Quantum Computing - Notes - v0.1.0

260236

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Preface

Every theory section in these notes has been taken from the sources:

- Course slides. [1]

About:

GitHub repository

These notes are an unofficial resource and shouldn't replace the course material or any other book on quantum computing. It is not made for commercial purposes. I've made the following notes to help me improve my knowledge and maybe it can be helpful for everyone.

As I have highlighted, a student should choose the teacher's material or a book on the topic. These notes can only be a helpful material.

Contents

1 Introduction

1.1 Complex Numbers recap

Complex Numbers play a fundamental role in quantum mechanics and quantum computing. For this reason, here is a brief summary of the most important concepts:

- **Definition of a Complex Number**. A complex number z is written as:

$$z = x + iy$$

  Where:

  - x is the **real part** ($\mathrm{Re}(z) = x$).
  - y is the **imaginary part** ($\mathrm{Im}(z) = y$).
  - i is the **imaginary unit**, satisfying $i^2 = -1$.

  A complex number can also be expressed in **polar form**:

$$z = re^{i\varphi}$$

  Where:

  - $r = |z| = \sqrt{x^2 + y^2}$ is the **modulus** (also called **magnitude**).
  - $\varphi = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$ is the **argument** (also called **phase angle**).

  Using Euler's formula:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

  We can rewrite z as:

$$z = r\left(\cos\varphi + i\sin\varphi\right)$$

- **Complex Conjugate**. The **Complex Conjugate** of z is:

$$\bar{z} = x - iy = re^{-i\varphi}$$

  Properties:

  - $z \cdot \bar{z} = |z|^2 = \left(\sqrt{x^2 + y^2}\right)^2 = x^2 + y^2$
  - The **conjugate reverses the sign of the imaginary part**.

- **Operations on Complex Numbers**

  - **Addition** and **Subtraction**:

$$\begin{aligned}(a + ib) + (c + id) &= (a + c) + i(b + d) \\ (a + ib) - (c + id) &= (a - c) + i(b - d)\end{aligned}$$

  - **Multiplication**. Using the distributive property:

$$(a + ib)(c + id) = ac + iad + ibc + i^2bd$$

    Since $i^2 = -1$, we get:

$$(ac - bd) + i(ad + bc)$$

- **Division**. To divide $\frac{z_1}{z_2}$, multiply by the conjugate of the denominator:

  $$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{c^2+d^2}$$

  Expanding:

  $$\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

- **Complex Exponentiation**. Using Euler's formula:

  $$e^{i\theta} = \cos\theta + i\sin\theta$$

  For integer powers:

  $$(e^{i\theta})^n = e^{i\cdot n\cdot\theta}$$

  For fractional exponents (roots):

  $$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{\frac{i(\varphi+2\pi k)}{n}}, \quad k = 0, 1, \ldots, n-1$$

- **Rotation Using Complex Numbers**. Multiplying by $e^{i\psi}$ rotates a complex number by an angle ψ:

  $$z' = ze^{i\psi}$$

  Since:

  $$e^{i\psi} = \cos\psi + i\sin\psi$$

  This means:

  $$\underbrace{(x+iy)}_{z}\underbrace{(\cos\psi + i\sin\psi)}_{e^{i\psi}}$$

  Expanding:

  $$(x\cos\psi - y\sin\psi) + i(x\sin\psi + y\cos\psi)$$

  Thus, the new coordinates are:

  $$x' = x\cos\psi - y\sin\psi, \quad y' = x\sin\psi + y\cos\psi$$

  Which is a standard **2D rotation matrix**:

  $$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

  And it rotates a point counterclockwise by an angle ψ in the 2D plane. This is important because rotations in the Bloch sphere (which represents qubits) are described by operations similar to this matrix.

- **Hermitian (Conjugate Transpose) of a Vector**. For a vector of complex numbers:

  $$\mathbf{z} = \begin{bmatrix} a \\ b \end{bmatrix}$$

  The **Hermitian conjugate** (denoted $\mathbf{z}^\dagger$ or $\mathbf{z}^H$) is:

  $$\mathbf{z}^H = \begin{bmatrix} \bar{a} & \bar{b} \end{bmatrix}$$

  Where $\bar{a}$ and $\bar{b}$ are the complex conjugates.

These concepts are fundamental because complex numbers describe quantum states. Also, Euler's formula provides a powerful tool for representing phase shifts. Finally, rotation and multiplication are key to quantum operations, and the hermitian conjugate is crucial in quantum mechanics.

