

# Quantum Computing - Notes - v0.3.1

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April 2025

## Preface

Every theory section in these notes has been taken from the sources:

- Course slides. [1]

About:

 [GitHub repository](#)



These notes are an unofficial resource and shouldn't replace the course material or any other book on quantum computing. It is not made for commercial purposes. I've made the following notes to help me improve my knowledge and maybe it can be helpful for everyone.

As I have highlighted, a student should choose the teacher's material or a book on the topic. These notes can only be a helpful material.

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# 1 Introduction

## 1.1 Complex Numbers recap

Complex Numbers play a fundamental role in quantum mechanics and quantum computing. For this reason, here is a brief summary of the most important concepts:

- **Definition of a Complex Number.** A complex number  $z$  is written as:

$$z = x + iy$$

Where:

- $x$  is the **real part** ( $\text{Re}(z) = x$ ).
- $y$  is the **imaginary part** ( $\text{Im}(z) = y$ ).
- $i$  is the **imaginary unit**, satisfying  $i^2 = -1$ .

A complex number can also be expressed in **polar form**:

$$z = re^{i\varphi}$$

Where:

- $r = |z| = \sqrt{x^2 + y^2}$  is the **modulus** (also called **magnitude**).
- $\varphi = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$  is the **argument** (also called **phase angle**).

Using Euler's formula:

$$e^{i\varphi} = \cos \varphi + i \sin \varphi$$

We can rewrite  $z$  as:

$$z = r (\cos \varphi + i \sin \varphi)$$

- **Complex Conjugate.** The **Complex Conjugate** of  $z$  is:

$$\bar{z} = x - iy = re^{-i\varphi}$$

Properties:

- $z \cdot \bar{z} = |z|^2 = \left(\sqrt{x^2 + y^2}\right)^2 = x^2 + y^2$
- The **conjugate reverses the sign of the imaginary part**.

- **Operations on Complex Numbers**

- **Addition and Subtraction:**

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

$$(a + ib) - (c + id) = (a - c) + i(b - d)$$

- **Multiplication.** Using the distributive property:

$$(a + ib)(c + id) = ac + iad + ibc + i^2bd$$

Since  $i^2 = -1$ , we get:

$$(ac - bd) + i(ad + bc)$$

- **Division.** To divide  $\frac{z_1}{z_2}$ , multiply by the conjugate of the denominator:

$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{c^2+d^2}$$

Expanding:

$$\frac{(ac+bd) + i(bc-ad)}{c^2+d^2}$$

- **Complex Exponentiation.** Using Euler's formula:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

For integer powers:

$$(e^{i\theta})^n = e^{i \cdot n \cdot \theta}$$

For fractional exponents (roots):

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{\frac{i(\varphi+2\pi k)}{n}}, \quad k = 0, 1, \dots, n-1$$

- **Rotation Using Complex Numbers.** Multiplying by  $e^{i\psi}$  rotates a complex number by an angle  $\psi$ :

$$z' = ze^{i\psi}$$

Since:

$$e^{i\psi} = \cos \psi + i \sin \psi$$

This means:

$$\underbrace{(x+iy)}_z \underbrace{(\cos \psi + i \sin \psi)}_{e^{i\psi}}$$

Expanding:

$$(x \cos \psi - y \sin \psi) + i(x \sin \psi + y \cos \psi)$$

Thus, the new coordinates are:

$$x' = x \cos \psi - y \sin \psi, \quad y' = x \sin \psi + y \cos \psi$$

Which is a standard **2D rotation matrix**:

$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

And it rotates a point counterclockwise by an angle  $\psi$  in the 2D plane. This is important because rotations in the Bloch sphere (which represents qubits) are described by operations similar to this matrix.

- **Hermitian (Conjugate Transpose) of a Vector.** For a vector of complex numbers:

$$\mathbf{z} = \begin{bmatrix} a \\ b \end{bmatrix}$$

The **Hermitian conjugate** (denoted  $\mathbf{z}^\dagger$  or  $\mathbf{z}^H$ ) is:

$$\mathbf{z}^H = [\bar{a} \quad \bar{b}]$$

Where  $\bar{a}$  and  $\bar{b}$  are the complex conjugates.

These concepts are fundamental because complex numbers describe quantum states. Also, Euler's formula provides a powerful tool for representing phase shifts. Finally, rotation and multiplication are key to quantum operations, and the hermitian conjugate is crucial in quantum mechanics.

## 1.2 Dirac's Notation

**Dirac Notation**, also called **bra-ket notation**, is a powerful mathematical framework used in quantum mechanics to **describe quantum states and their transformations**.

### 🔍 What is a Ket?

A **Ket**  $|v\rangle$  (is equal to the linear algebra annotation  $\vec{v}$ ) is a **column vector** in a Hilbert space, that **represents a quantum state**.

$$|v\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

Where  $a$  and  $b$  are complex numbers (amplitudes of the quantum state). In other words, a ket  $|v\rangle$  is a **state vector**, and what we usually require is that it has **unit norm**, meaning:

$$\langle v|v\rangle = 1$$

This ensures that the total probability of measurement outcomes is 1. So, a ket is a normalized column vector in a Hilbert space, meaning it has unit norm.

### 🔍 What is a Superposition and why is it related to the Ket?

**Superposition** is a fundamental principle of quantum mechanics that **applies to all quantum systems**, not just qubits.

#### Definition 1: Superposition

**Superposition** means that a **system can exist in several possible states at the same time until a measurement is made**.

More in general, a quantum state  $|\psi\rangle$  in a system with multiple possible states can exist in a **linear combination** of these states:

$$|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle + \dots + c_n |\psi_n\rangle$$

Where:

- $|\psi_1\rangle, |\psi_2\rangle, \dots, |\psi_n\rangle$  are **basis states** of the system.
- $c_1, c_2, \dots, c_n$  are **complex probability amplitudes**.
- The system is in **all states at the same time**, with the probability of measuring each state given by  $|c_i|^2$ .
- The state is **normalized**, meaning:

$$|c_1|^2 + |c_2|^2 + \dots + |c_n|^2 = 1$$

**Example 1: Single-Particle Systems (Quantum Mechanics)**

In general quantum mechanics, a particle can exist in **multiple positions simultaneously** as a wave function  $\Psi(x)$ :

$$|\Psi\rangle = \int \Psi(x) |x\rangle dx$$

- The particle is in a **superposition of all possible positions**  $|x\rangle$ .
- Measurement collapses the wave function to a single position.

**🔍 How Can a Particle Exist in Multiple States at Once?** This question touches the heart of quantum mechanics, where our everyday intuition breaks down. The **key idea** is that a **quantum particle** is not just a tiny ball, it is **a wave function that spreads across multiple possibilities at once**.

1. **Quantum Particles Are Waves, Not Just Points.** In classical physics, we think of particles as tiny, solid objects, like a small ball that always has a precise position and velocity.

In quantum mechanics, however, particles behave more like waves. These waves are described by a wave function  $\Psi(x)$ , which represents the **probability of finding the particle at different locations**.

The key idea is:

- The **wave function spreads across space**, meaning the **particle does not have a single location** before measurement.
  - Instead, it exists in a superposition of all possible locations.
2. **The Double-Slit Experiment: Proof That a Particle Can Be in Two Places at Once.** The double-slit experiment demonstrates that light and matter can exhibit behavior of both classical particles and classical waves. In 1927, Davisson and Germer and, independently, George Paget Thomson and his research student Alexander Reid demonstrated that electrons show the same behavior, which was later extended to atoms and molecules.

*Double Slit Experiment explained! by Jim Al-Khalili*



- 🔍 **What Happens in Classical Physics?** If we throw tiny balls at a screen with two slits, each ball will pass through one slit or the other. After many throws, **we get two lines** behind the slits, corresponding to the **two possible paths**.
- 🔍 **What Happens in Quantum Mechanics?** If we send a single **electron** (or photon) towards two slits, it **behaves like a wave**.

It **passes through both slits at the same time** and interferes with itself, creating an interference pattern. This means the electron was in a **superposition of passing through both slits at once**. If we try to measure which slit the electron goes through, the **superposition collapses**, and it behaves like a classical particle!

3. **Quantum Superposition: More Than Just Probability.** A common misconception is that a particle in superposition is just **an unknown state**, like a coin that is either heads or tails, but we just don't know which. This is wrong, because quantum superposition is much deeper.

A quantum state is a **combination of all possibilities**. *Until measurement*, the system is in **all possible states at once**.

Mathematically, for an electron in **two locations**  $x_1$  and  $x_2$ :

$$|\psi\rangle = a|x_1\rangle + b|x_2\rangle$$

- The electron is **literally in both places simultaneously**.
  - The coefficients  $a$  and  $b$  are **complex numbers representing the probability amplitudes**.
  - **Interference** between these amplitudes **creates quantum effects that cannot be explained by classical probability**.
4. **Superposition in Quantum Computing.** Quantum computing **directly uses** the fact that a particle can be in multiple states at once.
    - A **classical bit** can only be **0 or 1**.
    - A **qubit (quantum bit)** (we will explain this later) can be in a superposition:

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

This means a quantum computer can **perform many calculations simultaneously**. Therefore, **superposition allows quantum computers to process information exponentially faster than classical computers for certain tasks**.

5. **Why Don't We See Superposition in Everyday Life?** In our daily experience, objects are **not in multiple states at once** because of a process called **quantum decoherence**.

Quantum superposition is **fragile**. When a quantum system interacts with the environment (air, light, etc.), the superposition **collapses into one definite state**. This is why large objects (like humans or cars) do not appear in multiple places at once.

