varphi$$

$\overset{2^{\mathbb{N}}}{\forall} x.\varphi$ iff for countably many $a \in \mathcal{A}$ holds: $(\mathcal{A}, \mathcal{V}[x := a]) \not\models \varphi$

$FOL(\overset{2^{\mathbb{N}}}{\exists})$ and set cardinality

$$\mathcal{A} \models \overset{2^{\mathbb{N}}}{\exists}\, x.\mathbf{true} \text{ iff } \mathcal{A} \text{ is uncountable}$$

$$\mathcal{A} \models \overset{2^{\mathbb{N}}}{\forall}\, x.\mathbf{false} \text{ iff } \mathcal{A} \text{ is countable}$$

$FOL(\overset{2^{\mathbb{N}}}{\exists})$ and set cardinality

$$\mathcal{A} \models \overset{2^{\mathbb{N}}}{\exists} x.\mathbf{true} \text{ iff } \mathcal{A} \text{ is uncountable}$$

$$\mathcal{A} \models \overset{2^{\mathbb{N}}}{\forall} x.\mathbf{false} \text{ iff } \mathcal{A} \text{ is countable}$$

Consequence: Downward Löwenheim-Skolem theorem and Upward Löwenheim-Skolem theorem “from infinite to larger models” are violated.

$FOL(\overset{2^{\mathbb{N}}}{\exists})$ and calculi

- With a suitable additional axioms, $FOL(\overset{2^{\mathbb{N}}}{\exists})$ has a sound and complete calculus
- Hence, the compactness theorem holds.

Theorem

There is no set $\mathfrak{F}$ of $FOL(\overset{2^{\mathbb{N}}}{\exists})$-sentences over Voc_{nat} such that $(\mathbb{N}, succ, 0) \models \mathfrak{F}$ and all countable models for $\mathfrak{F}$ are isomorphic to $(\mathbb{N}, succ, 0)$.

Second order logic

Vocabulary has three kinds of variables:

- first-order variables $x, y, z, \ldots \in \mathit{Var}$ that represent elements of the domain of a structure,
- predicate variables $X, Y, Z, \ldots \in \mathit{PVar}$ that stand for relations over the domain,
- function variables $F, G, H, \ldots \in \mathit{FVar}$ that stand for functions over the domain.

$$t \quad ::= \quad \underbrace{x \mid f(t_1, \ldots, t_m)}_{\text{as in FOL}} \mid F(t_1, \ldots, t_m)$$

$$\phi \quad ::= \quad \underbrace{\mathsf{true} \mid P(t_1, \ldots, t_n) \mid t_1 = t_2 \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid \forall x.\phi}_{\text{as in FOL}}$$

$$X(t_1, \ldots, t_n) \mid \underbrace{\forall X.\phi \mid \forall F.\phi}_{\text{second-order quantification}}$$

Interpretation $\mathcal{I} = (\mathcal{A}, Val)$.

- valuation *Val* as in FOL, but additionally:
  - $Val(X) = X^{\mathcal{I}} \subseteq A^n$, for each n-ary predicate variable
  - $Val(F) = F^{\mathcal{I}} : A^m \to A$ for each n-ary function variable
- interpretation of an SOL formula as in FOL, but additionally:

$$\mathcal{I} \models X(t_1, \dots, t_n) \text{ iff } (t_1^{\mathcal{I}}, \dots, t_n^{\mathcal{I}}) \in X^{\mathcal{I}}$$

$$\mathcal{I} \models \forall X.\phi \text{ iff } (\mathcal{A}, Val[X := R]) \models \phi \text{ for all relations } R \subseteq A^n$$

$$\mathcal{I} \models \forall F.\phi \text{ iff } (\mathcal{A}, Val[F := f]) \models \phi \text{ for all functions}$$

$$f : A^m \to A$$

Leibnitz Axiom for Equality

Equality can be represented by SOL directly:

$$t_1 = t_2 \qquad \text{iff} \qquad \forall X . X(t_1) \leftrightarrow X(t_2)$$

$\Rightarrow$ obvious
$\Leftarrow$ As X is all-quantified, take $X^{\mathcal{I}} = \{t_1^{\mathcal{I}}\}$