1.2 Dirac's Notation

Dirac Notation, also called **bra-ket notation**, is a powerful mathematical framework used in quantum mechanics to **describe quantum states and their transformations**.

❓ What is a Ket?

A **Ket** $|v\rangle$ (is equal to the linear algebra annotation $\overrightarrow{v}$) is a **column vector** in a Hilbert space, that **represents a quantum state**.

$$|v\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

Where a and b are complex numbers (amplitudes of the quantum state). In other words, a ket $|v\rangle$ is a **state vector**, and what we usually require is that it has **unit norm**, meaning:

$$\langle v|v\rangle = 1$$

This ensures that the total probability of measurement outcomes is 1. So, a ket is a normalized column vector in a Hilbert space, meaning it has unit norm.

❓ What is a Superposition and why is it related to the Ket?

Superposition is a fundamental principle of quantum mechanics that **applies to all quantum systems**, not just qubits.

Definition 1: Superposition

Superposition means that a **system can exist in several possible states at the same time until a measurement is made**.

More in general, a quantum state $|\psi\rangle$ in a system with multiple possible states can exist in a **linear combination** of these states:

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle + \cdots + c_n|\psi_n\rangle$$

Where:

- $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle$ are **basis states** of the system.
- $c_1, c_2, \ldots, c_n$ are **complex probability amplitudes**.
- The system is in **all states at the same time**, with the probability of measuring each state given by $|c_i|^2$.
- The state is **normalized**, meaning:

$$|c_1|^2 + |c_2|^2 + \cdots + |c_n|^2 = 1$$

Example 1: Single-Particle Systems (Quantum Mechanics)

In general quantum mechanics, a particle can exist in **multiple positions simultaneously** as a wave function $\Psi(x)$:

$$|\Psi\rangle = \int \Psi(x)\,|x\rangle\,\mathrm{d}x$$

- The particle is in a **superposition of all possible positions** $|x\rangle$.
- Measurement collapses the wave function to a single position.

How Can a Particle Exist in Multiple States at Once? This question touches the heart of quantum mechanics, where our everyday intuition breaks down. The **key idea** is that a **quantum particle** is not just a tiny ball, it **is a wave function that spreads across multiple possibilities at once**.

1. **Quantum Particles Are Waves, Not Just Points**. In classical physics, we think of particles as tiny, solid objects, like a small ball that always has a precise position and velocity.

   In quantum mechanics, however, particles behave more like waves. These waves are described by a wave function $\Psi(x)$, which represents the **probability of finding the particle at different locations**.

   The key idea is:

   - The **wave function spreads across space**, meaning the **particle does not have a single location** before measurement.
   - Instead, it exists in a superposition of all possible locations.

2. **The Double-Slit Experiment: Proof That a Particle Can Be in Two Places at Once**. The double-slit experiment demonstrates that light and matter can exhibit behavior of both classical particles and classical waves. In 1927, Davisson and Germer and, independently, George Paget Thomson and his research student Alexander Reid demonstrated that electrons show the same behavior, which was later extended to atoms and molecules.

   Double Slit Experiment explained! by Jim Al-Khalili

   What Happens in Classical Physics? If we throw tiny balls at a screen with two slits, each ball will pass through one slit or the other. After many throws, **we get two lines** behind the slits, corresponding to the **two possible paths**.

   What Happens in Quantum Mechanics? If we **send a single electron** (or photon) towards two slits, it **behaves like a wave**.

It **passes through both slits at the same time** and interferes with itself, creating an interference pattern. This means the electron was in a **superposition of passing through both slits at once**.

If we try to measure which slit the electron goes through, the **superposition collapses**, and it behaves like a classical particle!

3. **Quantum Superposition: More Than Just Probability**. A common misconception is that a particle in superposition is just **an unknown state**, like a coin that is either heads or tails, but we just don't know which. This is wrong, because quantum superposition is much deeper.

   A quantum state is a **combination of all possibilities**. *Until measurement*, the system is in **all possible states at once**.

   Mathematically, for an electron in **two locations** x_1 and x_2:

   $$|\psi\rangle = a|x_1\rangle + b|x_2\rangle$$

   - The electron is **literally in both places simultaneously**.
   - The coefficients a and b are **complex numbers representing the probability amplitudes**.
   - **Interference** between these amplitudes **creates quantum effects that cannot be explained by classical probability**.