However, experiments (like the Double-Slit experiment) confirm that superposition is real at microscopic scales (electrons, photons, atoms, and even molecules).



The key **properties of Superposition** are:

1. **Linearity**: Any combination of valid quantum states is also a valid quantum state.
2. **Interference**: Quantum states in superposition can interfere, leading to constructive or destructive interference.
3. **Measurement Collapse**: When measured, the superposition collapses into a single outcome.
4. **Phase Information**: Unlike classical probabilities, quantum superpositions include complex phases that affect interference patterns.

### ❓ What is a Bra?

A **Bra**  $\langle v|$  (is equal to the linear algebra annotation  $\vec{v}^H$ ) is the **conjugate transpose (Hermitian conjugate) of the ket**  $|v\rangle$ .

Mathematically, if we start with the ket  $|v\rangle$ :

$$|v\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

The bra is obtained by:

1. **Transposing the ket** (switching it from column to a row).
2. **Taking the complex conjugate of each element.**

So the bra  $\langle v|$  is:

$$\langle v| = (\langle v|)^\dagger = [\bar{a} \quad \bar{b}]$$

A bra is a **mathematical object that allows us to compute inner products and measure probabilities.**

### ❓ But why do we need another topic called “Bra”?

A **bra**  $\langle v|$  does **not represent a physical** system by itself. Instead, it is a **mathematical tool** used to:

1. **Extract** information from a quantum state.
2. **Compute** inner products (which determine probabilities).
3. **Define** quantum operators and measurements.

The key idea of bras and kets working together is:

- A ket  $|v\rangle$  represents a quantum system.
- A bra  $\langle v|$  is like a *test function* that helps us extract measurable information from a quantum system.

When they are combined as  $\langle v|v\rangle = 1$ , we obtain a **probability amplitude**.

### Dirac's Notation: Multiplications

The following list shows how Dirac's notation is used to describe how kets and bras interact through inner products, matrix-vector multiplications, and operator applications:

- **Inner (Scalar) Product:**  $\langle x|y\rangle$

The **inner product** (or scalar product) between two quantum states  $|x\rangle$  and  $|y\rangle$  is written as:

$$\langle x|y\rangle$$

Where:

- The **bra**  $\langle x|$  is the **conjugate transpose** (row vector) of the ket  $|x\rangle$ .
- The **ket**  $|y\rangle$  is a **column vector**.
- The inner product is the **dot product of these two vectors**, resulting in a **scalar (complex number)**.

The **inner product tells us** how much two quantum states “overlap”.

- If  $\langle x|y\rangle = 0$ , the states are **orthogonal (completely different)**.
- If  $\langle x|y\rangle = 1$ , the states are **identical**.

#### Example 2: Inner (Scalar) Product

Suppose we have two quantum states:

$$\begin{aligned} |x\rangle &= \begin{bmatrix} a \\ b \end{bmatrix} \\ |y\rangle &= \begin{bmatrix} c \\ d \end{bmatrix} \end{aligned}$$

Then:

$$\langle x|y\rangle = [\bar{a} \quad \bar{b}] \begin{bmatrix} c \\ d \end{bmatrix} = \bar{a}c + \bar{b}d$$

- **Matrix-Ket Multiplication:**  $M|v\rangle$

A quantum system evolves by **applying a matrix** (operator)  $M$  to a **quantum state** (ket):

$$M|v\rangle$$

Where:

- $|v\rangle$  is a **column vector** (a **quantum state**).
- $M$  is a **matrix** (a **quantum operator**).

The result is a **new quantum state** (a transformed column vector).

Matrix-Ket Multiplication shows how **quantum gates (unitary matrices)** transform quantum states.

**Example 3: Matrix-Ket Multiplication**

Let's take:

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad |v\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

Then:

$$M |v\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}$$

- **Concatenated Multiplications:**  $\langle x|M|y\rangle$

A general quantum mechanical expression is:

$$\langle x|M|y\rangle$$

This is the **expected value or transition amplitude**, which means:

1. Apply the operator  $M$  to  $|y\rangle$  first:

$$M |y\rangle$$

Which gives a **new quantum state**.

2. Take the inner product with  $\langle x|$ :

$$\langle x| (M |y\rangle)$$

Which results in a scalar (complex number).

The result is a **scalar** that tells us the **probability amplitude of transitioning from  $|y\rangle$  to  $|x\rangle$  via  $M$** .

**Example 4**

Let's compute:

$$\langle x|M|y\rangle$$

Using:

$$|x\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \quad |y\rangle = \begin{bmatrix} c \\ d \end{bmatrix} \quad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

1. Compute  $M |y\rangle$ :

$$M |y\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} d \\ c \end{bmatrix}$$

2. Compute:  $\langle x| (M |y\rangle)$ :

$$\langle x| (M |y\rangle) = [\bar{a} \quad \bar{b}] \begin{bmatrix} d \\ c \end{bmatrix} = \bar{a}d + \bar{b}c$$

### 1.3 Single Qubits

A **Qubit** is a **two-level quantum system**, meaning it has only two basis states:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Any quantum state of a single qubit can be written as a **linear combination** (*superposition*) of these two basis states:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle \quad \alpha \wedge \beta \neq 0$$

Where:

- $\alpha, \beta$  are **complex numbers** called **Probability Amplitudes**. If either  $\alpha = 0$  or  $\beta = 0$ , the **state is NOT in superposition** (it is a pure basis state).
- The **normalization condition** holds:

$$|\alpha|^2 + |\beta|^2 = 1$$

To ensure total probability is 1.

Therefore, a single qubit is described as a **2D complex vector** in a Hilbert space.

#### Matrix Representation of Qubit States

Quantum states can be expressed in **matrix form** as **column vectors**:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

And, a general qubit state is:

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Where  $\alpha$  and  $\beta$  are complex numbers.

#### ? What is a Basis?

A **Basis** is a **set of vectors that define a coordinate system in which we describe quantum states**. Furthermore, basis should **always be orthonormal** (orthogonal and norm equal to one) because it ensures that quantum states are independent, complete, and allow meaningful probability calculations. For a **single qubit**, we typically use **two orthonormal basis states**  $|0\rangle$  and  $|1\rangle$ , forming the **computational basis**:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Any qubit state can be written as:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

Where  $\alpha$  and  $\beta$  are complex numbers satisfying  $|\alpha|^2 + |\beta|^2 = 1$ .

In other words, a **basis allows us to describe quantum states as linear combinations of simpler states**.

❓ **Why do we have a choice of different bases and why should we choose them?**

While the **computational basis**  $\{|0\rangle, |1\rangle\}$  is the *standard*, we are **not forced** to use it! We can choose **other bases** depending on the situation, and they help in different computations. This is because a **different basis simply provides a new way to describe the same quantum state**.

Another common basis is the **Hadamard basis**, defined as:

$$\begin{aligned} |+\rangle &= \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \\ |-\rangle &= \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \end{aligned}$$

The basis is also important because the **choice of basis affects the measurement results**. For example, let's take the qubit state:

$$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle$$

- If measured in the **computational basis**  $\{|0\rangle, |1\rangle\}$ , it has a 50% chance of collapsing to  $|0\rangle$  and 50% to  $|1\rangle$ .
- If measured in the **Hadamard basis**  $\{|+\rangle, |-\rangle\}$ , it **always collapses to  $|+\rangle$** . Then, the probability of collapsing into  $|+\rangle$  is 100%, and the probability of collapsing into  $|-\rangle$  is 0%.

*Proof.* We measure  $|\psi\rangle$  in the Hadamard basis, therefore we must express it using  $|+\rangle$  and  $|-\rangle$ .

Since:

$$|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

If we manipulate  $|\psi\rangle$  a bit, we can see this:

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \\ &= \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \\ &= |+\rangle \end{aligned}$$

And there is no component of  $|-\rangle$ .

QED

### 🔍 What happens when we measure a Qubit?

In classical computing, a bit is either 0 or 1. In quantum computing, a qubit exists in a **superposition** of  $|0\rangle$  and  $|1\rangle$ , but **when measured, it collapses into one of these basis states**. This is because the **measurement is probabilistic and destroys the superposition**.

If a qubit is in state:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

Measurement **forces the qubit to collapse** into either  $|0\rangle$  or  $|1\rangle$ . The **probability of each outcome** is given by the **squared magnitudes** of the coefficients:

$$P(0) = |\alpha|^2 \quad P(1) = |\beta|^2$$

**After measurement**, the qubit **loses superposition** and remains in the measured state.

#### Example 5: Qubit Measurement

Consider the qubit:

$$|\psi\rangle = \frac{3}{5} |0\rangle + \frac{4}{5} |1\rangle$$

- The probability of measuring  $|0\rangle$  is:

$$P(0) = \left(\frac{3}{5}\right)^2 = 0.36$$

- The probability of measuring  $|1\rangle$  is:

$$P(1) = \left(\frac{4}{5}\right)^2 = 0.64$$

If we **measure** the qubit:

- With 36% probability, it collapses to  $|0\rangle$ .
- With 64% probability, it collapses to  $|1\rangle$ .

After measurement, the qubit **remains in that state** until modified by another operation.

In the previous example, we can observe that the measurement **collapses** the quantum state **into one of the basis states** with a **probability determined by its amplitude**.