SOL and Natural Numbers

Characterization of the natural numbers based on the induction principle:

$$\phi_{\text{ind}} = \forall X.\Big(\big(X(0) \land \forall y.(X(y) \to X(succ(y)))\big) \to \forall y.X(y)\Big)$$

SOL and Natural Numbers

Characterization of the natural numbers based on the induction principle:

$$\phi_{\mathrm{ind}} = \forall X. \Big((X(0) \land \forall y.(X(y) \to X(succ(y)))) \to \forall y.X(y) \Big)$$

Two additional FOL-formulas necessary:

$$\varphi_1 = \forall x.succ(x) \neq 0$$
$$\varphi_2 = \forall x.\forall y.(succ(x) = succ(y) \to x = y$$

SOL and Natural Numbers

Characterization of the natural numbers based on the induction principle:

$$\phi_{\mathrm{ind}} = \forall X.\Big((X(0) \wedge \forall y.(X(y) \rightarrow X(succ(y)))) \rightarrow \forall y.X(y)\Big)$$

Two additional FOL-formulas necessary:

$$\varphi_1 = \forall x.succ(x) \neq 0$$
$$\varphi_2 = \forall x.\forall y.(succ(x) = succ(y) \rightarrow x = y$$

Other examples of SOL-definable structures: Well-ordering, periodic groups, ...

Characterization of infinite structures:

$$\psi_\infty \ = \ \exists F\Big(\underbrace{\forall x \forall y.(F(x) = F(y) \to x = y)}_{\text{injective}} \wedge \underbrace{\exists z \forall x.(F(x) \neq z)}_{\text{not surjective}}\Big)$$

The existence of an injective, but not surjective function $F : A \to A$ guarantees that A is infinite.

Characterization of infinite structures:

$$\psi_\infty \;=\; \exists F\Big(\underbrace{\forall x \forall y.(F(x) = F(y) \to x = y)}_{\text{injective}} \wedge \underbrace{\exists z \forall x.(F(x) \neq z)}_{\text{not surjective}}\Big)$$

The existence of an injective, but not surjective function $F : A \to A$ guarantees that A is infinite. Consequence of the definability of finite structures: The compactness property does not hold (same argumentation as for $FOL(\bigwedge)$

Characterization of countable structures (over *Voc* with a binary predicate variable $\sqsubseteq$)

$$\phi_{\mathsf{ctbl}} = \exists \sqsubseteq . \Big(\phi_{\mathsf{lin\_order}}(\sqsubseteq) \wedge \forall x \exists X. \big(\psi_{\mathrm{fin}}(X) \wedge \forall y. (y \sqsubseteq x \leftrightarrow X(y)) \big) \Big)$$

. . . and countable sets

Characterization of countable structures (over *Voc* with a binary predicate variable $\sqsubseteq$)

$$\phi_{\text{ctbl}} = \exists \sqsubseteq . \Big(\phi_{\text{lin\_order}}(\sqsubseteq) \wedge \forall x \exists X. \big(\psi_{\mathrm{fin}}(X) \wedge \forall y. \big(y \sqsubseteq x \leftrightarrow X(y) \big) \big) \Big)$$

Idea:

$$A \text{ is countable} \quad \text{iff} \quad \begin{cases} \text{there exists a linear order } \sqsubseteq \text{ on } A \\ \text{such that for each } a \in A \\ \text{the downward closure } A \downarrow a \text{ is finite} \end{cases}$$

$$A \downarrow a = \{ b \in A \,:\, b \sqsubseteq a \}$$

SOL and the Löwenheim-Skolem theorems

The existence of ψ_{fin} and ϕ_{ctbl} yields that non of the Löwenheim-Skolem theorems holds for SOL:

- There exists a SOL-sentence that has finite models of arbitrary size $n \geq 1$, but has no infinite model.
- There exists a SOL-sentence that has an infinite, but no uncountable model.
- There exists a satisfiable SOL-sentence that has no countable model.

Theorem
There is no sound and weakly complete deductive calculus for the set of valid SOL-sentences.