4. **Superposition in Quantum Computing**. Quantum computing **directly uses** the fact that a particle can be in multiple states at once.

   - A **classical bit** can only be **0 or 1**.
   - A **qubit (quantum bit)** (we will explain this later) can be in a superposition:
     $$|\psi\rangle = a|0\rangle + b|1\rangle$$

   This means a quantum computer can **perform many calculations simultaneously**. Therefore, **superposition allows quantum computers to process information exponentially faster than classical computers for certain tasks**.

5. **Why Don't We See Superposition in Everyday Life?** In our daily experience, objects are **not in multiple states at once** because of a process called **quantum decoherence**.

   Quantum superposition is **fragile**. When a quantum system interacts with the environment (air, light, etc.), the superposition **collapses into one definite state**. This is why large objects (like humans or cars) do not appear in multiple places at once.

   However, experiments (like the Double-Slit experiment) confirm that superposition is real at microscopic scales (electrons, photons, atoms, and even molecules).

The key **properties of Superposition** are:

1. **Linearity**: Any combination of valid quantum states is also a valid quantum state.
2. **Interference**: Quantum states in superposition can interfere, leading to constructive or destructive interference.
3. **Measurement Collapse**: When measured, the superposition collapses into a single outcome.
4. **Phase Information**: Unlike classical probabilities, quantum superpositions include complex phases that affect interference patterns.

❓ What is a Bra?

A **Bra** $\langle v|$ (is equal to the linear algebra annotation $\overrightarrow{v}^H$) is the **conjugate transpose (Hermitian conjugate) of the ket** $|v\rangle$.

Mathematically, if we start with the ket $|v\rangle$:

$$|v\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

The bra is obtained by:

1. **Transposing the ket** (switching it from column to a row).
2. **Taking the complex conjugate of each element**.

So the bra $\langle v|$ is:

$$\langle v| = (\langle v|)^\dagger = \begin{bmatrix} \bar{a} & \bar{b} \end{bmatrix}$$

A bra is a **mathematical object that allows us to compute inner products and measure probabilities**.

❓ But why do we need another topic called "Bra"?

A **bra** $\langle v|$ does not represent a physical system by itself. Instead, it is a **mathematical tool** used to:

1. **Extract** information from a quantum state.
2. **Compute** inner products (which determine probabilities).
3. **Define** quantum operators and measurements.

The key idea of bras and kets working together is:

- A ket $|v\rangle$ represents a quantum system.
- A bra $\langle v|$ is like a *test function* that helps us extract measurable information from a quantum system.

When they are combined as $\langle v|v\rangle = 1$, we obtain a **probability amplitude**.

Dirac's Notation: Multiplications

The following list shows how Dirac's notation is used to describe how kets and bras interact through inner products, matrix-vector multiplications, and operator applications:

- **Inner (Scalar) Product: $\langle x|y\rangle$**

  The **inner product** (or scalar product) between two quantum states $|x\rangle$ and $|x\rangle$ is written as:

  $$\langle x|y\rangle$$

  Where:

  - The **bra** $\langle x|$ is the **conjugate transpose** (row vector) of the ket $|x\rangle$.
  - The **ket** $|y\rangle$ is a **column vector**.
  - The inner product is the **dot product of these two vectors**, resulting in a **scalar (complex number)**.

  The inner product tells us **how much two quantum states "overlap"**.

  - If $\langle x|y\rangle = 0$, the states are **orthogonal** (completely different).
  - If $\langle x|y\rangle = 1$, the states are **identical**.

  Example 2: Inner (Scalar) Product

  Suppose we have two quantum states:

  $$\begin{aligned} |x\rangle &= \begin{bmatrix} a \\ b \end{bmatrix} \\ |y\rangle &= \begin{bmatrix} c \\ d \end{bmatrix} \end{aligned}$$

  Then:

  $$\langle x|y\rangle = \begin{bmatrix} \bar{a} & \bar{b} \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \bar{a}c + \bar{b}d$$

- **Matrix-Ket Multiplication: $M|v\rangle$**

  A quantum system evolves by **applying a matrix** (operator) M **to a quantum state** (ket):

  $$M|v\rangle$$

  Where:

  - $|v\rangle$ is a **column vector** (a quantum state).
  - M is a **matrix** (a quantum operator).

  The result is a **new quantum state** (a transformed column vector).

  Matrix-Ket Multiplication shows how **quantum gates (unitary matrices) transform quantum states**.