### ❓ What happens if we measure twice?

If the qubit collapses to  $|0\rangle$  in the first measurement, a second measurement in the same basis will return  $|0\rangle$  with probability 1. This is because the qubit is already in  $|0\rangle$  and has no component of  $|1\rangle$  left. Therefore, **repeating a measurement in the same basis always gives the same result.**

### 📖 Measurement as a Fundamental Axiom

The behavior of quantum measurement is **not derived from other principles**, it is an **axiom of quantum mechanics**:

1. Measurement collapses the quantum state.
2. The probability of each outcome is given by the squared amplitude.
3. A second measurement (in the same basis) gives the same result with probability 1.

Therefore, the measurement is a **fundamental rule** of quantum mechanics.

### ⚠️ Fundamental limitation of Quantum Computing

Unlike a classical bit, which can be only 0 or 1, a qubit can exist in any superposition:

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

Where  $a$  and  $b$  are complex numbers that satisfy  $|a|^2 + |b|^2 = 1$ . Since  $a$  and  $b$  can take infinitely many values, **a single qubit *theoretically* has an infinite number of possible states.**

❓ **Can a Qubit Store More than One Classical Bit?** One might hope that because a qubit has infinitely many states, it could **store and transmit more than one classical bit of information**. However, **this is not possible** because:

- ✗ **A single measurement only gives one classical bit.** Measuring  $|\psi\rangle$  forces it to collapse into  $|0\rangle$  or  $|1\rangle$ . The outcome follows probabilities  $P(0) = |a|^2$  and  $P(1) = |b|^2$ . Since the result is **just one binary outcome**, it **cannot reveal both  $a$  and  $b$  at the same time.**
- ✗ **Measurement Destroys the Quantum State.** Once we measure a qubit, its **original state is lost**. This means we cannot measure  $a$  and  $b$  separately, even if we repeat the measurement.

Unfortunately, this is a limitation, because a single qubit contains **infinite information theoretically**, but in practice, we can **only extract one classical bit per measurement**.

Furthermore, another problem is that we cannot copy a quantum state. The no-cloning theorem (explained later) states that it is **impossible to perfectly copy an arbitrary quantum state**. This has major consequences:

- **We cannot measure a qubit's twice.** In classical computing, we can copy and measure a bit multiple times. In quantum computing, copying is not possible. Once a qubit is measured, the original superposition is destroyed.
- *Why can't we just copy and measure?* Suppose we want to copy a qubit  $|\psi\rangle$  and measure both copies. **Quantum mechanics forbids perfect duplication of unknown quantum states.** This prevents duplicating quantum information and extracting more than one bit of classical information per qubit. A proof will be provided later with a better explanation.

### ☐ The state space of a Single Qubit

A single qubit exists in a two-dimensional complex Hilbert space:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

Where:

- $\alpha$  and  $\beta$  are complex numbers (probability amplitudes).
- The normalization condition ensures:

$$|\alpha|^2 + |\beta|^2 = 1$$

Since  $\alpha$  and  $\beta$  can take complex values, a **qubit is more than just a point** in 2D, it has **four real parameters** (two from each complex number). However, **due to normalization and global phase invariance, only two real parameters are needed to describe a qubit.**

Therefore, the **space of all possible qubit states is a continuous space, not just discrete values like classical bits.**



### ❓ Block Sphere representation of the qubit (and why)

The **bloch sphere** is a geometric representation of a qubit's state that helps visualize its properties. Since a general qubit state is:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

We can rewrite it using two angles  $\theta$  and  $\phi$  (as **spherical coordinates**):

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right) |0\rangle + e^{i\phi} \sin\left(\frac{\theta}{2}\right) |1\rangle \quad (1)$$

Where:

- $\theta$  is called **polar angle** (latitude). It determines how much of  $|0\rangle$  and  $|1\rangle$  are mixed (the qubit is not just in one state, but in both simultaneously).
  - When  $\theta = 0 \rightarrow |\psi\rangle = |0\rangle$   
It is a pure state.
  - When  $\theta = \pi \rightarrow |\psi\rangle = |1\rangle$   
It is a pure state.
  - When  $\theta = \frac{\pi}{2} \rightarrow |\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle + e^{i\phi} \frac{1}{\sqrt{2}} |1\rangle$   
It is called equal superposition.
- $\phi$  is called **relative phase** (longitude). It **controls the phase relationship** between  $|0\rangle$  and  $|1\rangle$ . Changing  $\phi$  **does not affect measurement probabilities**, but it affects interference when qubits interact with other qubits.

In general, on the block sphere:

- The **north pole**  $(0, 0, 1)$  is  $|0\rangle$ .
- The **south pole**  $(0, 0, -1)$  is  $|1\rangle$ .
- Any other point represents a **superposition state** (eq. 1, page 17).

So, the Bloch sphere shows **how a qubit evolves** under quantum operations, making it easier to understand **rotations**, **phase shifts**, and **measurements**.

❓ **Why is the bloch sphere important?** The bloch sphere **helps us visualize** superposition, phase, and quantum operations intuitively.

1. It gives a visual representation of qubit states. Classical bits are just points (0 or 1), whereas a qubit exists everywhere on the sphere.
2. It shows quantum gates as rotations. As we will see in the following pages, quantum gates rotate the qubit around the sphere. For example, the Hadamard gate rotates  $|0\rangle$  to  $|+\rangle$ , moving from the north pole to the equator.
3. It helps understand measurement. Measuring a qubit collapses it to either  $|0\rangle$  or  $|1\rangle$ , removing phase information.

### ❓ Unfortunately, the $e^{i\phi}$ factor does not affect the global sphere?

Wrong! Two quantum state vectors are considered **equivalent if they differ only by a global phase factor**. This means that if we have two quantum states:

$$\begin{aligned} |\psi\rangle &= \alpha |0\rangle + \beta |1\rangle \\ |\psi'\rangle &= e^{i\gamma} (\alpha |0\rangle + \beta |1\rangle) \end{aligned}$$

Where  $e^{i\gamma}$  is a **global phase factor** (a complex number with magnitude 1), then **these two states are physically identical**.

A **global phase** is a complex factor of the form:

$$e^{i\gamma} = \cos \gamma + i \sin \gamma$$

Which multiplies the entire quantum state but has no physical impact on measurement probabilities.

Therefore, a qubit state is **not changed** by multiplying it by a global phase factor  $e^{i\gamma}$ , meaning that:

$$|\psi\rangle \sim e^{i\gamma} |\psi\rangle$$

This means that the **block sphere represents only unique qubit states**, since the global phase doesn't affect the measurement.

#### Example 6: Identical Qubit States

Suppose we have two quantum states:

$$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle$$

And:

$$|\psi'\rangle = e^{i\pi} \left( \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right)$$

Since  $e^{i\pi} = -1$ , we can simplify:

$$|\psi'\rangle = -\frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle$$

Even though  $|\psi\rangle$  and  $|\psi'\rangle$  look different mathematically, they are physically the same because they only differ by a global phase factor  $e^{i\pi}$ .

The global phase does not affect the measurement results. In fact, the probabilities remain for both  $\psi$  and  $\psi'$ :

$$P(0) = \left| -\frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2} \quad P(1) = \left| -\frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

Since measurement gives the same results for both states, we consider them physically identical.

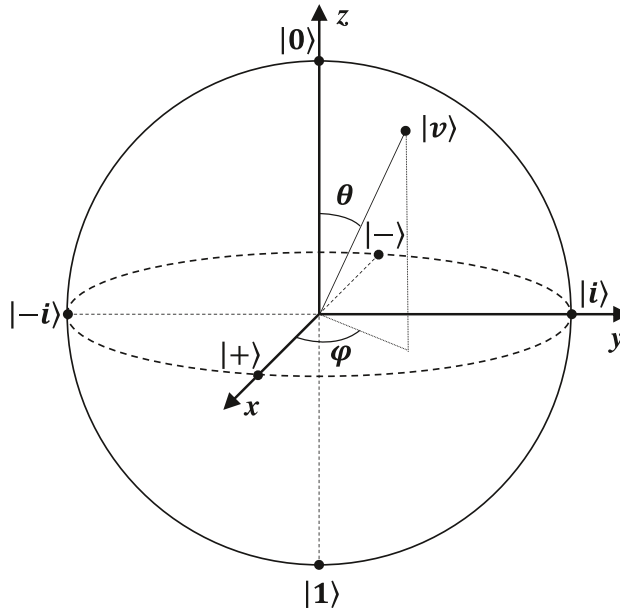


Figure 1: Bloch sphere representation of qubit.

## 2 Single Qubit Gates

### 2.1 Operations on Qubits

In quantum computing, **operations on qubits** are fundamental to information processing and computation. These operations can be broadly **classified** into:

- **Logic Gates**
- **Measurements**
- **Initialization Procedures**

Each of these plays a distinct role in the manipulation and evolution of quantum states, which are represented as **superpositions** of basis states  $|0\rangle$  and  $|1\rangle$ .

#### ≡ Logic Gates: Reversible Transformations

Quantum **Logic Gates** serve as the **building blocks of quantum circuits**. Unlike classical logic gates, quantum gates operate under the principles of quantum mechanics, notably **unitarity** and **reversibility**. These gates can act on a single qubit or on multiple qubits simultaneously, and their **primary function is to transform the state of the qubit(s)** involved.

**Key properties** of quantum logic gates include:

- They are **reversible**: this implies that **any operation performed by a gate can be undone by applying its inverse**. In quantum mechanics, reversibility is linked to the **requirement that gates be unitary matrices**.
- They **preserve the norm** of the quantum state, a **direct consequence of unitarity**.
- Gates **change the qubit's state** in a way that aligns with the **no-cloning** and **no-deleting theorems**; in other words, **information is neither duplicated nor destroyed, only transformed**.