This is a consequence of

Theorem ($\mathbf{SOL-VALID}$ is not recursively enumerable)
The set of valid SOL-sentences over some vocabulary with at least one binary predicate symbol is not recursively enumerable.

Proof

For every FOL-sentence ψ holds:

$$\begin{aligned} & \psi \in \text{FOL-VALID-FIN} \\ \text{iff } & \mathcal{A} \models \psi \text{ for all finite structures } \mathcal{A} \\ \text{iff } & \text{the SOL-sentence } \varphi_{\text{fin}} \to \psi \text{ is valid} \\ \text{iff } & \varphi_{\text{fin}} \to \psi \in \text{SOL-VALID} \end{aligned}$$

Proof

For every FOL-sentence ψ holds:

$$\begin{aligned} &\psi \in \text{FOL-VALID-FIN} \\ \text{iff } &\mathcal{A} \models \psi \text{ for all finite structures } \mathcal{A} \\ \text{iff } &\text{the SOL-sentence } \varphi_{\text{fin}} \to \psi \text{ is valid} \\ \text{iff } &\varphi_{\text{fin}} \to \psi \in \text{SOL-VALID} \end{aligned}$$

If SOL-VALID would be recursively enumerable, we could enumerate all valid SOL-formulas φ, and if it has the form $\varphi_{\text{fin}} \to \psi$, output ψ. Hence, FOL-VALID-FIN would be recursively enumerable. Contradiction to Trakhtenbrot's theorem.

SOL and reachability

Reachability in directed graphs is SOL-definable:

$$\phi_{\mathrm{reach}}(x,y) = \forall Z.\,\big(\,Z(x) \wedge \psi(Z) \;\rightarrow\; Z(y)\,\big)$$
$$\psi(Z) = \forall u \forall v.\,\big(\,Z(u) \wedge E(u,v) \;\rightarrow\; Z(v)\,\big)$$

.

SOL and reachability

Reachability in directed graphs is SOL-definable:

$$\begin{aligned}\phi_{\mathrm{reach}}(x, y) &= \forall Z. \left(Z(x) \land \psi(Z) \;\rightarrow\; Z(y) \right) \\ \psi(Z) &= \forall u \forall v. \left(Z(u) \land E(u, v) \;\rightarrow\; Z(v) \right)\end{aligned}$$

Idea: $\psi(Z)$ states that the unary predicate variable is closed under the edge relation E

Hence:

$$(\mathcal{G}, a, b) \models \phi_{\text{reach}}(x, y) \quad \text{iff} \quad b \text{ is reachable from } a \text{ in } \mathcal{G}$$

.

Definition (Existential SOL-formula)
An *existential SOL-formula*, simply called ESOL-formula, is a SOL-formula over some purely relational vocabulary of the form

$$\exists X_1 \ldots \exists X_n . \psi$$

where

- the X_i's are predicate variables of arbitrary arity,
- ψ a FOL-formula (possibly with equality) when $X_1, \ldots, X_n$ are viewed as predicate symbols and does not contain any other second-order variable.

Theorem (ESO-definable graph properties are in NP)
For each ESOL formula ϕ over Voc$_{graph}$, the problem ϕ-GRAPH belongs to NP. (Formula ϕ is viewed to be fixed.)

ESOL-definable graph properties

Theorem (ESO-definable graph properties are in NP)
For each ESOL formula ϕ over Voc$_{graph}$, the problem ϕ-GRAPH belongs to NP. (Formula ϕ is viewed to be fixed.)

Proof idea: Choose sets for existential quantified set variables non-deterministically. Apply PTIME algorithm for FOL-graph.

Weak SOL

- Syntax: the same as in normal SOL except there are no function variable.
- Semantics: quantification about finite relations only.