Example 3: Matrix-Ket Multiplication

Let's take:

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad |v\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

Then:

$$M|v\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}$$

- **Concatenated Multiplications: $\langle x|M|y\rangle$**

  A general quantum mechanical expression is:

  $$\langle x|M|y\rangle$$

  This is the **expected value or transition amplitude**, which means:

  1. Apply the operator M to $|y\rangle$ first:

     $$M|y\rangle$$

     Which gives a **new quantum state**.
  2. Take the inner product with $\langle x|$:

     $$\langle x|\left(M|y\rangle\right)$$

     Which results in a scalar (complex number).

  The result is a **scalar** that tells us the **probability amplitude of transitioning from $|y\rangle$ to $|x\rangle$ via M.**

Example 4

Let's compute:

$$\langle x|M|y\rangle$$

Using:

$$|x\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \qquad |y\rangle = \begin{bmatrix} c \\ d \end{bmatrix} \qquad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

1. Compute $M|y\rangle$:

   $$M|y\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} d \\ c \end{bmatrix}$$

2. Compute: $\langle x|\left(M|y\rangle\right)$:

   $$\langle x|\left(M|y\rangle\right) = \begin{bmatrix} \bar{a} & \bar{b} \end{bmatrix} \begin{bmatrix} d \\ c \end{bmatrix} = \bar{a}d + \bar{b}c$$

1.3 Single Qubits

A **Qubit** is a **two-level quantum system**, meaning it **has only two basis states**:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Any quantum state of a single qubit can be written as a **linear combination (*superposition*)** of these two basis states:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Where:

- α, β are **complex numbers** called **Probability Amplitudes**.
- The **normalization condition** holds:

  $$|\alpha|^2 + |\beta|^2 = 1$$

  To ensure total probability is 1.

Therefore, a single qubit is described as a **2D complex vector** in a Hilbert space.

Matrix Representation of Qubit States

Quantum states can be expressed in **matrix form** as **column vectors**:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

And, a general qubit state is:

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Where α and β are complex numbers.

What is a Basis?

A **Basis** is a **set of vectors that define a coordinate system in which we describe quantum states**. Furthermore, basis should **always be orthonormal** (orthogonal and norm equal to one) because it ensures that quantum states are independent, complete, and allow meaningful probability calculations. For a **single qubit**, we typically use **two orthonormal basis states** $|0\rangle$ and $|1\rangle$, forming the **computational basis**:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Any qubit state can be written as:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Where α and β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$.

In other words, a **basis allows us to describe quantum states as linear combinations of simpler states**.

❓ Why do we have a choice of different bases and why should we choose them?

While the **computational basis** $\{|0\rangle, |1\rangle\}$ is the *standard*, we are **not forced** to use it! We can choose **other bases** depending on the situation, and they help in different computations. This is because a **different basis simply provides a new way to describe the same quantum state**.

Another common basis is the **Hadamard basis**, defined as:

$$\begin{aligned} |+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{aligned}$$

The basis is also important because the **choice of basis affects the measurement results**. For example, let's take the qubit state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

- If measured in the **computational basis** $\{|0\rangle, |1\rangle\}$, it has a 50% chance of collapsing to $|0\rangle$ and 50% to $|1\rangle$.
- If measured in the **Hadamard basis** $\{|+\rangle, |-\rangle\}$, it **always collapses to** $|+\rangle$. Then, the probability of collapsing into $|+\rangle$ is 100%, and the probability of collapsing into $|-\rangle$ is 0%.

  Proof. We measure $|\psi\rangle$ in the Hadamard basis, therefore we must express it using $|+\rangle$ and $|-\rangle$.
  Since:

  $$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

  If we manipulate $|\psi\rangle$ a bit, we can see this:

  $$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ &= |+\rangle \end{aligned}$$

  And there is no component of $|-\rangle$. QED

What happens when we measure a Qubit?

In classical computing, a bit is either 0 or 1. In quantum computing, a qubit exists in a **superposition** of $|0\rangle$ and $|1\rangle$, but **when measured, it collapses into one of these basis states**. This is because the measurement is probabilistic and destroys the superposition.

If a qubit is in state:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Measurement **forces the qubit to collapse** into either $|0\rangle$ or $|1\rangle$. The **probability of each outcome** is given by the **squared magnitudes** of the coefficients:

$$P(0) = |\alpha|^2 \qquad P(1) = |\beta|^2$$

After measurement, the qubit **loses superposition** and remains in the measured state.