#### ≡ Measurement: Irreversible Collapse of State

**Measurement** is a fundamentally different operation from logic gates because it is **irreversible**. When a qubit is measured, **information about its state is extracted**, and as a consequence, the **qubit's quantum state collapses** to one of the basis states  $|0\rangle$  or  $|1\rangle$ . This collapse causes the **loss of superposition** and, if entanglement is involved, the **loss of entanglement** as well (explained later).

Important points:

- Measurement provides **probabilistic outcomes based on the amplitudes** of the quantum state.
- After measurement, the qubit is **no longer in a superposition**; it is deterministically in  $|0\rangle$  or  $|1\rangle$ .

- This operation is **non-unitary** and hence **non-reversible**.

### Initialization: Preparing Known States

The **Initialization** of qubits is the process of **setting them into a known, well-defined state**, most commonly  $|0\rangle$  or  $|1\rangle$ . **Initialization is essential because quantum algorithms require precisely defined input states.**

Key facts:

- Initialization can *often* be **implemented via measurement**: by measuring a qubit and then applying a gate if necessary, we can prepare it in the desired state.
- As with measurement, initialization **collapses the quantum state**, making it **irreversible**.
- While measurement discards the quantum coherence, **initialization ensures a clean starting point** for computations.

### Comparison between Initialization and Measurement

- **Purpose**
  - **Measurement**: To extract information about the qubit's state.
  - **Initialization**: To prepare the qubit in a known state ( $|0\rangle$  or  $|1\rangle$ ).
- **Reversibility**
  - **Measurement**: Irreversible, collapses superposition.
  - **Initialization**: Irreversible, collapses superposition to set a state.
- **Effect on Qubit**
  - **Measurement**: Collapses to  $|0\rangle$  or  $|1\rangle$  randomly (probabilistic).
  - **Initialization**: Collapses to a specific state, often  $|0\rangle$  (deterministic or controlled).
- **Information Gained**
  - **Measurement**: Yes, outcome of measurement is known.
  - **Initialization**: No, goal is not to gain information, just set state.
- **How Implemented**
  - **Measurement**: Via a measurement operator.
  - **Initialization**: Often via measurement + gate correction.

- **Post-operation Use**

- **Measurement:** Qubit may be discarded or reused depending on outcome.
- **Initialization:** Qubit is now ready for computation.

In other words:

- **Measurement** tells we **what state the qubit is** in.
- **Initialization** sets the **qubit** to a desired state so we can **start a computation**.

## 2.2 Quantum Logic Gates Overview

In quantum computing, quantum logic gates are the primary tools used to **manipulate qubits**. These gates perform **unitary transformations** on the state of one or more qubits, meaning they **preserve the norm** of the quantum state and are **reversible**. The **design of quantum algorithms** and the **execution of quantum circuits** rely entirely on the sequential application of these gates.

### ≡ Types of Quantum Gates

**Quantum Gates** can be classified into:

- **Single-Qubit Gates**: These gates operate on **individual qubits**, modifying their state in isolation.
- **Multiple-Qubit Gates**: These gates act on **two or more qubits simultaneously**, allowing for entanglement and complex correlations between qubits.

In this section we will focus on single qubit gates.

### ✚ Mathematical Framework: Qubit as Vectors, Gates as Matrices

A key distinction of quantum logic compared to classical logic is that **qubits are vectors**, while **quantum gates are matrices**.

- A **qubit's state**, as we discussed on page 12, is a **vector in a 2-dimensional complex vector space**, often written as:

$$|\psi\rangle = a|0\rangle + b|1\rangle \quad \text{with } |a|^2 + |b|^2 = 1$$

- A **Quantum Gate** that acts on a single qubit is a  $2 \times 2$  **complex matrix**, specifically a **unitary matrix**, denoted by  $U$ .

The **action of a gate**  $U$  on a qubit  $|\psi\rangle$  is simply **matrix-vector multiplication**:

$$|\psi'\rangle = U|\psi\rangle \tag{2}$$

This transforms the qubit into a **new quantum state**, while **ensuring** that **probabilities remain normalized**.

### ■ Unitarity: The Fundamental Constraint

A quantum gate **must be unitary**, which means:

$$U^\dagger U = U U^\dagger = I \tag{3}$$

Where:

- $U^\dagger$  is the **Hermitian transpose** (conjugate transpose) of  $U$ .
- $I$  is the **identity matrix**.

❓ **Why is it important?** Because **unitarity ensures reversibility**: every gate has an inverse, and information is never lost, only transformed. This is in contrast to measurement, which is non-unitary and irreversible.

❓ **Why must gates be unitarity?**

1. **No-Cloning Theorem**: It is impossible to create an identical copy of an unknown qubit. Unitary transformations respect this by not duplicating information.
2. **No-Deleting Theorem**: We cannot delete quantum information arbitrarily. Since unitarity gates are invertible, they do not erase information.
3. **Preservation of Probability**: The normalization condition  $|a|^2 + |b|^2 = 1$  must hold after any operation, and only unitarity matrices preserve this norm.

### 📖 Physical Intuition

In classical circuits, logic gates like AND, OR, NOT process bits deterministically. In quantum circuits:

- Gates **rotate** the quantum state on the **bloch sphere**.
- These rotations correspond to **unitary transformations**.
- The **physical implementation** of quantum gates may vary across hardware (e.g., superconducting qubits, trapped ions), but **mathematically**, the **same unitary matrix describes the gate**.

**Quantum gates do not correspond to physical gates** in the classical sense. They are **abstract operations** realized by **controlled interactions** in quantum systems.



## 2.3 Main Single-Qubit Gates

### 2.3.1 Identity Gate (I)

The **Identity Gate**, denoted  $I$ , is the most elementary gate in quantum computing. Its action on a qubit is trivial, yet it **serves important roles in quantum circuits**, particularly **when managing multi-qubit systems** or **timing synchronization**.

#### ✓ Matrix Representation

The Identity gate is represented by the  $2 \times 2$  **identity matrix**:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (4)$$

**Applying** the Identity gate to any qubit **leaves the state unchanged**. Formally, for a qubit in state  $|\psi\rangle$ :

$$I|\psi\rangle = |\psi\rangle$$

#### 📖 Physical Interpretation

Although it may appear useless at first glance, the Identity gate has **conceptual and practical utility**:

- ✓ It acts as a **placeholder** or “*do nothing*” operation when required by **circuit timing**.
- ✓ In **multi-qubit circuits**, it allows one **qubit to remain unchanged** while other qubits are operated on.
- ✓ In **matrix composition** of gates, it serves to **align dimensions** when performing **tensor products** (e.g.,  $I \otimes H$  applies Hadamard to the second qubit only).

#### ❓ Bloch Sphere Perspective

In the Bloch sphere (remember? page 19), the **identity gate does not rotate**. The **qubit’s position** on the sphere remains **fixed**, reinforcing the idea that no transformation has taken place.

### 2.3.2 Pauli-X (NOT) Gate

The **Pauli-X Gate**, often called the **quantum NOT gate**, is one of the *fundamental* single-qubit quantum gates. It is denoted  $X$  and **corresponds to the  $\sigma_X$  Pauli matrix** in quantum mechanics. This gate **flips the state** of a qubit, **similar to how a classical NOT gate inverts a bit**.

#### Remark: Pauli Matrix

In quantum mechanics, the **Pauli matrices** are a set of three  $2 \times 2$  **Hermitian and unitary matrices** that represent **spin operators** for spin- $\frac{1}{2}$  particles and are widely used in quantum computing to describe single-qubit gates. They are:

$$\begin{aligned}\sigma_X &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_Y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_Z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\end{aligned}$$

Key properties:

- Hermitian:  $\sigma_i^\dagger = \sigma_i$  (observable-related)
- Unitary:  $\sigma_i^\dagger \sigma_i = I$  (reversible transformations)
- Involution:  $\sigma_i^2 = I$
- Non-commuting:  $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$  (with  $\varepsilon$  the Levi-Civita symbol)

#### Matrix Representation

$$X = \sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (5)$$

This **matrix swaps** the  $|0\rangle$  and  $|1\rangle$  **basis states**:

$$X|0\rangle = |1\rangle \quad \text{and} \quad X|1\rangle = |0\rangle \quad (6)$$

#### Action on a General Qubit

Let's consider a qubit in the general state:

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

Applying the  $X$  gate:

$$X|\psi\rangle = aX|0\rangle + bX|1\rangle = a|1\rangle + b|0\rangle = b|0\rangle + a|1\rangle$$

As a result, the amplitudes  $a$  and  $b$  are **swapped**.

### 🔗 Geometric Interpretation: Rotation on the Bloch Sphere

The Pauli-X gate corresponds to a **rotation around the  $x$ -axis by  $\pi$  radians** (180°) on the Bloch sphere.

To understand this, recall the **general qubit state in spherical coordinates**:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right) |0\rangle + e^{i\phi} \sin\left(\frac{\theta}{2}\right) |1\rangle$$

After a  $\pi$  rotation around the  $x$ -axis:

- $\theta \rightarrow \pi - \theta$
- $\phi \rightarrow -\phi$

This **rotation mirrors the qubit across the  $x$ -axis**, flipping its position between  $|0\rangle$  and  $|1\rangle$ , and adjusting the phase accordingly.

### ⚙️ Key Properties of X Gate

1. **Involution.** Applying X twice returns the original state:

$$XX = X^2 = I$$

This **reflects reversibility** and **matches the intuition of two 180° rotations** around  $x$ -axis equaling no rotation.