Finite structures

For

$$\varphi_{\mathrm{fin}} = \exists X. \forall y. X(y)$$

holds:

$$\mathcal{A} \models_{\mathrm{weak}} \varphi_{\mathrm{fin}} \quad \text{iff} \quad \mathcal{A} \text{ is finite}$$

Finite structures

For

$$\varphi_{\text{fin}} = \exists X.\forall y.X(y)$$

holds:

$$\mathcal{A} \models_{\text{weak}} \varphi_{\text{fin}} \quad \text{iff} \quad \mathcal{A} \text{ is finite}$$

Even more, for every infinite structure $\mathcal{A}$ holds: $\mathcal{A} \not\models_{\text{weak}} \varphi_{\text{fin}}$ and $\mathcal{A} \models \varphi_{\text{fin}}$

Weak SOL and the natural numbers

Similar to $FOL(\overset{\infty}{\exists})$ we ask for a linear order, where the downward closure of every element is finite:

$$\phi^{W}_{\mathrm{nat}} \quad = \quad \phi_{\mathsf{lin\_order}} \quad \wedge \quad \underbrace{\forall x\, \exists y.\, x < y}_{\text{no maximal element}} \quad \wedge \quad \underbrace{\forall x\, \exists Y\, \forall y.\, (\, y \leq x \;\rightarrow\; Y(y)\,)}_{\text{the downward closure of each element is finite}}$$

We use, that there exists an SOL-formula $\psi_{\text{fin}}(X)$ for every n-ary predicate variable X stating that X is finite.

We use, that there exists an SOL-formula $\psi_{\mathrm{fin}}(X)$ for every n-ary predicate variable X stating that X is finite.
In a weak SOL formula φ replace every occurrence of $\forall X.\psi$ with $\forall X.(\psi_{\mathrm{fin}}(X) \to \psi')$ where ψ' is the replaced version of ψ.

Embedding Weak SOL into $FOL(\bigwedge)$

Idea: Replace predicate variables with disjunction of equivalences.

Embedding Weak SOL into $FOL(\bigwedge)$

Idea: Replace predicate variables with disjunction of equivalences. Consequence of the existence of an embedding of weak SOL into $FOL(\bigwedge)$: The Downward Löwenheim-Skolem theorem holds for weak SOL.

Monadic Second Order Logic

- Predicate variables are restricted to be unary
- Syntax and semantics cover full FOL
- We focus on finite words on structures

We assume a vocabulary $Voc_{\Sigma,\mathrm{graph}}$ denoting a graph with labels on the nodes.

$P_{\mathfrak{a}}(x)$	with $x \in Var$, $\mathfrak{a} \in \Sigma$
$E(x,y)$	with $x, y \in Var$
$x \in X$	with $x \in Var$ and X an unary predicate variable
$x = y$	with $x, y \in Var$

We assume a vocabulary $Voc_{\Sigma,\mathrm{graph}}$ denoting a graph with labels on the nodes.

$$
\begin{array}{ll}
P_{\mathfrak{a}}(x) & \text{with } x \in Var, \mathfrak{a} \in \Sigma \\
E(x, y) & \text{with } x, y \in Var \\
x \in X & \text{with } x \in Var \text{ and } X \text{ an unary predicate variable} \\
x = y & \text{with } x, y \in Var
\end{array}
$$

Examples for definability: Reachability, partitioning, singleton sets,...

MSO over finite words

We assume a vocabulary $Voc_{\Sigma,\text{graph}}$ denoting a graph with labels on the nodes.

$$
\begin{array}{ll}
P_{\mathfrak{a}}(x) & \text{with } x \in Var,\ \mathfrak{a} \in \Sigma \\
E(x, y) & \text{with } x, y \in Var \\
x \in X & \text{with } x \in Var \text{ and } X \text{ an unary predicate variable} \\
x = y & \text{with } x, y \in Var
\end{array}
$$

Examples for definability: Reachability, partitioning, singleton sets,...
A finite word w can be seen as graph where the nodes are natural numbers from $\{1, \ldots, |w|\}$ and the edges are $E(i, i+1)$ for every $i \in \{1, \ldots, n-1\}$.