Example 5: Qubit Measurement

Consider the qubit:

$$|\psi\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$$

- The probability of measuring $|0\rangle$ is:

$$P(0) = \left(\frac{3}{5}\right)^2 = 0.36$$

- The probability of measuring $|1\rangle$ is:

$$P(1) = \left(\frac{4}{5}\right)^2 = 0.64$$

If we **measure** the qubit:

- With 36% probability, it collapses to $|0\rangle$.
- With 64% probability, it collapses to $|1\rangle$.

After measurement, the qubit **remains in that state** until modified by another operation.

In the previous example, we can observe that the measurement **collapses** the quantum state **into one of the basis** states with a **probability determined by its amplitude**.

What happens if we measure twice?

If the qubit collapses to $|0\rangle$ in the first measurement, a second measurement in the same basis will return $|0\rangle$ with probability 1. This is because the qubit is already in $|0\rangle$ and has no component of $|1\rangle$ left. Therefore, **repeating a measurement in the same basis always gives the same result**.

Measurement as a Fundamental Axiom

The behavior of quantum measurement is **not derived from other principles**, it is an **axiom of quantum mechanics**:

1. Measurement collapses the quantum state.
2. The probability of each outcome is given by the squared amplitude.
3. A second measurement (in the same basis) gives the same result with probability 1.

Therefore, the measurement is a **fundamental rule** of quantum mechanics.

Fundamental limitation of Quantum Computing

Unlike a classical bit, which can be only 0 or 1, a qubit can exist in any superposition:

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

Where a and b are complex numbers that satisfy $|a|^2 + |b|^2 = 1$. Since a and b can take infinitely many values, **a single qubit *theoretically* has an infinite number of possible states**.

Can a Qubit Store More than One Classical Bit? One might hope that because a qubit has infinitely many states , it could **store and transmit more than one classical bit of information**. However, **this is not possible** because:

- ✗ **A single measurement only gives one classical bit**. Measuring $|\psi\rangle$ forces it to collapse into $|0\rangle$ or $|1\rangle$. The outcome follows probabilities $P(0) = |a|^2$ and $P(1) = |b|^2$. Since the result is **just one binary outcome**, it **cannot reveal both a and b at the same time**.
- ✗ **Measurement Destroys the Quantum State**. Once we measure a qubit, its **original state is lost**. This means we cannot measure a and b separately, even if we repeat the measurement.

Unfortunately, this is a limitation, because a single qubit contains **infinite information theoretically**, but in practice, we can **only extract one classical bit per measurement**.

Furthermore, another problem is that we cannot copy a quantum state. The no-cloning theorem (explained later) states that it is **impossible to perfectly copy an arbitrary quantum state**. This has major consequences:

- **We cannot measure a qubit's twice**. In classical computing, we can copy and measure a bit multiple times. In quantum computing, copying is not possible. Once a qubit is measured, the original superposition is destroyed.
- *Why can't we just copy and measure?* Suppose we want to copy a qubit $|\psi\rangle$ and measure both copies. **Quantum mechanics forbids perfect duplication of unknown quantum states**. This prevents duplicating quantum information and extracting more than one bit of classical information per qubit. A proof will be provided later with a better explanation.

The state space of a Single Qubit

A **single qubit exists in a two-dimensional complex Hilbert space**:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Where:

- α and β are complex numbers (probability amplitudes).
- The normalization condition ensures:

$$|\alpha|^2 + |\beta|^2 = 1$$

Since α and β can take complex values, a **qubit is more than just a point** in 2D, it has **four real parameters** (two from each complex number). However, due to normalization and global phase invariance, **only two real parameters are needed to describe a qubit**.

Therefore, the **space of all possible qubit states is a continuous space, not just discrete values like classical bits**.

❓ Block Sphere representation of the qubit (and why)

The **bloch sphere** is a geometric representation of a qubit's state that helps visualize its properties. Since a general qubit state is:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

We can rewrite it using two angles θ and ϕ (as **spherical coordinates**):

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle \tag{1}$$

Where:

- θ is called **polar angle** (latitude). It determines how much of $|0\rangle$ and $|1\rangle$ are mixed (the qubit is not just in one state, but in both simultaneously).
  - When $\theta = 0 \rightarrow |\psi\rangle = |0\rangle$
    It is a pure state.
  - When $\theta = \pi \rightarrow |\psi\rangle = |1\rangle$
    It is a pure state.
  - When $\theta = \frac{\pi}{2} \rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + e^{i\phi}\frac{1}{\sqrt{2}}|1\rangle$
    It is called equal superposition.
- ϕ is called **relative phase** (longitude). It **controls the phase relationship** between $|0\rangle$ and $|1\rangle$. Changing ϕ **does not affect measurement probabilities**, but it affects interference when qubits interacts with other qubits.