2. **Entanglement Tool.** The X gate is crucial in **entangling qubits** when combined with CNOT in multi-qubit circuits.
3. **Measurement Preparation.** Used to **flip the measurement outcome**, e.g., preparing  $|1\rangle$  from  $|0\rangle$ .

### 2.3.3 Pauli-Z (Phase Flip) Gate

The **Pauli-Z Gate**, also called the **Phase Flip Gate**, is one of the core **Pauli operators** used in quantum computing. It acts **only on the phase** of the qubit, **leaving the  $|0\rangle$  amplitude unchanged** but **flipping the sign of the  $|1\rangle$  amplitude**.

#### √\* Matrix Representation

$$Z = \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (7)$$

#### ≡ Action on a General Qubit

For a general state:

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

Applying  $Z$ :

$$Z|\psi\rangle = aZ|0\rangle + bZ|1\rangle = a|0\rangle - b|1\rangle$$

As a result,  $Z$  **flips the sign of the  $|1\rangle$  amplitude**, this is why it's called a **phase flip**. Finally, it's trivial to show that:

$$Z|0\rangle = |0\rangle \quad (\text{unchanged}), \quad Z|1\rangle = -|1\rangle \quad (\text{sign flip})$$

#### ❓ Geometric Interpretation: Rotation around the $z$ -axis

On the Bloch sphere, the Pauli-Z gate performs a  $\pi$  **radians ( $180^\circ$ ) rotation around the  $z$ -axis**. In spherical coordinates:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

After  $Z$  rotation:

- The angle  $\phi \rightarrow \phi + \pi$
- This phase shift **changes the sign of the complex amplitude for  $|1\rangle$**

Since **measurement probabilities depend on  $|a|^2$  and  $|b|^2$** , which **remain unchanged**, the  $Z$  gate **does not affect measurement outcomes** in the standard basis. It only affects **interference and future gate interactions**.

#### ≡ Interpretation and Use Cases

- **Phase Shift**:  $Z$  is a special case of a phase shift gate, shifting the phase by  $\pi$ .
- **No effect on  $|0\rangle$** : Useful for controlled operations, where only  $|1\rangle$  paths acquire a phase.

- **Circuit Simplification:** Often used in **error correction** and **phase-sensitive algorithms**.

### ≡ Key Properties

1. **Involution:**  $Z^2 = I$
2. **Diagonal Matrix:** Easy to compute and simulate.
3. **Commutates with Z and Phase gates**, but **anticommutes with X and Y**.

### 2.3.4 Pauli-Y Gate

The **Pauli-Y Gate**, denoted  $Y$ , is one of the three fundamental **Pauli matrices** in quantum mechanics. It is less intuitive than Pauli-X and Z, but equally important, especially for its role in complex rotations and quantum interference.

#### √\* Matrix Representation

$$Y = \sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Like the other Pauli gates,  $Y$  is **Hermitian** and **unitary**, which guarantees that it is **observable** (measurable) and **reversible**.



Given a qubit in state:

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

Apply the  $Y$  gate:

$$Y|\psi\rangle = aY|0\rangle + bY|1\rangle = a(-i|1\rangle) + b(i|0\rangle) = ib|0\rangle - ia|1\rangle$$

So the **amplitudes are swapped and multiplied by  $\pm i$** . This shows Pauli-Y **introduces a complex phase shift** alongside the swap.

#### 🔗 Geometric Interpretation: Rotation on the Bloch Sphere

The Pauli-Y gate corresponds to a **rotation around the  $y$ -axis by  $\pi$  radians** ( $180^\circ$ ). The effect on Bloch sphere coordinates:

- $\theta \rightarrow \pi - \theta$
- $\psi \rightarrow \pi - \psi$

This rotates the qubit across the  $y$ -axis, **altering both amplitudes and their phases**.

#### 📖 Pauli-Y as a Composite Gate

Interestingly,  $Y = iXZ$ . Let's verify this:

$$XZ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Now, multiply by  $i$ :

$$iXZ = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y$$

This identity confirms that **Pauli-Y can be viewed as a combination of Pauli-X and Pauli-Z** (page 28) **with a global phase factor  $i$** , which does not affect measurement outcomes.

### ≡ Key Properties of Y Gate

1. Involution:  $Y^2 = I$
2. Anticommutation:  $XY = -YX$
3. Phase Sensitive: introduces imaginary coefficients, unlike  $X$  and  $Z$ .

### 2.3.5 Phase Gate (S)

The **Phase Gate (S)** is a single-qubit gate that **adds a phase shift of  $\frac{\pi}{2}$  ( $90^\circ$ ) to the amplitude of the  $|1\rangle$  state**. It leaves  $|0\rangle$  **unchanged**, like the Pauli-Z gate, but instead of a sign flip ( $\pi$  phase), it **introduces a complex phase factor  $i$** .

#### √\* Matrix Representation

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad (8)$$

#### ☰ Action on a General Qubit

For a state:

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

Apply  $S$ :

$$S|\psi\rangle = a|0\rangle + ib|1\rangle$$

Only the  $|1\rangle$  **amplitude gains a phase of  $i$** , affecting interference but **not measurement probabilities** in the computational basis.

$$S|0\rangle = |0\rangle \quad (\text{unchanged}), \quad S|1\rangle = i|1\rangle \quad \left(\text{phase} + \frac{\pi}{2}\right)$$

#### 🕒 Geometric Interpretation: Rotation around $z$ -axis ( $\frac{\pi}{2}$ )

On the Bloch sphere,  $S$  performs a rotation of  $\frac{\pi}{2}$  radians around the  $z$ -axis. This rotation **shifts the phase angle  $\phi \rightarrow \phi + \frac{\pi}{2}$** , altering **how the qubit interferes** in later operations, especially when **Hadamard gates** are involved.

#### ↔ Relationship to Pauli-Z

Two applications of  $S$  equal  $Z$ :

$$S \cdot S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

Thus,  $S^2 = Z$ , this is a **useful identity** in circuit simplifications.

#### ✔ Applications and Significance

- **Superposition Control:**  $S$  **adjusts relative phases**, crucial in algorithms like Quantum Fourier Transform and Grover's Search.
- **Building Block:**  $S$  helps construct **higher-order phase gates**, such as the T gate, which introduces a  $\frac{\pi}{4}$  phase.
- **Error Correction:** Used in **stabilizer codes** for **fault-tolerant quantum computing**.



### ≡ Key Properties

1. **Unitary:**  $S^\dagger S = I$
2. **Not Hermitian:** Unlike Pauli gates,  $S \neq S^\dagger$
3. **Diagonal Matrix:** Efficient in simulation.

### 2.3.6 Hadamard Gate (H)

The **Hadamard Gate (H)** transforms a qubit from a definite state ( $|0\rangle$  or  $|1\rangle$ ) into a **superposition of both**. It is the **gateway to quantum parallelism**, allowing quantum computers to **process multiple possibilities simultaneously**.

#### √ Matrix Representation

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (9)$$

This matrix is **unitary** and **Hermitian**:  $H^\dagger = H$  and  $H^2 = I$ .

#### ≡ Action on Basis States

$$\begin{aligned} H|0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle \quad (\text{superposition state}) \\ H|1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle \end{aligned}$$

The  $|+\rangle$  and  $|-\rangle$  states are known as the **Hadamard basis**, or **superposition states**, with **equal probability amplitudes** for  $|0\rangle$  and  $|1\rangle$ .

#### 🌀 Bloch Sphere Interpretation

On the Bloch sphere, the **Hadamard gate performs two sequential rotations**:

1. Rotation around the  $y$ -axis by  $\frac{\pi}{2}$  radians ( $90^\circ$ ).
2. Followed by a rotation around the  $x$ -axis by  $\pi$  radians ( $180^\circ$ ).

This brings  $|0\rangle$  to the  $+x$ -axis ( $|+\rangle$ ) and  $|1\rangle$  to the  $-x$ -axis ( $|-\rangle$ ). As a result, **qubit moves from pole to equator**, entering **maximum superposition**.

#### ≡ Superposition and Measurement

After applying  $H$  to  $|0\rangle$ , the resulting qubit is:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Upon measurement:

- **Probability of 0:**

$$\left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

- **Probability of 1:**

$$\frac{1}{2}$$

This **equal probability** reflects **maximum uncertainty**, a hallmark of **quantum superposition**.

### ≡ Key Properties of H Gate

1. **Involution**:  $H^2 = I \rightarrow$  applying  $H$  **twice restores the original state**.
2. **Self-adjoint**:  $H = H^\dagger \rightarrow H$  is its own **inverse and adjoint**.
3. **Basis Change**:  $H$  converts between the **computational basis**  $\{|0\rangle, |1\rangle\}$  and the **Hadamard basis**  $\{|+\rangle, |-\rangle\}$ .

### ✓ Applications

- **Quantum Parallelism**: Prepares qubits for **simultaneous computation paths**.
- **Quantum Algorithms**: Found in Grover's Search, Quantum Fourier Transform, Shor's Algorithm.
- **Entanglement Creation**: Combined with CNOT,  $H$  creates Bell states.
- **Interference**: Enables **constructive and destructive interference** in algorithms.

## 2.4 Properties

Single-qubit gates exhibit a rich set of mathematical and geometric properties, all of which stem from their nature as unitary transformations on a two-level quantum system. Understanding these properties is key to **predicting gate behavior**, **designing quantum circuits**, and **analyzing quantum algorithms**.

