It remains to show that the languages $\mathfrak{L}(\phi)$ for all FOL-sentences ϕ over $Voc_{\Sigma,\leqslant}$ are star-free. This is shown in the following lemma using the Ehrenfeucht-Fraïssé theory:

Lemma 2.6.29 (FO-definable languages are star-free). *If* $L \subseteq \Sigma^*$ *is FO-definable then* L *is star-free.*

Proof. For technical reasons, we deal with an extended vocabulary $Voc^{\text{ext}}_{\Sigma,\leqslant}$ that extends $Voc_{\Sigma,\leqslant}$ by constant symbols c_{first} and c_{last} for the positions of the first and last element. The graph structures for finite words have to be extended with meanings for c_{first} and c_{last} in the obvious way. That is, if $w = a_1 \ldots a_n$ is a word of length $n \geqslant 1$ then:

$$\text{Graph}(w) \;=\; \big(\{1,\ldots,n\}, \leqslant, (P^w_a)_{a\in\Sigma}, c^w_{\mathit{first}}, c^w_{\mathit{first}}\big)$$

where

$$c^w_{\mathit{first}} \stackrel{\text{def}}{=} 1 \quad \text{and} \quad c^w_{\mathit{last}} \stackrel{\text{def}}{=} n$$

For the empty word, $\text{Graph}(\varepsilon) \;=\; \big(\{0\}, \leqslant, (P^\varepsilon_a)_{a\in\Sigma}, 0, 0\big)$.

We show by induction on k that each FOL[k]-sentence ϕ over the extended vocabulary $Voc^{\text{ext}}_{\Sigma,\leqslant}$ defines a star-free language. Recall that the FOL[k]-formulas are FOL-formulas where the quantifier rank is at most k. See Notation 1.6.20 on page 97.

Basis of induction. For $k = 0$, we consider a FOL[0]-sentence ϕ. Then, ϕ is quantifier-free and closed, and thus a propositional formula built by the atoms *true*, $P_a(c_{\mathit{first}})$, $P_a(c_{\mathit{last}})$, $t = t'$ and $t \leqslant t'$ where t and t' are variable-free terms, i.e., t, $t' \in \{c_{\mathit{first}}, c_{\mathit{last}}\}$. In fact, the languages of these atoms are star-free:

- *true*, $c_{\mathit{first}} = c_{\mathit{first}}$, $c_{\mathit{last}} = c_{\mathit{last}}$, $c_{\mathit{first}} \leqslant c_{\mathit{first}}$, $c_{\mathit{last}} \leqslant c_{\mathit{last}}$ and $c_{\mathit{first}} \leqslant c_{\mathit{last}}$, define the language Σ^*, which is given by the star-free expression $\overline{\emptyset}$.
- The language defined by $P_a(c_{\mathit{last}})$ is
$$\mathfrak{L}(\, P_a(c_{\mathit{last}})\,) \;=\; \big\{ wa : w \in \Sigma^* \big\}$$
is given by the star-free expression $\overline{\emptyset}\, a$.
- The language defined by $P_a(c_{\mathit{first}})$ is
$$\mathfrak{L}(\, P_a(c_{\mathit{first}})\,) \;=\; \big\{ aw : w \in \Sigma^* \big\}$$
is given by the star-free expression $a\,\overline{\emptyset}$.
- The languages defined by the atoms $c_{\mathit{last}} \leqslant c_{\mathit{first}}$, $c_{\mathit{first}} = c_{\mathit{last}}$ and $c_{\mathit{last}} = c_{\mathit{first}}$ agree with the star-free language
$$\{\varepsilon\} \cup \bigcup_{a\in\Sigma} \{a\}$$
which, for $\Sigma = \{a_1, \ldots, a_k\}$, are given by the star-free expression $\varepsilon + a_1 + \ldots + a_k$.

As boolean combinations of star-free languages are star-free, we get that all FOL[0]-sentences define a star-free language.

Step of induction. Let us now assume that $k \geqslant 0$ and that the languages $\mathfrak{L}(\psi)$ for all FOL[k]-sentences ψ over $Voc^{ext}_{\Sigma,\leqslant}$ are star-free. The goal is to show that the languages $\mathfrak{L}(\phi)$ of all FOL[k+1]-sentences ϕ over $Voc^{ext}_{\Sigma,\leqslant}$ are star-free. Again, as the class of star-free languages is closed under the boolean combinators (intersection and complementation) is suffices to consider the case of a FOL[k+1]-sentence

$$\phi \ = \ \exists x. \psi(x)$$

i.e., where ψ is a FOL[k](x)-formula (which means a FOL[k]-formula with *Free*$(\psi) \subseteq \{x\}$). Let

$$\mathfrak{T}_1, \ldots, \mathfrak{T}_s$$

be an enumeration of the rank-k types of $Voc_{\Sigma,\leqslant}$ and let $\theta_1, \ldots, \theta_s$ FOL[k]-sentences that define them. That is:

$$\mathfrak{T}_i \ = \ \{ \varphi : \varphi \text{ is a FOL[k]-sentence s.t. } \theta_i \Vdash \varphi \}$$