In general, on the block sphere:

- The **north pole** $(0, 0, 1)$ is $|0\rangle$.
- The **south pole** $(0, 0, -1)$ is $|1\rangle$.
- Any other point represents a **superposition state** (eq. 1, page 17).

So, the Bloch sphere shows **how a qubit evolves** under quantum operations, making it easier to understand **rotations**, **phase shifts**, and **measurements**.

❓ Why is the bloch sphere important? The bloch sphere **helps us visualize** superposition, phase, and quantum operations intuitively.

1. It gives a visual representation of qubit states. Classical bits are just points (0 or 1), whereas a qubit exists everywhere on the sphere.

2. It shows quantum gates as rotations. As we will see in the following pages, quantum gates rotate the qubit around the sphere. For example, the Hadamard gate rotates $|0\rangle$ to $|+\rangle$, moving from the north pole to the equator.

3. It helps understand measurement. Measuring a qubit collapses it to either $|0\rangle$ or $|1\rangle$, removing phase information.

❓ Unfortunately, the $e^{i\phi}$ factor does not affect the global sphere?

Wrong! Two quantum state vectors are considered **equivalent if they differ only by a global phase factor**. This means that if we have two quantum states:

$$\begin{aligned} |\psi\rangle &= \alpha|0\rangle + \beta|1\rangle \\ |\psi'\rangle &= e^{i\gamma}\left(\alpha|0\rangle + \beta|1\rangle\right) \end{aligned}$$

Where $e^{i\gamma}$ is a global phase factor (a complex number with magnitude 1), then **these two states are physically identical**.

A **global phase** is a complex factor of the form:

$$e^{i\gamma} = \cos\gamma + i\sin\gamma$$

Which multiplies the entire quantum state but has no physical impact on measurement probabilities.

Therefore, a qubit state is **not changed** by multiplying it by a global phase factor $e^{i\gamma}$, meaning that:

$$|\psi\rangle \sim e^{i\gamma}|\psi\rangle$$

This means that the **block sphere represents only unique qubit states**, since the global phase doesn't affect the measurement.

Example 6: Identical Qubit States

Suppose we have two quantum states:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

And:

$$|\psi'\rangle = e^{1\pi}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

Since $e^{i\pi} = -1$, we can simplify:

$$|\psi'\rangle = -\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Even though $|\psi\rangle$ and $|\psi'\rangle$ look different mathematically, they are physically the same because they only differ by a global phase factor $e^{i\pi}$. The global phase does not affect the measurement results. In fact, the probabilities remain for both ψ and ψ':

$$P(0) = \left|-\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} \qquad P(1) = \left|-\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Since measurement gives the same results for both states, we consider them physically identical.

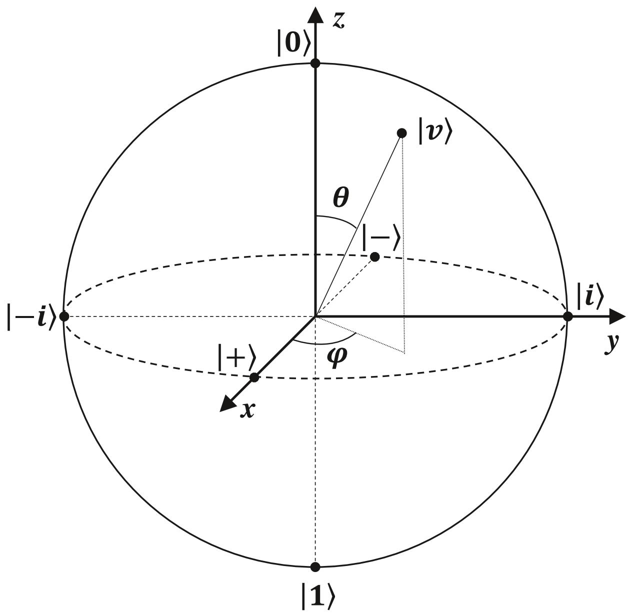

Figure 1: Bloch sphere representation of qubit.

References

[1] Cremonesi Paolo. Quantum computing. Slides from the HPC-E master's degree course on Politecnico di Milano, 2024.

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