### 1. All Quantum Gates Are Equivalent to Rotations

A profound insight in quantum computing is that **every single-qubit unitary gate corresponds to a rotation of the qubit's state vector on the Bloch sphere**.

- **Bloch Sphere** is a geometric representation of a qubit's state, where any point on the sphere represents a valid pure state.
- A **unitary operation** on qubit results in **rotating the vector** on this sphere **without changing its length** (preserving probability).

Some examples of rotations:

- **X Gate** → Rotation  $\pi$  around x-axis (section 2.3.2, page 26)
- **Z Gate** → Rotation  $\pi$  around z-axis (section 2.3.3, page 28)
- **Y Gate** → Rotation  $\pi$  around y-axis (section 2.3.4, page 30)
- **S Gate** → Rotation  $\frac{\pi}{2}$  around z-axis (section 2.3.5, page 32)
- **H Gate** → Rotation  $\frac{\pi}{2}$  around y, then  $\pi$  around x (section 2.3.6, page 34)

**🔗 Why is this important?** Quantum computation is fundamentally about **state rotations**, understanding gates as rotations helps visualize **interference**, **phase shifts**, and **entanglement creation**.

### 2. Reversibility: Applying a Gate Twice

Many single-qubit gates, especially the Pauli gates, exhibit the property of **involution**: applying the same gate twice **returns the qubit to its original state**. Mathematically it can be expressed as:

$$X^2 = Y^2 = Z^2 = I$$

This is due to the fact that **two 180° rotations around the same axis bring the vector back** to its original position on the Bloch sphere.

*Proof that X Gate has involution.*

$$X^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

These identities confirm that **Pauli gates are their own inverses**, and **all unitary gates are reversible**. QED

### 3. Global Phases Do Not Matter

Quantum gates may **introduce a global phase** (a constant complex factor  $e^{i\phi}$ ) to a state. Physically, **global phases are unobservable**, meaning they do **not affect measurement probabilities**. For example:

- States  $|\psi\rangle$  and  $e^{i\phi}|\psi\rangle$  are physically indistinguishable.
- The Y gate =  $iXZ$ , where  $i$  is a global phase that can be ignored in practice.

### 4. Composite Gates and Commutativity

- Gates can be **combined by matrix multiplication** (note: non commutative in general).
- **Order matters:**  $AB \neq BA$  for most gates.
- Exception: **Gate Z and Gate S are commuted** because they are both **diagonal phase gates**.

Property	Pauli Gates (X, Y, Z)	Phase Gate (S)	Hadamard (H)
Rotation Axis	x, y, z ( $\pi$ radians)	z ( $\frac{\pi}{2}$ radians)	y ( $\frac{\pi}{2}$ ) + x ( $\pi$ )
Involution $G^2 = I$	✓	✗ ( $S^2 = Z$ )	✓
Self-adjoint $G = G^\dagger$	✓	✗	✓
Unitary	✓	✓	✓
Phase Introduction	Z, S	$i$ for $ 1\rangle$	✓ (interference)
Creates Superposition?	✗ (X, Y, Z alone)	✗	✓

Table 1: Summary of Properties.

## 2.5 When Does a Gate Create Superposition?

In this section, our goal is to determine **what structural properties a quantum gate must have to create superposition** when applied to a **basis state** ( $|0\rangle$  or  $|1\rangle$ ).

Let's consider a generic  $2 \times 2$  gate  $A$ , given by:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$$

This matrix has a zero in the lower-left corner, suggesting no mixing from  $|0\rangle$  to  $|1\rangle$  via the first column. In other words:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow A|0\rangle = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ 0 \end{pmatrix}$$

That zero blocks (in the lower-left corner) blocks any  $|1\rangle$  component from arising when we apply  $A$  to  $|0\rangle$ . So no superposition is created from  $|0\rangle$ , it stays "pure". Note that to create a superposition, the output of  $A|0\rangle$  must contain both  $|0\rangle$  and  $|1\rangle$  components, so both entries are non-zero.

### Step 1: Apply $A$ to $|0\rangle$

$$A|0\rangle = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ 0 \end{pmatrix}$$

The output is purely  $|0\rangle$ , **no superposition**.

### Step 2: Apply $A$ to $|1\rangle$

$$A|1\rangle = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

We have both  $|0\rangle$  and  $|1\rangle$  components, **potential superposition**. But now the key insight: **superposition requires that both amplitudes (coefficients) are non-zero**. This only happens if  $a_{12} \neq 0$  and  $a_{22} \neq 0$ .

### Proof that Gate $A$ cannot create a superposition

*Proof that Gate  $A$  cannot create a superposition.* Quantum gates operate according to the principles of quantum mechanics, unitarity and reversibility. Therefore, Gate  $A$  must satisfy the following unitarity condition:

$$A^\dagger A = I$$

Let's simplify the discussion by assuming  $A$  is real, so  $A^\dagger = A^T$ . We compute  $A^T A$ :

$$\begin{aligned} A^T &= \begin{pmatrix} a_{11} & 0 \\ a_{12} & a_{22} \end{pmatrix} & A &= \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \\ A^T A &= \begin{pmatrix} a_{11}^2 & a_{11}a_{12} \\ a_{11}a_{12} & a_{12}^2 + a_{22}^2 \end{pmatrix} \end{aligned}$$

Set equal to identity:

$$A^T A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

From this, we get:

1.  $a_{11}^2 = 1 \Rightarrow a_{11} = \pm 1$
2.  $a_{11}a_{12} = 0 \Rightarrow a_{12} = 0$
3.  $a_{12}^2 + a_{22}^2 = 1 \Rightarrow a_{22} = \pm 1$

The superposition is eliminated by  $a_{12} = 0$ , because both columns must have non-zero elements. QED

In conclusion, **to create superposition, a gate must have non-zero elements in both columns**. In fact, gates like Hadamard H, with all non-zero elements, can create superposition. However, diagonal gates like Z or S cannot produce superposition.

## 2.6 Single-Qubit Quantum Circuits

A **Quantum Circuit** is a **mathematical model** for quantum computation, **composed of sequential operations applied to qubits**. These operations include initializations, unitary gates, and measurements. In this section, we focus on **circuits involving a single qubit**, which form the foundation for understanding multi-qubit systems.

### ≡ Circuit Elements

#### 1. Initialization

- Qubit is prepared in a **known state**, typically  $|0\rangle$  or  $|1\rangle$ .
- Essential **starting point** for any computation.

#### 2. Gates

- Single-qubit **unitary operations** (e.g., X, Z, H, S).
- Transform the qubit's state **reversibly**.

#### 3. Measurement

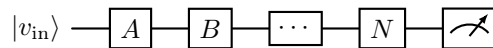
- **Irreversible process**.
- Collapses the qubit to  $|0\rangle$  or  $|1\rangle$  with probabilities dictated by the quantum state's amplitudes.

### ✂ Quantum Circuit Diagram: Visual Language

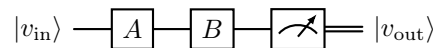
- **Horizontal lines** represent the **timeline of a single qubit**.

$$|v_{\text{in}}\rangle \text{ ————— } |v_{\text{out}}\rangle$$

- **Boxes** on the line represent **gates** (letters) or **measurements** (tachometer icon).



- **Double lines** (if present) represent **classical information** (e.g., measurement result).



- Left to right: The **circuit evolves over time** from **left (input)** to **right (output)**.



### Gate Sequence: Serial Gates

Consider two gates A and B applied in sequence:

$$|v_{\text{in}}\rangle \longrightarrow \boxed{A} \longrightarrow \boxed{B} \longrightarrow |v_{\text{out}}\rangle$$

1. First apply A, then B.
2. Mathematically, this is expressed as:

$$v_{\text{out}} = BAv_{\text{in}}$$

3. The **combined effect** of the two gates is equivalent to a single gate  $C = BA$ :

$$|v_{\text{in}}\rangle \longrightarrow \boxed{BA} \longrightarrow |v_{\text{out}}\rangle$$

$$v_{\text{out}} = Cv_{\text{in}} = BAv_{\text{in}}$$

**Matrix multiplication is from right to left**, which may seem counterintuitive. The **last gate applied** (B) is **on the left** of the product **BA**.

#### Example 1: Hadamard followed by Hadamard

Let's compute H followed by H to  $|0\rangle$ :

$$|0\rangle \longrightarrow \boxed{H} \longrightarrow \boxed{H} \longrightarrow |v_{\text{out}}\rangle$$

1. Apply H:

$$\begin{aligned} H|0\rangle &= \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \cdot \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ &= \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) = |+\rangle \end{aligned}$$

2. Applying H again, we **restore the original state due to the involution properties** ( $H^2 = I$ ):

$$H(H|0\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

## 2.7 Outer Product of Kets

In quantum mechanics, **Bra-Ket notation** (introduced by Dirac) is a concise and powerful tool for representing quantum states and operations. It uses **kets**  $|\psi\rangle$  for **vectors** (states), and **bras**  $\langle\psi|$  for their **Hermitian adjoints** (dual vectors).

Given two kets:

$$|a\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad |b\rangle = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

The **Outer Product**  $|a\rangle\langle b|$  is a **matrix**, formed by:

$$|a\rangle\langle b| = |a\rangle \otimes \langle b| = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} (\overline{b_1} \quad \overline{b_2}) = \begin{pmatrix} a_1\overline{b_1} & a_1\overline{b_2} \\ a_2\overline{b_1} & a_2\overline{b_2} \end{pmatrix} \quad (10)$$

Where the symbol  $\otimes$  is the **tensor product operator**. This operation **maps a vector to a matrix**.