Recall that FOL[k]$(\varnothing)$ denotes the set of all FOL[k]-sentence. The rank-k-type

$$\begin{aligned} & \text{FOL[k]}(\text{Graph}(w)) \\ = \ & \{ \varphi : \varphi \text{ is a FOL[k]-sentence s.t. Graph}(w) \models \varphi \} \\ = \ & \{ \varphi : \varphi \text{ is a FOL[k]-sentence s.t. } w \in \mathfrak{L}(\varphi) \} \end{aligned}$$

of a word structure agrees with the rank-k-type $\mathfrak{T}_i$ where θ_i holds for Graph(w), see Lemma 1.6.26 on page 99. That is:

$$w \in \mathfrak{L}(\theta_i) \quad \text{iff} \quad \text{Graph}(w) \models \theta_i \quad \text{iff} \quad \text{FOL[k]}(\text{Graph}(w)) \ = \ \mathfrak{T}_i$$

Hence:

$$\begin{aligned} \mathfrak{L}(\theta_i) \ &= \ \{ w \in \Sigma^* : \text{Graph}(w) \models \theta_i \} \\ &= \ \{ w \in \Sigma^* : \text{FOL[k]}(\text{Graph}(w)) = \mathfrak{T}_i \} \end{aligned}$$

Furthermore, the extension of the vocabulary by new constant symbols ensures that the empty word ε has its own rank-k-type. This follows from the fact that

$$\bigvee_{a \in \Sigma} P_a(c_{first})$$

is a FOL[0]-sentence which holds for each nonempty word, but not for the empty word. Thus, there is some index $k \in \{1, \ldots, s\}$ such that $\mathfrak{L}(\theta_k) = \{\varepsilon\}$ and $\varepsilon \notin \mathfrak{L}(\theta_i)$ for all $1 \leqslant i \leqslant s$ with $i \neq k$.

Let $\mathfrak{R}$ be the relation consisting of all pairs $(\mathfrak{T}_i, \mathfrak{T}_j)$ of rank-k-types such that for some nonempty word $v = \mathfrak{b}_1 \mathfrak{b}_2 \ldots \mathfrak{b}_r$ and word position $\ell \in \{1, \ldots, r\}$ where $(\text{Graph}(v), [x := \ell])$ is a model for $\psi(x)$, the first component $\mathfrak{T}_i$ is the rank-k-type of the prefix $\mathfrak{b}_1 \mathfrak{b}_2 \ldots \mathfrak{b}_\ell$ of ℓ and the second component $\mathfrak{T}_j$ is the rank-k-type of the (possibly empty) suffix $\mathfrak{b}_{\ell+1} \ldots \mathfrak{b}_r$ of v. Thus:

$$\begin{aligned} \mathfrak{R} \ \stackrel{\text{def}}{=} \ \{ \ (i,j) \in \{1, \ldots, s\}^2 \ : \ & \text{there exist } \mathfrak{b}_1 \mathfrak{b}_2 \ldots \mathfrak{b}_n \in \Sigma^+ \text{ and } \ell \in \{1, \ldots, r\} \text{ such that} \\ & (1) \ \ (\text{Graph}(\mathfrak{b}_1 \mathfrak{b}_2 \ldots \mathfrak{b}_r), [x := \ell]) \models \psi(x) \\ & (2) \ \ \text{FOL[k]}(\text{Graph}(\mathfrak{b}_1 \ldots \mathfrak{b}_\ell)) = \mathfrak{T}_i \\ & (3) \ \ \text{FOL[k]}(\text{Graph}(\mathfrak{b}_{\ell+1} \ldots \mathfrak{b}_r)) = \mathfrak{T}_j \qquad \} \end{aligned}$$

We will show that for each nonempty word $w = a_1\, a_2 \ldots a_n \in \Sigma^+$, the following statements (i) and (ii) are equivalent:

(i) $\mathrm{Graph}(w) \models \phi = \exists x.\psi(x)$

(ii) there exists $m \in \{1,\ldots,n\}$ such that $(i,j) \in \mathfrak{R}$ where

$$\begin{aligned}\mathfrak{T}_i &= \mathrm{FOL}[k]\big(\,\mathrm{Graph}(a_1\, a_2 \ldots a_m)\,\big)\\ \mathfrak{T}_j &= \mathrm{FOL}[k]\big(\,\mathrm{Graph}(a_{m+1} \ldots a_n)\,\big)\end{aligned}$$