Example formulas

$$\mathit{first}(x) = \bigvee_{\mathfrak{a} \in \Sigma} P_{\mathfrak{a}}(x) \wedge \neg \exists y . E(y, x)$$

denotes that x is the first position

Example formulas

$$first(x) = \bigvee_{\mathfrak{a} \in \Sigma} P_{\mathfrak{a}}(x) \wedge \neg \exists y. E(y, x)$$

denotes that x is the first position

$$last(x) = \bigvee_{\mathfrak{a} \in \Sigma} P_{\mathfrak{a}}(x) \wedge \neg \exists y. E(x, y)$$

denotes that x is the last position

Example formulas

$$\mathit{first}(x) = \bigvee_{\mathfrak{a} \in \Sigma} P_{\mathfrak{a}}(x) \wedge \neg \exists y. E(y, x)$$

denotes that x is the first position

$$\mathit{last}(x) = \bigvee_{\mathfrak{a} \in \Sigma} P_{\mathfrak{a}}(x) \wedge \neg \exists y. E(x, y)$$

denotes that x is the last position

$\bigvee_{\mathfrak{a} \in \Sigma} P_{\mathfrak{a}}(x)$ excludes the empty word

MSO-definability

A language $L \subseteq \Sigma^*$ is said to be MSO-definable, if there exists an MSO-sentence φ over $Voc_{\Sigma,\mathrm{graph}}$ such that

$$L \; = \; \{w \in \Sigma^* \; : \; \mathrm{Graph}(w) \models \varphi\}$$

A language $L \subseteq \Sigma^*$ is said to be MSO-definable, if there exists an MSO-sentence φ over $Voc_{\Sigma,\mathrm{graph}}$ such that

$$L = \{w \in \Sigma^* : \mathrm{Graph}(w) \models \varphi\}$$

Theorem (MSO, EMSO and regular languages)
Let $L \subseteq \Sigma^*$ be a language over finite words. Then, the following statements are equivalent:

(1) *L is regular.*
(2) *L is EMSO-definable.*
(3) *L is MSO-definable.*

Consequences

- Given an MSO formula φ, one can construct an NFA A with $L(\varphi) = L(A)$
- The set of satisfiable MSO formulas (over finite words) is decidable
- The set of valid MSO formula (over finite words) is decidable

FOL over finite words

FOL cannot express reachability. Therefore we assume a binary predicate $\leqslant$ with the semantics

$$x \leqslant y \text{ iff } y \text{ is reachable from } x$$

The standard edge relation can be established in straightforward manner:

$$E(x,y) = x < y \ \wedge \ \neg\exists z.(x < z \wedge z < y)$$

Star-free expression

… are given by the following grammar:

$$\alpha \quad ::= \quad \emptyset \;\Big|\; \varepsilon \;\Big|\; \mathfrak{a} \;\Big|\; \alpha_1 + \alpha_2 \;\Big|\; \alpha_1\,\alpha_2 \;\Big|\; \overline{\alpha}$$

Star-free expression

… are given by the following grammar:

$$\alpha \quad ::= \quad \emptyset \;\Big|\; \varepsilon \;\Big|\; \mathfrak{a} \;\Big|\; \alpha_1 + \alpha_2 \;\Big|\; \alpha_1\, \alpha_2 \;\Big|\; \overline{\alpha}$$

Every star-free language is induced by a star-free expression and vice versa.

Star-free expression

… are given by the following grammar:

$$\alpha \quad ::= \quad \emptyset \;\Big|\; \varepsilon \;\Big|\; \mathfrak{a} \;\Big|\; \alpha_1 + \alpha_2 \;\Big|\; \alpha_1 \, \alpha_2 \;\Big|\; \overline{\alpha}$$

Every star-free language is induced by a star-free expression and vice versa.
Examples: $L(\overline{\overline{\emptyset}\mathfrak{b}\overline{\emptyset}}) = L(\mathfrak{a}^*)$ is a starfree language.

FO-definable and star-free languages

Lemma (Star-free languages are FO-definable)
Each star-free language is FO-definable.

Proof. by induction, concatenation case is difficult/technical

FO-definable and star-free languages

Lemma (Star-free languages are FO-definable)
Each star-free language is FO-definable.

Proof. by induction, concatenation case is difficult/technical

Lemma (FO-definable languages are star-free)
If $L \subseteq \Sigma^*$ is FO-definable then L is star-free.

Proof. Induction over k for $FOL[k]$ sentences.