The **Inner Product**  $\langle b|a\rangle$  is a scalar:

$$\langle b|a\rangle = \overline{b_1}a_1 + \overline{b_2}a_2 \quad (11)$$

So:

- **Inner product**  $\rightarrow$  **projection**, result is number.
- **Outer product**  $\rightarrow$  **matrix**, result is operator.

### Associativity between outer and inner products

$$|a\rangle \cdot \langle b|c\rangle = \langle b|c\rangle \cdot |a\rangle \quad (12)$$

This is a fundamental identity in quantum mechanics, showing how outer products interact with inner products.

*Proof the Associativity.* Let's define:

$$|a\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad |b\rangle = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad |c\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

1. Compute inner product  $\langle b|c\rangle$ :

$$\langle b|c\rangle = \overline{b_1}c_1 + \overline{b_2}c_2 = \alpha$$

Where  $\alpha$  is a scalar value.

2. Multiply scalar  $\alpha$  with  $|a\rangle$ :

$$\alpha |a\rangle = \alpha \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \alpha \cdot a_1 \\ \alpha \cdot a_2 \end{pmatrix} = \begin{pmatrix} (\overline{b_1}c_1 + \overline{b_2}c_2) \cdot a_1 \\ (\overline{b_1}c_1 + \overline{b_2}c_2) \cdot a_2 \end{pmatrix} = (\overline{b_1}c_1 + \overline{b_2}c_2) \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

3. Compute outer product  $|a\rangle\langle b|$  (matrix):

$$|a\rangle\langle b| = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} (\overline{b_1} \quad \overline{b_2}) = \begin{pmatrix} a_1\overline{b_1} & a_1\overline{b_2} \\ a_2\overline{b_1} & a_2\overline{b_2} \end{pmatrix}$$

4. Multiply  $|a\rangle\langle b|$  by  $|c\rangle$ :

$$(|a\rangle\langle b|)|c\rangle = \begin{pmatrix} a_1\bar{b}_1 & a_1\bar{b}_2 \\ a_2\bar{b}_1 & a_2\bar{b}_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1(\bar{b}_1c_1 + \bar{b}_2c_2) \\ a_2(\bar{b}_1c_1 + \bar{b}_2c_2) \end{pmatrix}$$

Factor out the scalar  $(\bar{b}_1c_1 + \bar{b}_2c_2)$ :

$$(\bar{b}_1c_1 + \bar{b}_2c_2) \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \langle b|c\rangle \cdot |a\rangle$$

This proves that **outer product applied to a ket is equivalent to inner product  $\langle b|c\rangle$  scaled by  $|a\rangle$** . QED

## 2.8 Measurement

In quantum computing, **measurement** is a special type of operation that differs fundamentally from unitary gates. While gates perform reversible transformations, **measurement is irreversible**. It allows us to **gain information about the state of a qubit**, but at the price of **collapsing the quantum state**.

### √\* Measurement Operator: Matrix Form

A **measurement operator** is a **non-unitary, non-invertible** matrix. For a measurement **along a specific direction**, represented by a ket  $|k\rangle$ , the associated **projector** is:

$$M_k = |k\rangle\langle k| \quad (13)$$

This is called a **Projection Operator**, it projects any vector **onto the direction of**  $|k\rangle$ .

### ⚠ Effect of Measurement

Given a qubit  $|\psi\rangle$ , applying  $M_k$  yields the **(unnormalized) projected vector**:

$$|\psi_k\rangle = M_k |\psi\rangle = |k\rangle \langle k|\psi\rangle = \langle k|\psi\rangle |k\rangle$$

To obtain the **new normalized state**, we **divide by the square root of the probability**:

$$|\psi_k^{\text{norm}}\rangle = \frac{M_k \cdot |\psi\rangle}{\sqrt{\langle \psi | M_k | \psi \rangle}} = \frac{\langle k|\psi\rangle \cdot |k\rangle}{\sqrt{|\langle k|\psi\rangle|^2}} = |k\rangle$$

So after measurement, the qubit **collapses** to  $|k\rangle$ , the **eigenstate corresponding to the measurement outcome**.

#### Remark: $\langle \psi | M_k | \psi \rangle$

This is called a quadratic form in linear algebra, and in quantum mechanics it represents the **probability of the measurement outcome**  $k$  when **measuring the qubit**  $|\psi\rangle$  with **measurement operator**  $M_k$ . Suppose a qubit  $\psi$ :

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{with } |a|^2 + |b|^2 = 1$$

If we apply the measurement operator  $M_0$ :

$$M_0 |\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$$

And we finally calculate the bra  $\langle \psi |$ :

$$\langle \psi | \begin{pmatrix} a \\ 0 \end{pmatrix} = (\bar{a} \quad \bar{b}) \begin{pmatrix} a \\ 0 \end{pmatrix} = \bar{a} \cdot a = |a|^2$$

In other words, this is the probability of measuring 0:  $p_0$ .

### ✂ Probability of Outcome

The **probability of measuring the qubit in the state  $|k\rangle$**  is:

$$p_k = \langle \psi | M_k | \psi \rangle = |\langle k | \psi \rangle|^2 = |\overline{k}_1 \cdot \psi_1 + \overline{k}_2 \cdot \psi_2|^2 \quad (14)$$

This is the **Born rule**, a fundamental postulate of quantum mechanics. In quantum computing we have:

$$\begin{aligned} p_0 &= \langle \psi | M_0 | \psi \rangle = |\langle 0 | \psi \rangle|^2 = |1 \cdot \psi_1 + \cancel{0 \cdot \psi_2}|^2 \\ p_1 &= \langle \psi | M_1 | \psi \rangle = |\langle 1 | \psi \rangle|^2 = |\cancel{0 \cdot \psi_1} + 1 \cdot \psi_2|^2 \end{aligned}$$

### ☐ Standard Basis Measurement

In quantum computing, we usually measure in the **computational basis**  $\{|0\rangle, |1\rangle\}$ . The **projectors** are:

$$\begin{aligned} M_0 &= |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ M_1 &= |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \quad (15)$$

For a **general qubit**  $|\psi\rangle = a|0\rangle + b|1\rangle$ , the **probabilities** are:

$$p_0 = |a|^2 \quad p_1 = |b|^2 \quad (16)$$

After measurement, the qubit **collapses to  $|0\rangle$  with probability  $|a|^2$** , or to  **$|1\rangle$  with probability  $|b|^2$** .

### ★ Key Properties of Measurement Operator

- **Idempotent**: once applied, **applying it again does nothing**:

$$M_k^2 = M_k$$

A simple proof:

$$M_k^2 = |k\rangle\langle k| \langle k| \langle k| = |k\rangle\langle k| = M_k$$

This reflects that **after projection, the qubit is already in the state  $|k\rangle$** .

- **Non-unitary** and **Non-reversible**: measurement **destroys superposition** and **cannot be undone**.

### 3 Multiple Qubit Gates

#### 3.1 Multiple Qubit States

In quantum computing, the description of systems involving multiple qubits relies on a mathematical operation known as the **tensor product**. This operation enables us to **formally construct the joint state of two or more qubits starting from their individual states**.

##### ★ The Tensor Product

Suppose we have two qubits, each in its own superposition:

$$|v_A\rangle = a_0 |0\rangle + a_1 |1\rangle \quad \text{and} \quad |v_B\rangle = b_0 |0\rangle + b_1 |1\rangle$$

Our **goal** is to **describe the joint state of the system**, i.e., the probability amplitudes for:

- Both qubits being in state  $|0\rangle$
- Qubit  $A$  in  $|0\rangle$  and qubit  $B$  in  $|1\rangle$
- Qubit  $A$  in  $|1\rangle$  and qubit  $B$  in  $|0\rangle$
- Both in  $|1\rangle$

To construct the combined state mathematically, we use the **Tensor Product**  $\otimes$ . Give the vectors of  $v_A$  and  $v_B$ :

$$|v_A\rangle = a_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \quad |v_B\rangle = b_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

The tensor product  $|v_A\rangle \otimes |v_B\rangle$  gives:

$$\begin{aligned} |v_A\rangle \otimes |v_B\rangle &= (a_0 |0\rangle + a_1 |1\rangle) \otimes (b_0 |0\rangle + b_1 |1\rangle) \\ &= \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \otimes \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \\ &= a_0 b_0 |00\rangle + a_0 b_1 |01\rangle + a_1 b_0 |10\rangle + a_1 b_1 |11\rangle \\ &= a_0 b_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_0 b_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \\ &\quad a_1 b_0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_1 b_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= a_0 b_0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_0 b_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + a_1 b_0 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + a_1 b_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{bmatrix} \end{aligned} \tag{17}$$

This four-dimensional vector fully describes the joint system of two qubits in the computational basis. The tensor product **preserves the linearity** of the system and ensures that the **resulting state is still a valid quantum state**.

About the notation, the ket  $|00\rangle$  can be explicitly rewritten as the tensor product  $|0\rangle \otimes |0\rangle$ . These two notations are the same.

### % Measurement Probabilities

Each coefficient in the final four-dimensional vector corresponds to an **amplitude for a specific measurement outcome**. For instance:

- The probability of observing  $|00\rangle$  is  $|a_0b_0|^2$
- The probability of observing  $|01\rangle$  is  $|a_0b_1|^2$
- The probability of observing  $|10\rangle$  is  $|a_1b_0|^2$
- The probability of observing  $|11\rangle$  is  $|a_1b_1|^2$

This reflects the Born rule: the probability of an outcome is the square modulus of the corresponding amplitude (page 45).