Having established the equivalence of (i) and (ii), the remaining argument is as follows. The induction hypothesis applied to the FOL[k]-sentences θ_i yields that the languages $\mathfrak{L}(\theta_i)$ are star-free. Let

$$L \stackrel{\text{def}}{=} \bigcup_{(i,j)\in\mathfrak{R}} \mathfrak{L}(\theta_i)\,\mathfrak{L}(\theta_j)$$

As the languages $\mathfrak{L}(\theta_i)$ are star-free, so is the language L. The goal is now to show that L agrees with the language defined by $\exists x.\psi(x)$ excluding the empty word (which might or might not belong to $\mathfrak{L}(\exists x.\psi(x))$, i.e.:

$$\mathfrak{L}\big(\,\exists x.\psi(x)\,\big) \setminus \{\varepsilon\} \;=\; L \tag{*}$$

From (*) we conclude that

$$\mathfrak{L}\big(\,\exists x.\psi(x)\,\big) \;=\; \begin{cases} L \cup \{\varepsilon\} & : \text{ if } \mathrm{Graph}(\varepsilon) \models \exists x.\psi(x) \\ L & : \text{ otherwise} \end{cases}$$

is star-free. Establishing statement (*) amounts showing that for each word $w \in \Sigma^*$:

$$w \in L \quad \text{iff} \quad w \neq \varepsilon \text{ and } \mathrm{Graph}(w) \models \exists x.\psi(x)$$

"$\Longrightarrow$": Suppose $w = a_1 \ldots a_n \in L$. Then, $w \in \mathfrak{L}(\theta_i)\,\mathfrak{L}(\theta_j)$ for some pair $(i,j) \in \mathfrak{R}$. We first observe that $w \neq \varepsilon$. This follows from the fact that by condition (2) in the definition of $\mathfrak{R}$, the formula-set $\mathfrak{T}_i$ is the rank-k-type of some nonempty word. Hence, $n \geqslant 1$ and there exists $m \in \{1,\ldots,n\}$ such that

$$\begin{aligned} w_1 &\stackrel{\text{def}}{=} a_1\, a_2 \ldots a_m \;\in\; \mathfrak{L}(\theta_i)\\ w_2 &\stackrel{\text{def}}{=} a_{m+1} \ldots a_n \;\in\; \mathfrak{L}(\theta_j)\end{aligned}$$

But then $\mathrm{Graph}(w_1) \models \theta_i$ and $\mathrm{Graph}(w_2) \models \theta_j$, and therefore

$$\mathrm{FOL}[k](\,\mathrm{Graph}(w_1)\,) = \mathfrak{T}_i \;\text{ and }\; \mathrm{FOL}[k](\,\mathrm{Graph}(w_2)\,) = \mathfrak{T}_j$$

But then the equivalence of statements (i) and (ii) yields $\mathrm{Graph}(w) \models \exists x.\psi(x)$.

"$\Longleftarrow$": Let $w = a_1 a_2 \ldots a_n$ be a nonempty word with $\mathrm{Graph}(w) \models \exists x.\psi(x)$. Hence, by the equivalence of (i) and (ii) we obtain the existence of an index $m \in \{1,\ldots,n\}$ such that

$$(i,j) \in \mathfrak{R}$$

where $i,\ j \in \{1,\ldots,s\}$ with $\mathrm{FOL}[k]\big(\mathrm{Graph}(w_1)\big) = \mathfrak{T}_i$ and $\mathrm{FOL}[k]\big(\mathrm{Graph}(w_2)\big) = \mathfrak{T}_j$ where – as before – $w_1 = a_1 a_2 \ldots a_m$ and $w_2 = a_{m+1} a_2 \ldots a_n$. Then:

$$\mathrm{Graph}(w_1) \models \theta_i \quad \text{and} \quad \mathrm{Graph}(w_2) \models \theta_j$$

Hence, $w_1 \in \mathfrak{L}(\theta_i)$, $w_2 \in \mathfrak{L}(\theta_j)$ and therefore $w = w_1 w_2 \in \mathfrak{L}(\theta_i)\,\mathfrak{L}(\theta_j) \subseteq L$.

We now establish the equivalence of statements (i) and (ii).