### ≡ Normalization of Two-Qubit Quantum State

The normalization of a quantum state means that the **total probability of all possible outcomes must sum to 1**. The reason we need normalization is due to the **Born rule**: the probability of measuring a quantum state  $|\psi\rangle$  and finding it in basis state  $|i\rangle$  is  $|c_i|^2$ . Since total probability must equal 1, we must have:

$$\sum_i |c_i|^2 = 1$$

*Normalization of two-qubit quantum state.* Given two normalized single-qubit states:

$$|v_A\rangle = a_0|0\rangle + a_1|1\rangle \quad |v_B\rangle = b_0|0\rangle + b_1|1\rangle$$

We know that (because they are normalized, and the total probability of all possible outcomes must sum to 1):

$$|a_0|^2 + |a_1|^2 = 1 \quad |b_0|^2 + |b_1|^2 = 1$$

The combined state, via the tensor product, is:

$$|v_{AB}\rangle = a_0b_0|00\rangle + a_0b_1|01\rangle + a_1b_0|10\rangle + a_1b_1|11\rangle$$

The question is: **does this state remain normalized?** We need to verify that the sum of the squared moduli (i.e., squared magnitudes) of the coefficients equals 1:

$$|a_0b_0|^2 + |a_0b_1|^2 + |a_1b_0|^2 + |a_1b_1|^2 = 1$$

We use the property of modulus:

$$|a_ib_j|^2 = |a_i|^2 \cdot |b_j|^2$$

Hence:

- $|a_0 b_0|^2 = |a_0|^2 |b_0|^2$
- $|a_0 b_1|^2 = |a_0|^2 |b_1|^2$
- $|a_1 b_0|^2 = |a_1|^2 |b_0|^2$
- $|a_1 b_1|^2 = |a_1|^2 |b_1|^2$

We regroup the terms:

$$|a_0|^2 (|b_0|^2 + |b_1|^2) + |a_1|^2 (|b_0|^2 + |b_1|^2)$$

Factor out:

$$(|a_0|^2 + |a_1|^2) \cdot (|b_0|^2 + |b_1|^2)$$

Since both  $v_A$  and  $v_B$  are normalized:

$$|a_0|^2 + |a_1|^2 = 1 \quad |b_0|^2 + |b_1|^2 = 1$$

So:  $1 \cdot 1 = 1$ .

QED

The combined two-qubit state  $v_{AB} = v_A \otimes v_B$  is indeed **normalized**, provided the original qubits are normalized. This confirms that the **tensor product of two normalized states is itself a normalized state**, a crucial result that ensures consistency.



### 3.2 Introduction to Multiple Qubit Gates

In quantum computing, the transition from single-qubit to multi-qubit systems represents a fundamental leap in computational expressiveness. While single-qubit gates manipulate isolated quantum bits, the true power of quantum computation emerges only when we consider operations on multiple qubits. This is because **multi-qubit gates allow for the creation and manipulation of quantum entanglement**, a uniquely quantum phenomenon with no classical analog, and a critical resource for quantum advantage.

The behavior of multi-qubit systems is governed by the **tensor product** structure of quantum mechanics. That is, the state space of a system composed of multiple qubits is described by the tensor product of the state spaces of the individual qubits. For example, a system of two qubits is represented by a vector in a four-dimensional Hilbert space  $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$ , and similarly, an  $n$ -qubit system is described by a vector in  $\mathbb{C}^{2^n}$ . This **exponential growth in state space dimensionality is one of the key reasons why quantum computers can, in principle, outperform classical ones for certain tasks.**

**Multi-qubit gates** are therefore defined as **unitary transformations acting on these higher-dimensional vector spaces**. A simple extension of single-qubit operations to the multi-qubit context is given by applying gates in a factorized way (e.g., applying a Hadamard gate on the first qubit and an identity on the second corresponds to  $H \otimes I$ ). However, more powerful and interesting operations arise when gates do not decompose as tensor products of single-qubit gates. These non-separable gates, such as the Controlled-NOT (CNOT) gate, are capable of generating entangled states, and as such, are essential for universal quantum computation.

Thus, the **study of multi-qubit gates** is not merely a generalization of single-qubit logic; it is the **gateway to fundamentally quantum phenomena**. Understanding these gates, their representations, and their implications is central to designing quantum algorithms and circuits that go beyond classical capabilities.

### 3.3 Controlled NOT (CNOT) Gate

The **Controlled-NOT (CNOT)** gate is one of the most fundamental and widely used **two-qubit gates** in quantum computing. It serves as a canonical example of a **quantum gate that cannot be decomposed into single-qubit operations**, and it plays a central role in the generation of entanglement, error correction, and universal gate constructions.

#### ✂ How does it work?

The CNOT gate operates on two qubits:

- The **control qubit**  $|c\rangle$
- The **target qubit**  $|t\rangle$

Its defining action is **conditional flipping of the target qubit**. If the control qubit is in the state:

- $|1\rangle$ , the CNOT gate **applies the Pauli-X (NOT) gate** (page 26) to the target (so the target qubit flips and 0 becomes 1 and 1 becomes 0).
- $|0\rangle$ , the **target is left unchanged**.

This behavior can be expressed algebraically as:

$$\text{CNOT } |c\rangle |t\rangle = |c\rangle |t \oplus c\rangle \quad (18)$$

Where:

- Qubit Input: two-qubit basis state  $|c\rangle|t\rangle$  ( $|c\rangle |t\rangle = |ct\rangle$ ).
- CNOT operator: if  $c = 0$ , leave  $t$  unchanged, otherwise, flip  $t$ .
- XOR  $\oplus$  symbol: denotes addition modulo 2.

$$t \oplus c = \begin{cases} 0 & \text{if } t = c \\ 1 & \text{if } t \neq c \end{cases}$$

It flips  $t$  if and only if  $c = 1$ . The **Addition modulo 2**, often denoted as  $\oplus$ , is a simple binary operation where the result is the remainder after dividing the sum of two bits by 2. It's also known as the **exclusive OR (XOR)** operation.

$a$	$b$	$a \oplus b$
0	0	0
0	1	1
1	0	1
1	1	0

So,  $t \oplus c$  means: “flip the target qubit  $t$  if and only if the control qubit  $c$  is 1”.

- The control qubit  $|c\rangle$  stays the same.
- The target qubit  $|t\rangle$  becomes  $|t \oplus c\rangle$  (flipped if  $c = 1$ ).

### √ Matrix Representation

The matrix representation of the CNOT gate is:

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (19)$$

It operates on **two-qubit states** ordered as:

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

This matrix is clearly **unitary**, preserving the norm of quantum states, and it **interchanges the basis** vectors  $|10\rangle$  and  $|11\rangle$  while **leaving**  $|00\rangle$  and  $|01\rangle$  **unchanged**, consistent with its definition.

For example, let a general two-qubit quantum state written as a linear combination of basis states:

$$|\psi\rangle = c_0 |00\rangle + c_1 |01\rangle + c_2 |10\rangle + c_3 |11\rangle$$

Where:

$$c_0, c_1, c_2, c_3 \in \mathbb{C} \quad \text{and} \quad \sum |c_i|^2 = 1$$

Now apply the CNOT gate:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_3 \\ c_2 \end{bmatrix}$$

The amplitudes of the  $|10\rangle$  and  $|11\rangle$  components have been swapped. This corresponds to flipping the target qubit only when the control is  $|1\rangle$ :

$$|\psi\rangle = c_0 |00\rangle + c_1 |01\rangle + c_2 |11\rangle + c_3 |10\rangle$$

### ✓ Who is the Control Qubit?

We have discussed that the CNOT gate operates on two qubits: control  $|c\rangle$  and target  $|t\rangle$ . In the operations, *how can we understand which is the control qubit or the target qubit?* By **convention**, the **control qubit is the most significant qubit**, i.e., the **leftmost one** in the tensor product  $|c t\rangle$ . So:

- $|00\rangle$ : control = 0, target = 0
- $|01\rangle$ : control = 0, target = 1
- $|10\rangle$ : control = 1, target = 0
- $|11\rangle$ : control = 1, target = 1

Thus, in a state like:

$$|\psi\rangle = c_0 |00\rangle + c_1 |01\rangle + c_2 |10\rangle + c_3 |11\rangle$$

The **first qubit is the control**, and the **second is the target**.

### Generate Entanglement

An important aspect of the CNOT gate is its **ability to generate entanglement**. For instance, if the control qubit is prepared in a superposition state using a Hadamard gate  $H$ , such that the input to the CNOT is:

$$(H \otimes I) |00\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |10\rangle)$$

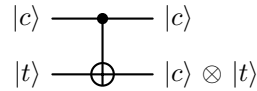
Then applying the **CNOT produces the entangled Bell state**:

$$\text{CNOT} \left( \frac{1}{\sqrt{2}} (|00\rangle + |10\rangle) \right) = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$$

This outcome exemplifies how **CNOT combined with single-qubit gates can prepare maximally entangled states**, a prerequisite for many quantum protocols including teleportation and superdense coding.

### Visual representation of a CNOT

Finally, in quantum circuit diagrams, the CNOT is represented with a solid dot on the control qubit and a  $\oplus$  symbol on the target qubit, connected by a vertical line. This visual notation helps emphasize the logical dependence between the two qubits.



## References

- [1] Cremonesi Paolo. Quantum computing. Slides from the HPC-E master's degree course on Politecnico di Milano, 2024.

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