"(i) $\Longrightarrow$ (ii)": Let $w = a_1 a_2 \ldots a_n$ be a nonempty word such that:

$$\mathrm{Graph}(w) \models \exists x.\psi(x)$$

Then there exists a word position $m \in \{1,\ldots,n\}$ such that:

$$(\mathrm{Graph}(w), [x := m]) \models \psi(x)$$

Let $w_1 \stackrel{\text{def}}{=} a_1 a_2 \ldots a_m$ and $w_2 \stackrel{\text{def}}{=} a_{m+1} \ldots a_n$. Let i and j be indices in $\{1,\ldots,s\}$ with

$$\mathrm{FOL}[k](\mathrm{Graph}(w_1)) = \mathfrak{T}_i \quad \text{and} \quad \mathrm{FOL}[k](\mathrm{Graph}(w_2)) = \mathfrak{T}_j.$$

Then, $(i,j) \in \mathfrak{R}$ by the definition of $\mathfrak{R}$.

"(ii) $\Longrightarrow$ (i)": Let $w = a_1 a_2 \ldots a_n \in \Sigma^+$ and $m \in \{1,\ldots,n\}$ and $i,\ j \in \{1,\ldots,s\}$ such that the following conditions (a), (b) and (c) are satisfied:

(a) $(i,j) \in \mathfrak{R}$

(b) $\mathrm{FOL}[k](\mathrm{Graph}(w_1)) = \mathfrak{T}_i$ where $w_1 = a_1 \ldots a_m$

(c) $\mathrm{FOL}[k](\mathrm{Graph}(w_1)) = \mathfrak{T}_j$ where $w_2 = a_{m+1} \ldots a_n$

Condition (a) yields the existence of a word $v = b_1 b_2 \ldots b_r \in \Sigma^+$ and a word position ℓ in $\{1,\ldots,r\}$ such that the following conditions (1), (2) and (3) hold:

(1) $(\mathrm{Graph}(v), [x := \ell]) \models \psi(x)$

(2) $\mathrm{FOL}[k](\mathrm{Graph}(v_1)) = \mathfrak{T}_i$ where $v_1 = b_1 b_2 \ldots b_\ell$

(3) $\mathrm{FOL}[k](\mathrm{Graph}(v_2)) = \mathfrak{T}_j$ where $v_2 = b_{\ell+1} \ldots b_r$

As the rank-k-types of w_1 and v_1 agree (both equal $\mathfrak{T}_i$ by conditions (b) and (2)), the word structures of w_1 and v_1 satisfy the same FOL[k]-sentences. The same holds for w_2 and v_2 which have the same rank-k type $\mathfrak{T}_j$; see conditions (c) and (3). Thus, by the Ehrenfeucht-Fraïssé Theorem (see Theorem 1.6.27 on page 101) we get:

$$\mathrm{Graph}(w_1) \cong_k \mathrm{Graph}(v_1) \quad \text{and} \quad \mathrm{Graph}(w_2) \cong_k \mathrm{Graph}(v_2)$$

Recall that $\mathcal{A} \cong_k \mathcal{B}$ denotes that structures $\mathcal{A}$ and $\mathcal{B}$ are k-round game equivalent for the Ehrenfeucht-Fraïssé game and that $\mathcal{A} \cong_k \mathcal{B}$ holds if and only if structures $\mathcal{A}$ and $\mathcal{B}$ satisfy the same FOL[k]-sentences.

Applying an argument similar to the one used in the concatenation lemma for linear orders (Lemma 1.6.41 on page 110) we obtain the k-round game equivalence of the pointed word structures:

$$(\text{Graph}(w), m) \;\cong_k\; (\text{Graph}(v), \ell)$$

The Ehrenfeucht-Fraïssé Theorem then yields that the interpretations

$$(\text{Graph}(w), [x := m]) \quad \text{and} \quad (\text{Graph}(v), [x := \ell])$$

fulfill the same $\text{FOL}[k](x)$-formulas. As $\psi(x) \in \text{FOL}[k](x)$, by (1) we get:

$$(\text{Graph}(w), [x := m]) \models \psi(x)$$

But this yields $\text{Graph}(w) \models \exists x.\psi(x) = \phi$.

□

This completes the proof of Theorem 2.6.27. In fact, since the star-free languages constitute a proper subclass of the class of regular languages we obtain that MSO over words is more powerful than FOL over words. An example for a regular language that is not star-free is L_{even}, the set of all finite words of even length. We may assume here the singleton alphabet $\Sigma = \{a\}$ and define L_{even} as the set of all words a^n where n is even.

Theorem 2.6.30 (MSO over finite words is strictly more expressive than FOL). *The regular language* L_{even} *is MSO-definable, but not FO-definable.*

Proof. The MSO-definability follows from the fact that L_{even} is regular as it is given by the regular expression

$$(aa)^*$$

The fact that L_{even} is not FO-definable follows by the observation that there is no FOL-sentence over the vocabulary of linear orders that characterizes the finite linear orders of even length. See Theorem 1.6.43 on page 112. □